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INSTANTONS ON $n\mathbb{CP}_2$

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0. Introduction

On a complex surface equipped with an Hermitian metric the splitting of the 2-forms into self-dual and anti-self-dual components is compatible with the splitting into forms of different types induced by the complex structure: $\Lambda^2_+ \otimes \mathbf{C} = \Lambda^{0,2} \oplus \Lambda^{2,0} \oplus \omega\Lambda^{0,0}$, and $\Lambda^2_- \otimes \mathbf{C} = \ker \omega\wedge : \Lambda^{1,1} \to \Lambda^{2,2}$, where ω is the positive $(1, 1)$-form defined by the metric and the complex structure. Thus a connection with anti-self-dual curvature on a unitary bundle over such a surface automatically acquires a compatible holomorphic structure by the Newlander-Nirenberg theorem. It is this key fact which underlies Donaldson's result [12] showing the equivalence of moduli of anti-self-dual connections and stable holomorphic bundles on an algebraic surface, a result of central importance in the evolving gauge-theoretic study of smooth 4-manifolds.

It is perhaps less well-known that the same fact can be used to describe moduli of *self*-dual Yang-Mills connections ("instantons") on oriented 4-manifolds without complex structures: let $\widetilde{\mathbb{C}}^2$ denote a modification of the complex plane consisting of n blow-ups and let ω be a positive $(1, 1)$-form on this space. An ω-anti-self-dual solution of the Yang-Mills equations is then a holomorphic bundle with hermitian connection whose curvature F satisfies $\omega \wedge F = 0$. If the solution has finite L^2 action and ω is suitable asymptotically flat, the bundle and connection extend to the one-point compactification by Uhlenbeck's theorem [30]. Since this one-point compactification is diffeomorphic to a connected sum of n copies of the reverse-oriented complex projective plane, flipping the orientation yields a self-dual solution of the Yang-Mills equations on this last space, that is, an instanton on $n\mathbb{CP}_2$.

There is a smooth orientation-reversing map $\bar{\pi} : \widetilde{\mathbb{P}}_2 \to n\mathbb{CP}_2$ collapsing the line L_∞ at infinity to a point y_∞ (an "antiholomorphic blow-down"). Under this map the instanton on $n\mathbb{CP}_2$ pulls back to an extension of the

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holomorphic bundle and connection on $\widetilde{\mathbb{C}}^2$ to $\widetilde{\mathbb{P}}_2$, and since the restriction of the connection to L_∞ is flat, the bundle is holomorphically trivial there. Thus there is a correspondence between instantons on $n\mathbb{CP}_2$ and certain holomorphic bundles on the blown-up complex projective plane, and the question then arises as to exactly which bundles occur in the latter category. This is answered by the following theorem, which is the main result of this paper:

Theorem 0.1. *Let X be a compact complex surface biholomorphic to a blow-up of $\mathbb{P}_2$ n times, and let $L_\infty \subset X$ be a rational curve with self-intersection $+1$. Let Y be a smooth 4-manifold diffeomorphic to $n\mathbb{CP}_2$ obtained by collapsing L_∞ to a point $y_\infty \in Y$ and reversing the orientation, and let $\overline{\pi} : X \to Y$ be the collapsing map. If g is any smooth metric on Y such that $\overline{\pi}^* g$ is compatible with the complex structure on X, then there is a one-to-one correspondence between*

(1) *equivalence classes of g-self-dual Yang-Mills connections on a unitary bundle E_{top} over Y, and*

(2) *equivalence classes of holomorphic bundles E on X topologically isomorphic to $\overline{\pi}^* E$ whose restriction to L_∞ is holomorphically trivial and is equipped with a compatible unitary structure.*

(As in [6], a unitary structure on a holomorphic bundle B over L_∞ is a holomorphic isomorphism $\phi : B \to \sigma^* \overline{B}^*$ where $\sigma : L_\infty \to L_\infty$ is a fixed-point-free antiholomorphic involution (the antipodal map). ϕ must satisfy $(\sigma^* \overline{\phi})^* = \phi$ and induce a positive form on holomorphic sections of B over L_∞.)

In the case where $n = 0$ or 1, the theorem has already been proved in [11] and [19] respectively, at least for the standard metrics on S^4 and $\mathbb{CP}_2$. These metrics have self-dual Weyl curvature, so the twistor spaces for them are integrable and the instantons correspond to certain holomorphic bundles on the twistor spaces [2]. Techniques of complex analysis can be used to classify these bundles [1], [6], [13] which can then be compared directly with classifications of bundles on $\mathbb{P}_2$ and its blow-up at a point [4], [7] to arrive at the result above (further details are given in §3).

In general, it is known that $n\mathbb{CP}_2$ admits self-dual metrics for any n [14], [21], [24], [25]. For some of those metrics there is a complex hypersurface in the twistor space biholomorphic to $\widetilde{\mathbb{P}}_2$, and the holomorphic bundles of the theorem are then just the restriction of the bundles on the twistor space to the hypersurface, as in the case that $n = 0, 1$. Of course, a self-dual metric is certainly not required here, and it should be noted that there are many metrics satisfying the hypotheses of the theorem: for example, choose any metric which is conformally flat in a neighborhood of y_∞, so the twistor space for this metric is integrable near L_∞ and the pullback of the metric is certainly compatible with the complex structure there. The corresponding Hermitian form can then be glued to any other on the rest of the complex surface using a cut-off function.

Note that Theorem 0.1 implies that all instanton moduli spaces associated with any such metric must be smooth away from reducible connections: by Serre duality $H^2(\operatorname{End} E) = H^0(\operatorname{End} E \otimes K)^*$ for any holomorphic bundle E, where K is the canonical bundle. If E is trivial on L_∞ then it is trivial on all nearby lines, and since $K \simeq \mathcal{O}(-3)$ there, section of $H^0(\operatorname{End} E \otimes K)$ must vanish on all such nearby lines; this implies it vanishes on an open set and hence is identically zero.

The theorem could equally well be stated in terms of *based* connections and bundles as in [11] and [19], but it turns out that for the analytical proof given here (as opposed to the algebraic approach of those references) the statement given is slightly more natural. As in [13] and [6], it is a simple matter to incorporate other gauge groups into the description having once dealt with the case of unitary instantons.

It will be apparent from the proof that the techniques used here should be applicable to a wider range of situations than just blow-ups of the complex plane. However, it should also be noted that a complex surface with one end biholomorphic to the complement of a compact set in $\mathbb{C}^2$ can be compactified by adding a $\mathbb{P}_1$ at infinity to give a compact surface containing a rational curve with self-intersection $+1$. By V.4.3 of [5], such a surface must be a blow-up of either a Hirzebruch surface or $\mathbb{P}_2$, and in fact it is not hard to see that such a surface must be exactly a blow-up of $\mathbb{P}_2$.

In one direction the proof of theorem 0.1 is as indicated earlier: the pullback of a self-dual connection A on a bundle over $Y = n\mathbb{CP}_2$ is an anti-self-dual connection with respect to $\overline{\pi}^* g$, and because the latter metric is compatible with the complex structure and is nondegenerate off L_∞ the curvature of $\overline{\pi}^* A$ is of type $(1,1)$ off L_∞, hence everywhere. The pullback connection is flat on L_∞ so a trivialization at $y_\infty \in Y$ pulls back to a holomorphic trivialization and a natural unitary structure along L_∞. The bulk of the proof is thus to construct an anti-self-dual connection on a holomorphic bundle over $X = \widetilde{\mathbb{P}}_2$, where anti-self-duality is with respect to the degenerate metric $\overline{\pi}^* g$ (since a smooth connection on a bundle over Y which is anti-self-dual with respect to $\overline{\pi}^* g$ off L_∞ and which is flat along that line is, after a unitary change of gauge, the pullback of a self-dual connection from Y by the removable singularities theorem [30]).

The approach used here is a modification of that which was used in [8], i.e., minimizing a useful function measuring the size of the self-dual curvature. Because of the degeneracy of the metric, the analysis is performed on the noncompact manifold $Z = \widetilde{\mathbb{P}}_2 - L_\infty = \widetilde{\mathbb{C}}^2$. This change to a noncompact setting has both benefits and costs: on the one hand, the nonlinear part of the problem is greatly simplified by virtue of the fact that it is not hard to obtain a priori bounds preventing curvature from "bubbling-off"; on the other, the linear part of the theory is more delicate and it is fortunate that it is possible to fall back on the work of Lockhardt and McOwen [22] dealing precisely with elliptic theory in this type of context; the works of Uhlenbeck [29], [30], [15] and of Taubes [28] also provide key results.

In §1 below, notation is established, preliminary material is introduced, and the linear part of the proof of the main result is given; §2 deals with the nonlinear part. On the whole, it is assumed in these sections that the reader is familiar with [12] or [8] or some other such paper concerning the existence of Hermitian-Einstein connections on stable bundles, of which there are now a number in the literature. The third section is devoted to providing some examples, including an explicit computation of the moduli space $\mathscr{M}$ of SU(2) 1-instantons on $n\mathbb{CP}_2$.

The referee has pointed out that results similar to and more general than those presented here have recently been obtained by Bando [3].

1. Proof of Theorem 0.1: Linear aspects

If Z is a complex surface equipped with a positive $(1, 1)$-form ω, the problem of constructing an ω-anti-self-dual Hermitian connection on a holomorphic bundle E over Z can be treated equivalently as the problem of finding an anti-self-dual unitary connection on a smooth unitary bundle E_{top} such that the $(0, 1)$ component of the connection induces the holomorphic structure E. This is the approach taken in both [12] and [8] where more details can be found. Given one integrable connection A_0 inducing E, every other such connection A is determined by a unique complex automorphism g of E_{top} with $\bar\partial_A = g^{-1} \circ \bar\partial_0 \circ g$ and $\partial_A = g^* \circ \partial_0 \circ g^{*-1}$, usually written as $A = g \cdot A_0$. If $h := g^* g$, the curvature of A has the form $F(g \cdot A_0) = g[F(A_0) + \bar\partial_0(h^{-2}\partial_0 h)]g^{-1}$, and the substitution $g \mapsto ug$, $g \mapsto gu$ for unitary u amount to unitary changes of gauge of A, A_0 respectively. To find an anti-self-dual Hermitian connection on E is to therefore to solve the equation

$$\omega \wedge [F(A_0) + \overline{\partial}_0(h^{-1}\partial_0 h)] = 0. \tag{1.1}$$

If Z has finite volume $\frac{1}{2}\int_Z \omega^2$ and the entities on the left of (1.1) can be integrated, various obstructions become apparent: if ω is Kähler there is a topological obstruction coming from the degree of the first Chern class, and of a more subtle nature, there is a holomorphic obstruction which is the condition of *stability* (cf. [13], [8]). An indecomposable bundle on a blow-up of $\mathbb{P}_2$ trivial on L_∞ can be far from stable with respect to any Hermitian metric on the blow-up (though it should be added that some notion of stability does lurk in the background in that the direct image of any such bundle onto $\mathbb{P}_2$ is always a semistable torsion-free sheaf; see [23, p. 210]). This absence of stability is the main reason for performing the analysis on manifolds with infinite volume.

If $s = s^* \in \Gamma(\operatorname{End} E_{\text{top}})$ and $g_t := 1 + ts$ is invertible for small t then $F(g_t \cdot A_0) = F(A_0) + t(\overline{\partial}_0\partial_0 - \partial_0\overline{\partial}_0)s + O(t^2)$ so the linearization at A_0 of the operator implicit in (1.1) is given by $s \mapsto R_0 s := i *_\omega \omega \wedge (\overline{\partial}_0\partial_0 - \partial_0\overline{\partial}_0)s$. Once appropriate function spaces have been found between which this operator acts as an isomorphism, solving (1.1) boils down to an exercise in techniques of nonlinear analysis; heat equation or continuity method approaches should work just as well as the functional-minimizing method used here.

Having dealt with these preliminary observations, the task of setting up the initial conditions for the body of this paper can be commenced.

Let g be a smooth metric on $Y = n\mathbb{CP}_2$ with the properties required by Theorem 0.1. Fix geodesic normal coordinates $\{y^a\}$ in a neighborhood U of y_∞ so the blowing-up map $\overline{\pi}$ is given by inversion $y^a \mapsto y^a/|y|^2$, taking $U - \{y_\infty\}$ onto a deleted neighborhood $\overline{U}$ of the line L_∞ in $X = \widetilde{\mathbb{P}}_2$. According to the discussion of pp. 121–122 of [15], if (r, θ) are corresponding geodesic polar coordinates and $r =: \exp(-\tau)$, the rescaled metric $\tilde{g} = g/r^2$ is asymptotic to the standard product metric on the cylinder $\mathbb{R}_+ \times S^3$, and the smoothness of g at y_∞ implies that all derivatives of $\tilde{g}$ with respect to (τ, θ) have exponential decay as $\tau \to \infty$. Since the complex structure defining $Z = \widetilde{\mathbb{C}}^2 \subset X$ extends smoothly across L_∞, the same exponential decay is enjoyed by the derivatives of the Kähler form $\tilde{\omega}$ determined by $\tilde{g}$ and this complex structure. If $\omega := \tilde{\omega}/r^2$ is the corresponding positive $(1, 1)$ form on Z, then it follows that holomorphic coordinates (z^0, z^1) on $\overline{U} - L_\infty$ can be found so that $\lim_{|z|\to\infty} |z|^k |d_0^k(\omega - \omega_0)| = 0$ for any k, where $|z|^2 := |z^0|^2 + |z^1|^2$,

$\omega_0 := \frac{i}{2}\partial\overline{\partial}|z|^2$ and d_0 denotes full covariant exterior differentiation with respect to the flat metric in these coordinates.

This decay is somewhat stronger than is required for the purposes of the proof and indeed will be weakened somewhat. If ω_0 is a fixed Kähler metric on Z, equal to (the pullback of) $\frac{i}{2}\partial\overline{\partial}|z|^2$ on the complement of a compact set, the following pair of conditions

$$\lim_{|z|\to\infty}[|\omega-\omega_0|+|z|\,|d_0\omega|+|z|^2|d_0^2\omega|]=0 \tag{1.2}$$

and

$$\lim_{|z|\to\infty}|z|^{2+\epsilon_0}i\overline{\partial}\partial\omega=0 \quad \text{for some } \epsilon_0>0. \tag{1.3}$$

By the discussion of the last paragraph, both conditions are certainly satisfied for a form ω obtained from a smooth metric on Y satisfying the hypotheses of Theorem 0.1. Unless otherwise stated, all norms used here will be those induced by the metric ω rather than ω_0 (throughout, terminology is abused by identifying Hermitian metrics and positive $(1,1)$-forms).

In addition to the metrics, some initial conditions must also be established for the connections. Let A_0 be a smooth integrable unitary connection on a bundle $E=E_{\text{top}}$ over Z and suppose that A_0 has finite L^2 action $\int_Z|F(A_0)|^2\omega^2$. Using the technique of broken Hodge gauges [30], the finite action condition enables the construction of a gauge for the connection such that both the bundle and the gauged connection extend to the 1-point compactification $\overline{Y}=n\overline{\mathbb{P}}_2$ of Z—see [27] (of course, $\overline{Y}$ is simply Y with the opposite orientation, but it is helpful to retain this notation as it suggests compactification, the noncompact manifold Z now being regarded as the fundamental manifold). The extended connection is only L_1^2 in a neighborhood of the point at infinity though if A_0 satisfies the Yang-Mills equations (at least in a neighborhood of infinity), elliptic regularity implies smoothness of the extension (see [29]). It will be assumed until further notice that the connection A_0 satisfies the condition

$$\int_Z|\widehat{F}(A_0)|^p(1+|z|^2)^\gamma\omega_0^2<\infty \quad \text{for some } p>4 \text{ and } \gamma>p-2, \tag{1.4}$$

where $\widehat{F}:=*_\omega\,\omega\wedge F$; this will provide sufficient regularity in a neighborhood of infinity for the purposes required here, and is a condition which is easily removed at the end of the proof.

Now set $\varphi:=(1+|z|^2)^{1/2}$ and for $p>0$, $\delta\in\mathbb{R}$ introduce the weighted Sobolev spaces $L^p_{k,\delta}$ which are by definition the completion of $C_0^\infty(Z)$

under the norm

$$(1.5) \qquad \|f\|_{L^p_{k,\delta}} := \left\{ \sum_{\alpha \le k} \int_Z (\varphi^{\alpha+\delta} |d_0^\alpha f|)^p \varphi^{-4} \omega_0^2 \right\}^{1/p} .$$

(After changing to polar coordinates in $\mathbb{C}^2$ and substituting $\tau = \log r$ for $r \geq 1$ as before, the pullback of $\varphi^{-2}\omega_0$ agrees with the standard product metric $d\tau^2 + d\theta^2$ on the cylinder $[1, \infty) \times S^3$, so the sum in (1.5) is equivalent to

$$\sum_{\alpha \le k} \int_{S^3 \times [1, \infty)} |\tilde{d}_0^\alpha f|^p e^{p\delta\tau} \, d\tau \, dV_{S^3}$$

with $|\cdot|$ and $\tilde{d}_0$ determined by the standard product metric.) In terms of these weighted norms, the hypotheses on $F(A_0)$ are equivalent to the conditions $|F(A_0)| \in L^2_{0,2}$ and $|\widehat{F}(A_0)| \in L^p_{0,2+\delta}$ for some $p > 4$ and $\delta > 0$.

Analogues of the usual Sobolev inequalites for these spaces are given in Lemma 5.2 of [28], where it is shown that if $\delta > 0$ then $L^p_{1,\delta}$ is continuously embedded in $L^q_{0,\delta}$ for $p \in [2, 4)$ and $q = 4p/(4-p)$ and that if $p > 4$ then $\lim_{|z| \to \infty} |\varphi^\delta f| = 0$ for any function $f \in L^p_{1,\delta}$. Moreover, Taubes also proves that if $df \in L^p_{0,1+\delta}$ for some $p \geq 2$ then there is a constant $\tilde{f} \in \mathbb{C}$ such that $f - \tilde{f} \in L^p_{1,\delta}$; this will be of considerable use subsequently.

For current purposes, the "standard" metric on $\overline{Y}$ is (that induced by) $\varphi^{-4}\omega_0$ agreeing with the standard metric on S^4 in a neighborhood of infinity. Using the corresponding Riemannian volume form, it is straightforward to show that there are constants $C(p, \delta) > 0$ such that

$$\|f\varphi^{-1+\delta+4/p}\|_{L^p_1(\overline{Y})} \leq C(p, \delta) \|f\|_{L^p_{1,\delta}}, \qquad f \in C_0^\infty(Z).$$

Since $\|f\varphi^\alpha\|_{L^q(\overline{Y})} = \|f\|_{L^q_{0,\alpha-4/q}}$ it follows from the usual Sobolev embedding theorem for $\overline{Y}$ that if $p > 4$ then $f \mapsto \varphi^{-1+\delta+4/p} f$ defines a compact mapping $L^p_{1,\delta} \to C^0(\overline{Y})$. For $p < 4$ and $q \leq 4p/(4-p)$ the embedding theorem gives a continuous inclusion $L^p_{1,\delta} \to L^q_{0,-1+\delta+4/p-4/q}$, and this is a compact mapping if $q < 4p/(4-p)$. Since any negative power of φ is integrable with respect to $\varphi^{-4}\omega_0^2$, it follows from Hölder's inequality that $L^p_{0,\delta} \subset L^{p'}_{0,\delta'}$ whenever $p \geq p'$ and $\delta > \delta'$ (for any p, δ), and therefore if $1 < p < 4$, $1 < q < 4p/(4-p)$, and $\delta' < \delta$ then $L^p_{1,\delta}$ is compactly embedded in $L^q_{0,\delta'}$.

Before placing appropriate Sobolev structures on sections of E and its associated bundles, it is convenient to pick a useful gauge for the initial connection A_0, as mentioned above. By Corollary 23 of [12], the L^2 bound on $F(A_0)$ and the L^p bound (1.4) on the self-dual curvature combined with the asymptotic behavior (1.2) of ω imply that there exist "good" gauges for A_0 on the annuli $2^n < |z| < 2^{n+1}$ and these gauges can be patched together using the technique of [30]. Indeed, the hypotheses on $F(A_0)$ imply that there is a compact set $K \subset Z$ and a trivialization of E on $Z - K$ such that, in this trivialization, the connection forms for A_0 lie in $L^p_{1,1+\delta}$ for some $\delta > 0$. Consequently, the embedding results given above imply that in this gauge, the bundle E extends to $\overline{Y}$ and the connection A_0 extends continuously to this bundle.

For form-valued sections of the bundle E and associated bundles, define the weighted Sobolev spaces as in (1.5), replacing d_0 there by the covariant derivative induced by the connection A_0. Kato's inequality $|d|s|| \leq |d_A s|$ implies that the Sobolev inequalities remain valid for bundle-valued functions as above, and in addition, the asymptotic decay of the connection forms for A_0 in the above-mentioned gauge implies that if $p \geq 4$, $\delta > 0$ and $s \in \Gamma(E)$ has $d_0 s \in L^p_{0,1+\delta}$, then there is a section $s_0 \in \Gamma(Z - K, E)$ which is constant in this trivialization such that $s - s_0 \in L^p_{1,\delta}(Z - K)$; simply stated, $s = s_0$ at infinity.

As in [8], let $\Lambda := *_\omega \omega \wedge : \Lambda^{1,1} \to \Lambda^0$, and let P be the second-order linear elliptic operator on functions $P := i\Lambda\overline{\partial}\partial$. If $d_0 = \partial_0 + \overline{\partial}_0$ is the covariant exterior derivative from A_0, let R_0 denote the operator $i\Lambda(\overline{\partial}_0\partial_0 - \partial_0\overline{\partial}_0)$, so R_0 is a bounded map from $L^p_{2,\delta}$ into $L^p_{0,\delta+2}$. By (1.2), the operator $\varphi^2 R_0$ has principal symbol which is asymptotic to a translation-invariant elliptic operator in the sense of §6 of [22]. By Theorems 6.1 and 1.1 of that reference, it follows that R_0 is a Fredholm map between these spaces for all $p > 1$ for all but a discrete set of $\delta \in \mathbb{R}$ and by Lemma 7.3 of the same reference, the index is independent of p.

More generally, if $\delta > 0$ and $p \geq 2$ are given, the set (indeed, group) $\mathcal{G} = \mathcal{G}(\delta, p)$ of complex automorphisms g of E such that $d_0 g \in L^p_{1,1+\delta}$ acts on the integrable connections on E in the usual way, each such connection defining a holomorphic structure isomorphic to the given one. For any connection $A = g\cdot A_0$ with $g \in \mathcal{G}$ it follows from the Sobolev inequalities and Hölder's inequality that the corresponding operator $R = R_g$ defined as above replacing A_0 by A also defines a continuous map $L^q_{2,\delta'} \to L^q_{0,2+\delta'}$. Indeed, the Sobolev embedding results stated above imply that the operators R_0 and R_g differ by a compact mapping $L^q_{2,\delta'} \to L^q_{0,2+\delta'}$ so one is Fredholm iff the other is and they have the same index.

It will now be shown that if δ is properly chosen, then the index of each of these operators is zero and that each defines an isomorphism between the relevant spaces. The key ingredients here are the Sobolev embedding results mentioned above, and the classical Maximum Principle.

If u is a C^2 function on Z satisfying $Pu \leq 0$ and $U \Subset Z$ is open, then since P annihilates the constants the Maximum Principle implies that u must attain its maximum over U on ∂U. If, in addition, u lies in $L^p_{1,\delta}$ for some $p \geq 2$ and some $\delta > 0$ then since $\lim_{|z|\to\infty} u = 0$ by Lemma 5.2 of [28], it must be the case that $u \leq 0$ everywhere on Z. If $s \in \Gamma(E)$, then since $P|s|^2 = \frac{1}{2}\langle Rs, s\rangle + \frac{1}{2}\langle s, Rs\rangle - |ds|^2_\omega \leq |s|\,|Rs|$ it follows from the same argument that the kernel of any of the operators R acting on $L^p_{2,\delta}$ is zero whenever $p \geq 2$ and δ is positive.

For $p \geq 2$ and noncritical $\delta > 0$ the operator R thus has nonpositive index. By Lemma 7.3 of [22] the index is independent of p, so the problem of determining when the index is zero is reduced to the case $p = 2$ where Hilbert space techniques apply. Clearly, the index of R is the same as that of $R_\alpha := \varphi^{2\alpha} R : L^2_{2,\delta} \to L^2_{0,2+\delta-2\alpha}$, the latter operator being more easily dealt with if α is appropriately chosen. The adjoint of R_α is given by $R^*_\alpha(s) = \varphi^{4-2\delta} *_0 [i(\overline{\partial}\partial - \partial\overline{\partial})(\rho\varphi^{2(\delta-\alpha)} s\omega)]$, where $\omega^2 =: \rho\omega_0^2$ (so $\rho \to 1$ at infinity) and $*_0$ is the Hodge $*$-operator for ω_0 acting on 4-forms. In particular,

$$R^*_\delta(s) = \varphi^{4-2\delta} *_0 [i(\overline{\partial}\partial - \partial\overline{\partial})(\rho s\omega)] \in L^2_{0,\delta} \quad \text{for } s \in L^2_{2,2-\delta}. \tag{1.7}$$

By direct calculation,

$$\begin{aligned} i\overline{\partial}\partial(\langle s, s\rangle\omega) - \langle s, s\rangle(i\overline{\partial}\partial\omega) = {} & \frac{1}{2}\langle i(\overline{\partial}\partial - \partial\overline{\partial})(s\omega) - 2si\overline{\partial}\partial\omega, s\rangle \\ & + \frac{1}{2}\langle s, i(\overline{\partial}\partial - \overline{\partial}\partial)(s\omega) - 2si\overline{\partial}\partial\omega\rangle \\ & - |ds|^2_\omega \omega^2 . \end{aligned} \tag{1.8}$$

The operator on the left of this equation annihilates the constants and therefore obeys the Maximum Principle. Replacing s by ρs in (1.8), it follows that the kernel of the operator $\check{R}$ defined by $L^2_{2,2-\delta} \ni s \mapsto *_0[i(\overline{\partial}\partial - \partial\overline{\partial})(\rho s\omega) - 2\rho si\overline{\partial}\partial\omega] \in L^2_{0,4-\delta}$ is zero if $2-\delta$ is positive, and therefore its index is nonpositive for such δ. By (1.3) and the Sobolev embedding theorem, multiplication by $i\overline{\partial}\partial\omega$ defines a compact mapping $L^2_{2,\delta} \to L^2_{0,\delta+2}$ so $\varphi^{4-2\delta}\check{R}$ and R^*_δ differ by compact operator and therefore have the same index. Combining all this information, it follows that if $0 < \delta < 2$ the operator R has index 0 and defines an isomorphism $L^p_{2,\delta} \to L^p_{0,2+\delta}$ for all $p \geq 2$, except for the finite set of $\delta \in [0, 2]$ where it fails to be Fredholm. (As a brief aside, note that (1.3) could have been replaced with the condition that $i\bar{\partial}\partial\omega$ should be everywhere nonnegative to obtain the same result.)

This completes the task of assembling the facts required for the linear part of the proof.

2. Proof of Theorem 0.1: Nonlinear aspects

The following lemma, providing an a priori bounds for sequences of Hermitian connections, demonstrates the generally useful (and not unexpected) fact that bubbling of curvature for such sequences can only occur if the Hermitian metrics degenerate.

Lemma 2.1. *Let $B_R \subset \mathbb{C}^2$ be the ball of radius R centered at 0. Let ω be a smooth positive $(1, 1)$-form on B_4 and let A_0 be a smooth integrable connection on the trivial bundle over B_4. If $C_0 > 0$ and $p > 4$ are given, then there is a constant $C = C(\omega, A_0)$ such that, for any matrix of smooth functions g on B_4 satisfying*

(a) $\|F(g \cdot A_0)\|_{L^2(B_4)} \leq C_0$,

(b) $\|\Lambda_\omega F(g \cdot A_0)\|_{L^p(B_4)} \leq C_0$, *and*

(c) $\|\log \operatorname{tr} h\|_{L^1(B_4)} + \|\log \operatorname{tr} h^{-1}\|_{L^1(B_4)} \leq C_0$ *(for $h := g^* g$), it follows that*

$$\|\operatorname{tr} h\|_{C^0(B_3)} + \|\operatorname{tr} h^{-1}\|_{C^0(B_3)} \leq C \tag{2.2)(a}$$

and

$$\|h\|_{L^p_2(B_1)} \leq C(\|\Lambda_\omega F(g \cdot A_0)\|_{L^p(B_2)} + \|h\|_{L^2(B_2)}). \tag{2.2)(b}$$

(All norms on forms are with respect to ω, and $\operatorname{tr} h = \operatorname{tr} g^* g = |g|^2$ of course.)

Proof. By the Cauchy-Schwarz inequality,

$$i\Lambda_\omega \bar{\partial}\partial \log |g|^2 \leq |\Lambda_\omega F(g \cdot A_0)| + |\Lambda_\omega F(A_0)|$$

and similarly for $i\Lambda_\omega \bar{\partial}\partial \log |g^{-1}|^2$. By Theorem 9.20 of [18] the L^p bound (b) on the central component of the curvature and the L^1 bounds (c) imply uniform C^0 bounds on $\log |h|$, $\log |h^{-1}|$ over any given compact subset of B_4, giving (2.2)(a).

If (2.2)(b) fails then there is a sequence of automorphisms $\{g_i\}$ satisfying (a), (b), (c) and (2.2)(a) with

$$\|h_i\|_{L_2^p(B_1)}/(\|\Lambda_\omega F(g_i \cdot A_0)\|_{L^p(B_2)} + \|h_i\|_{L^2(B_2)}) \to \infty .$$

If A_0 and $\{g_i\}$ were defined on a compact surface and satisfied these conditions, the desired contradiction would follow from the argument on pp. 645–646 of [8] which shows that a subsequence of $\{g_i \cdot A_0\}$ must be uniformly bounded in L_1^p and converge strongly in C^0 (that argument relies heavily on the proof of Lemma 19 in [12], to give due credit). Given such bounds on the connection forms, standard linear elliptic estimates applied to the equation

$$i\Lambda(\overline{\partial}\partial h_j - \overline{\partial} h_j \wedge h_j^{-1}\partial h_j) = i\Lambda(g_j^* F(A_j)g_j - h_j F(A_0))$$

give a uniform L_2^p bound on h_j precisely of the form (2.2)(b).

To convert the local problem to one on a compact surface, it can first be assumed without loss of generality that A_0 is the trivial flat connection. By Sedlacek's method [26], the bound (a) on the L^2 norm of $F(g_i \cdot A_0)$ implies that there is a finite set $S \subset \overline{B}_3$ where the curvature is concentrating. If $K \Subset B_3 - S$, it follows from the proof of Corollary 23 of [12] and the L^p bound (b) on the central component of the curvature that there is a subsequence of $\{g_i A_0\}$ and gauge transformations so that the gauged subsequence converges weakly in L_1^p and strongly in C^0 over K; ellipticity of the $\overline{\partial}$-operator then implies that the corresponding automorphisms $\{g_{i_j}\}$ will converge weakly in L_2^p and strongly in C^1 on any $K' \Subset K$. In particular, if $K = B_r - \overline{B}_{r'}$ is an annulus with $2 < r' < r < 3$ and $\overline{K} \cap S = \varnothing$, choose a cut-off function ρ such that $\rho \equiv 1$ on $B_{r'}$ and $\operatorname{supp} d\rho \subset B_r$ and replace g_i by $\exp(\rho \log(h_i^{1/2}))$. After extending ω smoothly to a positive $(1, 1)$-form on $\mathbb{P}_2$, the entire problem can now be transferred to the compact setting of a sequence of integrable connections on the trivial bundle over $\mathbb{P}_2$, all satisfying (a), (b), (c) with B_4 replaced by $\mathbb{P}_2$ and with C_0 replaced by a different constant. By the argument mentioned above, there are in fact no bad points for the new sequence, implying that the connections in the subsequence of the original sequence are bounded in $L_1^p(B_{r'})$, thus giving the desired contradiction. q.e.d.

With the above lemma in hand, the nonlinear part of the proof of the main result can now be presented.

Fix $p > 4$ and $0 < \delta < 2$ such that $\widehat{F}(A_0) \in L^p_{0,\delta+2}$ and such that $R_0 : L^p_{2,\delta} \to L^p_{0,\delta+2}$ is an isomorphism; thus $R = R_g$ defines an isomorphism between these spaces for any $g \in \mathcal{G} = \mathcal{G}(p, \delta)$. For later purposes, δ

should be chosen to be less than $\sqrt{2}$. Two easy observations simplify the problem of solving (1.1) for $g \in \mathscr{G}$. First, taking the trace yields the equation $P \log \det h = -\operatorname{tr} i\widehat{F}(A_0)$ for the determinant of $h = g^* g$. By choice of function spaces and the discussion above, there is a unique solution $u \in L^p_{2,\delta}$ to $Pu = -\operatorname{tr} i\widehat{F}(A_0)$ so it may therefore be supposed without loss of generality that A_0 already satisfies $\operatorname{tr} \widehat{F}(A_0) \equiv 0$, and all $g \in \mathscr{G}$ to be considered may be assumed to have unit determinant.

Second, note that the solution will not be unique without the imposition of a boundary condition at infinity, corresponding to the choice of unitary structure on L_∞. This is fixed as follows: if r is the rank of the bundle and h_0 is a positive self-adjoint unimodular matrix of rank r, let $\mathscr{G}(h_0) := \{g \in \mathscr{G} | g^* g = h_0 \text{ at } \infty,\ \det g \equiv 1\}$. If $g_1, g_2 \in \mathscr{G}(h_0)$ both satisfy $\widehat{F}(g_i \cdot A_0) = 0$, first make unitary transformations replacing g_i by $g_i^* g_i^{1/2}$. Then for $g := g_2 g_1^{-1}$ and $A := g_1 \cdot A_0$ it follows that $\widehat{F}(A) = 0 = \widehat{F}(g \cdot A)$, implying $P \log \operatorname{tr} g^* g \leq 0$ and hence that $\operatorname{tr} g^* g \leq r$ everywhere by the Maximum Principle. Since $g^* g$ is positive with unit determinant, it follows that $g^* g = 1$ everywhere, implying $g_1 = g_2$ since both are positive. Thus, modulo unitary transformations (i.e., gauge transformations), a solution is uniquely determined by this boundary condition.

Now fix a positive unimodular matrix h_0 and define the functional

$$J(g) := \|\widehat{F}(g \cdot A_0)\|^p_{L^p_{0,2+\delta}} = \int_Z |\widehat{F}(g \cdot A_0)|^p (1 + |z|^2)^{[(2+\delta)p-8]/2} \omega_0^2, \qquad g \in \mathscr{G}(h_0).$$

Choose a minimizing sequence $\{g_i\} \subset C^\infty \cap \mathscr{G}(h_0)$ for J and set $h_i := g_i^* g_i$. Using the Cauchy-Schwarz inequality, $P \log\langle g_i, g_i\rangle = i\Lambda\bar{\partial}\partial \log \operatorname{tr} h_i \leq |\widehat{F}(A_0)| + |\widehat{F}(g_i \cdot A_0)|$. Since P is an isomorphism onto $L^p_{0,\delta+2}$, there is a unique function $v_i \in L^p_{2,\delta}$ such that $Pv_i = |\widehat{F}(A_0)| + |\widehat{F}(g_i \cdot A_0)|$, and the $L^p_{2,\delta}$ norm of v_i is uniformly bounded by a constant which is independent of i; from the Sobolev inequalities it follows $\varphi^\delta v_i$ is uniformly bounded in $C^0(Z)$. Since $P(\log \operatorname{tr} h_i - v_i) \leq 0$ the Maximum Principle implies that $\log \operatorname{tr} h_i - v_i$ attains its maximum at infinity (where it has the value $\log \operatorname{tr} h_0$) and therefore the functions $\log \operatorname{tr} h_i$ are also bounded in $C^0(Z)$ independent of i; since $\det h_i \equiv 1$, the same is true of the inverses h_i^{-1}.

The fact that it is possible to bound the the sequence $\{h_i\}$ of unimodular metrics uniformly in $C^0(Z)$ is the key fact which prevents all of the difficulties encountered in [8] from arising. Returning now to the problem in hand, since $L^p_{0,2+\delta} \subset L^2(Z, \omega_0)$ the curvatures $F_i = F(g_i \cdot A_0) =$

$g_i(F_0 + \overline{\partial}_0(h_i^{-1}\partial_0 h_i))g_i^{-1}$ are uniformly bounded in $L^2(Z, \omega)$. If the annulus $A(k, k+1) := \{2^k \le |z| \le 2^{k+1}\}$ is dilated to the uniform size $A(1, 2)$, the pulled-back rescaled metric $\widetilde{\omega}$ is close to the standard flat metric by (1.2), and this improves as k increases. Similarly, in the gauge for A_0 constructed earlier, the pulled-back connection approaches the trivial flat connection in L_1^p as $k \to \infty$. The local a priori estimate given by Lemma 2.1 (which applies to the annulus by applying it to a fixed finite cover by balls) then translates back into a bound of the form

$$\|h_i - h_0\|_{L^p_{2,\delta}(A(k,k+1))} \le C\|\widehat{F}(g_i \cdot A_0)\|_{L^p_{2,\delta}(A(k-1,k+2))}$$

for some constant C independent of i and k. Here, h_0 has been smoothly extended to Z so as to be constant in a neighborhood of infinity. Thus there is an a priori bound on $h_i - h_0$ in $L^p_{2,\delta}$ and a subsequence $\{h_{i_j} - h_0\}$ can be found converging weakly in this space and strongly in $C^1(\overline{Y})$ to a limit h_∞, and the automorphism $g_\infty := h_\infty^{1/2} \in \mathscr{G}(h_0)$ minimizes the functional J.

The rest of the argument is now as in [8]: by the results of §1, the operator R corresponding to the connection $g_\infty \cdot A_0$ defines an isomorphism onto $L^p_{0,\delta+2}$. If $s \in L^p_{2,\delta} \otimes \operatorname{End} E_{\text{top}}$ satisfies $Rs = i\widehat{F}(A_\infty)$ then s must be self-adjoint and trace-free; moreover, $s \in C^0(\overline{Y})$ by the embedding theorem and $\lim_{|z|\to\infty} |s| = 0$. Thus $1-ts$ is invertible for sufficiently small t and $\det(1-ts) = 1+O(t^2)$, and it follows that $g_t := [\det(1-ts)]^{-1/r}(1-ts)$ lies in $\mathscr{G}(h_0)$ for such t. Since $F(g_t \cdot A_\infty) = F(A_\infty) - t(\overline{\partial}\partial - \partial\overline{\partial})s + O(t^2)$, it follows that $\widehat{F}(g_t \cdot A_\infty) = (1-t)\widehat{F}(A_\infty) + O(t^2)$ and therefore $F(g_\infty \cdot A_0) = 0$ else the functional J is not minimized at $t = 0$.

It remains only to remove the extra assumption (1.4) concerning the decay of the central component of the curvature of A_0. As noted at the beginning of §1, the fact that $F(A_0)$ lies in $L^2(Z, \omega_0)$ is sufficient to imply, by the techniques of [30], that there is a gauge transformation such that the gauged connection extends to a connection on a bundle over $\overline{Y}$; see, e.g., [27]. The metric defined by $\varphi^{-4}\omega_0$ extends smoothly to Y and is flat in a neighborhood of y_∞ so the twistor space for this metric is integrable in a neighborhood W of L_∞, being isomorphic to a neighborhood of a linearly embedded line in $\mathbb{P}_3$. If the holomorphic bundle E defined by A_0 together with its unitary structure can be extended to W (or some subneighborhood of L_∞ in W) then the extended bundle is trivial on all lines near to L_∞ and standard flat-space twistor theory yields a corresponding connection in a neighborhood of y_0 which is self-dual with respect to the flat metric in this neighborhood and whose pullback $\widetilde{A}_0$ to X induces E near L_∞. By Lemma D2 of [15], the full curvature tensor $F(\widetilde{A}_0)$ then lies in $L^p_{0,2+\delta}$ for any $\delta < \sqrt{2}$, so the assumption (1.4) is satisfied for the connection $\widetilde{A}_0$.

The proof that E does extend amounts to solving the equations term-by-term and proving convergence in some neighborhood: let $W_0 := W \cap X$ and $\sigma : W \to W$ be the real structure induced from $\mathbb{P}_3$ so $L_\infty = W_0 \cap \sigma W_0$ and $W_1 := W_0 \cup \sigma W_0$ is defined by a degenerate section of $\mathscr{O}(2)$. The bundle E extends to W_1 by setting $E_x := \overline{E}^*_{\sigma x}$ for $x \in \sigma W_0$, using the given unitary structure to identify the two vector spaces along L_∞. Since W can be covered by two Stein sets, there are no obstructions to the formal extension of E from W_1 to W. By simple extension theory, at each step the extension of the bundle from one formal neighborhood to the next can be chosen so that the unitary structure $E \simeq \sigma^* \overline{E}^*$ also extends to that neighborhood. Thus the bundle with unitary structure extends to all formal neighborhoods of W_1 in W, and then Wavrik's theorem [31] can be applied to give a genuine (convergent) extension of the single transition function for E in W_1 to a neighborhood in W whilst preserving the unitary structure.

This completes the proof of Theorem 0.1.

3. Examples

For $n = 0$, Theorem 0.1 gives an analytical proof of the result of Donaldson [11] on the correspondence between the instantons on S^4 with its standard metric and holomorphic bundles on $\mathbb{P}_2$ trivial on the line at infinity. Similarly, for $n = 1$, the result gives an analytical proof of the results of King [19] describing the same correspondence for instantons on $\mathbb{CP}_2$ and holomorphic bundles on the blow-up of $\mathbb{P}_2$ at a point. For general interest and to some extent for later purposes, the case of instantons on $\mathbb{CP}_2$ will be discussed in a little more detail.

Instantons on $\mathbb{CP}_2$ are studied in [6] (see also [13]) where it is shown that gauge equivalence classes of self-dual unitary connections on a bundle over $\mathbb{CP}_2$ are in one-to-one correspondence with isomorphism classes of certain monads on the twistor space $\mathbb{F} := \{(z, w) \in \mathbb{P}_2 \times \mathbb{P}_2^* | z \cdot w = 0\}$ for $\mathbb{P}_2$. Explicitly, if $\mathscr{O}(p, q) := \pi_1^* \mathscr{O}_{\mathbb{P}_2}(p) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_2^*}(q)$ and the bundle on $\mathbb{P}_2$ has total Chern class $r - lH - kH \cdot H$ where H is the first Chern class of the Hopf bundle, then the monads are of the form

$$(3.1) \qquad \mathbb{F}: \quad 0 \to K_1(-1, 0) \xrightarrow{a} N \oplus \overline{K}_2^*(-1, 1) \xrightarrow{b} \overline{K}_1^*(0, 1) \to 0,$$

where K_1, K_2 and N are complex vector spaces of dimension $k + \frac{1}{2}l(l+1)$, $k+\frac{1}{2}l(l-1)$ and $n+k+\frac{1}{2}l(l+3)$ respectively (cf. [6, (3.7)]). Further, the unitary structure of the instanton is encoded in the fact that the map b is determined from a using the antiholomorphic involution $\sigma : (z, w) \mapsto (\overline{w}, \overline{z})$ on $\mathbb{F}$ and definite Hermitian forms on N, K_1 (the twistor projection can be identified with $\mathbb{F} \ni (z, w) \mapsto [z \times \overline{w}] \in \mathbb{CP}_2$). Instantons for other compact groups are described by embedding the group in $\mathrm{U}(*)$ and imposing holomorphic restrictions on the corresponding monads. The connection and curvature of the instanton on $\mathbb{P}_2$ corresponding to a monad of the form (3.1) can be explicitly calculated using the methods described in §4 of [6], though the calculation is rather involved.

For the $\mathrm{SU}(2)$ 1- instantons, the vector spaces K_1, K_2 are one-dimensional and it is shown in §5 of [6] that the corresponding monads are determined up to isomorphism by the homomorphism $\mathcal{O}(-1, 0) \xrightarrow{a} \mathcal{O}(-1, 1)$ modulo multiplication by an element of $\mathrm{U}(1)$. Each such homomorphism is given by multiplication by $R \cdot w$ for some $R \in \mathbb{C}^3$, and nonsingularity of the monad forces $R \cdot \overline{R} < 1$, thereby identifying $\mathcal{M}$ with the open unit ball in $\mathbb{C}^3$ modulo the multiplicative action of $\mathrm{U}(1)$.

The blow-up of $\mathbb{P}_2$ at a point is isomorphic to the Hirzebruch surface H_1 (the projectivization of the bundle $\mathcal{O} \oplus \mathcal{O}(-1)$ over $\mathbb{P}_1$), and by the results of [7], isomorphism classes of holomorphic bundles on H_1 which are trivial on the line at infinity are also described by certain monads. Specifically, if $\mathcal{O}(0, 1)$ is the pullback of the Hopf bundle on $\mathbb{P}_1$ and $\mathcal{O}(1, 0)$ is the dual of the tautological line bundle associated to the projectivization $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ then under the identification $H_1 = \widetilde{\mathbb{P}}_2$ the bundle $\mathcal{O}(1, 0)$ is identified with the pullback of the Hopf bundle from $\mathbb{P}_2$ and $\mathcal{O}(1, -1)$ with the line bundle of the exceptional line. These identifications are explicitly realized by viewing H_1 as a hypersurface in $\mathbb{F}$: if $A \in \mathbb{C}^3$ is nonzero, $H_1 \simeq \{(z, w) | A \cdot w = 0\}$, with projection onto first factor realizing H_1 as the blow-up of $\mathbb{P}_2$ at $[A]$ and projection onto second factor realizing it as a $\mathbb{P}_1$ bundle over the line $A \cdot w = 0$ in $\mathbb{P}_2^*$; the twistor projection collapses the line at infinity $z \cdot \overline{A} = 0$ to the point $[\overline{A}] \in \mathbb{CP}_2$ at infinity.

Using the lemma of §1 of [7], a simple calculation as in §2 of that reference shows that isomorphism classes of holomorphic vector bundles on H_1 which are trivial on the line at infinity and have total Chern class $r - l(x-y) + kxy$ for $x = c_1(\mathcal{O}(1, 0))$ and $y = c_1(\mathcal{O}(0, 1))$ are in one-to-one correspondence with isomorphism classes of monads of the form

$$(3.2) \qquad H_1 : \quad 0 \to K_1(-1, 0) \xrightarrow{a} N \oplus K_2(-1, 1) \xrightarrow{b} K_3(0, 1) \to 0,$$

where K_1, K_2, K_3 and N are complex vector spaces of dimension $k + \frac{1}{2}l(l+1)$, $k+\frac{1}{2}l(l-1)$, $k+\frac{1}{2}l(l+1)$ and $r+k+\frac{1}{2}l(l+3)$ respectively, i.e., the same as those in (3.1) (cf. [7] $(2.2)\otimes\mathcal{O}(-1, 1)$). There is no longer a relationship between the maps a, b other than the condition that $ba = 0$.

The fact that these two monad descriptions have the same form is the manifestation of Theorem 0.1 in algebraic form for the case of the Fubini-Study metric on $\mathbb{P}_2$. The restriction of (3.1) to H_1 represents the pulling back of an instanton on $\mathbb{P}_2$ to $\overline{\mathbb{P}_2\sharp\overline{\mathbb{P}}_2}$, and the construction of the hermitian connection satisfying the anti-self-duality equations is represented by the construction of an extension of (3.2) to a monad on $\mathbb{F}$ of the form (3.1). For a detailed discussion on these matters, see [19].

For $n > 1$, a monad description of holomorphic bundles on the blowup of $\mathbb{P}_2$ at n points is unavailable. However, it is not difficult to analyze the structure of moduli of bundles on the blow-up of a complex surface in terms of bundles on the surface itself, a task which is done in [9]. The net result is a description of bundles on $\widetilde{\mathbb{P}}_2$ almost as detailed as that which exists for bundles on $\mathbb{P}_2$. In the simplest nontrivial case of rank 2 holomorphic bundles E with with $c_1(E) = 0$ and $c_2(E) = 1$ on $\widetilde{\mathbb{P}}_2$ a direct calculation of the moduli space is possible, as will now be indicated. According to the Theorem 0.1, such bundles, when trivial and equipped with unitary structures on L_∞, are in one-to-one correspondence with the SU(2) 1-instantons on $n\mathbb{CP}_2$.

Let E be such a bundle on $\widetilde{\mathbb{P}}_2$. By the Riemann-Roch formula the holomorphic Euler characteristic of E is $\chi(E) = 1$, and since $H^2(E) = H^0(E^* \otimes K)^*$ either E or $E^* \otimes K$ must have a nonzero holomorphic section, where K denotes the canonical bundle; the latter is not possible since E is trivial on L_∞ and hence on the generic line, to each of which the restriction of K is negative. Thus E can be expressed as an extension $0 \to \mathcal{O} \to E \to \mathcal{S} \to 0$ for some rank 1 sheaf $\mathcal{S}$ on $\widetilde{\mathbb{P}}_2$, and taking the maximal normal extension of $\mathcal{O}$ in E gives the commutative diagram with exact rows

$$\begin{array}{lccccccccc} & 0 & \longrightarrow & \mathcal{O} & \longrightarrow & E & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \\ (3.3) & & & \downarrow & & \| & & \downarrow & & \\ & 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & \mathcal{S}/\tau(\mathcal{S}) & \longrightarrow & 0. \end{array}$$

Here L is the line bundle defined by the lower row, and since $\det E$ is trivial it follows that $\mathcal{S}/\tau(\mathcal{S}) = L^* \otimes \mathcal{S}'$ for some torsion-free sheaf $\mathcal{S}'$ with $\mathcal{S}'^{**} = \mathcal{O}$.

Since L has a holomorphic section and E is trivial on L_∞ it follows that L is a tensor product of line bundles associated with the components of the exceptional divisor; i.e., $c_1(H)\cdot c_1(L) = 0$ where H denotes the pullback of the Hopf bundle from $\mathbb{P}_2$. Again using the Riemann-Roch formula it follows that $1 = \chi(E) = \chi(L) + \chi(L^* \otimes \mathcal{S}') = -c_1(L)^2 + h^0(\mathcal{S}'^{**}/\mathcal{S}')$. Since $c_1(L)^2 \leq 0$ with equality iff L is trivial, there are only two possiblities, namely (1) $L = \mathcal{O}$ and $\mathcal{S}' = m_x$ for some $x \in \widetilde{\mathbb{P}}_2$, i.e., E has the form

$$0 \to \mathcal{O} \to E \to m_x \to 0 \tag{3.4}$$

where m_x is the maximal ideal in $\mathcal{O}_x$; or (2) $S' = \mathcal{O}$ and $L = \mathcal{O}(L_i)$ for some $1 \leq i \leq n$ where L_i is the ith exceptional line, i.e., E has the form

$$0 \to \mathcal{O}(L_i) \to E \to \mathcal{O}(-L_i) \to 0. \tag{3.5}$$

Extensions of the form (3.4) are classified by the point $x \in \widetilde{\mathbb{P}}_2$, and moreover it must be the case that $x \notin L_\infty$ else the triviality of E on the line is violated. For each $i = 1, \cdots, n$, $\mathcal{O}(L_i)|_{L_i} = \mathcal{O}(-1)$ and $H^1(\widetilde{\mathbb{P}}_2, \mathcal{O}(2L_i)) \overset{\text{restr.}}{\to} H^1(L_i, \mathcal{O}(-2)) = \mathbb{C}$ is an isomorphism; it follows that up to isomorphism there are only two distinct bundles of the form (3.5), namely the trivial and the nontrivial extension. The latter restricts to the trivial bundle on L_i and is therefore a pullback from the blow-up of $\mathbb{P}_2$ at $n-1$ points; the trivial extension is the reducible bundle $\mathcal{O}(L_i) \oplus \mathcal{O}(-L_i)$.

Whether L in (3.3) is either $\mathcal{O}$ or $\mathcal{O}(L_i)$, the endomorphism group of E always contains the composition ψ given by $E \to L^* \otimes \mathcal{S}' \hookrightarrow L^* \to L \to E$; since $\psi^2 = 0$ it follows $\lambda\mathbf{1} + \lambda'\psi$ is an isomorphism for any $\lambda \neq 0$. Apart from the case of the reducible bundles (i.e., when the lower row of (3.3) splits) the automorphism group of E consists exactly of these morphisms. By triviality of E on L_∞ the section of $E \otimes L^*$ is nowhere vanishing there, and it follows that the quotient of the set of unitary structures on $E|_{L_\infty}$ by the group of holomorphic automorphisms is either a point or $\mathbb{R}_+$, according to whether E is respectively a direct sum $\mathcal{O}(L_i) \oplus \mathcal{O}(-L_i)$ or is indecomposable.

For $n = 1$ the global structure of the moduli space is more easily analyzed using the monad descriptions given earlier. Using the same notation as before, it is easy to check that the holomorphic bundle $E(R)$ on H_1 corresponding to the instanton $R \in B_1 - \{0\}$ is nontrivial on the exceptional line $L_0 = \{[z] = [A]\}$ iff $R \cdot \overline{A} = 0$; in terms of (3.3), these are the bundles of the form (3.4) for $x \in L_0$. The bundle $E(R)$ is an extension

$0 \to \mathscr{O}(L_0) \to E \to \mathscr{O}(-L_0) \to 0$ iff $R = \lambda A$ for some $\lambda \in \mathbb{C}$, where $\mathscr{O}(L_0) = \mathscr{O}(1, -1)$ now. Apart from the reducible connection, these last connections can be regarded as those which are "centered" at the point at infinity.

In the general case when $n \geq 1$, the map on moduli spaces induced by pulling back holomorphic bundles to the blow-up gives an injection of $\mathscr{M}_{n-1} \hookrightarrow \mathscr{M}_n$ onto an open set. The complement of the image of this map consists of the bundles which are on the new exceptional line L_n. Any bundle which is near to one of these must be trivial on all the other exceptional lines, and is therefore a pullback from the space with the first $n-1$ exceptional lines collapsed; thus a neighborhood of the new reducible is isomorphic to a neighborhood of the reducible connection in the case $n = 1$, i.e., isomorphic to a cone on $\mathbb{P}_2$.

Thus, for a large family of metrics on $n\mathbb{CP}_2$ a realization of Donaldson's famous moduli space [10] has been obtained. It has the correct boundary, and in addition, the description shows that each such moduli space is connected and is smooth away from the cone points.

This paper concludes in a somewhat speculative vein with some remarks intended to indicate part of the motivation for the work here. If Y is any simply connected 4-manifold with positive intersection form, $X := \mathbb{P}_2 \sharp \overline{Y}$ admits almost complex structures, and after removing a point, each such structure is homotopic to an integrable one [20]. The methods of this paper and the theorems of Uhlenbeck indicate that the removal of a point from a manifold does not cause irrevocable damage to gauge theory on that manifold. It might therefore be hoped that gauge theory, together with techniques such as those used here, can shed further light on the differential topology of general smooth definite 4-manifolds.

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