J. DIFFERENTIAL GEOMETRY
37 (1993) 603-638

MORSE INEQUALITIES FOR PSEUDOGROUPS OF LOCAL ISOMETRIES

JESÚS A. ALVAREZ LÓPEZ

Abstract

For complete pseudogroups of local isometries with compact space of orbits, the method of Witten is used to prove Morse inequalities for the invariant cohomology. An inequality is also proved for the cohomology of the space of orbit closures. These results are applied to the basic cohomology of Riemannian foliations, relating the tautness character to basic functions with no degenerate critical leaf closures.

Introduction

Let $\mathscr{H}$ be a complete pseudogroup of local isometries of a Riemannian manifold M such that the space of $\mathscr{H}$-orbits, $M/\mathscr{H}$, is compact. In this paper we prove Morse inequalities for the invariant cohomology $H(M)_{\mathscr{H}}$ (the cohomology of the complex $(A(M)_{\mathscr{H}}, d)$ of invariant differential forms).

Definition. An $\mathscr{H}$-orbit closure F is called a critical orbit closure of a function $f \in C^{\infty}(M)_{\mathscr{H}}$ if F contains critical points of f. F is called a nondegenerate critical orbit closure if F is the disjoint union of nondegenerate critical submanifolds. In this case, the index of F is well defined as the index of any of its connected components, and denoted by $m_F(f)$ (or simply m_F). The function f is called a nondegenerate $\mathscr{H}$-Morse function if all of its critical orbit closures are nondegenerate. For such a function, let $\operatorname{Crit}_{\mathscr{H}}(f)$ be the set of its critical orbit closures.

(See e.g. [4] or [6] for the degenerate Morse theory that will be used in this paper.)

If f is a nondegenerate $\mathscr{H}$-Morse function, then $\operatorname{Crit}_{\mathscr{H}}(f)$ is a discrete subset of the space of $\mathscr{H}$-orbit closures, $M/\overline{\mathscr{H}}$. Thus $\operatorname{Crit}_{\mathscr{H}}(f)$ is finite because $M/\mathscr{H}$ is compact. The existence of nondegenerate $\mathscr{H}$-Morse functions follows easily from the case solved by A. G. Wasserman [24], where $\mathscr{H}$ is generated by an action of a compact Lie group.

Received November 29, 1990 and, in revised form, April 20, 1992.
Key words and phrases. Pseudogroup of local isometries, Morse function, invariant cohomology, Riemannian foliation, tautness.

If F is a nondegenerate critical orbit closure of a function $f \in C^\infty(M)_{\mathscr{H}}$, then the quadratic form $H_F f$, defined by the Hessian of f on the normal bundle $N_F = (TM|_F)/TF$, is nondegenerate. So $H_F f$ yields a decomposition of N_F as direct sum of the subbundles, $N_{F,+}$ and $N_{F,-}$, where $H_F f$ is respectively positive and negative definite. The index m_F is thus the fiber dimension of $N_{F,-}$. The $\mathscr{H}$-line bundle over F of orientations of $N_{F,-}$ will be denoted by $\mathscr{O}_F$. (See §2 for the definition of $\mathscr{H}$-vector bundle.)

The indices of the critical orbit closures of a nondegenerate $\mathscr{H}$-Morse function f give some information about $H(M)_{\mathscr{H}}$. But clearly f cannot give any information about the cohomological contribution from each orbit closure. We prove the following result that establishes Morse inequalities relating the dimensions of the spaces $H^j(M)_{\mathscr{H}}$ to some numbers whose definition combines the invariant cohomology of the critical orbit closures and the corresponding indices. If the orbits are dense, these Morse inequalities are trivial equalities.

Theorem A. *Let $\mathscr{H}$ be a complete pseudogroup of local isometries of an n-dimensional Riemannian manifold M with $M/\mathscr{H}$ compact. Let f be a nondegenerate $\mathscr{H}$-Morse function on M, $\beta_j(\mathscr{H})$ (or simply β_j) $=$ $\dim H^j(M)_{\mathscr{H}}$, and*

$$\nu_j(\mathscr{H}, f)(\text{or simply } \nu_j) = \sum_{F\in \mathrm{Crit}_{\mathscr{H}}(f)} \dim H^{j-m_F}(F, \mathscr{O}_F)_{\mathscr{H}}.$$

Then we have the inequalities

$$\begin{aligned} \beta_0 &\le \nu_0, \\ \beta_1 - \beta_0 &\le \nu_1 - \nu_0, \\ \beta_2 - \beta_1 + \beta_0 &\le \nu_2 - \nu_1 + \nu_0, \end{aligned}$$

etc., and the equality

$$\sum_{j=0}^{n}(-1)^j \beta_j = \sum_{j=0}^{n}(-1)^j \nu_j.$$

The proof of Theorem A is an adaptation of the method of Witten [25], especially as it is shown in [21, Chapter 12]. The general arguments of differential operators used in [21] can be easily adapted. Indeed, by results of A. El Kacimi Alaoui [10], the general study of $\mathscr{H}$-invariant transversely elliptic operators on invariant sections can be reduced to the study of invariant transversely elliptic operators on $O(n)$-manifolds. Then, as in the proof of Witten, the nondegenerate $\mathscr{H}$-Morse function is used to modify the Laplacian on invariant forms so that its "$\mathscr{H}$-smoothing kernel" concentrates around the critical orbit closures, whose cohomological contribution is thus obtained by a local study. The most special part of the proof is made in this local analysis, where the Haefliger description of a neighborhood of each orbit closure is strongly used [13] (see also [19]).

This idea of Witten is also used in [5] to prove the degenerate Morse inequalities of Bott [6]. But the local analysis of [5] around the nondegenerate critical submanifolds is more complicated. Indeed, another modification of the de Rham complex has to be used so that the Betti numbers of the manifold can be compared to the Betti numbers of the L^2-cohomology of the Witten complex around each critical submanifold. In our case, since the invariant de Rham complex of each critical orbit closure is of finite dimension, we can find a large enough dilation of the metric in the transverse direction so that the error terms in the tangential direction are small. We thus get the Morse inequalities by approximation.

Similar tools will be used to prove the following estimation of the dimension of $H^1(M/\overline{\mathscr{H}})$.

Theorem B. *Let $\mathscr{H}$ be a complete pseudogroup of local isometries of a Riemannian manifold M with $M/\mathscr{H}$ compact. Let f be a nondegenerate $\mathscr{H}$-Morse function on M. Then* $\dim H^1(M/\overline{\mathscr{H}})$ *is less than or equal to than the number of critical orbit closures F of f with $m_F(f) = 1$ and $N_{F,-}$ $\mathscr{H}$-trivial.*

(See §2 for the definition of $\mathscr{H}$-triviality.)

These theorems can be applied to the cases of isometric Lie group actions, and holonomy pseudogroups of Riemannian foliations. For isometric Lie group actions on compact manifolds, Theorem A is a special case of the degenerate Morse inequalities, and Theorem B gives an inequality for the cohomology of orbit spaces. For Riemannian foliations on compact manifolds, Theorem A gives Morse inequalities for the basic cohomology, and Theorem B gives an inequality for the cohomology of the space of leaf closures. Moreover, since the tautness character of the foliation depends on its basic cohomology ([17], [1] and [2]), Theorems A and B have some consequences relating this property to basic functions with no degenerate critical leaf closures.

By the reduction of transversely elliptic operators on $(M, \mathscr{H})$ to transversely elliptic ones on some compact $O(n)$-manifold W, these Morse inequalities can be considered as Morse inequalities on W. Nevertheless, the de Rham complex of W does not correspond to the de Rham complex of M. If G is a compact Lie group, the same method can be used to prove Morse inequalities for the invariant cohomology of differential complexes on compact G-manifolds, assuming these complexes have a nice behavior with respect to the Koszul Slice Theorem. But in this paper we are only interested on the invariant de Rham complex.

Another possible way of studying the Morse theory for a pseudogroup $\mathscr{H}$ of local isometries is the following. By using a nondegenerate $\mathscr{H}$-Morse function and the Haefliger description of a neighborhood of each orbit closure, it could be possible to obtain a description of the pseudogroup up to "equivariant homotopy equivalence". Theorems A and B should follow from this description. Moreover this method could be used to answer the following question: When is it possible to find a Riemannian foliation whose holonomy pseudogroup is the given $\mathscr{H}$? A. Haefliger has solved this problem in a neighborhood of each orbit closure [13], so hopefully a nondegenerate $\mathscr{H}$-Morse function could be used to build up a foliation with the Haefliger local models around the critical orbit closures.

For a Lie group G, the equivariant Morse theory of a G-manifold M is studied in [4] (see also [16] for more applications). It is a Morse theory for the equivariant cohomology $H_G(M)$, which is defined as the cohomology of $E\times_G M$, where E is the universal G-bundle. At least when G acts freely on M, we have $H_G(M) \cong H(M/G)$. These inequalities also follow from Theorem A when the action is free. An interesting problem is to adapt our method to obtain some type of Morse inequalities for the cohomology of more general orbit spaces.

The paper has the following distribution. In §1 we describe some facts about modified Laplacians. §2 contains the analysis of invariant differential operators on invariant sections. In §3 we describe the invariant forms on a neighborhood of each orbit closure. A proof of the existence of nondegenerate $\mathscr{H}$-Morse functions is given in §4. Theorems A and B are respectively proved in §§5 and 6. Some consequences of these theorems are shown in §7. The above results are applied to Riemannian foliations in §8, and finally, some examples are studied in §9.

1. Modified Laplacians

Let (M, g) be a Riemannian manifold, and $(A(M), d)$ its de Rham graded differential algebra. Let $\mathscr{B} \subseteq A(M)$ be a C^∞-closed graded differential subalgebra containing the constant functions such that

$$\mathscr{B} \neg \mathscr{B} \subseteq \mathscr{B}, \tag{1.1}$$

$$\delta(\mathscr{B}) \subseteq \mathscr{B}, \tag{1.2}$$

where $\alpha \neg \beta = -\iota_X \beta$ for $\alpha, \beta \in A(M)$, X being a smooth section of

$\bigwedge TM$ that is g-dual of α, i.e., $\alpha = g(X, \cdot)$. From (1.1) we also have $g(\mathcal{B}, \mathcal{B}) \subseteq \mathcal{B}^0$, where g also denotes the induced metric on the fibers of $\bigwedge TM^*$.

Any strictly positive linear functional $I: \mathcal{B}^0 \to \mathbb{R}$ (for $f \geq 0$ we have $f \neq 0$ iff $I[f] > 0$) defines an inner product $\langle \cdot, \cdot \rangle_I$ on $\mathcal{B}$ in a standard way, $\langle \alpha, \beta \rangle_I = I[g(\alpha, \beta)]$. The Hilbert space completion of $(\mathcal{B}, \langle \cdot, \cdot \rangle_I)$ will be denoted by $L^2_I(\mathcal{B})$. Analogously we define $L^p_I(\mathcal{B})$ $(1 \leq p \leq \infty)$ generalizing the usual definitions of L^p-spaces. It is proved in [2] that $\delta: \mathcal{B} \to \mathcal{B}$ has a $\langle \cdot, \cdot \rangle_I$-adjoint operator $d_I: \mathcal{B} \to \mathcal{B}$ iff

$$\exists \gamma_I \in \mathcal{B}^1 \text{ such that } \langle \gamma_I, \alpha \rangle_I = -I[\delta(\alpha)] \ \forall \alpha \in \mathcal{B}^1. \tag{1.3}$$

Moreover, if (1.3) holds, then $d_I = d - \gamma_I \wedge$, and the operator $\delta_I = \delta - \gamma_I \lrcorner$ is $\langle \cdot, \cdot \rangle_I$-adjoint of d on $\mathcal{B}$.

In this paper we need a more general situation. Let E be a flat Riemannian vector bundle over M, and $(A(M, E), d)$ the graded differential space of E-valued forms on M (see e.g. [7]). Now we consider a C^∞-closed graded differential subspace $\mathcal{E} \subseteq A(M, E)$ such that

$$g(\mathcal{E}, \mathcal{E}) \subseteq \mathcal{B}^0. \tag{1.4}$$

Here g also denotes the induced metric on the fibers of $\bigwedge TM^* \otimes E$. We thus have an inner product $\langle \cdot, \cdot \rangle_I$ on $\mathcal{E}$ defined as above, yielding the spaces $L^p_I(\mathcal{E})$. Suppose that the exterior derivative has a $\langle \cdot, \cdot \rangle_I$-adjoint operator δ_I on $\mathcal{E}$. We will use the notation $D_I = d + \delta_I$ and $\Delta_I = D_I^2$.

These definitions have a natural behavior with respect to products. For $i = 1, 2$, let (M_i, g_i) be a Riemannian manifold, $\mathcal{B}_i \subseteq A(M_i)$ a C^∞-closed graded differential subalgebra that contains the constant functions and satisfies (1.1) and (1.2), I_i a strictly positive continuous linear functional on $\mathcal{B}_i^0$ satisfying (1.3), E_i a flat Riemannian vector bundle on M_i, and $\mathcal{E}_i \subseteq A(M_i, E_i)$ a C^∞-closed graded differential subspace satisfying (1.4). We have the flat Riemannian vector bundle $E = \mathrm{pr}_1^* E_1 \otimes \mathrm{pr}_2^* E_2$ on $M = M_1 \times M_2$, where pr_1 and pr_2 are the canonical projections of M onto each factor. There is a canonical injection of graded differential spaces

$$\begin{aligned} &\mathcal{E}_1 \otimes \mathcal{E}_2 \to A(M, E), \\ &(\alpha_1 \otimes s_1) \otimes (\alpha_2 \otimes s_2) \mapsto (\mathrm{pr}_1^* \alpha_1 \wedge \mathrm{pr}_2^* \alpha_2) \otimes (\mathrm{pr}_1^* s_1 \otimes \mathrm{pr}_2^* s_2). \end{aligned} \tag{1.5}$$

The C^∞-closure $\mathcal{E}$ of its image satisfies (1.4) with respect to the product metric $g = g_1 \times g_2$.

In particular we have a canonical injection of $\mathcal{B}_1 \otimes \mathcal{B}_2$ into $A(M)$, whose C^∞-closure, $\mathcal{B}$, is a graded differential subalgebra that contains the constant functions and satisfies (1.1) and (1.2). Moreover $I_1 \otimes I_2 \colon \mathscr{B}_1^0 \otimes \mathscr{B}_2^0 \to \mathbb{R} \otimes \mathbb{R} \equiv \mathbb{R}$ has a strictly positive continuous linear extension $I \colon \mathscr{B}^0 \to \mathbb{R}$ such that

$$I[f] = I_1[x \mapsto I_2[f(x, \cdot)]] = I_2[y \mapsto I_1[f(\cdot, y)]].$$

Proposition 1.1. *In the above situation, there exists δ_I on $\mathscr{C}$, and D_I is the continuous extension to $\mathscr{C}$ of $D_{I_1} \otimes \mathrm{id} + w \otimes D_{I_2}$ by the injection* (1.5), *where w denotes the degree involution.*

The proof of this result is a standard computation.

2. Analysis of invariant differential operators on invariant sections

Let $\mathscr{H}$ be a complete pseudogroup of local isometries of an n-dimensional Riemannian manifold (M, g), $M/\mathscr{H}$ the space of $\mathscr{H}$-orbits, and $M/\overline{\mathscr{H}}$ the space of $\mathscr{H}$-orbit closures. For any subspace S of differential forms on M, $S_{\mathscr{H}}$ will denote the space of $\mathscr{H}$-invariant differential forms in S, and $S_{\imath=0}$ the space of forms $\alpha \in S$ such that $\imath_X \alpha = 0$ for any vector field X that is tangent to the $\mathscr{H}$-orbit closures. If $\mathscr{H}$ is generated by the action of a Lie group H, then S_H will be used instead of $S_{\mathscr{H}}$. $A_{cp}(M)$ will denote the space of differential forms α on M such that the projection of $\mathrm{supp}(\alpha)$ to $M/\overline{\mathscr{H}}$ is compact. Finally, as usual, $A_c(M)$ will be the space of differential forms on M with compact support.

A vector bundle $\pi \colon E \to M$ will be called an $\mathscr{H}$-vector bundle when each $h \in \mathscr{H}$ can be lifted to an isomorphism of vector bundles

$$\tilde{h} \colon \pi^{-1}(\mathrm{Dim}(h)) \to \pi^{-1}(\mathrm{Im}(h))$$

such that

(i) $(h_1 h_2)^\sim = \tilde{h}_1 \tilde{h}_2$,
(ii) $(\mathrm{id}_M)^\sim = \mathrm{id}_E$,
(iii) $(h|_U)^\sim = \tilde{h}|_{\pi^{-1}(U)}$ for any open subset $U \subseteq \mathrm{Dom}(h)$.

Clearly the direct sum and tensor product of $\mathscr{H}$-vector bundles are $\mathscr{H}$-vector bundles in a canonical way. Let $\widetilde{\mathscr{H}}$ be the pseudogroup on E generated by the diffeomorphisms $\tilde{h}$ $(h \in \mathscr{H})$. If E has an $\widetilde{\mathscr{H}}$-invariant structure, then it will be called an $\mathscr{H}$-structure on E. For instance we have the definition of $\mathscr{H}$-triviality, $\mathscr{H}$-orientability, $\mathscr{H}$-flatness, $\mathscr{H}$-connection, $\mathscr{H}$-Riemannian structure, and $\mathscr{H}$-Hermitian structure.

We will use the following notation. $C^\infty(E)$ and $C_c^\infty(E)$ will be the spaces of smooth sections of E, and smooth sections with compact support, respectively. $C^\infty(E)_{\mathscr{H}}$ will be the space of $\mathscr{H}$-invariant sections of E, i.e., the sections $s \in C^\infty(E)$ satisfying $sh = \tilde{h}s$ for all $h \in \mathscr{H}$. $\mathrm{Diff}_{\mathscr{H}}(E)$ will be the space of $\mathscr{H}$-invariant differential operators on $C^\infty(E)$, and $\mathrm{Diff}^*_{\mathscr{H}}(E)$ the space of their restrictions to $C^\infty(E)_{\mathscr{H}}$. We will say that any $R \in \mathrm{Diff}_{\mathscr{H}}(E)$ is transversely elliptic when its leading symbol is an isomorphism at each $\xi \in TM^*$ which vanishes on vectors that are tangent to the orbit closures. (This is a generalization of the definition given in [3].) Moreover, if (R, E) is a differential complex [11, §1.5], we will say that it is a transversely elliptic complex when its leading symbol is exact at each ξ as above.

If $N \subseteq M$ is an $\mathscr{H}$-invariant submanifold, we define an $\mathscr{H}$-vector bundle E over N as an $\mathscr{H}_N$-vector bundle, where $\mathscr{H}_N$ is the restriction of $\mathscr{H}$ to N. We also define $\mathscr{H}$-structures over E as $\mathscr{H}_N$-structures, and $C^\infty(E)_{\mathscr{H}}$ will be used instead of $C^\infty(E)_{\mathscr{H}_N}$.

Assume that $\mathscr{H}$ is complete [13]. Then we have the following description of its structure, which is due to E. Salem [22]. (It is an adaptation of the description of Riemannian foliations due to P. Molino [18].) Let $\pi: \widehat{M} \to M$ be the $O(n)$-principal bundle of orthonormal frames of M with the Levi-Civita connection. Then $\mathscr{H}$ canonically defines a complete pseudogroup $\widehat{\mathscr{H}}$ on $\widehat{M}$, that preserves the canonical parallelisms of $\widehat{M}$, and whose orbit closures are the fibers of an equivariant surjective submersion $\pi_b: \widehat{M} \to W$, where W is an $O(n)$-manifold. Moreover, for each point $x \in W$ there is a Lie group G, a dense subgroup $\Gamma \subseteq G$, and an open neighborhood U of x such that the restriction of $\widehat{\mathscr{H}}$ to $\pi_b^{-1}(U)$ is equivalent to the pseudogroup generated by the Γ-action on $G \times U$, acting by left translations on G and trivially on U. So π_b corresponds to pr_2 by this equivalence. The Lie algebra of this Lie group does not depend on the chosen point if $M/\mathscr{H}$ is connected, and is called the structural Lie algebra of $\mathscr{H}$.

The C^∞-closed graded differential $\mathbb{R}$-subalgebra $A(M)_{\mathscr{H}} \subseteq A(M)$ contains the constant functions, and satisfies (1.1) and (1.2). The canonical injections

$$A(M)_{\mathscr{H}} \xrightarrow{\pi^*} A(\widehat{M})_{\widehat{\mathscr{H}}, O(n)} \xleftarrow{\pi_b^*} A(W)_{O(n)}$$

define an isomorphism of graded differential spaces

$$A(M)_{\mathscr{H}, \imath=0} \cong A(W)_{O(n), \imath=0}. \tag{2.1}$$

In particular

$$C^\infty(M)_{\mathscr{H}} \cong C^\infty(W)_{O(n)}. \tag{2.2}$$

Moreover π and π_b define an identity

$$M/\overline{\mathscr{H}} \equiv W/O(n). \tag{2.3}$$

Hence $M/\mathscr{H}$ is compact iff so is W. In this case we can define a strictly positive continuous linear functional $I: C^\infty(M)_{\mathscr{H}} \to \mathbb{R}$ by integrating on W with respect to any $O(n)$-invariant volume element. By integrating on W with respect to this volume element, we also have the usual inner product $\langle\cdot, \cdot\rangle_W$ on the space of sections of any Hermitian vector bundle over W. In this case, $L^p_I(E)_{\mathscr{H}}$ will be used instead of $L^p_I(C^\infty(E)_{\mathscr{H}})$.

By the Salem description of $\mathscr{H}$, the study of transversely elliptic operators on invariant sections can be reduced to the case of compact Lie group actions. Namely, we have the following.

Theorem 2.1. *In the above situation, for each $\mathscr{H}$-Hermitian vector bundle $\pi: E \to M$, there is a canonically associated $O(n)$-Hermitian vector bundle $\pi': E' \to W$, and a canonical isomorphism of algebras*

$$\mathrm{Diff}^*_{\mathscr{H}}(E) \cong \mathrm{Diff}^*_{O(n)}(E').$$

This isomorphism preserves transverse ellipticity, and adjointness with respect to $\langle\cdot, \cdot\rangle_I$ and $\langle\cdot, \cdot\rangle_W$ if $M/\mathscr{H}$ is compact.

The proof of this theorem will be a consequence of Proposition 2.2, where we consider a slightly more general setting. A description of E' and the stated isomorphism will be given.

Suppose that there exists a surjective Riemannian submersion $p: M \to N$ whose fibers are equal to the $\mathscr{H}$-orbit closures. Moreover suppose that for each point $x \in N$ there is a Lie group H, a dense subgroup $\Lambda \subseteq H$, and an open neighborhood V of x such that the restriction of $\widehat{\mathscr{H}}$ to $p^{-1}(V)$ is equivalent to the pseudogroup generated by the Λ-action on $H \times V$, acting by left translations on H and trivially on V. We have this situation for $\widehat{\mathscr{H}}$ as we saw, and also for pseudogroups generated by free actions. Let

$$pE = \{s \in C^\infty(E|_{p^{-1}(y)})_{\mathscr{H}} \text{ such that } y \in N\}.$$

As in [10, Proposition 2.7.2], it can be proved that pE is a Hermitian vector bundle over N in a canonical way. Let $\mathscr{G}$ be another complete pseudogroup of local isometries of M commuting with the elements of $\mathscr{H}$, and let $\mathscr{G}_1$ be the pseudogroup generated by $\mathscr{G}$ and $\mathscr{H}$. The elements of $\mathscr{G}$ can be projected by p, defining a pseudogroup $p\mathscr{G}$ of local diffeomorphisms of N. Assume that E is also a $\mathscr{G}$-Hermitian vector bundle such that the elements of $\widetilde{\mathscr{G}}$ commute with the elements of $\widetilde{\mathscr{H}}$, and let F be a $p\mathscr{G}$-Hermitian vector bundle over N. Then, canonically, p^*F

and pE are $\mathcal{G}_1$- and $p\mathcal{G}$-Hermitian vector bundles respectively. Moreover there are canonical isomorphisms

$$\begin{aligned} &p^*pE \cong E, \\ (2.4)\quad &(x, s) \mapsto s(x) \text{ if } (x, s) \in p^*pE, \text{ i.e. } s \in C^\infty(E|_{p^{-1}p(x)})_{\mathcal{H}}, \end{aligned}$$

(2.5)

$$\begin{aligned} &pp^*F \cong F, \\ &s \mapsto f \text{ if } s \in C^\infty(p^*F|_{p^{-1}(y)})_{\mathcal{H}} \text{ and } s(x) = (x, f) \\ &\qquad \text{for any } x \in p^{-1}(y), \end{aligned}$$

of $\mathcal{G}_1$- and $p\mathcal{G}$-Hermitian vector bundles respectively.

Any $R \in \mathrm{Diff}_{\mathcal{G}_1}(E)$ canonically defines a differential operator $pR \in \mathrm{Diff}_{p\mathcal{G}}(pE)$ as in [10, Proposition 2.7.7], whose order is less than or equal to the order of R. On the other hand, for any $S \in \mathrm{Diff}_{p\mathcal{G}}(F)$ there is an operator $p^*S \in \mathrm{Diff}_{\mathcal{G}_1}(p^*F)$ of the same order, defined as in [10, 2.8], such that pp^*S corresponds to S by the canonical isomorphism $C^\infty(F) \cong C^\infty(pp^*F)$ defined by (2.5), and whose leading symbol satisfies

$$\begin{aligned} &\sigma_L(p^*S)(x, p^*\xi)(x, \eta) = (x, \sigma_L(S)(p(x), \xi)(\eta)), \\ &\sigma_L(p^*S)(x, \zeta)(x, \eta) = 0, \end{aligned}$$

for $x \in M$, $\xi \in T_{p(x)}N^*$, $\zeta \in T_xM^*$ vanishing on $\ker(p_{*x})^\perp$, and $\eta \in F_{p(x)}$. Thus p^*S depends on the chosen $\mathcal{H}$-invariant metric on M. It is easy to check that $R|_{C^\infty(E)_{\mathcal{H}}}$ corresponds to $p^*pR|_{C^\infty(p^*pE)_{\mathcal{H}}}$ by the isomorphism $C^\infty(p^*p(E))_{\mathcal{H}} \cong C^\infty(E)_{\mathcal{H}}$ defined by (2.4). Hence the following result follows.

Proposition 2.2. *The maps $R \mapsto pR$ and $S \mapsto p^*S$ induce isomorphisms*

$$\mathrm{Diff}^*_{\mathcal{G}_1}(E) \cong \mathrm{Diff}^*_{p\mathcal{G}}(pE) \quad \text{and} \quad \mathrm{Diff}^*_{p\mathcal{G}}(F) \cong \mathrm{Diff}^*_{\mathcal{G}_1}(p^*F).$$

By these isomorphisms, restrictions of $p\mathcal{G}$-transversely elliptic complexes correspond to restrictions of $\mathcal{G}_1$-transversely elliptic ones.

Theorem 2.1 follows by taking $E' = \pi_b\pi^*E$, and the isomorphism $\mathrm{Diff}^*_{\mathcal{H}}(E) \cong \mathrm{Diff}^*_{O(n)}(E')$ is induced by the map $R \mapsto \pi_b\pi^*R$ of $\mathrm{Diff}_{\mathcal{H}}(E)$ to $\mathrm{Diff}_{O(n)}(E')$.

Theorem 2.1 has the following consequence.

Corollary 2.3. *If $M/\mathcal{H}$ is compact, for each $R \in \mathrm{Diff}_{\mathcal{H}}(E)$ there exists some $R^{*_I} \in \mathrm{Diff}_{\mathcal{H}}(E)$ which is $\langle\cdot, \cdot\rangle_I$-adjoint of R on $C^\infty(E)_{\mathcal{H}}$. Moreover, if (R, E) is a differential complex of order r, then it is a transversely* *elliptic complex iff* $R_j^{*_I} R_j + R_{j-1} R_{j-1}^{*_I}$ *is a transversely elliptic operator of order* $2r$ *for all* j.

Proof. The existence of R^{*_I} follows from Theorem 2.1 by taking formal adjoints on W. The remainder of the corollary can be obtained by similar arguments to those in [11, Lemma 1.5.1].

Example 2.4. If E is a flat $\mathscr{H}$-vector bundle over M, then we also have the graded differential space of invariant E-valued forms $(A(M, E)_{\mathscr{H}}, d)$ on M, and its cohomology $H(M, E)_{\mathscr{H}}$. By Corollary 2.3 there exists $d_I, \delta_I \in \mathrm{Diff}_{\mathscr{H}}(\bigwedge TM^* \otimes E)$ whose restriction to $A(M, E)_{\mathscr{H}}$ are $\langle\cdot, \cdot\rangle_I$-adjoint of δ and d respectively. Moreover $D_I = d_I + \delta_I$, and $\Delta_I = D_I^2$ are transversely elliptic since $(d, \bigwedge TM^* \otimes E)$ is an elliptic complex. So $(M, g, A(M)_{\mathscr{H}}, I)$ satisfies (1.3), and the operators d_I and δ_I are modifications of d and δ as we saw in §1. The class defined by γ_I in $H^1(M)_{\mathscr{H}}$ depends only on $\mathscr{H}$ [2], and will be denoted by $\xi(\mathscr{H})$. From [2] it follows that if $(M, g, A(M)_{\mathscr{H}}, I')$ satisfies (1.3) for some strictly positive linear functional I' on $C^\infty(M)_{\mathscr{H}}$, then I' is defined by integrating on W with respect to some $O(n)$-invariant volume element in the same way as I was defined. Thus I' is continuous. Moreover all such functionals I' define the same class $\xi(\mathscr{H})$.

By using Theorem 2.1 and some of the tools of its proof, we get the following.

Proposition 2.5. *Any* $Q \in \mathrm{Diff}^*_{\mathscr{H}}(E)$ *is* $C^\infty(M)_{\mathscr{H}}$*-linear iff it is the restriction of some zero order operator of* $\mathrm{Diff}_{\mathscr{H}}(E)$.

Proof. Clearly, Q is $C^\infty(M)_{\mathscr{H}}$-linear if it is the restriction of some zero order operator of $\mathrm{Diff}_{\mathscr{H}}(E)$.

To prove the reciprocal statement, by Theorem 2.1 and Example 2.4 we can assume that M is compact and $\mathscr{H}$ generated by the action of a compact Lie group G. Take any $R \in \mathrm{Diff}_G(E)$ whose restriction to $C^\infty(E)_G$ is Q. Let $V \subseteq M$ be the open subset of regular G-orbits, and $p\colon V \to G\backslash V$ the orbit space projection, which is a principal bundle. Since Q is $C^\infty(V)_G$-linear on $C^\infty(E|_V)_G$, pR is $C^\infty(G\backslash V)$-linear, and thus of order zero.

Let ∇ be the covariant derivative of any connection on E. (∇ may not be G-invariant. Indeed there may not be any G-connection on E.) Then R can be given as a sum of compositions of smooth sections of $\mathrm{End}(E)$, and covariant derivatives with respect to vector fields on M which are tangent to the G-orbits on V; otherwise pR would not be of order zero. But since V is dense on M, these vector fields are tangent to the G-orbits on M. The space of such vector fields are $C^\infty(M)$-generated by the fundamental vector fields of the G-action, as can be easily checked. Hence R can be given as follows. Let $\mathfrak{g}^-$ be the Lie algebra of right invariant vector fields on G, and $X_1, \cdots, X_p$ linearly independent elements of $\mathfrak{g}^-$ that generate a complement of the kernel of the induced infinitesimal action $\mathfrak{g}^- \to C^\infty(TM)$, $X \mapsto X^*$. For multi-indices $J = (j_1, \cdots, j_p)$, let $\nabla_J = \nabla_1^{j_1} \cdots \nabla_p^{j_p}$, where $\nabla_i = \nabla_{X_i^*}$. Then there are sections $f_J \in C^\infty(\mathrm{End}(E))$, $f_J = 0$ for almost all J, such that

$$R = \sum_J f_J \nabla_J .$$

By using a G-invariant partition of unity argument, the result follows if, for each $x \in M$, Q is the restriction of some G-invariant zero order operator on some G-invariant neighborhood U_x of x. By Koszul Slice Theorem, there is a representation of the isotropy group G_x on $\mathbb{R}^m$ ($m = \operatorname{codim} Gx$) such that U_x can be chosen equivariantly diffeomorphic to $G \times_{G_x} \mathbb{R}^m$ (the quotient space of the "diagonal" G_x-action on $G \times \mathbb{R}^m$ given by $b(a, v) = (ab^{-1}, bv)$). The G-action on $G \times_{G_x} \mathbb{R}^m$ is induced by the G-action on $G \times \mathbb{R}^m$, acting by left translations on G and trivially on $\mathbb{R}^m$. We will identify U_x to $G \times_{G_x} \mathbb{R}^m$. Let $\rho\colon G \times \mathbb{R}^m \to U_x$ be the quotient space projection, and $\widetilde{\nabla}$ the pullback of ∇ to $\rho^*(E|_{U_x})$. For each multi-index J as above, let $\widetilde{\nabla}_J = \widetilde{\nabla}_1^{j_1} \cdots \widetilde{\nabla}_p^{j_p}$, where $\widetilde{\nabla}_i = \widetilde{\nabla}_{(X_i, 0)}$, and

$$\widetilde{R} = \sum_J \rho^* f_J \widetilde{\nabla}_J .$$

Since $X_1^*, \cdots, X_p^*$ are linearly independent at each point in V, and $\rho_*(X_i, 0) = X_i^*$, there are G-invariant metrics on $\rho^{-1}(U_x \cap V)$ such that each $(X_i, 0)$ is orthogonal to the fibers of ρ. It is easy to check that $\widetilde{R} = \rho^* R$ on $\rho^{-1}(U_x \cap V)$ for any of such metrics. Since $\rho^{-1}(U_x \cap V)$ is dense on $G \times \mathbb{R}^m$, we get that $\widetilde{R}$ is G-invariant on M, and canonically associated to R. Moreover, since the vector fields $(X_i, 0)$ are tangent to the G-orbits in $G \times \mathbb{R}^m$, $\mathrm{pr}_2 \widetilde{R}$ is $C^\infty(\mathbb{R}^m)$-linear, and thus of order zero. So, over U_x, Q corresponds to the restriction of the zero order operator $\rho(\mathrm{pr}_2^* \mathrm{pr}_2 \widetilde{R})$ by the isomorphism defined by (2.4) and (2.5).

Remark. The proof does not follow easily from the identity $R \equiv \pi \pi_b^* \pi_b \pi^* R$ on invariant sections, for it seems to be difficult to find an $\widehat{\mathscr{H}}$-invariant metric on $\widehat{M}$ so that $\pi^* R$ is $C^\infty(\widehat{M})_{\widehat{\mathscr{H}}}$-linear, and thus $\pi_b \pi^* R$ of order zero.

The object of the remainder of this section is to canonically associate an elliptic operator to an invariant transversely elliptic one on invariant sections. So the study of these operators can be reduced to the study of elliptic ones. We shall prove the following.

Theorem 2.6. *Let $\mathscr{H}$ be a complete pseudogroup of local isometries of a Riemannian manifold M such that $M/\mathscr{H}$ is compact. Let I be a strictly positive linear functional on $C^\infty(M)_\mathscr{H}$ such that* (1.3) *is satisfied. Let E be an $\mathscr{H}$-Hermitian vector bundle, and R an $\mathscr{H}$-invariant transversely elliptic differential operator on $C^\infty(E)$. Then we have the orthogonal decomposition* [10]

$$C^\infty(E)_\mathscr{H} = \operatorname{Ker}(R|_{C^\infty(E)_\mathscr{H}}) \oplus \operatorname{Im}(R^{*_I}|_{C^\infty(E)_\mathscr{H}}).$$

If $R = R^{*_I}$, then R defines a selfadjoint operator in $L^2_I(C^\infty(E)_\mathscr{H})$ with discrete spectrum. For any Schwartz function ϕ on $\mathbb{R}$ there is some $k \in C^\infty(E \otimes E^*)_{\mathscr{H}\times\mathscr{H}}$, that will be called the $\mathscr{H}$-smoothing kernel of $\phi(R)$, such that

$$(\phi(R)(s))(x) = (I \otimes \mathrm{id})[y \mapsto k(x, y)s(y)]$$

for any $s \in C^\infty(E)_\mathscr{H}$ and any $x \in M$, where $I \otimes \mathrm{id}\colon C^\infty(M)_\mathscr{H} \otimes E_x \equiv C^\infty(M, E_x)_\mathscr{H} \to \mathbb{R} \otimes E_x \equiv E_x$. $\phi(R)$ is a trace class operator, and

$$\operatorname{Tr}(\phi(R)) = I[x \mapsto \operatorname{Tr} k(x, x)].$$

Finally, the pointwise norm $|k(x, y)|$ $(x, y \in M)$ is estimated by the operator norm of $\phi(R)\colon L^1_I(C^\infty(E)_\mathscr{H}) \to L^\infty_I(C^\infty(E)_\mathscr{H})$.

By Theorem 2.1 and Example 2.4, we can consider the case where $\mathscr{H}$ is generated by a left isometric action of a compact Lie group G on M to prove Theorem 2.6. Let E be a G-Hermitian vector bundle over M, R a transversely elliptic operator of $\operatorname{Diff}_G(E)$, and $\mathfrak{g}$ the Lie algebra of G with the corresponding metric. We have the representation of $\mathfrak{g}$ on $C^\infty(E)$ given by

$$(Xs)(x) = \lim_{t\to 0} \frac{1}{t}(g_t s(g_t^{-1}x) - s(x))$$

for $X \in \mathfrak{g}$, $s \in C^\infty(E)$, and $x \in M$, where $g_t = \exp(tX)$. Let $X_1, \cdots, X_q$ be an orthonormal frame of $\mathfrak{g}$, and $\omega_1, \cdots, \omega_q$ the dual coframe. Let $d_{E,\mathfrak{g}}$ and $\delta_{E,\mathfrak{g}}$ be the first order differential operators on $C^\infty(E \otimes \bigwedge \mathfrak{g}^*) \equiv C^\infty(E) \otimes \bigwedge \mathfrak{g}^*$ defined by

$$d_{E,\mathfrak{g}}(s \otimes \alpha) = \sum_{i=1}^q (X_i s) \otimes (\omega_i \wedge \alpha) + s \otimes d_G(\alpha),$$

$$\delta_{E,\mathfrak{g}}(s \otimes \alpha) = -\sum_{i=1}^q (X_i s) \otimes \iota_{X_i}\alpha + s \otimes \delta_G(\alpha),$$

for $s \in C^\infty(E)$ and $\alpha \in \Lambda\mathfrak{g}^*$. Clearly $d_{E,\mathfrak{g}}$ and $\delta_{E,\mathfrak{g}}$ do not depend on the chosen orthonormal frame of $\mathfrak{g}$. Let also $D_{E,\mathfrak{g}} = d_{E,\mathfrak{g}} + \delta_{E,\mathfrak{g}}$.

For any $s \in C^\infty(E)$, any $\alpha \in \bigwedge\mathfrak{g}^*$, and any $x \in M$, we clearly have that $(D_{E,\mathfrak{g}}(s \otimes \alpha))|_{Gx}$ depends only on $(s \otimes \alpha)|_{Gx}$. So $D_{E,\mathfrak{g}}$ defines an operator $D_{E,\mathfrak{g},x}$ on $C^\infty(E|_{Gx} \otimes \bigwedge\mathfrak{g}^*)$. Let $\tau\colon G \to Gx$ be defined by $\tau(a) = ax$. We also have the canonical isomorphism

$$E|_{Gx} \otimes \bigwedge\mathfrak{g}^* \cong \tau(\tau^*(E|_{Gx}) \otimes \bigwedge TG^*). \tag{2.6}$$

But since $\tau^*(E|_{Gx})$ has a global G-invariant frame,

$$C^\infty(\tau^*(E|_{Gx}) \otimes \bigwedge TG^*) \cong C^\infty(\tau^*(E|_{Gx}))_G \otimes A(G). \tag{2.7}$$

Then the following result follows easily.

Lemma 2.7. *$D_{E,\mathfrak{g},x}$ corresponds to $\tau(\mathrm{id} \otimes D_G)$ by* (2.6) *and* (2.7), *where $D_G = d_G + \delta_G$.*

The operators $R \otimes \mathrm{id}$ and $R^* \otimes \mathrm{id}$ are well defined on $C^\infty(E \otimes \bigwedge\mathfrak{g}^*) \equiv C^\infty(E) \otimes \bigwedge\mathfrak{g}^*$, and both of them commute with $D_{E,\mathfrak{g}}$ since R and R^* are G-invariant. Then, on the space

$$C^\infty((E \otimes \bigwedge\mathfrak{g}^*) \oplus (E \otimes \bigwedge\mathfrak{g}^*)) \equiv C^\infty(E \otimes \bigwedge\mathfrak{g}^*) \oplus C^\infty(E \otimes \bigwedge\mathfrak{g}^*),$$

we have the operator

$$R_G = \begin{pmatrix} R \otimes \mathrm{id} & (D_{E,\mathfrak{g}})^r \\ (D_{E,\mathfrak{g}})^r & -R^* \otimes \mathrm{id} \end{pmatrix},$$

where r is the order of R.

Lemma 2.8. *R_G is an elliptic operator that preserves the subspace $C^\infty(E)_G \oplus C^\infty(E)_G$, and such that*

$$R_G|_{C^\infty(E)_G \oplus C^\infty(E)_G} = \begin{pmatrix} R & 0 \\ 0 & -R^* \end{pmatrix}.$$

Moreover, if R is formally selfadjoint, then so is R_G.

Proof. Since $R \otimes \mathrm{id}$ and $R^* \otimes \mathrm{id}$ commute with $D_{E,\mathfrak{g}}$, the leading symbol of $R_G R_G^*$ is

$$\sigma_L(R_G R_G^*) = \begin{pmatrix} \sigma_L(R)\sigma_L(R)^* \otimes \mathrm{id} + \sigma_L(D_{E,\mathfrak{g}})^{2r} & 0 \\ 0 & \sigma_L(R)\sigma_L(R)^* \otimes \mathrm{id} + \sigma_L(D_{E,\mathfrak{g}})^{2r} \end{pmatrix}.$$

A similar expression also holds for $\sigma_L(R_G^* R_G)$. Therefore, the transverse ellipticity of R and Lemma 2.7 imply the ellipticity of $R_G R_G^*$ and $R_G^* R_G$ with standard arguments. Thus R_G is elliptic.

The remainder of the proposition is obvious because $D_{E,\mathfrak{g}}(C^\infty(E)_G) = 0$ and $D_{E,\mathfrak{g}}$ is formally selfadjoint. q.e.d.

By averaging the G-action we get a retraction $\eta_0\colon C^\infty(E) \to C^\infty(E)_G$ with commutes with R and R^*. Then we can define a retraction η of $C^\infty(E \otimes \bigwedge \mathfrak{g}^*) \oplus C^\infty(E \otimes \Lambda\mathfrak{g}^*)$ onto $C^\infty(E)_G \oplus 0 \equiv C^\infty(E)_G$ as the composition

$$C^\infty(E \otimes \textstyle\bigwedge \mathfrak{g}^*) \oplus C^\infty(E \otimes \bigwedge \mathfrak{g}^*) \to C^\infty(E) \oplus C^\infty(E) \\ \to C^\infty(E)_G \oplus C^\infty(E)_G \to C^\infty(E)_G,$$

where the first map is the direct sum of the canonical projections, the second one is $\eta_0 \oplus \eta_0$, and the last one is the first canonical projection. Clearly η is an orthogonal projection, and Lemma 2.7 yields $\eta R_G = R\eta$ and $\eta R_G^* = R^*\eta$.

Since R_G is elliptic, we have the orthogonal decomposition

$$C^\infty(E \otimes \textstyle\bigwedge \mathfrak{g}^*) \oplus C^\infty(E \otimes \bigwedge \mathfrak{g}^*) = \operatorname{Ker}(R_G) \oplus \operatorname{Im}(R_G^*).$$

By applying η to both sides of this equality we get [10]

$$C^\infty(E)_G = \operatorname{Ker}(R|_{C^\infty(E)_G}) \oplus \operatorname{Im}(R^*|_{C^\infty(E)_G}).$$

Suppose that R is also formally selfadjoint. Then so is R_G and the operator $\exp(itR_G)$ preserves $C^\infty(E)_G$. Thus R is essentially selfadjoint in $L^2(E)_G$ by [9, Lemma 2.1]. Moreover its spectrum is given by the eigenvalues of R_G whose corresponding eigensections are η-invariant.

For any Schwartz function ϕ on $\mathbb{R}$, let k be the smoothing kernel of $\phi(R_G)$. Define $k_G \in C^\infty(E \otimes E^*)$ by

$$k_G(\cdot, y) = (\eta \otimes \mathrm{id})(k(\cdot, y)).$$

It is easy to check that k_G is invariant by the induced actions of $G \times G$ on $M \times M$ and $E \times E$. Moreover

$$(\phi(R)(s))(x) = \int_M k_G(x, y)s(y)\operatorname{vol}(y)$$

for $s \in C^\infty(E)_G$ and $x \in M$.

The stated formula for the trace of $\phi(R)$ follows arguing as in the proof of the formula for the trace of a smoothing operator in terms of its smoothing kernel (see e.g. [21, (6.9) and (6.10)]).

It is easy to check that $L^1(C^\infty(E)_G)^* = L^\infty(C^\infty(E)_G)$, yielding that the pointwise norm $|k_G(x, y)|$ $(x, y \in M)$ is estimated by the operator norm of $\phi(R): L^1(C^\infty(E)_G) \to L^\infty(C^\infty(E)_G)$. So the proof of Theorem 2.6 is completed.

3. Differential forms on a neighborhood of an orbit closure

Let $\mathscr{H}$ and (M, g) be as in the above section. First, we shall recall the way of obtaining the Haefliger description of a neighborhood of an orbit closure by using Salem description of $\mathscr{H}$ [13], [19]. Second, we shall use this result to obtain a description of the space $\mathscr{H}$-invariant differential forms on this neighborhood.

Take some $x \in M$ and some $z \in \widehat{M}$ such that $\pi(z) = x$. Suppose that z is adapted to the orbit closure $F = \overline{\mathscr{H}x}$ at x, i.e., the first r vectors of z generate T_xF $(r = \dim F)$. The orbit closure $E = \overline{\widehat{\mathscr{H}}z}$ is a principal bundle over F with structural Lie group $H \subseteq O(n)$. All the points of E are orthonormal frames adapted to F, so $H \subseteq O(r) \times O(n')$, where $n' = n - r$. Let $\pi_Q: Q \to F$ be the $O(n')$-principal bundle of orthonormal frames of $TF^\perp$, and P the canonical projection of E to Q. Then $\pi_P = \pi_Q|_P$ is an H'-principal bundle over F, where H' is the projection of H to $O(n')$. Moreover $\mathscr{H}$ canonically defines a pseudogroup $\mathscr{H}_P$ of local diffeomorphisms of P. The Levi-Civita connection on $\widehat{M}$ restricts to a connection on E, which projects to a connection on P. The corresponding algebraic connection will be denoted by $\omega: \mathfrak{H}^* \to A^1(P)$.

Let $\mathscr{H}_p \times \mathrm{id}$ be the pseudogroup of local diffeomorphisms of $P \times \mathbb{R}^{n'}$ generated by products of local diffeomorphisms in $\mathscr{H}_P$ and restrictions of the identity map of $\mathbb{R}^{n'}$. $\mathscr{H}_P \times \mathrm{id}$ canonically defines a pseudogroup $\mathscr{H}_P \times_{H'} \mathrm{id}$ of local diffeomorphisms of $P \times_{H'} \mathbb{R}^{n'}$.

Theorem 3.1 [13], [19]. *In the above situation, there exists an $\mathscr{H}$-invariant open neighborhood U of F such that the restriction of $\mathscr{H}$ to U is equivalent to the pseudogroup $\mathscr{H}_P \times_{H'} \mathrm{id}$ on $P \times_{H'} \mathbb{R}^{n'}$, which in turn is equivalent to its restriction to $P \times_{H'} B$ for any open ball $B \subseteq \mathbb{R}^{n'}$ centered at the origin.*

To describe the equivalence of Theorem 3.1, we shall use the following lemma whose proof is an easy exercise.

Lemma 3.2. *Let $\mathscr{G}$ be a pseudogroup of local homeomorphisms of a locally compact topological space X. If $X/\mathscr{G}$ is compact, then there is a relatively compact open subset $X_0 \subseteq X$ intersecting all the orbits of $\mathscr{G}$.*

Let $M_0 \subseteq M$ be the open subset given by the above result, and $\mathscr{H}_0$ the restriction of $\mathscr{H}$ to M_0. Since M_0 intersects all the $\mathscr{H}$-orbits, the inclusion $M_0 \subseteq M$ generates an equivalence between $\mathscr{H}_0$ and $\mathscr{H}$.

Let $F_0 = F \cap M_0$, and $P_0 = P|_{F_0}$. Also let V be a tubular neighborhood of radius λ of the zero section in $(TF_0)^{\perp}$, and B the ball in $\mathbb{R}^{n'}$ of radius λ centered at the origin. Since M_0 is relatively compact, we can take λ small enough so that the exponential map of M is a diffeomorphism of V onto some open subset $U_0 \subseteq M$. Then the composition of the canonical identity $P_0 \times_{H'} B \equiv V$ with the exponential map defines a diffeomorphism Φ of $P_0 \times_{H'} B$ onto U_0. Moreover the restriction of $\mathscr{H}_P \times_{H'} \mathrm{id}$ to $P_0 \times_{H'} B$ corresponds by Φ to the restriction of $\mathscr{H}$ to U_0. Then, since $P_0 \times_{H'} B$ and U_0 respectively intersect all the orbits in $P \times_{H'} B$ and $U = \mathscr{H}(U_0)$, Φ generates an equivalence between $\mathscr{H}_P \times_{H'} \mathrm{id}$ and the restriction of $\mathscr{H}$ to U.

Since Φ generates an equivalence of pseudogroups, Φ^* defines an isomorphism of graded differential algebras

$$A(U)_{\mathscr{H}} \cong A(P \times_{H'} B)_{\mathscr{H}_P \times_{H'} \mathrm{id}} \subseteq A(P \times_{H'} \mathbb{R}^{n'})_{\mathscr{H}_P \times_{H'} \mathrm{id}}. \tag{3.1}$$

Now $A(P \times_{H'} \mathbb{R}^{n'})_{\mathscr{H}_P \times_{H'} \mathrm{id}}$ can be described as follows.

Proposition 3.3. *Let $\mathscr{L}$ be an H'-normed vector space of dimension one. Then there are isomorphisms of graded algebras*

$$A(P \times_{H'} \mathbb{R}^{n'})_{\mathscr{H}_P \times_{H'} \mathrm{id}}$$

$$\cong A(P \times \mathbb{R}^{n'})_{\mathscr{H}_P \times \mathrm{id}, H', \imath'=0} \tag{3.2}$$

$$\cong (A(P)_{\mathscr{H}_P} \otimes A(\mathbb{R}^{n'}))_{H', \imath'=0} \tag{3.3}$$

$$\cong (A(P)_{\mathscr{H}_P, \imath'=0} \otimes \bigwedge \mathfrak{H}'^* \otimes A(\mathbb{R}^{n'}))_{H', \imath'=0} \tag{3.4}$$

$$\cong (A(P)_{\mathscr{H}_P, \imath'=0} \otimes A(\mathbb{R}^{n'}))_{H'} \tag{3.5}$$

$$\cong (A(P, \mathscr{L})_{\mathscr{H}_P, \imath'=0} \otimes A(\mathbb{R}^{n'}, \mathscr{L}))_{H'}. \tag{3.6}$$

Here we consider the H'-invariance and interior products $\imath'$ defined by the H'-actions on $P \times H' \times \mathbb{R}^{n'}$ and $P \times \mathbb{R}^{n'}$ respectively given by $a(z, b, v) = (za^{-1}, ba^{-1}, av)$ and $a(z, v) = (za^{-1}, av)$ for $a, b \in H'$, $z \in P$, and $v \in B$.

Proof. If $\rho: P \times B \to P \times_{H'} B$ denotes the quotient space projection, ρ^* clearly defines (3.2).

The canonical injection of $A(P) \otimes A(\mathbb{R}^{n'})$ into $A(P \times \mathbb{R}^{n'})$ defines an isomorphism of $A(P)_{\mathscr{H}_P} \otimes A(\mathbb{R}^{n'})$ onto $A(P \times \mathbb{R}^{n'})_{\mathscr{H}_P \times \mathrm{id}}$ since $\mathscr{H}_P$ has dense orbits, and thus also defines (3.3).

(3.4) is induced by the isomorphism [12, §8.4, Theorem I]

$$(3.7)\qquad A(P)_{\imath'=0}\otimes\bigwedge\mathfrak{H}'^{*}\cong A(P),\qquad \alpha\otimes\gamma\mapsto\alpha\wedge\omega_{\wedge}\gamma,$$

where $\omega_{\wedge}\colon\bigwedge\mathfrak{H}'^{*}\to A(P)$ is the canonical extension of the algebraic connection $\omega\colon\mathfrak{H}'^{*}\to A^{1}(P)$.

Let $\eta\colon P\times H'\times\mathbb{R}^{n'}\to P\times H'\times\mathbb{R}^{n'}$ be the diffeomorphism defined by $\eta(z,a,v)=(za^{-1},a,av)$. Let T and T' be the H'-actions on $P\times H'\times\mathbb{R}^{n'}$ defined by

$$T_b(z,a,v)=(zb^{-1},ab^{-1},bv),\qquad T'_b(z,a,v)=(z,ba,v).$$

We have

$$\eta T_b\eta^{-1}(z,a,v)=(z,ab^{-1},v),$$
$$\eta T'_b\eta^{-1}(z,a,v)=(zb^{-1},ba,bv).$$

Therefore we get a commutative diagram

$$\begin{array}{ccc} A(P\times H'\times\mathbb{R}^{n'}) & \xrightarrow{\eta^{*}} & A(P\times H'\times\mathbb{R}^{n'})\\ \uparrow & & \uparrow\\ (A(P)_{\mathscr{H}_P,\imath'=0}\otimes\bigwedge\mathfrak{H}'^{*}\otimes A(\mathbb{R}^{n'}))_{H',\imath'=0} & \longrightarrow & (A(P)_{\mathscr{H}_P,\imath'=0}\otimes A(\mathbb{R}^{n'}))_{H'} \end{array}$$

where the vertical arrows denote the canonical injections, and the lower horizontal arrow is an isomorphism. Hence (3.5) follows.

Since the induced H'-action on $\mathscr{L}\otimes\mathscr{L}$ is trivial, the canonical isomorphism

$$A(P,\mathscr{L})_{\mathscr{H}_{P'},\imath'=0}\otimes A(\mathbb{R}^{n'},\mathscr{L})\cong A(P)_{\mathscr{H}_P,\imath'=0}\otimes A(\mathbb{R}^{n'})$$

preserves H'-invariant elements, and thus defines (3.6). q.e.d.

(3.1)–(3.6) define isomorphisms

$$(3.8)\qquad \begin{aligned} A(U)&\cong(A(P,\mathscr{L})_{\mathscr{H}_P,\imath'=0}\otimes A(B,\mathscr{L}))_{H'}\\ A_{cp}(U)&\cong(A(P,\mathscr{L})_{\mathscr{H}_P,\imath'=0}\otimes A_c(B,\mathscr{L}))_{H'}. \end{aligned}$$

In particular

$$(3.9)\qquad C^{\infty}(U)_{\mathscr{H}}\cong C^{\infty}(B)_{H'},\qquad C^{\infty}_{cp}(U)_{\mathscr{H}}\cong C^{\infty}_{c}(B)_{H'}.$$

Now we will prove that there are special choices of metrics and strictly positive functionals that behave nicely with respect to (3.8) and (3.9).

Proposition 3.4. *Let g' denote both any H'-invariant positive definite inner product on $\mathbb{R}^{n'}$, and the induced Riemannian metric on $\mathbb{R}^{n'}$. Let g_P be any $\mathscr{H}_P$-invariant metric on P. Then there is an $\mathscr{H}$-invariant metric on U that, on differential forms, corresponds to $g_P \times g'$ by (3.8). Moreover there is an $O(n)$-invariant volume element on W such that I on $C^\infty_{cp}(U)_{\mathscr{H}}$ corresponds to I' on $C_c(B)_{H'}$ by (3.9), where I' is the integration functional with respect to the g'-volume element.*

Proof. Let $g_{H'}$ be any bi-invariant metric on H'. Consider the metric $g_1 = \eta^*(g_P \times g_{H'} \times g')$ on $P \times H' \times B$, where η is the diffeomorphism defined in the proof of Proposition 3.3. Let g_2 be the Riemannian metric on $P \times B$ defined in the following way. For $(z, v) \in P \times B$, we have $T_{(z,v)}(P \times B) \equiv T_z P \oplus T_v B$, and $T_{(z,e,v)}(P \times H' \times B) \equiv T_z P \oplus T_e H' \oplus T_v B$, where e denotes the identity element of H'. Let X, X' be horizontal vectors in $T_z P$, $Y, Y' \in \mathfrak{H}'$, Y^*, Y'^* the corresponding fundamental vector fields on P, and $Z, Z' \in T_v B$. Define

$$g_2((X + Y^*_z, Z), (X' + Y'^*_z, Z)) = g_1((X, Y_e, Z), (X', Y'_e, Z')).$$

g_2 is $\mathscr{H} \times$ id- and H'-invariant (with respect to the diagonal H'-action). So g_2 defines an $\mathscr{H} \times_{H'}$ id-invariant metric on $P \times_{H'} B$. It is easy to check that, on differential forms, the corresponding $\mathscr{H}$-invariant metric g on U corresponds to $g_P \times g'$ by (3.8).

For $x \in F$, take any $z \in \pi^{-1}(x)$ and let $E = \overline{\widehat{\mathscr{H}} z}$ as before. Also let $x' = \pi_b(z)$ and $F' = O(n)x'$. The maps

$$\pi_*: T_z E^\perp \cap \operatorname{Ker}(\pi_{*z})^\perp \to T_x F^\perp,$$
$$\pi_{b^*}: T_z E^\perp \cap \operatorname{Ker}(\pi_{*z})^\perp \to T_{x'} F'^\perp,$$

are clearly isomorphisms. So $O(n)_{x'}$ acts isometrically on $(\mathbb{R}^{n'}, g')$, and by Koszul Slice Theorem, $\pi_b \pi^{-1}(U)$ is equivariantly diffeomorphic to $O(n) \times_{O(n)_{x'}} B$. It is easy to check that H' is the image of the homomorphism $O(n)_{x'} \to O(n')$ given by the above action. So we get an isomorphism

$$(3.10) \qquad C^\infty(\pi_b \pi^{-1}(U))_{O(n)} \cong C^\infty(B)_{H'}, \qquad f \mapsto f',$$

defined by $\rho^* f = \mathrm{pr}_2^*(f')$, where $\rho: O(n) \times B \to \pi_b \pi^{-1}(U)$ is the induced projection. Furthermore the above equivariant diffeomorphism can be chosen so that (2.2), (3.9), and (3.10) define a commutative diagram

$$\begin{array}{ccc} C^\infty_{cp}(U)_{\mathscr{H}} & \longrightarrow & C^\infty_c(\pi_b\pi^{-1}(U))_{O(n)} \\ & \searrow \quad \swarrow & \\ & C^\infty_c(B)_{H'} & \end{array}$$

Therefore the last statement of the proposition follows directly from the following lemma.

Lemma 3.5. *Let g' be any positive definite inner product on $\mathbb{R}^m$, and G a compact Lie group, and suppose that there is an isometric action of a closed subgroup $H \subseteq G$ on $(\mathbb{R}^m, g')$. Let $U = G \times_H \mathbb{R}^m$ with the canonical left action of G, and $\rho\colon G \times \mathbb{R}^m \to U$ the canonical projection. Consider the isomorphism $C^\infty_c(U)_G \cong C^\infty_c(\mathbb{R}^m)_H$, $f \mapsto f'$, defined by $\rho^* f = \mathrm{pr}_2^*(f')$. Then there is a G-invariant metric g on U such that*

$$\int_U f \operatorname{vol} = \int_{\mathbb{R}^m} f' \operatorname{vol}'$$

for all $f \in C^\infty_c(U)_G$, where vol *and* vol$'$ *respectively denote the g- and g'-volume elements.*

Proof. Consider a bi-invariant metric on $O(m)$ such that the canonical map $O(m) \to O(m)/O(m-1) \equiv \mathbb{S}^{m-1}$ is a Riemannian submersion onto the unit sphere with the restriction of the Riemannian metric defined by g' on $\mathbb{R}^m$, which will be also denoted by g'. Choose a bi-invariant metric g_G on G so that the homomorphism $H \to O(m)$, given by the H-action on $\mathbb{R}^m$, is a Riemannian submersion onto its image. Then, for any $v \in \mathbb{S}^{m-1}$, the map $H \to \mathbb{S}^{m-1}$, $b \mapsto bv$, is a Riemannian submersion onto its image. Consider the diffeomorphism $\eta\colon H \times \mathbb{R}^m \to H \times \mathbb{R}^m$ given by $\eta(b, v) = (b, bv)$, and let $g_1 = \eta^*(g_H \times g')$, where g_H is the restriction of g_G to H. Define a metric g_2 on $G \times \mathbb{R}^m$ in the following way. For $(a, v) \in G \times \mathbb{R}^m$, we have the canonical identity $T_{(a,v)}(G \times \mathbb{R}^m) \equiv T_aG \oplus T_v\mathbb{R}^m$. Let $X, X' \in T_a(aH)^\perp$, $Y, Y' \in T_a(aH)$, and $Z, Z' \in T_v\mathbb{R}^m$. Then $(L_{a*}^{-1}Y, Z), (L_{a*}^{-1}Y', Z') \in T_eH \oplus T_v\mathbb{R}^m \equiv T_{(e,v)}(H \times \mathbb{R}^m)$, where e is the identity element of G. Define

$$g_2((X+Y,Z), (X'+Y',Z')) = g_G(X, X') + g_1((L_{a*}^{-1}Y, Z), (L_{a*}^{-1}Y', Z')).$$

It is easy to check that g_2 is invariant by the diagonal action of H, and defines a G-invariant metric g on U so that ρ is a Riemannian submersion.

Fix any $v \in \mathbb{R}^m$, and let $x = \rho(e, v)$. The formula

$$\operatorname{Vol}(Gx) = \operatorname{Vol}(Hv)\operatorname{Vol}(G)/\operatorname{Vol}(H) \tag{3.11}$$

is a direct consequence of the following ones:

(3.12) $$\mathrm{Vol}(Hv) = \|v\|'^{d}\mathrm{Vol}(H)/\mathrm{Vol}(H_v),$$

(3.13) $$\mathrm{Vol}(Gx) = \|v\|'^{d}\mathrm{Vol}(G)/\mathrm{Vol}(G_x) = \|v\|'^{d}\mathrm{Vol}(G)/\mathrm{Vol}(H_v),$$

where $d = \dim(Hv)$, and $\|\cdot\|'$ is the g'-norm. (3.12) and (3.13) can be proved as follows.

Let $\mathfrak{g}^-$, $\mathfrak{H}^-$, and $\mathfrak{H}_v^-$ be the Lie algebras of right invariant vector fields on G, H, and H_v respectively. There are canonical inclusions $\mathfrak{H}_v^- \subseteq \mathfrak{H}^- \subseteq \mathfrak{H}^-$.

To compute $\mathrm{Vol}(Hv)$, consider the fiber bundle $\tau\colon H \to Hv$ defined by $\tau(b) = bv$. For any $w = bv \in Hv$, we have $\tau^{-1}(w) = H_w = bH_v b^{-1}$, which has the same volume as $H_v = \tau^{-1}(v)$. We get

$$\mathrm{Ker}(\tau_{*b}) = \{X_b | X \in \mathrm{Ad}_b(\mathfrak{H}_v^-)\}, \qquad \mathrm{Ker}(\tau_{*b})^{\perp} = \{X_b | X \in \mathrm{Ad}_b(\mathfrak{H}_v^-)^{\perp}\}.$$

Moreover it is easy to check that

$$\|\tau_b(X_b)\| = \|v\|'\|X_b\|$$

for any $X \in \mathrm{Ad}_b(\mathfrak{H}_v^-)^{\perp}$, yielding (3.12).

To compute $\mathrm{Vol}(Gx)$, consider the fiber bundle $\sigma\colon G \to Gx$ given by $\sigma(a) = ax$. Then $\sigma = \rho\tilde{\sigma}$, where $\tilde{\sigma}\colon G \to G \times \mathbb{R}^m$ is given by $\tilde{\sigma}(a) = (a, v)$. For any $y = ax \in Gx$, we have $\sigma^{-1}(y) = G_y = aG_x a^{-1}$, which has the same volume as $G_x = \sigma^{-1}(x)$. Since $G_x = H_v$, as can be easily checked, we get

$$\mathrm{Ker}(\sigma_{*a}) = \{X_a | X \in \mathrm{Ad}_a(\mathfrak{H}_v^-)\}, \qquad \mathrm{Ker}(\sigma_{*a})^{\perp} = \{X_a | X \in \mathrm{Ad}_a(\mathfrak{H}_v^-)^{\perp}\}.$$

Moreover,

$$\mathrm{Ker}(\rho_{*(a,v)}) = \{(\psi_*(Y)_a, Y_v^*) | Y \in \mathfrak{H}^-\},$$

where $\psi\colon G \to G$ is the inversoin map, and, for $Y \in \mathfrak{H}^-$, Y^* is the corresponding fundamental vector field defined by the H-action on $\mathbb{R}^m$.

Take any $X \in \mathrm{Ad}_a(\mathfrak{H}^-)^{\perp}$, and any $X' \in \mathrm{Ad}_a(\mathfrak{H}^- \cap \mathfrak{H}_v^{\perp})$. Then

$$X_a \perp T_a(aH) = \{\psi_*(Y)_a | Y \in \mathfrak{H}^-\}.$$

So $\tilde{\sigma}_*(X_a) = (X_a, O_v)$ is g_2-orthogonal to $\mathrm{Ker}(\rho_{*(a,v)})$, yielding

$$\|\sigma_*(X_a)\| = \|\tilde{\sigma}_*(X_a)\| = \|X_a\|.$$

On the other hand,

$$X'_a \in T_a(aH) \cap T_a(aH_v)^{\perp} = \{\psi_*(Y)_a | Y \in \mathfrak{H}^- \cap \mathfrak{H}_v^{-\perp}\}.$$

Thus $X'_a = \psi_*(Y)_a$ for some $Y \in \mathfrak{H}^- \cap \mathfrak{H}_v^{-\perp}$, and therefore

$$\tilde{\sigma}_*(X'_a) = (X'_a, O_v) = (\psi_*(Y)_a, Y_v^*) + (O_a, -Y_v^*),$$

with $(\psi_*(Y)_a, Y_v^*) \in \operatorname{Ker}(\rho_{*(a,v)})$. Moreover it can be easily checked that

$$\eta_*(O_a, -Y_v^*) = (O_a, -(\operatorname{Ad}_a Y)^*_{av}),$$
$$\eta_*(\psi_*(Z)_a, Z_v^*) = (\psi_*(Z)_a, O_{av}) \quad \text{for all } Z \in \mathfrak{H}^-.$$

Thus $(O_a, -Y_v^*)$ is g_2-orthogonal to $\operatorname{Ker}(\rho_{*(a,v)})$, yielding

$$\|\sigma_*(X'_a)\| = \| - (\operatorname{Ad}_a Y)^*_{av}\| = \|av\|'\|(\operatorname{Ad}_a Y)_e\| = \|v\|'\|X'_a\|.$$

Furthermore, $\tilde{\sigma}_*(X_a)$ is g_2-orthogonal to $\tilde{\sigma}_*(X'_a)$. So (3.13) follows.

Now let $\pi_1: U \to G\backslash U$ and $\pi_2: \mathbb{R}^m \to H\backslash\mathbb{R}^m$ be the canonical projections onto the orbit spaces of the above actions. Let $V_1 \subseteq U$ and $V_2 \subseteq \mathbb{R}^m$ be the open subsets of regular H- and G-orbits respectively. There is a canonical identity $G\backslash U \equiv H\backslash\mathbb{R}^m$ so that $\pi_1(V_1) = \pi_2(V_2)$ $(= W)$, which is a manifold. Let g_w be the unique Riemannian metric on W so that $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are Riemannian submersions. Let μ be the measure on $G\backslash U$ that is concentrated on W, where it is given by the g_W-volume element. For $f \in C_c^\infty(U)_G$, let $\overline{f}$ be the unique continuous function on $G\backslash U$ such that $\pi_1^*(\overline{f}) = f$. Then $\pi_2^*(\overline{f}) = f'$, and therefore

$$\begin{aligned}\int_U f \operatorname{vol}_g &= \int_{G\backslash U} \operatorname{Vol}(\pi_1^{-1}(\theta))\overline{f}(\theta)\, d\mu(\theta)\\ &= \frac{\operatorname{Vol}(G)}{\operatorname{Vol}(H)} \int_{G\backslash U} \operatorname{Vol}(\pi_2^{-1}(\theta))\overline{f}(\theta)\, d\mu(\theta) \quad (\text{by } (3.11))\\ &= \frac{\operatorname{Vol}(G)}{\operatorname{Vol}(H)} \int_{\mathbb{R}^m} f' \operatorname{vol}'.\end{aligned}$$

Hence the result follows by multiplying this g by a positive constant.

4. Existence of nondegenerate $\mathscr{H}$-Morse functions

For any $\mathscr{H}$-orbit closure F, any $x \in F$, and any transversal Σ of F at x, it is easy to check that F is a nondegenerate critical orbit closure of some $f \in C^\infty(M)_{\mathscr{H}}$ iff x is a nondegenerate critical point of $f|_\Sigma$.

The following result assures the existence of nondegenerate $\mathscr{H}$-Morse functions.

Theorem 4.1. *Let k be a nonnegative integer. Then in the above situation, any function in $C^\infty(M)_{\mathscr{H}}$ can be uniformly C^k-approximated by a nondegenerate $\mathscr{H}$-Morse function.*

Here "uniform C^k-approximation" means uniform approximation of all the ith derivatives for $i \leq k$.

Proof. Take any function $f \in C^\infty(M)_{\mathscr{H}}$. Using the Salem description of $\mathscr{H}$ (§2), the proof of this theorem can be reduced as follows. If s is a local section of $\widehat{M}$ defined around some point $x \in M$, and $\Sigma \subseteq \mathrm{Dom}(s)$ is a transversal to $\overline{\mathscr{H}x}$ at x, then it is easy to see that $\Sigma_W = \pi_b s(\Sigma) \subseteq W$ is a transversal to the $O(n)$-orbit of $\pi_b(s(x))$. Moreover $(\pi_b s)^*(f|_\Sigma) = f_W|_{\Sigma_W}$, where f_W corresponds to f by (2.2). Therefore the proof follows directly from the case solved in [24], where $\mathscr{H}$ is generated by a compact Lie group action on a compact manifold.

5. Proof of Theorem A

Suppose that ϕ is a positive even Schwartz function on $\mathbb{R}$ with $\phi(0) = 1$. Then $\phi(D_I)$ is of trace class, and let

$$\mu_j = \mathrm{Tr}(\phi(D_I)|_{A^j(M)_{\mathscr{H}}}).$$

The following proposition follows with the same arguments as in [19, (12.3)] by using Theorem 2.6, especially its Hodge type decomposition.

Proposition 5.1. *We have the inequalities*

$$\beta_0 \leq \mu_0,$$
$$\beta_1 - \beta_0 \leq \mu_1 - \mu_0,$$
$$\beta_2 - \beta_1 + \beta_0 \leq \mu_2 - \mu_1 + \mu_0,$$

etc., and the equality

$$\sum_{j=1}^{n}(-1)^j \beta_j = \sum_{j=1}^{n}(-1)^j \mu_j.$$

For any nondegenerate $\mathscr{H}$-Morse function f on M, and any real number s, let

$$d_s = e^{-sf} d e^{sf} = d - s\, df\wedge,$$
$$\delta_{I,s} = e^{sf}\delta_I e^{-sf} = \delta_I - s\, df\neg = \delta - (\gamma_I + s\, df)\neg.$$

Let $D_{I,s} = d_s + \delta_{I,s} = D_I + sH$, where $H = df\wedge - df\neg$, and let $\Delta_{I,s} = D_{I,s}^2$. With the same arguments as in [21, (12.8)], we get the following.

Lemma 5.2. (i) *H^2 is the endomorphism given by multiplication by $|df|^2$.*

(ii) *$HD_{I,s} + D_{I,s}H$ is an endomorphism of order zero.*

For any fixed $\varepsilon > 0$ we have the following.

Lemma 5.3. *There is a positive even Schwartz function ϕ on $\mathbb{R}$ such that $\phi(0) = 1$, the Fourier transform $\hat{\phi}$ has compact support, and*

$$|\mathrm{Tr}(\phi(D_{I_F})|_{A^j(F,\mathscr{O}_F)_{\mathscr{H}}}) - \dim H^j(F, \mathscr{O}_F)_{\mathscr{H}}| < \varepsilon$$

for all $F \in \mathrm{Crit}_{\mathscr{H}}(f)$ and all j, where I_F is the identity map on $C^\infty(F)_{\mathscr{H}} \equiv \mathbb{R}$.

Proof. Clearly we always have

$$\dim H^j(F, \mathscr{O}_F)_{\mathscr{H}} \leq \mathrm{Tr}(\phi(D_{I_F})|_{A^j(F,\mathscr{O}_F)_{\mathscr{H}}})$$

for any positive even Schwartz function ϕ with $\phi(0) = 1$.

Choose a positive even Schwartz function ψ such that $\psi(0) = 1$ and $\hat{\psi}$ has compact support. For each $u \in \mathbb{R}$, let ψ_u be the Schwartz function defined by $\psi_u(x) = \psi(ux)$. Then we have

$$(\psi_u)^\wedge(\xi) = \frac{1}{u}\hat{\psi}(\xi/u).$$

Since each $A^j(F, \mathscr{O}_F)_{\mathscr{H}}$ is of finite dimension, and F is an $\mathscr{H}$-orbit closure, the result follows by taking $\phi = \psi_u$ for u large enough. q.e.d.

Since the support of $\hat{\phi}$ is compact, it is contained in some interval $[-\rho, \rho]$ for some large enough $\rho > 0$. The following lemma can be proved with the same arguments as in [21, (12.10)], by using Theorem 2.6 and Lemma 5.2.

Lemma 5.4. *On the product of M and the complement of a 2ρ-neighborhood of the union of the critical orbit closures of f, the smoothing $\mathscr{H}$-kernel of $\phi(D_{I,s})$ tends uniformly to zero as $s \to \infty$.*

Even though ρ is fixed, by dilating the metric transversely to the critical orbit closures, the 2ρ-neighborhood of the critical orbit closures will be made small. So, as in [21, Chapter 12], the above result will be used to obtain the trace of $\phi(D_{I,s})$ as the sum of the contributions from the critical orbit closures.

For any fixed $F \in \mathrm{Crit}_{\mathscr{H}}(f)$, we will use the notation of §3. Let $f' \in C^\infty(B)$ be the function that corresponds to $f|_U$ by (3.7). The origin 0 is a nondegenerate critical point of f'. Taking Morse coordinates (x^j) on some H'-invariant open neighborhood B' of 0 in B, we have

$$f' = \frac{1}{2}\sum_j \lambda_j (x^j)^2.$$

We can assume $B' = B$. The number of negative λ_j's is equal to the index of f' at 0, which will be denoted simply by m.

Consider on B the flat Euclidean metric g' with respect to the Morse coordinates (x^j). For each $r > 0$ small enough, let B_r be the g'-ball of radius r in B centered at 0, and $U_r = \Phi(P \times_{H'} B_r)$. Since f' is H'-invariant, clearly g' is also H'-invariant. Moreover g' is the restriction of a flat Euclidean metric on $\mathbb{R}^{n'}$, which will be also denoted by g'.

For each real number $u > 0$, we can also take the coordinates $y^j = ux^j$ on B, for which

$$f' = \frac{1}{2}\sum_j \frac{\lambda_j}{u^2}(y^j)^2 .$$

Therefore we can choose the Morse coordinates (x^j) so that the closure of $B_{5\rho}$ in $\mathbb{R}^{n'}$ is contained in B. We can suppose that the Morse coordinates on B are the restriction of the standard coordinates (x^j) on $\mathbb{R}^{n'}$. Then f' is the restriction to B of the H'-invariant function

$$\frac{1}{2}\sum_j \lambda_j (x^j)^2$$

on $\mathbb{R}^{n'}$, which will be also denoted by f'. Since f' is H'-invariant, we have $H' \subseteq O(m) \times O(n' - m)$. Let H'_- be the projection of H' to $O(m)$. The projection $H' \to H'_-$ defines an isometric action of H' on $\mathscr{L} = \bigwedge^m \mathbb{R}^m$. We thus have the isomorphisms (3.6) and (3.8) for this particular choice of $\mathscr{L}$.

Let g_P be any H'- and $\mathscr{H}_P$-invariant metric on P, and g_F the $\mathscr{H}$-invariant metric defined by g_P on F. Let g be the $\mathscr{H}$-invariant metric on U given by Proposition 3.4 for these choices of g' and g_P. Let d' and δ' respectively denote the differential and g'-codifferential operators defined on both $A(\mathbb{R}^{n'})$ and $A(\mathbb{R}^{n'}, \mathscr{L})$. For $s > 0$, let also $d'_s = d' - s\,df' \wedge$, $\delta'_s = \delta' - s\,df' \lrcorner$, $D'_s = d'_s + \delta'_s$, and $\Delta'_s = D'^2_s$ defined on both $A(\mathbb{R}^{n'})$ and $A(\mathbb{R}^{n'}, \mathscr{L})$. Then we have [21, (12.13)]

$$\Delta'_s = \sum_j \left(- \left(\frac{\partial}{\partial x^j}\right)^2 + s^2\lambda_j^2 (x^j)^2 + s\lambda_j [dx^j \lrcorner, dx^j \wedge] \right) .$$

Δ'_s is an essentially selfadjoint operator with discrete spectrum [21, (12.14)]. For any integer $k \geq 0$, let $e^k_{0,s}, e^k_{1,s}, \cdots$ be an orthonormal frame of $L^2(A^k_c(\mathbb{R}^{n'}))$ given by eigenforms of Δ'_s, and $\mu^k_{0,s} \leq \mu^k_{1,s} \leq \cdots$ the corresponding eigenvalues. By [21, (12.14) and (12.15)] we have

$\mu^m_{0,s} = 0$, and can take

$$e^m_{0,s} = \left(s^{n'/2} \pi^{-n'/4} \prod_j \lambda_j^{1/2} \right) \exp\left(-s \sum_j \lambda_j (x^j)^2/2 \right) dx^1 \wedge \cdots \wedge dx^m ,$$

where we assume that the first m of the λ_j's are negative. Moreover $\mu^k_{i,s}$ is of order s for $(k, i) \neq (m, 0)$. If l is a normalized generator of $\mathscr{L}$ then $\tilde{e}^k_{i,s} = e^k_{i,s} \otimes l$ $(i = 0, 1, 2, \cdots)$ is an orthonormal frame of $L^2(A_c(\mathbb{R}^{n'}, \mathscr{L}))$ given by eigenforms of Δ'_s, whose corresponding eigenvalues are also the numbers $\mu^k_{i,s}$.

Let

$$A = (A(P, \mathscr{L})_{\mathscr{H}_P, \iota'=0} \otimes A_c(\mathbb{R}^{n'}, \mathscr{L}))_{H'} ,$$

$$A' = A(P, \mathscr{L})_{\mathscr{H}_P, H', \iota'=0} \otimes A_c(\mathbb{R}^{n'}, \mathscr{L})_{H'} \subseteq A ,$$

and let I_P be the identity map on $C^\infty(P)_{\mathscr{H}_P} \equiv \mathbb{R}$. $(A(P, \mathscr{L})_{\mathscr{H}_P, \iota'=0},$ $\langle \cdot, \cdot \rangle_{I_P})$ is an H'-Riemannian vector space of finite dimension (since $\mathscr{H}_P$ has dense orbits). Then

$$\mathscr{V} = A(P, \mathscr{L})_{\mathscr{H}_P, \iota'=0} \otimes \bigwedge T\mathbb{R}^{n'*} \otimes \mathscr{L} ,$$

and

$$\mathscr{V}' = A(P, \mathscr{L})_{\mathscr{H}_P, H', \iota'=0} \otimes \bigwedge T\mathbb{R}^{n'*} \otimes \mathscr{L}$$

are finite rank H'-Riemannian graded vector bundles over $\mathbb{R}^{n'}$, and there are canonical identities

$$A \equiv C^\infty_c(\mathscr{V})_{H'} , \qquad A' \equiv C^\infty_c(\mathscr{V}')_{H'} .$$

Let $\mathfrak{H}$ and $\mathfrak{H}'$ be the Hilbert spaces of H'-invariant L^2-bounded sections of $\mathscr{V}$ and $\mathscr{V}'$ respectively, with the inner product defined by g'.

Let $\nabla : A(P)_{\mathscr{H}_P} \to A(P)_{\mathscr{H}_P, \iota'=0}$ and $\chi : \mathfrak{H}'^* \to A^2(P)_{\mathscr{H}_P, \iota'=0}$ respectively denote the covariant derivative and the curvature of the algebraic connection ω [12, Chapter VIII, §§8.5 and 8.6]. For a pair of dual bases e_ν, e^*_ν of $\mathfrak{H}'$ and $\mathfrak{H}'^*$, consider the operators

$$d_\theta = \sum_\nu w \theta_{e_\nu} \otimes e^*_\nu \wedge , \qquad h_\chi = \sum_\nu w \chi(e^*_\nu) \wedge \otimes \iota_{e_\nu}$$

on $A(P)_{\mathscr{H}_P, \iota'=0} \otimes \bigwedge \mathfrak{H}'^*$, where w denotes the degree involution, and θ_{e_ν} the Lie derivative with respect to the fundamental vector field associated to e_ν. Then

$$w \otimes d_{\mathfrak{H}'} + d_\theta + h_\chi + \nabla_{\iota'=0} \otimes \mathrm{id}$$

corresponds by (3.7) to the de Rham derivative on $A(P)_{\mathscr{H}_P}$ [12, Chapter VIII, §8.7, Theorem II]. Let

$$R = d_\theta + h_\chi + \nabla_{\iota'=0} \otimes \mathrm{id},$$

and let S be the operator on A that corresponds to $R \otimes \mathrm{id}$ by (3.4)–(3.6). Clearly the operator $w \otimes d' + S$ on A corresponds to the de Rham derivative on $A(P \times \mathbb{R}^{n'})_{\mathscr{H}_P \times \mathrm{id}, H', \iota'=0}$ by (3.2)–(3.6).

Lemma 5.5. *For any* $\psi \in C^\infty(\mathbb{R}^{n'})$, *we have*

$$(w \otimes d'\psi\wedge)(S + S^*) + (S + S^*)(w \otimes d'\psi\wedge) = 0,$$
$$(w \otimes d'\psi\neg)(S + S^*) + (S + S^*)(w \otimes d'\psi\neg) = 0.$$

Proof. Since $(d'\psi\wedge)^* = d'\psi\neg$, it is enough to prove only the first equality. Clearly η^* is the identity on $1 \otimes 1 \otimes A(\mathbb{R}^{n'})_{H', \iota'=0} \equiv A(\mathbb{R}^{n'})_{H', \iota'=0}$, and thus $g_1 = g_P \times g_{H'} \times g' \equiv g'$ on this subspace of $A(P \times H' \times \mathbb{R}^{n'})$. Since $d'\psi \in A(\mathbb{R}^{n'})_{H', \iota'=0}$, the first equality is equivalent to

(5.1) $(w \otimes w \otimes d'\psi\wedge)((R + R^*) \otimes \mathrm{id}) + ((R + R^*) \otimes \mathrm{id})(w \otimes w \otimes d'\psi\wedge) = 0$

on

$$(A(P)_{\mathscr{H}_P, \iota'=0} \otimes \bigwedge \mathfrak{H}'^* \otimes A(\mathbb{R}^{n'}))_{H', \iota'=0},$$

where R^* denotes the adjoint of R on the finite dimensional Riemannian vector space

$$A(P)_{\mathscr{H}_P, \iota'=0} \otimes \bigwedge \mathfrak{H}'^*.$$

Hence (5.1) follows from the fact R is homogeneous of degree one.

Lemma 5.6. $(w \otimes D')(S + S^*) + (S + S^*)(w \otimes D')$ *on* A *is the restriction of a zero order operator of* $\mathrm{Diff}_{H'}(\mathscr{V})$.

Proof. By Proposition 2.5, it is enough to prove that $(w \otimes D') \cdot (S + S^*) + (S + S^*)(w \otimes D')$ is $C^\infty(\mathbb{R}^{n'})_{H'}$-linear. Take any $\psi \in C^\infty(\mathbb{R}^{n'})_{H'}$, and let Ψ be the operator of multiplication by ψ on $A(\mathbb{R}^{n'}, \mathscr{L})$. Then

$$d'\Psi = d'\psi \wedge + \Psi d', \qquad \delta'\Psi = d'\psi\neg + \Psi\delta'.$$

Since $S + S^*$ is $C^\infty(\mathbb{R}^{n'})_{H'}$-linear, we get

$$\begin{aligned}
&((w \otimes D')(S + S^*) + (S + S^*)(w \otimes D'))(\mathrm{id} \otimes \Psi)\\
&\quad = (\mathrm{id} \otimes \Psi)((w \otimes D')(S + S^*) + (S + S^*)(w \otimes D'))\\
&\qquad + (w \otimes (d'\psi \wedge + d'\psi\neg))(S + S^*) + (S + S^*)(w \otimes (d'\psi \wedge + d'\psi\neg))\\
&\quad = (\mathrm{id} \otimes \Psi)((w \otimes D')(S + S^*) + (S + S^*)(w \otimes D'))
\end{aligned}$$

by Lemma 5.5. q.e.d.

Consider the positive symmetric operator $T_s = (w \otimes D'_s + S + S^*)^2$ on A. By using the main result of [9] and averaging the H'-action on $C_c^\infty(\mathscr{V})$, we see that T_s is an essentially selfadjoint operator on $\mathfrak{H}$, so $T_s = \mathrm{id} \otimes \Delta'_s + \widetilde{S}$, where

$$\widetilde{S} = (w \otimes D'_s)(S + S^*) + (S + S^*)(w \otimes D'_s) + (S + S^*)^2$$

is the restriction of a zero order operator which does not depend on s (by Lemmas 5.5 and 5.6), and thus defines a continuous operator on $\mathfrak{H}$. From [15, Theorem 3.4] (see also [14]), and Lemma 5.2, it also follows that, for s large enough, T_s has a discrete spectrum, and the corresponding eigenspaces in $\mathfrak{H}$ are of finite dimension. So $\phi(\sqrt{T_s})$ is a trace class operator on $\mathfrak{H}$.

Let Ψ be the operator of multiplication on $\mathfrak{H}$ by some H'-invariant nonnegative smooth function ψ on $\mathbb{R}^{n'}$ with compact support and $\psi(0) = 1$. Clearly $\Psi\phi(\sqrt{T_s})$ preserves $\mathfrak{H}'$ and $\mathfrak{H}'^{\perp}$.

Lemma 5.7. $\lim_{s\to\infty} \mathrm{Tr}(\Psi\phi(\sqrt{T_s})|_{\mathfrak{H}'^{\perp}}) = 0$.

Proof. Let

$$f_{i,s} = \sum_{j,k} \alpha_{i,j,k,s} \otimes \tilde{e}^k_{j,s} \qquad (i = 0, 1, 2, \cdots)$$

be an orthonormal frame of $\mathfrak{H}'^{\perp}$ given by the eigenforms of T_s. For each i and each s, $\alpha_{i,j,k,s} \neq 0$ for some $(j, k) \neq (0, m)$ since $1 \otimes \tilde{e}^m_{0,s} \in A'$. Thus we have

$$\begin{aligned}\langle T_s f_{i,s}, f_{i,s}\rangle &= \sum_{j,k} (\|\alpha_{i,j,k,s}\|^2 \mu^k_{j,s} + \langle \widetilde{S} f_{i,s}, f_{i,s}\rangle) \\ &\geq \inf\{\mu^k_{j,s} | (j, k) \neq (0, m)\} - \|\widetilde{S}\|,\end{aligned}$$

which is of order s. So

$$0 \leq \mathrm{Tr}(\Psi\phi(\sqrt{T_s})|_{\mathfrak{H}'^{\perp}}) \leq \sum_i \phi(\langle T_s f_{i,s}, f_{i,s}\rangle)\|\Psi\|,$$

which converges to zero as $s \to \infty$. q.e.d.

Lemma 5.8. $\lim_{s\to\infty} \mathrm{Tr}(\Psi\phi(\sqrt{T_s})|_{\mathfrak{H}'^{\perp}}) = \mathrm{Tr}(\phi(D_{I_F})|_{A^{m-k}(F, \mathscr{O}_F)_{\mathscr{H}}})$.

Proof. Since $\pi_P^* \mathscr{O}_F$ can be canonically identified with the trivial line bundle $P \times \mathscr{L}$, π_P^* defines a canonical isomorphism of graded differential spaces

$$A(F, \mathscr{O}_F)_{\mathscr{H}} \cong A(P, \mathscr{L})_{\mathscr{H}_P, H', \iota'=0}. \tag{5.2}$$

Moreover $\langle\cdot,\cdot\rangle_{I_F}$ corresponds to the restriction of $\langle\cdot,\cdot\rangle_{I_P}$ by (5.2), and thus D_{I_F} corresponds to the restriction of D_{I_P}.

Because F is an $\mathscr{H}$-orbit, for any integer p, $A^p(F, \mathscr{O}_F)_{\mathscr{H}}$ is of finite dimension, say equal to d. Let $f_1, \cdots, f_d$ be an orthonormal frame of $A^p(F, \mathscr{O}_F)_{\mathscr{H}}$ given by the eigenforms of D_{I_F}, and let $\theta_1, \cdots, \theta_d$ be the corresponding eigenvalues. Since $S|_{A'} = d \otimes \mathrm{id}$, we get

$$T_s|_{A'} = \Delta_{I_P} \otimes \mathrm{id} + \mathrm{id} \otimes \Delta'_s$$

by Lemma 1.1. So

$$T_s(\pi_P^* f_i \otimes \tilde{e}^q_{j,s}) = (\theta_i + w(f_i)\sqrt{\mu^q_{j,s}})^2 \pi_P^* f_i \otimes \tilde{e}^q_{j,s},$$

which yields

$$\text{(5.3)} \qquad \mathrm{Tr}(\Psi\phi(\sqrt{T_s})|_{\mathfrak{H}'^{p,q}}) = \sum_{i=1}^{p}\sum_{j=1}^{\infty} \phi(\theta_i + w(f_i)\sqrt{\mu^q_{j,s}})\langle \Psi\tilde{e}^q_{j,s}, \tilde{e}^q_{j,s}\rangle,$$

where $\mathfrak{H}'^{p,q}$ is the closure in $\mathfrak{H}'$ of

$$A^p(P, \mathscr{L})_{\mathscr{H}_P, H', \iota'=0} \otimes A^q(\mathbb{R}^{n'})_{H', \iota'=0}.$$

If $q = m$, (5.3) converges to

$$\sum_{i=1}^{d} \phi(\theta_i) = \mathrm{Tr}(\phi(D_{I_F})|_{A^p(F, \mathscr{O}_F)_{\mathscr{H}}})$$

as $s \to \infty$. If $q \neq m$, (5.3) converges to zero as $s \to \infty$. Thus the result follows. q.e.d.

For $r > 0$, let

$$A_r = (A(P, \mathscr{L})_{\mathscr{H}_P, \iota'=0} \otimes A_c(B_r, \mathscr{L}))_{H'},$$

and let $\mathfrak{H}_r$ be the closure of A_r in $\mathfrak{H}$. If we take $\mathrm{supp}(\psi) \subseteq B_r$, then $\Psi\phi(\sqrt{T_s})(\mathfrak{H}) \subseteq \mathfrak{H}_r$, and thus $\mathrm{Tr}(\Psi\phi(\sqrt{T_s})) = \mathrm{Tr}(\Psi\phi(\sqrt{T_s})|_{\mathfrak{H}_r})$.

Lemma 5.9. *We have*

$$\phi(T_s)(A_{3\rho}) \subseteq A_{4\rho}, \qquad \phi(D_{I,s})(A(U_{3\rho})_{\mathscr{H}}) \subseteq A(U_{4\rho})_{\mathscr{H}}.$$

Moreover $\phi(D_{I,s})\colon A(U_{3\rho})_{\mathscr{H}} \to A(U_{4\rho})_{\mathscr{H}}$ *corresponds to* $\phi(T_s)\colon A_{3\rho} \to A_{4\rho}$ *by* (3.8).

Proof. Since $\hat{\phi}$ is even and supported in $[-\rho, \rho]$, the result follows from Proposition 3.6 with a finite propagation speed argument similar to that given in [21, (12.10)].

End of the proof of Theorem A. For the fixed $\varepsilon > 0$, choose ϕ, and ρ as above. Choose also an $\mathscr{H}$-invariant metric g on M such that each critical orbit closure F has a 4ρ-neighborhood $U_{F,4\rho}$ of the type given by Theorem 3.1, so that the neighborhoods $U_{F,4\rho}$ are disjoint to each other, and the restriction of g to each $U_{F,4\rho}$ is as before.

Since $\beta_j = \dim H(A(M), d_s)$ for all s, the β_j's satisfy the inequalities of Proposition 5.1 with respect to the numbers μ_j^s given as the traces of the operators

$$\phi(D_{I,s})|_{L_I^2(A^j(M)_{\mathscr{H}})}. \tag{5.4}$$

By Theorem 3.6, each μ_j^s is the image by I of the trace of the smoothing $\mathscr{H}$-kernel of the operator (5.4) over the diagonal. So, using Lemma 5.4 we obtain

$$\lim_{s\to\infty} \mu_j^s = \sum_{F\in \mathrm{Cirt}_{\mathscr{H}}(f)} \lim_{s\to\infty} \mathrm{Tr}(\Psi_F \phi(D_{I,s})|_{L_I^2(A^j(M)_{\mathscr{H}})}),$$

where Ψ_F is the operator of multiplication by a nonnegative smooth $\mathscr{H}$-invariant function on M that is equal to 1 on $U_{F,2\rho}$ and supported in $U_{F,3\rho}$. Hence, by Lemmas 5.7, 5.8, and 5.9, the trace of $\Psi_F\phi(D_{I,s})$ on $L_I^2(A^j(M)_{\mathscr{H}})$ converges to the sum of the traces of the operators

$$\phi(D_{I_F})|_{A^{j-m_F}(F,\mathscr{O}_F)_{\mathscr{H}}} \qquad (F\in \mathrm{Crit}_{\mathscr{H}}(f)),$$

from which the result follows by Lemma 5.3 since $\varepsilon > 0$ is arbitrary.

6. Proof of Theorem B

We will use the same notation as in the above section.

There is a filtration of $A(M)_{\mathscr{H}}$ by differential ideals

$$F^k A(M)_{\mathscr{H}} = \bigoplus_r \{\alpha \in A^r(M)_{\mathscr{H}} | \iota_X\alpha = 0 \text{ for } X = X_1\wedge\cdots\wedge X_{r-k+1}, \text{ if the vector fields } X_i \text{ are tangent to the } \mathscr{H}\text{-orbit closures}\}.$$

The associated spectral sequence (E_i, d_i) converges to $H(M)_{\mathscr{H}}$. From (2.1) we have

$$E_1^{\cdot,0} \equiv A(M)_{\mathscr{H},\iota=0} \cong A(W)_{O(n),\iota=0}.$$

So

$$E_2^{\cdot,0} \equiv H(A(M)_{\mathscr{H},\iota=0}, d) \cong H(M/\overline{\mathscr{H}}) \tag{6.1}$$

by (2.3) and the result of [23], yielding a canonical injection of $H^1(M/\overline{\mathscr{H}})$ into $H^1(M)_{\mathscr{H}}$.

By the Hodge type decomposition of Theorem 2.6 applied to $D_{I,s}$, and using $d_s(C^\infty(M)_{\mathscr{H}}) \subseteq A^1(M)_{\mathscr{H},\imath=0}$, we obtain
(6.2)
$$\operatorname{Ker}(d_s) \cap A^1(M)_{\mathscr{H},\imath=0} = (\operatorname{Ker}(D_{I,s}) \cap A^1(M)_{\mathscr{H},\imath=0}) \oplus d_s(C^\infty(M)_{\mathscr{H}}).$$
Hence, in consequence of (6.1) we get

$$\operatorname{Ker}(D_{I,s}) \cap A^1(M)_{\mathscr{H},\imath=0} \cong H^1(M/\overline{\mathscr{H}}). \tag{6.3}$$

Let $F \in \operatorname{Crit}_{\mathscr{H}}(f)$. From (3.8) for $\mathscr{S} = \mathbb{R}$ with the trivial H'-action it follows that

$$A_{cp}(U)_{\mathscr{H},\imath=0} \cong A_c(B)_{H',\imath'=0}, \tag{6.4}$$

so that $\Psi e^m_{0,s} \in A^1_c(B)_{H',\imath'=0}$ iff both $m=1$ and $H'_- = \{\mathrm{id}\}$. Otherwise, $\Psi e^m_{0,s} \perp A^1_c(B)_{H',\imath'=0}$. Thus, with similar arguments to those in the proof of Lemma 5.8, and using Lemma 5.9, we get

$$\lim_{s\to\infty} \operatorname{Tr}(\Psi\phi(D_{I,s})|_{A^1_{cp}(U_{3p})_{\mathscr{H}}}) = \begin{cases} 1, & \text{if } m=1 \text{ and } H'_- = \{\mathrm{id}\}, \\ 0, & \text{otherwise}. \end{cases} \tag{6.5}$$

When $m=1$, clearly $H'_- = \{\mathrm{id}\}$ iff $N_{F,-}$ is $\mathscr{H}$-trivial.

For each $F \in \operatorname{Crit}_{\mathscr{H}}(f)$, take an operator of multiplication Ψ_F as at the end of the proof of Theorem A. Then

$$\begin{aligned} \dim H^1(M/\overline{\mathscr{H}}) &\le \lim_{s\to\infty} \operatorname{Tr}(\phi(D_{I,s})|_{A^1(M)_{\mathscr{H},\imath=0}}) \quad \text{(by (6.3))} \\ &= \lim_{s\to\infty} \sum_{F\in\operatorname{Crit}_{\mathscr{H}}(f)} \operatorname{Tr}(\Psi_F\phi(D_{I,s})|_{A^1(M)_{\mathscr{H},\imath=0}}) \quad \text{(by Lemma 5.4)}. \end{aligned}$$

Hence Theorem B follows from (6.5).

7. Some consequences of Theorems A and B

Suppose $M/\mathscr{H}$ is connected in this section. From [2] we have that $\xi(\mathscr{H}) = 0$ iff both the structural Lie algebra $\mathfrak{g}$ of $\mathscr{H}$ is unimodular, and the $\widehat{\mathscr{H}}$-orbit closures are minimal submanifolds for some $\widehat{\mathscr{H}}$-invariant metric on $\widehat{M}$. The following corollaries of Theorems A and B give conditions for having these properties.

Corollary 7.1. *If there is a nondegenerate $\mathscr{H}$-Morse function f such that $\nu_1(\mathscr{H}, f) < \nu_0(\mathscr{H}, f)$, then $\xi(\mathscr{H}) = 0$. In particular, this is true if $\nu_1(\mathscr{H}, f) = 0$.*

Proof. From Theorem A it follows that $\beta_1(\mathscr{H}) < \beta_0(\mathscr{H}) = 1$, so that $\beta_1(\mathscr{H}) = 0$, yielding $\xi(\mathscr{H}) = 0$.

Corollary 7.2. *Suppose that there is a nondegenerate $\mathscr{H}$-Morse function on M such that $N_{F,-}$ is $\mathscr{H}$-trivial for any critical orbit closure F of index one. Then the $\widehat{\mathscr{H}}$-orbit closures are minimal submanifolds for some $\widehat{\mathscr{H}}$-invariant metric on M.*

Proof. If $\mathfrak{g}$ is not unimodular, then the statement is true without assuming any other hypothesis [2].

On the other hand, if $\mathfrak{g}$ is unimodular, $\xi(\mathscr{H})$ can be considered as an element of $H^1(M/\overline{\mathscr{H}})$ [2]. But $H^1(M/\overline{\mathscr{H}}) = 0$ by Theorem B. So $\xi(\mathscr{H}) = 0$, and $\widehat{\mathscr{H}}$ thus satisfies the stated property.

Example 7.3. Consider the $O(n)$-action on $\mathbb{S}^n$ defined by the restriction of the $O(n)$-action on $\mathbb{R}^{n+1} \equiv \mathbb{R}^n \times \mathbb{R}$, acting canonically on $\mathbb{R}^n$ and trivially on $\mathbb{R}$. Let $\mathscr{H}$ be a complete pseudogroup of local isometries of a Riemannian manifold M. If $M/\overline{\mathscr{H}}$ is homeomorphic to $G\backslash\mathbb{S}^n$ with $n \geq 2$, where G is any closed subgroup of $O(n)$, then the $\widehat{\mathscr{H}}$-orbit closures are minimal submanifolds for some $\widehat{\mathscr{H}}$-invariant metric on $\widehat{M}$ (by Corollary 7.2). This is not true for $n = 1$ (see Example 9.1).

Proposition 7.4. *Suppose that M is $\mathscr{H}$-orientable. Let f be any nondegenerate $\mathscr{H}$-Morse function on M. Then $\nu_n(\mathscr{H}, f) > 0$ iff $\mathfrak{g}$ is unimodular.*

Proof. If $\mathfrak{g}$ is unimodular, the $H^n(M)_{\mathscr{H}} \neq 0$ [2]. So $\nu_n > 0$ by Theorem A.

Reciprocally, assume $\nu_n > 0$. For any $F \in \mathrm{Crit}_{\mathscr{H}}(f)$, we have $m_F \leq \operatorname{codim} F$, so $n - m_F \geq \dim F$. Moreover the equalities hold iff f reaches a local maximum at the points of F. So, since $\nu_n > 0$, there exists an $F \in \mathrm{Crit}_{\mathscr{H}}(f)$ such that $m_F = \operatorname{codim} F$, and

$$H^{\dim F}(F, \mathscr{O}_F)_{\mathscr{H}} \neq 0. \tag{7.1}$$

For such an F, $\mathscr{O}_F$ is isomorphic to the $\mathscr{H}$-line bundle of orientations of F, because M is $\mathscr{H}$-orientable. Hence (7.1) implies the unimodularity of $\mathfrak{g}$ [2]. q.e.d.

A nondegenerate $\mathscr{H}$-Morse function will be said to be perfect if $\beta_j(\mathscr{H}) = \nu_j(\mathscr{H}, f)$ for all j.

Corollary 7.5. *If M is $\mathscr{H}$-orientable, $\xi(M) \neq 0$, and $\mathfrak{g}$ is unimodular, then there is no perfect nondegenerate $\mathscr{H}$-Morse function on M.*

Proof. We have $\beta_n = 0$ since M is $\mathscr{H}$-orientable and $\xi(\mathscr{H}) \neq 0$ [2]. On the other hand, Proposition 7.4 yields $\nu_n > 0$ for any nondegenerate $\mathscr{H}$-Morse function.

8. Application to Riemannian foliations

Let $\mathcal{F}$ be a Riemannian foliation of codimension q on a compact connected manifold M, and $(A_b(\mathcal{F}), d_b)$ its basic complex [20]. Then the holonomy pseudogroup $\mathcal{H}$ of $\mathcal{F}$ is a complete pseudogroup of local isometries of some manifold T with $T/\mathcal{H}$ compact and connected, and there is a canonical identity $(A(T)_{\mathcal{H}}, d_T) \equiv (A_b(\mathcal{F}), d_b)$. So $H(T)_{\mathcal{H}}$ is isomorphic to the basic cohomology $H_b(\mathcal{F})$ of $\mathcal{F}$. Moreover there is a natural homeomorphism of $M/\overline{\mathcal{H}}$ onto the space of leaf closures $M/\overline{\mathcal{F}}$. In this case $\xi(\mathcal{F})$ will be used instead of $\xi(\mathcal{H})$, and the structural Lie algebra $\mathfrak{g}$ of $\mathcal{H}$ is called the structural Lie algebra of $\mathcal{F}$.

As in the introduction, for a basic function we can define critical leaf closures, nondegenerate critical leaf closures, and the corresponding indexes. We also have the obvious definition of a nondegenerate $\mathcal{F}$-Morse function f, for which we have $\mathrm{Crit}_{\mathcal{F}}(f)$, $\nu_j(\mathcal{F}, f)$, and $N_{F,-}$ for any $F \in \mathrm{Crit}_{\mathcal{F}}(f)$. Then Theorem A yields a relation between the basic Betti numbers of $\mathcal{F}$, $\beta_{b,j}(\mathcal{F})$ (or simply $\beta_{b,j}$) $= \dim H_b^j(\mathcal{F})$, and the numbers $\nu_j(\mathcal{F}, f)$ for a nondegenerate $\mathcal{F}$-Morse function f on M. Theorem B also gives an estimation of $\dim H^1(M/\overline{\mathcal{F}})$.

The above relations have some consequences about tautness. A foliation $\mathcal{F}$ on some manifold M is said to be taut when there exists some Riemannian metric on M for which the leaves are minimal submanifolds. When $\mathcal{F}$ is Riemannian and M compact, this property has the following cohomological characterizations. First, X. Masa [17] has proved that $\mathcal{F}$ is taut iff $H_b^q(\mathcal{F}) \neq 0$ when $\mathcal{F}$ is transversally oriented. We also have that $\mathcal{F}$ is taut iff $\xi(\mathcal{F}) = 0$ [1]. Moreover $\mathfrak{g}$ is unimodular iff the restriction of $\mathcal{F}$ to some leaf closure is taut, and in this case the restriction of $\mathcal{F}$ to any leaf closure is also taut [2]. Corollaries 7.1 and 7.2 have the following consequences about tautness.

Corollary 8.1. *If there exists a nondegenerate $\mathcal{F}$-Morse function f such that $\nu_1(\mathcal{F}, f) < \nu_0(\mathcal{F}, f)$, then $\mathcal{F}$ is taut. In particular $\mathcal{F}$ is taut if $\nu_1(\mathcal{F}, f) = 0$.*

Corollary 8.2. *Suppose that there is some nondegenerate $\mathcal{F}$-Morse function such that $N_{F,-}$ is $\mathcal{H}$-trivial for any critical leaf closure F of index one. Then the following properties are equivalent:*

(i) *$\mathcal{F}$ is taut.*
(ii) *$\mathfrak{g}$ is unimodular.*
(iii) *The restriction of $\mathcal{F}$ to some leaf closure is taut.*
(iv) *The restriction of $\mathcal{F}$ to any leaf closure is taut.*

The hypotheses of Corollary 8.2 are satisfied when $M/\overline{\mathcal{F}}$ is homeomorphic to the orbit space $G\backslash\mathbb{S}^n (n \geq 2)$ of Example 7.3.

A nondegenerate $\mathcal{F}$-Morse function f will be said to be perfect when $\beta_{b,j}(\mathcal{F}) = \nu_j(\mathcal{F}, f)$ for all j. Corollary 7.5 has the following consequence.

Corollary 8.3. *If $\mathcal{F}$ is transversally orientable, not taut, and its restriction to some leaf closure is taut, then there is no perfect nondegenerate $\mathcal{F}$-Morse function on M.*

9. Examples

Example 9.1. This example of foliation is due to Y. Carrière [8]. Let A be a matrix in $\mathrm{SL}(2, \mathbb{Z})$ of trace greater than 2. Then A has two real irrational eigenvalues, λ and $1/\lambda$. The translates of the eigenspace of λ define a flow $\mathcal{F}_0$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The induced flow in $\mathbb{T}^2 \times \mathbb{R}$ is invariant by the transformation

$$h_A: ((x, y) + \mathbb{Z}^2, t) \mapsto (A(x, y) + \mathbb{Z}^2, t + 1),$$

so it defines a flow $\mathcal{F}$ on the hyperbolic torus $\mathbb{T}^3_A = (\mathbb{T}^2 \times \mathbb{R})/h_A$. $\mathcal{F}$ is a Lie $\mathfrak{g}$-folation, where $\mathfrak{g}$ is the Lie algebra of the affine group [8]. Its basic fibering is the fibration $\pi_b: \mathbb{T}^3_A \to \mathbb{S}^1$ defined by the canonical projection of $\mathbb{T}^2 \times \mathbb{R}$ onto $\mathbb{R}$. So π_b^* is an isomorphism of $C^\infty(\mathbb{S}^1)$ onto $A^0_b(\mathcal{F})$ such that nondegenerate $\mathcal{F}$-Morse functions on $\mathbb{T}^3_A$ correspond to nondegenerate Morse functions on $\mathbb{S}^1$. Indeed, if θ is a nondegenerate critical point of any $f \in C^\infty(\mathbb{S}^1)$, then $F = \pi_b^{-1}(\theta)$ is a nondegenerate leaf closure of $\pi_b^* f$ and $m_F(\pi_b^* f)$ is equal to the index of f at θ. Moreover each fiber with the restriction of $\mathcal{F}$ is a copy of $(\mathbb{T}^2, \mathcal{F}_0)$, whose basic cohomology is $H^0_b(\mathcal{F}_0) \cong H^1_b(\mathcal{F}_0) = \mathbb{R}$ and $H^j_b(\mathcal{F}_0) = 0$ for $j \neq 0, 1$.

Take for instance any nondegenerate Morse function f on $\mathbb{S}^1$ with two critical poins, θ_0 and θ_1, whose indexes are 0 and 1 respectively. Then clearly we have $\nu_0(\mathcal{F}, \pi_b^* f) = \nu_2(\mathcal{F}, \pi_b^* f) = 1$, and $\nu_1(\mathcal{F}, \pi_b^* f) = 2$. So the first and third transverse Morse inequalities are equalities in this case, but the second one is the strict inequality

$$\nu_1 - \nu_0 = 1 > 0 = \beta_{b,1} - \beta_{b,0}.$$

Indeed, since $\mathcal{F}$ is not taut (because $H^2_b(\mathcal{F}) = 0$), and its restriction to any leaf closure is taut, by Corollary 8.3 there is no perfect nondegenerate $\mathcal{F}$-Morse function.

Example 9.2. Let $\theta_1, \cdots, \theta_{n+1}$ be positive real numbers. Consider the flow on $\mathbb{C}^{n+1} - \{0\}$ given by the uniparametric group of transformations

$$A_t(z_1, \cdots, z_{n+1}) = (e^{i\theta_1 t} z_1, \cdots, e^{i\theta_{n+1} t} z_{n+1}) \qquad (t \in \mathbb{R}).$$

These transformations preserve the unit sphere

$$\mathbb{S}^{2n+1} = \{z = (z_j) \mid |z| = \sum_j |z_j|^2 = 1\} \subset \mathbb{C}^{n+1},$$

yielding a Riemannian flow $\mathscr{F}$ on $\mathbb{S}^{2n+1}$ [10]. The basic cohomology of $\mathscr{F}$ is computed in [10] by means of a long exact sequence involving $H_b(\mathscr{F})$ and $H(\mathbb{S}^{2n+1})$, obtaining

$$H_b^j(\mathscr{F}) \cong \begin{cases} \mathbb{R}, & \text{if } j \text{ is even and } 0 \leq j \leq 2n, \\ 0, & \text{otherwise}. \end{cases}$$

Let $B^{2n} \subseteq \mathbb{C}^n$ be the open unit ball centered at the origin 0. Then, for $1 \leq k \leq n+1$, $\mathscr{F}$ has the transversal $j_k \colon B^{2n} \to \mathbb{S}^{2n+1}$ defined by

$$j_k(z) = (z_1, \cdots, z_{k-1}, \sqrt{1 - |z|^2}, z_{k+1}, \cdots, z_n)$$

for $z = (z_1, \cdots, z_n)$. The j_k's define a complete transversal of $\mathscr{F}$ on the disjoint union of $n+1$ copies of B^{2n}.

Let $f \colon \mathbb{S}^{2n+1} \to \mathbb{R}$ be defined by

$$f(z_1, \cdots, z_{n+1}) = |z_1|^2 + 2|z_2|^2 + \cdots + (n+1)|z_{n+1}|^2.$$

It is easy to check that

$$\begin{aligned}(j_k^* f)(z_1, \cdots, z_n) = k &- (k-1)|z_1|^2 - (k-2)|z_2|^2 - \cdots - |z_{k-1}|^2 \\ &+ |z_k|^2 + 2|z_{k+1}|^2 + \cdots + (n-k+1)|z_n|^2.\end{aligned}$$

So each $j_k^* f$ is a nondegenerate Morse function having 0 as the unique critical point, which has index $2(k-1)$.

The leaf closure of $\mathscr{F}$ containing $(z_1, \cdots, z_{n+1})$ is a torus of dimension equal to the number of $\mathbb{Q}$-independent θ_k's corresponding to nonzero z_k's. Hence, for any choice of the θ_k's, the leaf L_k of $\mathscr{F}$ containing $j_k(0)$ is closed and diffeomorphic to $\mathbb{S}^1$. Therefore f is a nondegenerate $\mathscr{F}$-Morse function with $\mathrm{Crit}_{\mathscr{F}}(f) = \{L_1, \cdots, L_{n+1}\}$ and $m_{L_k}(h) = 2(k-1)$.

Moreover the basic cohomology of each $\mathcal{F}|_{L_k}$ is isomorphic to the cohomology of a point. Thus $\nu_j(\mathcal{F}, f) = \beta_{b,j}(\mathcal{F}, f)$ for all j; i.e., the Morse inequalities are equalities in this case.

Acknowledgments

I would like to thank A. El Kacimi Alaoui, G. Hector, S Hurder and P. Tondeur for very helpful comments. I also would like to thank the referees for very helpful remarks.

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UNIVERSIDADE DE SANTIAGO DE COMPOSTELA, SPAIN