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AN EXOTIC MENAGERIE

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0. Introduction

Among the most peculiar examples in topology are the *exotic* $\mathbb{R}^4$'s. These are smooth 4-manifolds which are homeomorphic to a Euclidean 4-space $\mathbb{R}^4$ but not diffeomorphic to it. Exotic $\mathbb{R}^4$'s represent a phenomenon which is fundamentally unique to dimension 4, since it is a central result of smoothing theory that, for $n \neq 4$, any smooth manifold homeomorphic to $\mathbb{R}^n$ must be diffeomorphic to it. Consequently, exotic $\mathbb{R}^4$'s provide counterexamples to various basic conjectures about the extension of high-dimensional topology to dimension 4. For example, in contrast with high dimensions, exotic smooth structures on 4-manifolds cannot be adequately analyzed via an obstruction theory (since $\mathbb{R}^4$ is contractible).

There are two main approaches to constructing and distinguishing exotic $\mathbb{R}^4$'s, and these yield manifolds with rather different properties—exotic $\mathbb{R}^4$'s of one type are much "larger" than those of the other type. Both constructions rely on work of Freedman and Donaldson, and the roots of both constructions can be traced to work of Casson [4]. We will discuss these constructions in chronological order.

The original exotic $\mathbb{R}^4$ resulted from an observation of Freedman, who noted that in the presence of his own work [8] and Donaldson's nonexistence theorem [6], a certain construction of Casson yielded a manifold homeomorphic to $\mathbb{R}^4$, whose end was not diffeomorphic to $S^3 \times \mathbb{R}$. In fact, it was easily seen that this exotic $\mathbb{R}^4$ contained a compact, codimension-zero submanifold which could not be smoothly embedded in S^4. Subsequently, it was shown that more than one such example existed [12]—in fact, infinitely many [13]. Then Taubes [22] solved a problem in gauge theory posed by Freedman, which yielded an uncountable family $\{R_t \mid t \in (0, 1)\}$ for which R_t embedded in $R_{t'}$ if and only if $t \leq t'$. In [13], this was extended to a two-parameter family $\{R_{s,t} \mid s, t \in (0, 1)\}$ with natural

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inclusions $R_{s,t} \hookrightarrow R_{s',t'}$ for $s \leq s'$ and $t \leq t'$, but with no embedding existing if $s > s'$ or $t > t'$.

This last nonembedding result can be strengthened via the following definition: We say $R \leq R'$ if any compact, smooth, codimension-zero submanifold of R embeds in R'. We call R and R' *compactly equivalent*, $R \sim R'$, if $R \leq R' \leq R$, i.e., if R and R' have the same compact submanifolds. The set of all compact equivalence classes of manifolds homeomorphic to $\mathbb{R}^4$ is partially ordered by $\leq$, and admits a metrizable topology with countable basis [15]. It follows from [13] that $R_{s,t} \leq R_{s',t'}$ if and only if $s \leq s'$ and $t \leq t'$. In particular, no two members of this family are compactly equivalent.

In a more recent development, Freedman introduced the second type of exotic $\mathbb{R}^4$ construction by expanding another of Casson's arguments. This construction relied on the failure of the h-cobordism theorem (see, for example, [7]). The resulting manifold R' embedded in $\mathbb{R}^4$ (with its usual smooth structure), and therefore R' was compactly equivalent to $\mathbb{R}^4$. In fact, DeMichelis and Freedman [5] constructed a one-parameter family $\{R'_t \mid t \in (0, 1)\}$ with natural inclusions $R'_t \subset R'_{t'} \subset \mathbb{R}^4$ for $t \leq t'$, and showed that the family contained uncountably many diffeomorphism types (with the cardinality of the continuum in ZFC set theory). Thus, the compact equivalence class of $\mathbb{R}^4$ contains uncountably many exotic $\mathbb{R}^4$'s.

In the present article, we describe various phenomena involving exotic $\mathbb{R}^4$'s and related examples. Much of this is related to the DeMichelis-Freedman paper, to which we refer for details when necessary. Our article consists of four essentially independent sections. In the first section, we show how to combine the work of DeMichelis and Freedman with earlier results involving the larger exotic $\mathbb{R}^4$'s to obtain an uncountable (one-parameter) family of distinct compact equivalence classes of exotic $\mathbb{R}^4$'s, each of which contains an uncountable family analogous to that of DeMichelis and Freedman. This suggests the following open question: Do all compact equivalence classes have uncountably many representatives? Our argument fails to extend to the second parameter of [13]. In fact, it is at least plausible that the *universal* $\mathbb{R}^4$ of Freedman and Taylor [10] is unique in its compact equivalence class, but this is presently an open problem.

The second section deals with a related question: Which topological 4-manifolds admit uncountably many diffeomorphism types of smooth structures? Furuta and Ohta [11] have conjectured that for a closed, simply-connected topological 4-manifold M, $M - \text{pt.}$ admits uncountably many smoothings, and they have proven most cases of this. Our main theorem in §2 states that for *any* topological 4-manifold M (possibly noncompact, with boundary, nonsimply connected or even nonorientable), $M - \text{pt.}$ admits uncountably many smoothings. It seems natural to conjecture that any noncompact 4-manifold can be smoothed in uncountably many distinct ways. (By Quinn [21], such a manifold is always smoothable.) Some caution is required, however, since we can at least imagine the possibility of a manifold M so complicated that in any smoothing all of its ends are forced to be universal (in the sense of [10]). For such a manifold, the smoothings would be classified up to isotopy by the Kirby-Siebenmann obstruction group $H^3(M, \partial M; \mathbb{Z}_2)$, which might be finite or even trivial.

In §3, we address a problem posed in [5]: Can an exotic $\mathbb{R}^4$ be described *explicitly* (for example, by a handlebody picture)? How simple a description can be obtained? So far, the best type of candidate for this is the type described in [5]. Such an exotic $\mathbb{R}^4$ has the following description: Begin with a certain type of ribbon link in ∂B^4. Remove tubular neighborhoods of the ribbon disks in B^4. We could recover B^4 by gluing in 2-handles along meridians of the deleted ribbon disks. Instead, however, we glue in Casson handles [4] or "pinched regular neighborhoods of convergent towers" [9] (henceforth, referred to as *Freedman handles*). For sufficiently complex Casson or Freedman handles, the interior of the result will be an exotic $\mathbb{R}^4$, of a type called a *ribbon* $\mathbb{R}^4$ in [5]. DeMichelis and Freedman make no attempt to estimate the complexity of the ribbon link or the Freedman handles, but suggest that such a task might be possible. Their examples are constructed by analyzing (any) simply connected, nontrivial five-dimensional h-cobordism, and the complexity of the associated link (in particular, the number of components) is determined by the complexity of the intersection of ascending and descending 2-spheres in the middle level. In §3, we observe that a certain nontrivial h-cobordism of Akbulut [2] has the simplest possible pattern of intersections: There are unique ascending and descending spheres, and these have a unique pair of extra intersections. Furthermore, the complement of the union of these spheres is simply connected. It follows [5] that an exotic ribbon $\mathbb{R}^4$ can be constructed from the obvious ribbons for the 2-component link shown in Figure 1, next page (which is [5, Figure 3.8]).

§4 describes a certain group action which yields several unusual branched-covering involutions on exotic $\mathbb{R}^4$'s. The first such interesting involution was constructed by Freedman (unpublished) in 1985. An exotic ribbon $\mathbb{R}^4$ $R' \subset S^4$ was shown to admit a smooth (branched-covering)

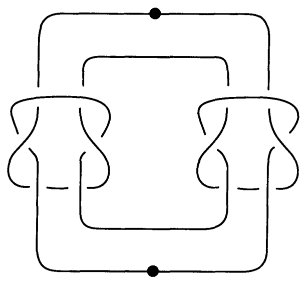

FIGURE 1

involution which was topologically standard (i.e., R' was equivariantly homeomorphic to $\mathbb{R}^2 \times \mathbb{R}^2$ with the involution acting as $180°$ rotation on the first factor and trivially on the second). The quotient of R' by this involution was *diffeomorphic* to $\mathbb{R}^4$. Thus, R' was realized as a 2-fold branched cover of $\mathbb{R}^4$ along a smooth, topologically standard $\mathbb{R}^2$. A seemingly unrelated example was exhibited in [5]: the *end* of R' was shown to admit an involution which could not be smoothly extended over R'. We will show how both of these examples fit into a more general picture, and simultaneously provide more examples with different properties. In the process, we will uncover a startling relation between the two seemingly different methods of constructing exotic $\mathbb{R}^4$'s.

For perspective, it should be noted that exotic $\mathbb{R}^4$'s which admit involutions or other group actions are easy to construct, although these easy examples will not have the special properties described above. For example, if G is any finite subgroup of $O(4)$, then G acts on many exotic $\mathbb{R}^4$'s. Alternatively, we may take G to be any group acting smoothly on $\mathbb{R}^3$, whose action on some open subset is properly discontinuous. (For example, consider Euclidean or hyperbolic isometries.) To construct such an exotic G-space, first let G act on $\mathbb{R}^4$ $(= \mathbb{R}^3 \times \mathbb{R})$. Then equivariantly end-sum with copies of any fixed exotic $\mathbb{R}^4$, adding one copy for each group element. (End-sum is the noncompact analog of boundary sum. It is compatible with $\leq$ in the sense that $R_1 \leq R_2$ and $R_3 \leq R_4$ imply $R_1 \natural R_3 \leq R_2 \natural R_4$. See [13] for details.) The resulting manifold R_1 will be an exotic $\mathbb{R}^4$ (since no exotic $\mathbb{R}^4$ has an inverse under end-sum [13].) The G-action on R_1 will be topologically equivalent to the G-action on $\mathbb{R}^4$ (since the end-sums are topologically trivial). If the quotient $\mathbb{R}^4/G$ is diffeomorphic to $\mathbb{R}^4$ (as in the case of the standard involution), the induced quotient space R_2 will be an exotic $\mathbb{R}^4$. This yields (for example) topologically standard involutions inducing 2-fold branched coverings

$R_1 \to R_2$ between exotic $\mathbb{R}^4$'s. Note, however, that the following constraints are built into the construction: (1) R_1 and R_2 are both exotic. (2) $R_1 \geq R_2$. (In fact, R_1 is an end-sum of copies of R_2.) (3) R_1 is compactly equivalent to $\mathbb{R}^4$ if and only if R_2 is.

We will produce examples which violate all of these constraints. We call an exotic $\mathbb{R}^4$ *small* if it is compactly equivalent to $\mathbb{R}^4$ and *large* otherwise. Freedman's example with R_1 a small exotic $\mathbb{R}^4$ and R_2 standard violates (1) only. We will exhibit an example with R_1 *large* and R_2 standard, violating both (1) and (3). We will also obtain an example with R_1 small (exotic) and R_2 large, violating (2) and (3). By combining these examples with the techniques of the previous paragraph, we obtain smooth, topologically standard 2-fold branched coverings $R_1 \to R_2$, where R_1 and R_2 can be independently chosen to be large or small. If R_2 is small, it may be chosen to be either standard or nonstandard (by summing with a small exotic $\mathbb{R}^4$ if necessary). It is still an open question whether can we have R_1 standard and R_2 exotic. In particular, can we violate (1), (2), and (3) simultaneously?

Our examples fit into a larger group action. We will construct R^*, a small exotic $\mathbb{R}^4$ which embeds in S^4. R^* is not known to be ribbon, but it is *complementary* to a ribbon $\mathbb{R}^4$ R', in the sense that the construction provides related embeddings R', $R^* \hookrightarrow S^4$ with $R' \cup R^* = S^4$. (It follows that $R' \cap R^*$ is homeomorphic to $S^3 \times \mathbb{R}$. In our case, it will be an exotic $S^3 \times \mathbb{R}$ with no smooth S^3 representing its homology.) R^* will admit a smooth $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ action which is topologically standard (equivariantly homeomorphic to the action on $\mathbb{R}^3 \times \mathbb{R}$ given by 180° rotations about the three axes in the first factor). Two of the three nontrivial elements of G will have quotients diffeomorphic to $\mathbb{R}^4$, and the quotient R^*_G of R^* by the entire group action will also be standard. (In fact, both of the induced 2-fold branched coverings of R^*_G will be smoothly standard, although they must somehow clash with each other.) In contrast, the remaining element of G will have quotient R^*_x which is a *large exotic* $\mathbb{R}^4$. The 2-fold branched coverings $R^* \to R^*_x \to R^*_G$ provide our new examples. (The other involutions on R^* provide other examples with the same behavior as Freedman's example.)

We will simultaneously construct a smooth $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action on the connected sum $S^2 \times S^2 \# R'$, with R' a ribbon $\mathbb{R}^4$ complementary to R^*. Topologically this will come from a well-known action on $S^2 \times S^2$ by removing a fixed point. Again, two of the three quotients by involutions will be smoothly standard ($\mathbb{R}^4$ and $\overline{\mathbb{C}P}^2 - \text{pt.}$), as will the quotient by the entire group ($\mathbb{R}^4$). (The two corresponding branched coverings over this latter $\mathbb{R}^4$ will also be standard.) The remaining quotient will be an *exotic* $\mathbb{C}P^2 - \text{pt.}$ which contains no smooth $\mathbb{C}P^1$. If we restrict the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action to the end of $S^2 \times S^2 \# R'$, we obtain an action on the end of R'. One of the involutions of the end extends over all of R' (by surgering out the $S^2 \times S^2$), yielding Freedman's branched covering $R' \to \mathbb{R}^4$. The other two involutions cannot extend, by [5]. (One of these is explicitly described in [5]; the other is similar.) We will examine the quotients of these two actions. The one from [5] has quotient which is the end of the exotic $\mathbb{C}P^2 - \text{pt.}$ and cannot be the end of any exotic $\mathbb{R}^4$. The other quotient, however is the end of the *standard* $\overline{\mathbb{C}P}^2 - \text{pt.}$, so it is also the end of the standard $\mathbb{R}^4$ (even though the involution itself cannot extend over R'). We will give an explicit ribbon description of R' in which the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action of its end is clearly visible, as is Freedman's involution of all of R'. (In fact, Freedman's involution can also be seen as rotation about the y-axis in Figure 1.)

1. Even more exotic $\mathbb{R}^4$'s

The goal of this section is to prove the following:

Theorem 1.1. *There is a one-parameter family of distinct compact equivalence classes of exotic $\mathbb{R}^4$'s (with the order type of $(0, 1)$ under $\leq$), such that each class contains an uncountable family (with the cardinality of the continuum in ZFC) of distinct diffeomorphism types.*

The proof relies on two ingredients: [13] and [5]. First, we consider the two-parameter family of distinct compact equivalence classes given in [13]. The construction of this begins with R (denoted R_Γ in [13]), an exotic $\mathbb{R}^4$ which embeds in $\mathbb{C}P^2$. This is exotic because it contains a certain compact, codimension-zero submanifold X which cannot embed in S^4, or in any negative definite manifold. (If such an embedding existed, we could construct a counterexample to Donaldson's Theorem [6] on nonexistence of 4-manifolds with nonstandard, definite intersection forms.) We identify R topologically with $\mathbb{R}^4$, and consider the family of all open balls of radius $r > N$, where N is chosen large enough so that X lies in each ball. Since each ball is an open subset of R, it inherits a smooth structure, resulting in a one-parameter family $\{R_t | t \in (0, 1)\}$ of exotic $\mathbb{R}^4$'s with $R_t \subset R_{t'}$ for $t \leq t'$. Using Taubes' extension [22] of Donaldson's

Theorem to open 4-manifolds with "periodic" ends, it can be shown that $R_t \le R_{t'}$ if and only if $t \le t'$. To obtain the second parameter, we let $\overline{R}_t$ denote R_t with reversed orientation (so that $\overline{R}_t \subset \overline{\mathbb{C}P}^2$) and let $R_{s,t}$ be the end-sum $R_s \natural \overline{R}_t$. Again, Taubes' work implies $R_{s,t} \le R_{s',t'}$ if and only if $s \le s'$ and $t \le t'$.

The other ingredient for our construction is the family $R'_t \subset \mathbb{R}^4$ of [5]. This construction begins with R', an exotic $\mathbb{R}^4$ which embeds in $\mathbb{R}^4$. The exotic nature of R' is detected via the failure of the smooth five-dimensional h-cobordism theorem, which in turn is detected by Donaldson's invariants [7]. The family $\{R'_t\}$ is constructed from R' using a radial family of topological balls, as above. Uncountably many diffeomorphism types are distinguished using an extension of Donaldson's invariants to manifolds with periodic ends.

For convenience, we choose our h-cobordism carefully. We want to use a Donaldson invariant with simple behavior under blow-up. The invariant of Kotschick which appears in [5] depends on a chamber structure after blow-ups, so we switch to a different invariant which has been studied by Mrowka [17], and choose our h-cobordism accordingly. Let B be a simply connected, nonspin elliptic surface with $b_+ = 3$. (A simple example would be $K3 \# \overline{\mathbb{C}P}^2$.) Let Q be the connected sum of $\pm\mathbb{C}P^2$ with the same intersection pairing as B. Then by Wall's work [24], B and Q are h-cobordant. We use this h-cobordism to construct R'_t.

We distinguish B and Q using the Donaldson invariant of [17]. For any M in the homotopy type of B, we define the invariant as follows: For any class $\eta \in H^2(M; \mathbb{Z}_2)$ with $\eta^2 \equiv 2 \pmod 4$ there is a unique SO(3) bundle P over M with $w_2(P) = \eta$ and $p_1(P) = -6$. For a generic metric on M, the moduli space of anti-self-dual connections on P is a compact manifold of dimension $-2p_1 - 3(1 + b_+) = 0$. The Donaldson invariant $\gamma_M(\eta) \in \mathbb{Z}$ is the number of points in this space, counted with suitable sign. For Q we have $\gamma_Q(\eta) = 0$ for all η, since Q splits as a connected sum of two pieces with $b_+ > 0$. However, for B there are classes η with $\gamma_B(\eta) \neq 0$. If we extend η trivially to a class in $H^2(B \# n\overline{\mathbb{C}P}^2; \mathbb{Z}_2)$, we have $\gamma_{B \# n\overline{\mathbb{C}P}^2}(\eta) = \gamma_B(\eta)$.

Now we construct our required family of exotic $\mathbb{R}^4$'s. Let $R'_{s,t}$ denote the end-sum $R'_s \natural \overline{R}_t$. Since $R'_s \sim \mathbb{R}^4$, it follows immediately that $R'_{s,t} \sim \overline{R}_t$. Since $R_t \le R_{t'}$ if and only if $t \le t'$, we have $R'_{s,t} \le R'_{s',t'}$ if and only if $t \le t'$. (In addition, $R'_{s,t}$ embeds in $R'_{s',t'}$ if $t \le t'$, since the given embeddings $R'_s \subset \mathbb{R}^4$ and $R_t \subset R_{t'}$ can be assumed to be "shaved" in the sense of [13].) Thus, t parametrizes a family of compact equivalence classes which are ordered like $(0, 1)$ under $\leq$, and for each t, s parametrizes a family within the given equivalence class.

It remains to show that for any fixed t, $\{R'_{s,t} | s \in (0, 1)\}$ contains uncountably many diffeomorphism types. We generalize the method of [5]. According to this method, a certain compact, codimension-zero submanifold $K \subset R'$ is constructed. We may assume $K \subset R'_s$ for all s. DeMichelis and Freedman showed that for $s > s'$ there is no diffeomorphism (or even embedding) $(R'_s, K) \to (R'_{s'}, K)$ which is the identity on K. Since there can be only countably many embeddings of K in R'_s (up to isotopy) it follows that there are only countably many manifolds R'_s diffeomorphic to any given one, and their theorem follows immediately. To generalize the argument to our case, it suffices to prove the following:

Lemma 1.2. *If $s > s'$, there is no embedding $(R'_{s,t}, K) \hookrightarrow (R'_{s',t^*}, K)$ restricting to the identity on K.*

Remark. This suggests the utility of defining a relation $\leq$ on *pairs* (R, K).

Proof. Suppose such an embedding exists. We have ("shaved") embeddings $R'_{s'} \subset R'_s$ (with compact closure) and $\overline{R}_t \subset \overline{\mathbb{C}P}^2$. Combining these yields an embedding $R'_{s',t^*} \hookrightarrow R'_s \# \overline{\mathbb{C}P}^2$ with compact closure, restricting to the identity on K. Using the hypothesized embedding, we obtain $R'_s \subset R'_{s,t} \hookrightarrow R'_{s',t} \hookrightarrow R'_s \# \overline{\mathbb{C}P}^2$, yielding an embedding $d\colon R'_s \hookrightarrow R'_s \# \overline{\mathbb{C}P}^2$ with compact closure, and with $d|K$ the identity.

We now apply the argument of [5], carrying along the extra $\overline{\mathbb{C}P}^2$. The construction of R' gives us embeddings $i\colon R' \hookrightarrow B$ and $j\colon R' \hookrightarrow Q$, such that the given h-cobordism from B to Q defines a diffeomorphism $B - i(K) \approx Q - j(K)$. Let B_n denote $B \# n\overline{\mathbb{C}P}^2$. Restricting i gives $i_0\colon R'_s \hookrightarrow B_0$. Inductively define $i_n\colon R'_s \hookrightarrow B_n = B_{n-1} \# \overline{\mathbb{C}P}^2$ by $i_n = (i_{n-1} \# id_{\overline{\mathbb{C}P}^2}) \circ d$. Since $\operatorname{Im} d$ has compact closure, the maps $i_0, \cdots, i_n$ determine n successive rings on $B_n - i(K)$. Each ring is diffeomorphic to $R'_s \# \overline{\mathbb{C}P}^2 - \operatorname{Im} d$. As in [5] we may put Riemannian structures on the manifolds B_n so that the rings are all isometric, and the structures on the various B_n's agree elsewhere. We obtain a limiting Riemannian manifold B_∞ by throwing away the part of B_n containing K and adding an infinite periodic end (with infinitely generated homology). Now, using the diffeomorphism $B - i(K) \approx Q - j(K)$, we create an analogous structure for Q: Metrize $Q_n = Q \# n\overline{\mathbb{C}P}^2$ by identifying $Q_n - j(K)$ with $B_n - i(K)$

and extending the metric over $j(K)$. The limiting Q_∞ will be isometric to B_∞.

Let $\eta \in H^2(B; \mathbb{Z}_2)$ be a class for which $\gamma_B(\eta)$ is defined and nonzero. Then η extends trivially to classes $\eta \in H^2(B_n; \mathbb{Z}_2)$ and $\eta \in H^2(B_\infty; \mathbb{Z}_2)$ (and the latter is compactly supported). Via the diffeomorphism $B - i(K) \approx Q - j(K)$ we obtain corresponding classes η' in Q_n and Q_∞. (Note that K is separated from the cohomology of B_n by a flat topological S^3.) Since the work of Taubes explicitly allows negative definite 2-homology in periodic ends, the arguments of [5] go through with little change in our case to show that $\gamma_{B_\infty}(\eta)$ and $\gamma_{Q_\infty}(\eta')$ are well defined. (For example, the closed 2-form ψ used in the proof of Point 2 in [5] is self-dual, so it is still trivial in H^2_{DR} in the presence of negative definite 2-homology.) Furthermore, $\gamma_{B_\infty}(\eta) = \gamma_{B_n}(\eta) = \gamma_B(\eta) \neq 0$. Similarly, $\gamma_{Q_\infty}(\eta') = \gamma_{Q_n}(\eta') = 0$. (This vanishes for *any* class η' in Q_n.) However, there is an isometry $B_\infty \to Q_\infty$ sending η to η', so $\gamma_{B_\infty}(\eta) = \gamma_{Q_\infty}(\eta')$, and we have the required contradiction. q.e.d.

Remark. We also see that the family R'_s (or $R'_{s,t}$, t fixed) has uncountably many diffeomorphism types of *ends*. (We say two exotic $\mathbb{R}^4$'s have diffeomorphic ends if there is a diffeomorphism outside of suitably large compact sets.) This follows from the observation that there are only countably many exotic $\mathbb{R}^4$'s which can have a given end. As a corollary, there are uncountably many diffeomorphism types of exotic $S^3 \times \mathbb{R}$'s which embed in $\mathbb{R}^4$ and do not admit any smooth S^3's generating their homology (or even any homology spheres lacking nontrivial representations $\pi_1 \to \mathrm{SO}(3)$).

2. Exotic punctured manifolds

Next, we prove:

Theorem 2.1. *Let M be a connected, topological 4-manifold (possibly noncompact, possibly with boundary). Then $M -$ pt. admits uncountably many nondiffeomorphic smooth structures (with the cardinality of the continuum).*

Proof. First, suppose M is noncompact. Then by Quinn [21] (see also [11]), M is smoothable. Fix a smoothing and consider the family $\{M \# R_t\}$, where $\{R_t\}$ is any family of exotic $\mathbb{R}^4$'s with uncountably many diffeomorphism types of ends (for example, from §1 or [13]). Since these manifolds are all homeomorphic to $M -$ pt., it suffices to show that there are uncountably many diffeomorphism types. This is clear, except for one technicality: We must beware of diffeomorphisms which permute the ends of $M - \mathrm{pt}$. But M can have only countably many ends which are topologically collared by $S^3 \times \mathbb{R}$, so for uncountably many values of t no end of M will be diffeomorphic to that of R_t.

Now suppose M is compact, and that the Kirby-Siebenmann obstruction $ks(M)$ vanishes. Then for sufficiently large n, $M\#nS^2 \times S^2$ is smoothable. (See, for example, [11, §8.6].) Take $n \geq 3$, choose a smoothing, and let N denote the resulting smooth manifold. Let $U \subset N$ be the open submanifold consisting of $(\#nS^2 \times S^2) - \mathrm{pt}$. By Casson [4], we can represent the homology of U by an open set $V \subset U$ consisting of $2n$ Casson handles attached to a 4-ball along n 0-framed Hopf links. Similarly, if N' denotes $K3\#(n-3)S^2 \times S^2$, we can find such an open set $V' \subset N'$ representing the hyperbolic part of the intersection form of N'. Since any two Casson handles have a common refinement, we may assume there is a diffeomorphism $\varphi : V \to V'$. Because any Casson handle is homeomorphic to an open 2-handle [8], V is homeomorphic to $(\#nS^2 \times S^2) - \mathrm{pt}$. Let $K \subset V$ be a compact, topologically embedded copy of $(\#nS^2 \times S^2) - \mathring{B}^4$ with flat boundary, and let $W = N - K$, $W' = N' - \varphi(K)$. Clearly, W and W' are smooth manifolds with diffeomorphic ends (via φ). W is homeomorphic to $M - \mathrm{pt.}$, since the closure of $U - K$ in W has the proper homotopy type of $B^4 - \mathrm{pt.}$ and is therefore homeomorphic to it. W' is homeomorphic to $|E_8|\#|E_8| - \mathrm{pt.}$ where $|E_8|$ denotes Freedman's closed, simply-connected manifold with intersection form (negative definite) E_8. Now identify $V - K$ topologically with $\mathring{B}^4 - \mathrm{pt.}$ (with K corresponding to pt.), and let $W_t (0 < t < 1)$ denote the manifold obtained from W by deleting the closed ball of radius t from $B^4 - \mathrm{pt.} \subset W$. Each W_t is still homeomorphic to $M - \mathrm{pt.}$, but no two of these manifolds can be diffeomorphic, by a standard argument. If W_s and W_t were diffeomorphic, with $s < t$, we could stack together an infinite sequence of copies of the topological annulus $W_s - W_t$ to obtain a smooth manifold homeomorphic to $S^3 \times \mathbb{R}$ with a periodic end in the sense of Taubes [22]. We could then splice this onto W' to obtain a smoothing of $|E_8|\#|E_8| - \mathrm{pt.}$ with a periodic end. This would violate Taubes' generalization of Donaldson's Theorem to end-periodic manifolds. (See [13] or [22] for details).

Finally, consider the case with M compact and $ks(M) \neq 0$. Let Σ denote the Poincaré homology sphere, with its usual orientation as the boundary of a (negative definite) E_8 plumbing P. Let $\overline{\Sigma}$ denote Σ with the opposite orientation. Then $\overline{\Sigma}$ bounds a contractible topological manifold Δ [F]. (See also [11].) Δ embeds in S^4 (realized as the double of Δ), so we may assume Δ is embedded in M (preserving a preassigned orientation if M is orientable). Let M_0 denote the compact manifold $M - \operatorname{int}\Delta$. Since the Kirby-Siebenmann obstruction adds under gluing in dimension 4 (see [11], Section 10.2B), and since $P \cup_\Sigma \Delta = |E_8|$, we have $ks(\Delta) \neq 0$ and $ks(M_0) = 0$. In particular, we may apply the previous argument to M_0. We obtain a smoothing of $M_0 - \text{pt.}$ with end diffeomorphic (preserving any specified orientation) to the end of a smooth manifold W_0' homeomorphic to $|E_8| \# |E_8| - \text{pt.}$ Next, we smooth $\Delta - \text{pt.}$, obtaining an end diffeomorphic to that of a smooth $W_1' = P \cup_\Sigma \Delta - \text{pt.}$ which is homeomorphic to $|E_8| - \text{pt.}$ Combining these smoothings, we obtain a smooth W homeomorphic to $M - 2$ points, with (disconnected) end diffeomorphic to the end of $W' = W_0' \# W_1'$ (homeomorphic to $\#3|E_8| - 2$ points). Let $l' \subset W'$ be a smooth, properly embedded line connecting the two ends of W'. (Such an l' is unique up to smooth isotopy.) For a suitably chosen $U' \subset W'$ containing l' and a neighborhood of the ends, we can find a neighborhood U of the end of W, which is diffeomorphic to U'. Let l be the image of l' in U. Then $W - l$ and $W' - l'$ have diffeomorphic ends, $W - l$ is homeomorphic to $M - \text{pt.}$, and $W' - l'$ is homeomorphic to $\#3|E_8| - \text{pt.}$ Now we can form a radial family of smoothings W_t of $M - \text{pt.}$ as before, and no two can be diffeomorphic, by Taubes' Theorem applied to $\#3|E_8| - \text{pt.}$ q.e.d.

3. A simple exotic ribbon $\mathbb{R}^4$

We now verify that an exotic ribbon $\mathbb{R}^4$ can be constructed, starting with the obvious ribbon disks for the 2-component link given in Figure 1. According to [5] (see also §4) it suffices to find a (simply connected) nontrivial h-cobordism with exactly two handles (a 2–3 pair) such that the ascending sphere A and descending sphere D in the middle level have only one extra pair of intersections, and such that $A \cup D$ has simply connected complement.

It is worth noting that nontrivial h-cobordisms with exactly two handles are easy to construct. There are many known examples of compact, simply connected, smooth 4-manifolds M and M' which are homeomorphic but not diffeomorphic. Frequently, M and M' are known to become diffeomorphic after sum with a single $S^2 \times S^2$ (see, for example, [19], [20]). Thus, if a five-dimensional 2-handle is added (with correct framing in the spin case) on top of each of $M \times I$ and $M' \times I$, the resulting top boundaries will be diffeomorphic. If we turn one of these manifolds upside

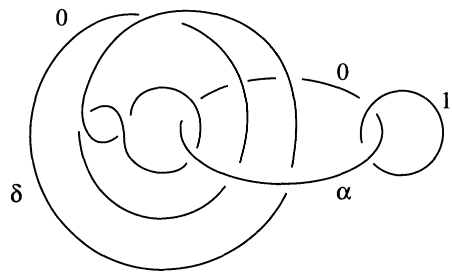

FIGURE 2

down and glue it on top of the other one, we will obtain a cobordism from M to M' with a unique 2-handle/3-handle pair. In general, this will not be an h-cobordism, but if either M has an indefinite intersection form or $b_2(M) \leq 8$ (and ∂M is empty or a homology sphere) then Wall's work [23] shows that any automorphism of the intersection form of the middle level $M \# S^2 \times S^2$ is realized by a diffeomorphism. Composing the gluing map with a suitable diffeomorphism, we can arrange for the two handles to algebraically cancel, yielding an h-cobordism .

It is a much more delicate matter to minimize the intersections between A and D. For this, we must work with an explicit example. The simplest known example seems to be that of Akbulut [2]. (Other examples can be constructed from [16], but the simplest such example appears to have two extra pairs of intersections.) Akbulut constructs two compact manifolds with boundary, Q_1 and Q_2, which are homeomorphic but not diffeomorphic and have the same homotopy type and intersection form as $\mathbb{C}P^2 - \mathring{B}^4$. Akbulut's Figure 10 implicitly describes a suitable h-cobordism (rel ∂) from Q_1 to Q_2. (A similar h-cobordism using contractible manifolds is described explicitly by Akbulut in [1], but the former example seems better suited to the present purpose.)

Figure 2 shows the middle level P of Akbulut's h-cobordism. P is obtained from B^4 by adding three 2-handles as shown. Since the curves α and δ are each unknotted, they bound embedded disks in S^3. By pushing the interiors of these disks into B^4 and adding the cores of the corresponding 2-handles we obtain spheres A and D, respectively, in P. Surgery on A (i.e., adding a dot to α) yields Q_1, and surgery on D yields Q_2. Thus, we have diffeomorphisms $Q_1 \# S^2 \times S^2 \approx P \approx Q_2 \# S^2 \times S^2$, and we have exhibited an h-cobordism from Q_1 to Q_2 with middle level P, one pair of handles, and ascending and descending spheres A and D, respectively. (Note that the intersection number of A and D is 1, so that the two handles algebraically cancel, as required.)

The required properties of A and D in P are easy to verify. To check that A and D intersect transversely in three points, simply draw them in an $S^3 \times I$ collar of ∂B^4, with time representing the I coordinate. To see that $A \cup D$ has simply connected complement in P, first observe that D has simply connected complement. (Since δ is unknotted, the complement of D has abelian π_1. But the 2-handle attached to α kills the generator in homology.) Thus, $\pi_1(P-(A \cup D))$ is generated by meridians of A. But the 2-handle attached with framing 1 kills these meridians.

To construct the required ribbon $\mathbb{R}^4$, apply the procedure of [5]. Since $A \cup D$ already has simply connected complement, we do not need to add intersections to $A \cup D$. Since there are only two extra intersections, we need to add only two Casson or Freedman handles; this can be done automatically ([4] or [9, Chapter 7]). The reduction to Figure 1 follows as in [5]; also see the end of §4. It now seems feasible to complete the program suggested in [5]—to build Freedman handles and explicitly compute their ramification using the height-raising algorithms of [9]. This would provide the first completely explicit description of an exotic $\mathbb{R}^4$.

Remarks. (1) For carrying out the above program, it may be useful to recall that Q_1 and Q_2 remain nondiffeomorphic after connected sum with any number of copies of $\mathbb{C}P^2$ (with its usual orientation) [2]. Thus, it suffices to work in $P \# k\mathbb{C}P^2$ for any k (by summing the entire h-cobordism with $\mathbb{C}P^2 \times I$). For example, we may construct a framed transverse sphere for A with a unique self-intersection. Begin with the embedded sphere in P determined by the 1-framed 2-handle. Blow up $\mathbb{C}P^2$ to raise the homological self-intersection number of this sphere to 2, then add a $(+)$-self-intersection.

(2) The fact that our $\mathbb{R}^4$ is exotic follows directly from nontriviality of the h-cobordism. (See, for example, [18].) For sharper results, it is useful to recall that this h-cobordism can be extended to a nontrivial h-cobordism of *closed* manifolds by gluing a certain product $M_1 \times I$ onto the lateral boundary [2]. Since the two corresponding closed 4-manifolds are distinguished by Donaldson's invariants, the argument of [5] shows that any homology sphere in the end of the exotic $\mathbb{R}^4$ must have a nontrivial representation $\pi_1 \to SO(3)$.

4. Some exotic G-spaces

We begin with a basic group action. Let $M = S^2 \times S^2$ and $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. There is a natural orientation-preserving G-action on M which possesses

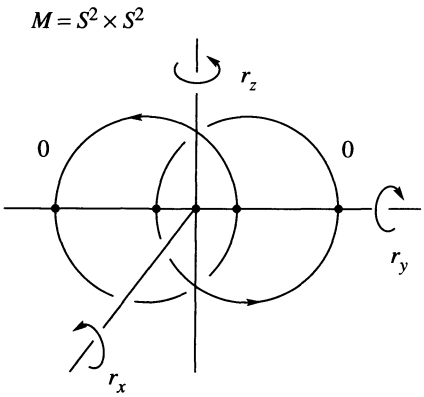

FIGURE 3

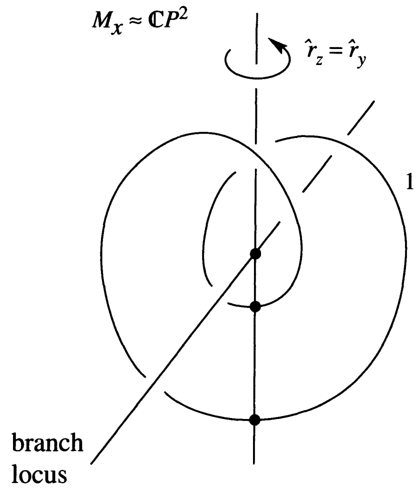

FIGURE 4

fixed points. The three involutions r_x, r_y, and r_z are visible in the Kirby calculus description of M in Figure 3 as 180° rotations about the three coordinates axes. If M is represented by $\mathbb{C}P^1 \times \mathbb{C}P^1$, the action will be given in affine coordinates by $r_x(z, w) = (w, z)$, $r_y(z, w) = (\overline{z}, \overline{w})$, and $r_z(z, w) = (\overline{w}, \overline{z})$. We use M_x, M_y, M_z, and M_G to denote the quotients of M by r_x, r_y, r_z, and G, respectively, and we use similar notation for quotients of equivariant subsets of M. M_x is well known to be $\mathbb{C}P^2$, with the branch locus of the ramified covering given by the quadric $S^2 \subset \mathbb{C}P^2$. This is seen in Figure 4 (cf. Figure 3). Since the framings in Figure 3 are given by the plane of the paper, the same will be true for Figure 4. Thus, the framing is $+1$, due to the $(+)$ crossing of the curve. Note that r_y and r_z descend to the same involution $\hat{r}_y = \hat{r}_z$ of M_x. M_z is easily seen to be $\overline{\mathbb{C}P}^2$, the mirror image of M_x. (Imagine the two circles of Figure 3 as lying in perpendicular planes, and rotate 90° about the y-axis.) M_z is shown in Figure 5. By the method of [3], M_y is seen to be S^4, with branch locus a standardly embedded T^2 (Figure 6). Since the remaining involution $\hat{r}_x$ on M_y is the standard involution fixing an unknotted S^2, we have $M_G = S^4$. (This can also be seen from Figure 4 or 5, where the remaining involution $\hat{r}_y$ is seen to be complex conjugation on $\pm\mathbb{C}P^2$.)

In [5] an exotic ribbon $\mathbb{R}^4$ $R' \subset S^4$ is constructed, together with a proper h-cobordism from R' to itself. Let N denote the middle level

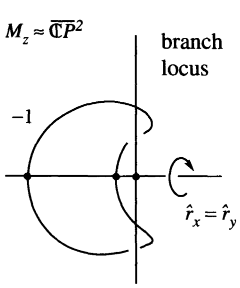

FIGURE 5

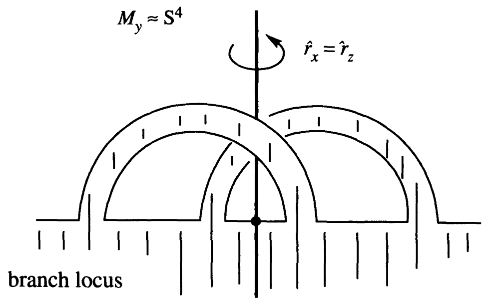

FIGURE 6

of this h-cobordism (Y^- in the notation of [5]). As in §3, we assume the h-cobordism has only one 2-handle/3-handle pair, so that N is diffeomorphic to $S^2 \times S^2 \# R'$. N is shown explicitly as the interior of the 4-manifold in Figure 7, next page (which is [5, Figure 3.2]). The wiggly circles denote Casson handles. Here, we assume the ascending and descending spheres have only one extra pair of intersections. We could easily deal with extra pairs of intersections by extending Figure 7 vertically (choosing the handedness of the clasps so as to preserve the involution r_x), but we will see that this is unnecessary. (For example, by §3, there is an exotic R' as given in Figure 7.) Our constructions will also be stable under additional ramification of the Casson handles.

There is an obvious embedding $N \subset M$. First, note that if we replace the Casson handles in Figure 7 by honest 2-handles, they will (G-equivariantly) cancel the 1-handles, yielding $M - \mathring{B}^4$ as in Figure 3. Since any Casson handle has a standard smooth embedding (rel ∂) in a 2-handle, we obtain our embedding $N \subset M$. The G-action on $M - \mathring{B}^4$ is clearly visible in Figure 7 (as rotations about the three axes), and we would like this action to preserve N. By adding ramification if necessary, we may assume the two Casson handles are the same, so it suffices to arrange each Casson handle and its embedding into a 2-handle to be $\mathbb{Z}_2$-symmetric. This is easily accomplished by constructing the Casson handle as follows: Begin with a standard open 2-handle with the obvious involution (reversing orientation on the attaching circle). Equivariantly add a pair of identical Casson handles along the attaching circle to obtain a Casson handle with an involution. This clearly has a standard $\mathbb{Z}_2$-equivariant embedding in a

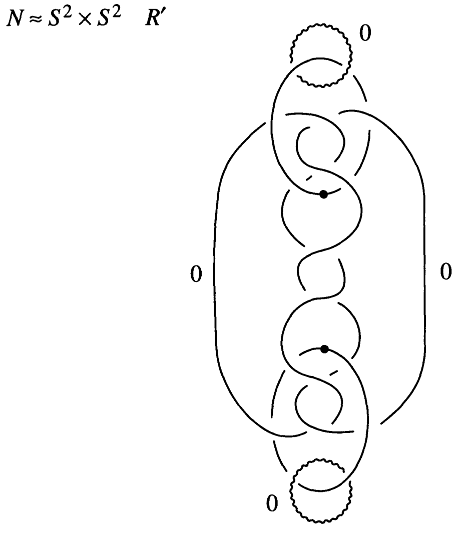

FIGURE 7

larger 2-handle. Inserting these Casson handles in Figure 7, we obtain N as a G-equivariant subset of $M - \mathring{B}^4$. Note that topologically, each Casson handle with $\mathbb{Z}_2$-action is simply an open 2-handle with the standard involution. It immediately follows that N is G-equivariantly homeomorphic to $M - B^4$. In particular, N_x, N_y, N_z, and N_G are homeomorphic to $\mathbb{C}P^2 - B^4$, $\mathbb{R}^4$, $\overline{\mathbb{C}P}^2 - B^4$, and $\mathbb{R}^4$, respectively.

Our goal is to prove the following theorems.

Theorem 4.1. *The G-action on $N \approx S^2 \times S^2 \# R'$ has the following characteristics*:

(1) *N_y and N_G are diffeomorphic to $\mathbb{R}^4$, and N_z is diffeomorphic to* $\overline{\mathbb{C}P}^2$ – pt. *The induced involutions on N_y and N_z are smoothly standard, i.e., equivariantly diffeomorphic to the corresponding involutions on M_y –* pt. (*rotation* $\times$ id *on* $\mathbb{R}^2 \times \mathbb{R}^2$) *and* M_z – pt. (*complex conjugation on* $\overline{\mathbb{C}P}^2$ – pt.).

(2) *N_x is an* ***exotic*** $\mathbb{C}P^2$ – pt. *It has no smoothly embedded sphere generating its homology.*

(3) (*Freedman*) *The involution r_y on N induces* (*by surgery*) *a smooth involution of R' which is topologically standard. The quotient is diffeomorphic to $\mathbb{R}^4$, so R' is realized as a 2-fold branched cover of $\mathbb{R}^4$ along a smooth, topologically standard $\mathbb{R}^2$.*

(4) *The involutions* r_x *and* r_z *restrict to involutions of the end of* R' *which cannot be extended to diffeomorphisms of* R'. (*For* r_x, *this is proved in* [5].) *The quotient of* r_z *on the end of* R' *is diffeomorphic to the end of* $\mathbb{R}^4$, *but the corresponding quotient of* r_x *cannot be the end of any manifold homeomorphic to* $\mathbb{R}^4$ (*or even any negative definite manifold*).

Theorem 4.2. *There is a* G*-equivariant open subset* R^* *of* M *with* $N \cup R^* = M$ *such that* R^* *is a small exotic* $\mathbb{R}^4$. *Surgery on* N *yields an embedding of* R^* *in* S^4 *with* $R' \cup R^* = S^4$ *and* $R' \cap R^*$ *an exotic* $S^3 \times \mathbb{R}$ *containing no smooth* S^3 *generating its homology. The* G*-action on* R^* *is topologically standard* (*i.e., equivariantly homeomorphic to* $\mathbb{R}^3 \times \mathbb{R}$ *with* G *acting on the first factor by* $180°$ *rotations of the three axes*). *The action has the following characteristics*:

(1) R^*_y, R^*_z, *and* R^*_G *are all diffeomorphic to* $\mathbb{R}^4$. *The induced involutions on* R^*_y *and* R^*_z *are smoothly standard* (*so the branched coverings* $R^*_y, R^*_z \to R^*_G$ *are standard*).

(2) R^*_x *is a* ***large exotic*** $\mathbb{R}^4$. *It embeds in* $M_x \approx \mathbb{C}P^2$, *but has a compact submanifold which cannot embed smoothly in any negative definite 4-manifold.*

Remarks. Clearly, each of the three exotic $\mathbb{R}^4$'s appearing above (R', R^*, and R^*_x) can be described as a 2-fold branched cover of $\mathbb{R}^4$ along a smooth, topologically standard $\mathbb{R}^2$. The G-manifolds N and R^* both lie in one-parameter families as in §1. Within these families, no two of the resulting N_x's or R^*_x's will be diffeomorphic, since their ends will all be distinct as in [13]. The manifolds R' will realize uncountably many diffeomorphism types, by [5], and the N's will be uncountable, by the remark at the end of §1. It is not known whether the above theorems can be assumed to hold for all parameter values, although the assertions about quotients of r_x hold for all values, and all of Theorem 4.1 can be assumed for values in a Cantor set (cf. [5, Theorem 3.2]).

The proof of Theorem 4.2(1) depends on the notion of *cellularity.* A subset X of the interior of an n-manifold W is called (*smoothly*) *cellular* if it is a nested intersection of smooth (compact) n-balls. For such an X, there is a diffeomorphism from $W - X$ to $W -$ pt. with support contained in a preassigned neighborhood of X. If $D \subset \operatorname{int} W$ is an *almost smooth* 2-disk (i.e., a topological 2-disk which is a smooth submanifold except at a single interior point) then D is cellular. This is easily proven by constructing isotopies with support in arbitrarily small neighborhoods of D, which shrink D radially into small n-balls.

Proof of Theorems 4.1 *and* 4.2. If CH is any Casson handle, Freedman [8, Addendum A to Theorem 1.1] proves that there is a homeomorphism $D^2 \times \mathbb{R}^2 \to CH$ which is smooth near the boundary and sends the core disk $D^2 \times 0$ onto an almost smooth disk D. If CH is standardly embedded in a 2-handle H, we verify that $D \subset H$ is topologically ambiently isotopic (rel ∂) to the core of H. Since D is flat, with unknotted boundary in ∂H, it suffices (by the s-cobordism theorem with $\pi_1 \cong \mathbb{Z}$ [9]) to show that $\pi_1(H - D) \cong \mathbb{Z}$. But $\pi_1(CH - D) \cong \mathbb{Z}$, and inclusion maps this group onto $\pi_1(H - D)$, since $H - CH$ is two-dimensional (a generalized cone on a Whitehead continuum). The result follows immediately.

We construct $R^* \subset M$ by constructing its complement $N^* \subset N$. Figure 7 shows a compact manifold L with two Casson handles attached. Let $L_0 \subset L$ be a compact submanifold obtained from L by removing a G-equivariant collar of the boundary. For each Casson handle, we attach a 2-disk to L_0 as follows: By our $\mathbb{Z}_2$-equivariant construction, the Casson handle is composed of two identical Casson handles glued together along part of their attaching regions. In each of these Casson handles we locate an almost smooth core disk. These fit together to give a $\mathbb{Z}_2$-equivariant core for the larger Casson handle. We ambiently attach these topological disks to L_0 to obtain a G-equivariant compactum N^* which is a deformation retract of N. Now we let $R^* = M - N^*$. Thus, R^* is a smooth, open G-manifold. Topologically, R^* is obtained from the 4-handle of M by trivially adding an open collar. (See the previous paragraph.) Thus, R^* is equivariantly homeomorphic to the interior of the 4-handle, so the G-action on R^* is topologically standard, as required. R^* is complementary to R' since we may surger the $S^2 \times S^2$ out of $N = S^2 \times S^2 \# R'$ by using an S^2 contained in N^* to reduce $M = N \cup R^*$ to $S^4 = R' \cup R^*$. By [5], $R' \cap R^*$ $(= N - N^*)$ has no smooth 3-spheres carrying H_3. (See the last remark of our §3.) Thus, R^* is a small exotic $\mathbb{R}^4$ with no 3-spheres near infinity (or even homology spheres which lack representations $\pi_1 \to \mathrm{SO}(3)$).

Now we analyze the various quotient spaces, beginning with N_y. This space is shown in Figure 8. (Compare with Figure 6.) Ignoring the branch locus, we find that N_y is obtained from $S^1 \times B^3$ by adding a Casson handle along $S^1 \times$ pt. and deleting the boundary. Thus, N_y is diffeomorphic to the interior of a Casson handle. But any such manifold is known to be diffeomorphic to $\mathbb{R}^4$. (The interior V of a Casson handle can be written as a nested union $\bigcup T_n$, where each T_n is a regular neighborhood of a finite 1-complex, and each map $T_n \hookrightarrow T_{n+1}$ is trivial in π_1. It follows that

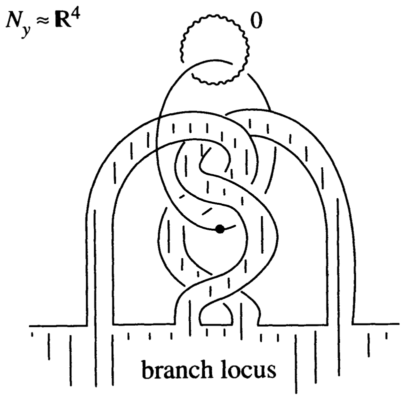

FIGURE 8

V can be written as a nested union of smooth balls, so it is diffeomorphic to $\mathbb{R}^4$.)

The surgery of N to R' is equivariant under r_y. We simply trade $S^2 \times D^2$ (with an involution fixing $S^1 \times I$) for $D^3 \times S^1$ (with involution fixing $D^2 \times S^0$). In N_y, the corresponding operation cuts out a 4-ball and glues in another 4-ball, so N_y is preserved, but the branch locus is changed by a surgery. In Figure 7, the surgery is accomplished by changing one 0-framed circle to a dotted circle. The corresponding operation in Figure 8 is to ambiently surger the branch locus along one of the two obvious trivial disks in B^4 (whose boundary runs over a 1-handle of the branch locus). Thus, the new branch locus appears as a ribbon disk in Figure 8. Topologically, the 1-handle and Casson handle cancel, showing that the branch locus is topologically equivalent to the standard $\mathbb{R}^2 \subset \mathbb{R}^4$. Thus, the 2-fold branched cover $N \to N_y$ has been surgered to a map $R' \to \mathbb{R}^4$ satisfying (3) of Theorem 4.1.

The above description of r_y also allows us to identify the manifold R_y^*. Since $R^* = M - N^*$, we obtain R_y^* as $M_y - N_y^*$, where M_y is diffeomorphic to S^4. N_y^* consists of a 2-disk in S^4 which is smooth away from two points, together with a tubular neighborhood of its boundary. We may delete this tubular neighborhood from N_y^* without changing the complement R_y^*. Similarly, we may shrink the disk to obtain a pair of almost smooth disks connected by a smooth arc. Since almost smooth disks are cellular, we see that R_y^* is obtained from S^4 minus two points by deleting a smooth curve connecting the punctures. Such a curve in a 4-manifold cannot be knotted, so R_y^* is diffeomorphic to $\mathbb{R}^4$.

The manifolds N_G and R_G^* are now easy to identify. The remaining involution $\hat{r}_z$ on N_y is rotation about the z-axis in Figure 8. This is just the given involution on the interior of our $\mathbb{Z}_2$-equivariant Casson handle. By construction, the quotient of this involution is again the interior of a Casson handle. Thus N_G is diffeomorphic to $\mathbb{R}^4$. In fact, the involution on N_y is smoothly standard. To see this, note that when we strip the boundary off of our $\mathbb{Z}_2$-equivariant Casson handle, we are left with $\mathbb{R}^4$ with the standard involution, equivariantly end-summed with two standard $\mathbb{R}^4$'s. R_G^* is diffeomorphic to $\mathbb{R}^4$, since N_G^* is essentially an almost smooth disk. In addition, the branched covering $R_y^* \to R_G^*$ is the standard one. This follows immediately from the $\mathbb{Z}_2$-equivariance of our identification $R_y^* \approx \mathbb{R}^4$.

In contrast with our previous discussion, the analysis of N_x and R_x^* yields a surprising connection with the *other* method of constructing exotic $\mathbb{R}^4$'s. In particular, R_x^* turns out to be a *large* exotic $\mathbb{R}^4$ which embeds in $\mathbb{C}P^2$ but has a compact submanifold X which does not embed in any negative definite manifold. We begin by recalling some details of the construction of such an exotic $\mathbb{R}^4$ [12]. The construction begins with the usual handle structure on $\mathbb{C}P^2$: A 2-handle attached to a $+1$-framed unknot in ∂B^4 forms a Hopf bundle, to which a 4-handle attaches. Inside the 2-handle, we find a suitably ramified Casson handle. This, together with the interior of the 4-ball, forms an exotic open Hopf bundle U which admits no smooth embedded 2-sphere generating its homology. If $R \subset \mathbb{C}P^2$ denotes any open subset homeomorphic to $\mathbb{R}^4$ such that $R \cup U = \mathbb{C}P^2$, then R will have the same end as a certain open manifold V with nonstandard, negative definite intersection form. It follows easily from Donaldson's Theorem that R is a large $\mathbb{R}^4$, with the required X any compact submanifold of R containing $\mathbb{C}P^2 - U$.

Now consider N_x, as given by Figure 9. (Compare with Figure 4.) Ignoring the branch locus, we see a Whitehead link with a Casson handle attached. But this is simply a Casson handle attached to a 1-framed unknot in ∂B^4, where the first stage has a single $(+)$ self-intersection and is drawn explicitly. Thus, N_x is diffeomorphic to U, for a suitably chosen Casson handle. (We obtain suitable ramification by additionally ramifying the 0-framed Casson handle if necessary. We can ramify the first stage of the 1-framed Casson handle by ramifying the link in Figure 7, but this is unnecessary: By [14] it suffices to have only one $(+)$ link at the first stage.) The embedding $N_x \subset M_x$ is the embedding $U \subset \mathbb{C}P^2$. Clearly,

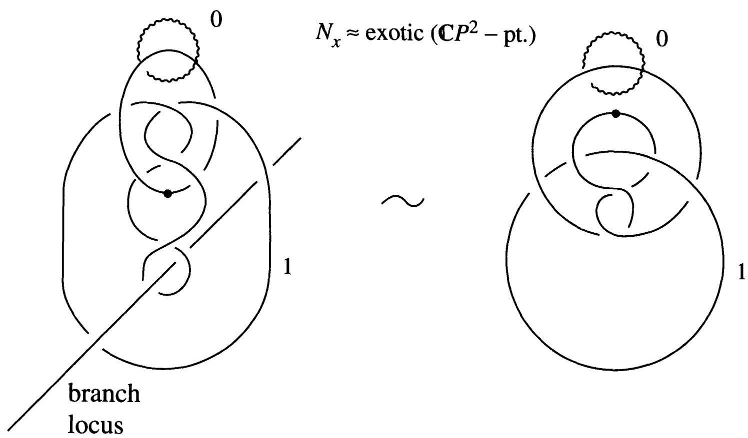

FIGURE 9

$R^*_x \cup N_x = \mathbb{C}P^2$, so R^*_x is a large exotic $\mathbb{R}^4$, and the map $R^* \to R^*_x$ is a topologically standard 2-fold branched cover from a small exotic $\mathbb{R}^4$ to a large one. Note that r_x on N is the involution described in [5], which acts on the end of R' but cannot extend smoothly over R'. We see that the quotient of this action on the end of R' is the end of N_x, an exotic $\mathbb{C}P^2 - \mathrm{pt}$. It appears, inside out, near the end of the large exotic $\mathbb{R}^4$ $R = R^*_x$. It cannot be the end of any exotic $\mathbb{R}^4$ (or any negative definite manifold), otherwise we could glue together this manifold and a suitable V to contradict Donaldson's Theorem.

Finally, we analyze N_z and R^*_z. To identify N_z, we refer back to Figure 7. The 1-skeleton here is a boundary sum of two $S^1 \times D^3$'s, with r_z the obvious involution which reverses both S^1's. The quotient of this is B^4. The two $\mathbb{Z}_2$-symmetric Casson handles each collapse to a single Casson handle which is attached to B^4 along *half* of its attaching circle. In N_z each of these Casson handles represents a standard $\mathbb{R}^4$ end-summand, so the only nontrivial topology in N_z comes from the two standard 2-handles in Figure 7. These collapse to a single 2-handle as in Figure 5, so N_z is diffeomorphic to $\overline{\mathbb{C}P}^2 - \mathrm{pt}$. The remaining involution $\hat{r}_y$ on N_z (with quotient $N_G \approx \mathbb{R}^4$) is the one shown in Figure 5. Thus, the branched covering $N_z \to N_G$ is (smoothly) the map $\mathbb{C}P^2 - \mathrm{pt.} \to \mathbb{R}^4$ obtained from the standard map $\mathbb{C}P^2 \to S^4$ (induced by complex conjugation) by deleting a point in the branch locus. Note that since r_z interchanges the

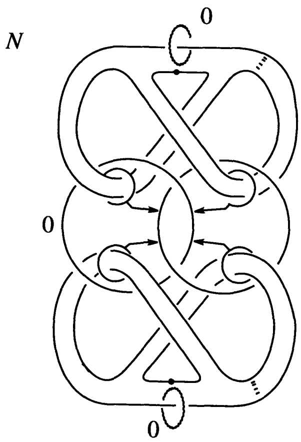

FIGURE 10

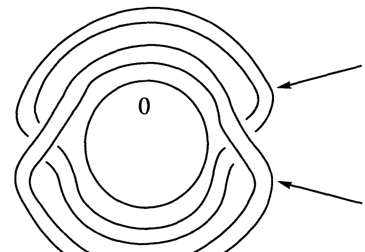

FIGURE 11

two S^2's of N, it gives an involution of the end of R' which cannot extend smoothly over R'. (By the same argument applied to r_x in [5], we can change the diffeomorphism type of a certain manifold by cutting out an embedded R' and regluing it via the map r_z.) However, this involution is much different from the one determined by r_x: its quotient is the end of $N_z \approx \overline{\mathbb{C}P}^2$ − pt., i.e., the *standard* end $S^3 \times \mathbb{R}$. The compactum $N_z^* \subset M_z \approx \overline{\mathbb{C}P}^2$ is obtained from a smoothly standard Hopf bundle by gluing a pair of almost smooth disks to it, along arcs in the disk boundaries. Thus $R_z^* = M_z - N_z^*$ is standard, as is its involution $\hat{r}_y$, by reasoning similar to that for R_y^*. q.e.d.

We now exhibit R' explicitly as a ribbon $\mathbb{R}^4$, so that the G-action on the end is visible. It is routine to verify that Figure 7 is G-equivariantly isotopic to Figure 10. (It may be easiest to work backwards.) We can obtain R' from this by surgering out the $S^2 \times S^2$—specifically, we can change one component of the Hopf link to a dotted circle and cancel it with the other component, and interpret the remaining circles as ribbon disks in B^4. This must be done with care if we wish to retain the symmetry. First, we restrict attention to the end of N by interpreting Figure 10 as (3-manifold) $\times\ \mathbb{R}\ \cup$ (Casson handles). We G-equivariantly slide the four ribbons over the 2-handles as shown by the arrows to free the Hopf link which we then erase. (To preserve the symmetry, the four parallels of the left 2-handle should be drawn as in Figure 11; the right handle should be opposite.) The result is equivariantly isotopic to Figure 12, which is essentially Figure 3.7 of [5] redrawn so that the full $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry is

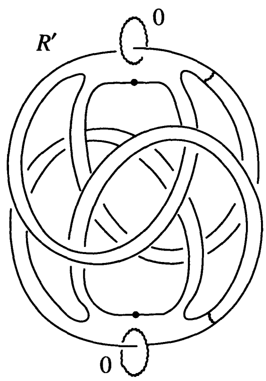

FIGURE 12

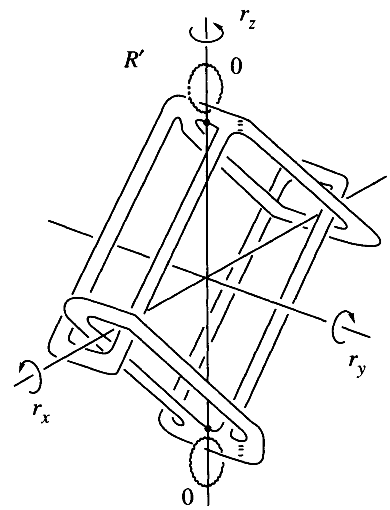

FIGURE 13

visible. This exhibits the end of R' with its G-action. To see all of R' with its r_y action we return to Figure 10 and break the other symmetries. The entire 4-manifold N is given here as $S^2 \times S^2 - B^4$ with two slice disks removed and replaced by Casson handles. Each disk is unknotted, so it is obtained by ambient surgery on the obvious genus 1 Seifert surface. Thus, it is exhibited as a ribbon disk, and we may choose the ribbon to be given by the appropriate dotted arc in Figure 10. (We have chosen these arcs to preserve the r_y-symmetry.) We $\mathbb{Z}_2$-equivariantly slide these two bands over the 2-handle as indicated in the right half of Figure 10. Then we surger out the $S^2 \times S^2$, eliminating the Hopf link. We necessarily obtain Figure 12 as before, but now it represents all of R' as the complement of a pair of ribbon disks in $\mathring{B}^4$ (with the indicated ribbons) union Casson handles. The r_y-symmetry on R' is clear. Thus, Figure 12 simultaneously shows the G-action on the end of R' and the extension of r_y over all of R'. Figure 13 shows the same thing from a different perspective. (Compare with [5, Figure 3.7].) Figure 12 is r_y-equivariantly isotopic to Figure 1 with the obvious ribbon disks, and with r_y still appearing as rotation about the y-axis. (The latter is Freedman's original description of his involution—although he needed extra ramification, since our §3 was unavailable then. Note that we may see directly that the quotient is diffeomorphic to $\mathbb{R}^4$ with a topologically standard branch locus. In fact, calculation shows that the quotient is again described by the link of Figure 1, with one component giving the (unknotted) slice disk and the other giving the branch locus.)

Remarks. (1) Figure 1 also exhibits an orientation reversing symmetry, the significance of which is unclear.

(2) It is instructive to observe how our construction of r_y fails for r_x (as guaranteed by Theorem 4.1). We could have chosen our ribbon disks in Figure 10 to preserve r_x, but the handle slides would have been blocked. In fact, Figure 12 possesses no ribbon disks with r_x symmetry. Otherwise, we could mod out the r_x action on S^4 to obtain a forbidden embedding of the end of N_x in S^4. (Alternatively, we may check that the quotient of Figure 12 by r_x is a trefoil knot, which is not slice.)

(3) Figures 12 and 13 also give a rough description of R^* with its G-action, and the r_y-equivariant splitting $S^4 = R' \cup R^*$. The complement in S^4 of the ribbon complement is obtained from B^4 by *adding* 2-handles along the dotted circles, and R^* is obtained from this by removing certain r_z-invariant topological cocores of the 2-handles.

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