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DEFORMATIONS OF FLAT CONFORMAL STRUCTURES ON A HYPERBOLIC 3-MANIFOLD

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Abstract

We show that a particular closed hyperbolic 3-manifold with a totally geodesic hypersurface of genus two admits a real two-dimensional family of flat conformal deformations that are distinct from the deformations obtained by bending along the totally geodesic hypersurface. The construction is quite general and can be applied to other not necessarily hyperbolic manifolds; it follows from a more general theory of bending hyperbolic cone manifolds along totally geodesic hyperplanes intersecting at the singular set.

1. Introduction

It is well known by Mostow's rigidity theorem [16] that the hyperbolic structure on a closed hyperbolic manifold M of dimension $n \geq 3$ is rigid. On the other hand, in the category of flat conformal structures there may be nontrivial deformations. (Recall that a flat conformal structure on M is a maximal atlas of charts modelled on subsets of S^n such that, locally, the transition functions are restrictions of the group of conformal automorphisms of S^n.) The most obvious deformations are those that correspond to "bending" along complete, totally geodesic hypersurfaces; these have been studied by several authors (see for example [2], [10], [11], [13] and [14]). At the same time, Apanasov has constructed a different type of deformations called "stamping deformations" and studied the general problem of the deformation space of flat conformal structures in a series of papers (see [1]–[6]).

In this paper, we construct a different type of deformation for simply connected conformally flat three-manifolds. The problem with applying this to obtain deformations of closed hyperbolic three-manifolds is that we need to be able to apply our deformations in an equivariant manner to the universal cover of the three-manifolds. We show that this is possible for a particular closed three-manifold X, thus obtaining a real two-dimensional

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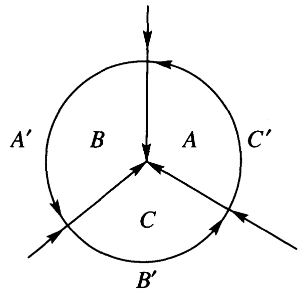

FIGURE 1

family of flat conformal deformations on X which are different from the usual "bending" deformations. The construction is quite general and it seems likely that many other hyperbolic three-manifolds admit similar deformations. It is also a natural generalization of the bending and stamping deformations and can be applied to manifolds of higher dimensions, for example, the Gromov-Thurston examples constructed in §3.7 of [9].

To state our result more precisely, we first describe the three-manifold X which is constructed as follows:

Take two tetrahedra with faces A, B, C and D and A', B', C' and D' respectively. Glue the two faces D and D' together and then glue the remaining faces according to Figure 1 so that the arrows on the edges match. (Figure 1 is a schematic picture where the tetrahedra have been flattened so that the vertex where A', B' and C' meet is moved to ∞.)

The resulting complex has one edge and one vertex, the link of the vertex is a surface of genus 2, and removing a neighborhood of the vertex one obtains a manifold N with boundary. X is obtained by doubling N along its boundary.

We shall see in §2 that the manifold X admits a hyperbolic structure with a totally geodesic hypersurface of genus two (cf. §3.2 of [19]). Our main result is the following:

Theorem 1. *X admits a real two-dimensional family of flat conformal deformations about the hyperbolic structure* (*thought of as a flat conformal structure*). *These structures are distinct from the one-parameter family of deformations obtained by bending the hyperbolic structure along the totally geodesic hypersurface.*

Remark. In fact, the family of deformations constructed forms an incomplete two-dimensional cone about the hyperbolic structure.

On the other hand, applying our construction to the examples of Gromov-Thurston in §3.7 of [9], we get

Corollary 1. *For $n \geq 4$, there exists a sequence of manifolds W_i of dimension n such that the dimension of the deformation space of flat conformal structures on W_i approaches ∞ as $i \to \infty$ and such that each W_i is doubly branch covered by a hyperbolic manifold V_i.*

The basic idea in our construction is to bend along intersecting totally geodesic hypersurfaces, or, to put it in another way, to generalize bending from codimension-one pieces to pieces with higher codimensions. It turns out that to bend along intersecting totally geodesic hypersurfaces simultaneously, it is necessary for the intersection of the hypersurfaces to have a cone-type singularity of positive curvature. At the same time, the bending parameters or weights are no longer independent; certain geometric conditions must be satisfied (these are equivalent to spherical polygonal conditions on the cross section of the intersection). A set of weights on the hypersurfaces is said to be *admissible* if they satisfy the conditions or equivalently, if bending along the hypersurfaces by these weights gives a flat conformal structure.

Theorem 1 is proven by showing that there is a one-parameter family of hyperbolic cone structures on X together with a collection of totally geodesic hypersurfaces intersecting at the singular set such that for each of the singular structures on X there is a one-parameter family of *admissible* weights on the totally geodesic hypersurfaces. The proof of Corollary 1 is a simple parameter count for the admissible weights on an appropriate set of intersecting totally geodesic hypersurfaces of each of the W_i.

This paper is organized as follows:

In §2 we construct a one-parameter family of deformations of cone hyperbolic structures on the three-manifold X. In §3 we give a brief account of some of the general techniques used in the study of conformally flat manifolds including the canonical stratification and the pleated image associated to a conformally flat manifold. This is applied in §4 to show how bending is done along intersecting geodesic hypersurfaces. In §5, we apply the results of §2 and §4 to our three-manifold X to prove Theorem 1. §6 gives some other examples of how our deformations can be applied to other manifolds, in particular, Corollary 1 is proven.

2. Hyperbolic cone structures on X

Let X be the closed three-manifold constructed in §1. In this section, we prove the following.

Proposition 1. *There exists a one-parameter family X_α of deformations of hyperbolic cone structures on X, parametrized by α, $0 < \alpha < \pi/3$. The*

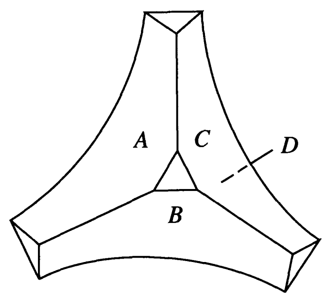

FIGURE 2. THE TRUNCATED HYPERBOLIC SIMPLEX

singular set V is an embedded S^1 in X, and has positive curvature for $0 < \alpha < \pi/6$. For $\alpha = \pi/6$, the structure is nonsingular. For each cone structure X_α there are four totally geodesic hypersurfaces (with boundary) A, B, C and D such that $A \cap B \cap C \cap D = V$.

Proof. It is easy to construct a regular truncated hyperbolic simplex such that the dihedral angles between the adjacent hexagonal faces are all α, $0 < \alpha < \pi/3$, and the dihedral angles between the adjacent triangular and hexagonal faces are all $\pi/2$. (This can be easily seen by using the projective model for H^3 and expanding a regular tetrahedron from one where the vertices touch the sphere at infinity to one where the adjacent faces touch at the sphere at infinity, and then truncating by $v_i^\perp$ where v_i are the vertices of the tetrahedron; see Figure 2.)

We name the hexagonal faces A, B, C and D and identify two isometric copies according to the pattern given in §1. The edges where the hexagonal faces meet are all identified and we obtain a hyperbolic cone manifold with a singular, totally geodesic boundary. The cone angle at the singular set is 12α; it meets the boundary (the singular surface formed by the triangular faces) at two points. We get a nonsingular hyperbolic manifold if and only if $\alpha = \pi/6$. Doubling along the boundary, we obtain a cone hyperbolic structure on X, with singular set an embedded S^1 in X. We denote the singular set by V, and if $0 < \alpha < \pi/6$, V has positive curvature. The double of the faces A, B, C and D (which we will also call A, B, C and D for convenience) are totally geodesic codimension-one hypersurfaces whose intersection is V. This proves the proposition.

3. The canonical stratification and the pleated image

In this section, we recall briefly some general properties of a conformally flat manifold. The basic ideas were probably due to Thurston for the dimension-two case, the generalizations to higher dimensions can be found in [1], [2], [5], [11] and [13].

Let U be a simply connected conformally flat manifold and let $\overline{U}$ be the completion of U with respect to the pullback of the spherical metric on S^n to U by the developing map. A *round ball* in U is a subset of U which is mapped diffeomorphically to a open, geometric round ball in S^n by the developing map. This notion is well defined since the group of conformal automorphisms of S^n preserves the set of balls in S^n. A round ball is said to be *maximal* if it is maximal with respect to inclusion. If $U \not\cong S^n$ or R^n then there is at least one maximal round ball; in fact, it is not difficult to see that there is either one or infinitely many maximal balls. If B is a maximal round ball in U, then clearly, $\partial B \cap \partial U \neq \varnothing$. We define $\partial B \cap \partial U$ to be the *ideal boundary* of B, denoted by Λ_B. Endowing B with the Poincaré metric of constant curvature -1, we can form the convex hull of Λ_B in B, denoted by $C(B)$. Note that $C(B) \subseteq B$ and thus $C(B) \subseteq U$. Also, $C(B) = \varnothing$ if Λ_B has only one point; otherwise $\dim C(B) = k$, $1 \leq k \leq n$, and $C(B)$ inherits a hyperbolic structure from the Poincaré metric on B. We have the following general lemma which shows that the $C(B)$ form a geometric partition or stratification of U.

Lemma 1 (Thurston, etc.). *Let U be a simply connected flat conformal manifold of dimension ≥ 2, $U \not\cong S^n$ or R^n and let Ω be the collection of maximal balls of U. For $B \in \Omega$, let $C(B)$ be the convex hull of its ideal boundary. Then*

(a) $\bigcup_{B \in \Omega} C(B) = U$,

(b) $C(B) \cap C(B') = \varnothing$ *if* $B \neq B'$.

The lemma was proved for dimension two by Thurston. In the special case when U is a isomorphic to a subset of S^2, a proof can be found in [8] and for the general case, the proof can be found in [5], [11] and [13].

Thus every point $x \in U$ lies in the convex hull of the ideal boundary of a unique maximal ball or equivalently, the convex hulls of the maximal balls form a partition of U into hyperbolic pieces. Clearly, if M is a conformally flat manifold and $\widetilde{M}$ its universal cover is not conformally equivalent to S^n or E^n, then the canonical stratification of $\widetilde{M}$ passes down to a stratification of M since the stratification is invariant under the covering transformations.

We can map U to a pleated hypersurface in H^{n+1} as follows:

Let B be a maximal ball in U and let $\operatorname{dev}(C(B))$ and $\operatorname{dev}(\Lambda_B)$ be the image in S^n of $C(B)$ and Λ_B respectively by the developing map. Regarding S^n as the boundary of H^{n+1} we can take the convex hull of $\operatorname{dev}(\Lambda_B)$ in H^{n+1}; we denote it by $C'(B)$. $\operatorname{dev}(C(B))$ and $C'(B)$ are isometric (using the Poincaré metric on $\operatorname{dev}(C(B))$ and the induced metric from H^{n+1} on $C'(B)$). There is a (isometric) map from $\operatorname{dev}(C(B))$ to $C'(B)$ which is essentially the nearest point map; the point $x \in \operatorname{dev}(C(B))$ is taken to the point $x' \in C'(B)$ if there is a horosphere in H^{n+1} at x tangent to $C'(B)$ at x'.

The *pleated map* p from U to H^{n+1} is defined to be the composition of the developing map with the above map. Since every point in U lies in the convex hull of a unique maximal ball, this is well defined up to composition by the automorphisms of S^n.

The image $p(U)$ of the pleated map is called the pleated image. The map p and the pleated image $p(U)$ have many interesting properties; for example, the induced metric on $p(U)$ from H^{n+1} gives a pseudometric on U. For dimension two, the metric on $p(U)$ is isometric to the hyperbolic metric if the pleated image has dimension 2. This is in general not true in higher dimensions. The geometry of the pleated image can be quite complicated; for example, the sectional curvature of $p(U)$ may vary and in fact, the pleated image may be singular.

An important special case is when U is conformally equivalent to a subset of S^n not equal to S^n or R^n. In this case, the pleated image is the boundary of the convex hull of $S^n - U$ in H^{n+1}. (Points in the convex hull are interior points if they have an open neighborhood strictly contained inside the convex hull; otherwise, they are boundary points. Note that there may be no interior points; for example, if $U = S^3 - \{w, x, y, z\}$ where w, x, y, z are in general position.)

4. Bending hyperbolic structures along intersecting hypersurfaces

In this section we specialize the results of the previous section to show how to bend a hyperbolic cone structure along totally geodesic hypersurfaces intersecting at the singular set to obtain nonsingular flat conformal structures. For simplicity, we only consider this for dimension $n = 3$ although analogous results hold for higher dimensions (see also [1] and [3]).

We start by recalling the standard bending deformation along a totally geodesic hypersurface (see [10], [11], [13] and [14] for more details). Using the Poincaré ball model, we identify hyperbolic three-space H^3 with an open B^3 in $S^3 \cong (\mathbb{R}^3 \cup \infty)$ and let H^2 be a hyperbolic plane in H^3. The plane separates H^3 into two components and intersects the sphere at infinity in a geometric circle. Bending along the plane then corresponds to bulging out one of the components of the H^3 by an angle α. Actually, in the conformal picture, it is more accurate to describe this as inserting a lens of angle α. When $\alpha < \pi$, the resulting structure is $U = B_0 \cup B_\alpha$, the union of two intersecting balls.

The bending terminology arises by looking at the picture of the pleated image of this set in H^4. This is the boundary of the convex hull of $S^3 - U$ in H^4, which is two hyperplanes in H^4 intersecting at a bend or pleat such that the dihedral angle between them is $\pi - \alpha$, or equivalently, the angle between their outward pointing normals at the pleat is α.

The maximal balls of U are B_0, B_α and B_t, $0 < t < \alpha$, where B_t are the maximal balls in U whose ideal boundaries are all $\partial B_0 \cap \partial B_\alpha$. Under the pleated map, the points in $C(B_0)$ and $C(B_\alpha)$ are mapped to the two hyperplanes, and the points in $C(B_t)$ are all mapped to the pleat.

The space of maximal balls in this case is isomorphic to a closed interval; using the bending measure, we can put a metric on this so that the interval has length α. We note also that, in this case, the pleated image $p(U)$ is isometric to H^3 since it has extrinsic curvature 0 in H^4.

Now, if we have a set $\{F_i\}$ of nonintersecting, totally geodesic hypersurfaces in H^3, then we can obviously extend the above argument and bend along each F_i by α_i to obtain a flat conformal structure. Thus, if we have a hyperbolic manifold M with nonintersecting totally geodesic hypersurfaces, by passing to the universal cover, we can bend along the lifts of the totally geodesic hypersurfaces in an equivariant way to obtain deformations of flat conformal structures on M. The underlying structure on the pleated image is hyperbolic and the structure obtained is determined by the hypersurfaces and their bending data.

We next examine the case of bending a cone hyperbolic structure along intersecting hypersurfaces. This time we will start with the conformally flat manifold U which we will identify with its developing image since the developing map will be a diffeomorphism from U to its image. It is defined as follows:

Let U be the union of three open balls B_1, B_2 and B_3 in S^3 such that $B_1 \cap B_2 \cap B_3 \neq \varnothing$ and $B_i \not\subseteq B_j$ for $i \neq j$. Then $\partial B_1 \cap \partial B_2 \cap \partial B_3 = \{x, y\}$ where x and y are two points in $S^3 = R^3 \cup \infty$ and by a conformal transformation of S^3, we may assume that $x = (0, 0, 0)$, $y = \infty$ so that,

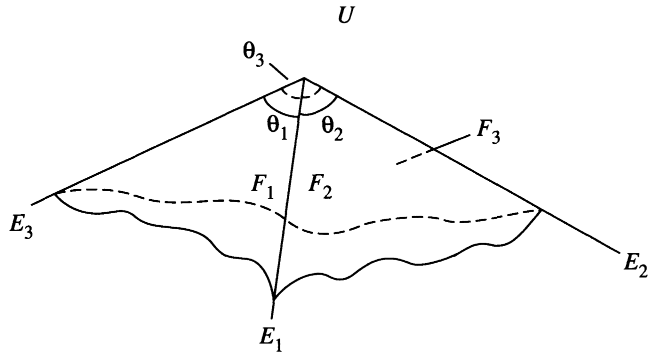

FIGURE 3. THE UNION OF THREE HALF-SPACES WITH NONEMPTY INTERSECTION

in fact, our three balls are three half-spaces passing through the origin and U is union of these half-spaces. Thus U consists of all points in R^3 above an infinite triangular pyramid with vertex at $(0, 0, 0)$. Let the faces of the pyramid be F_1, F_2 and F_3, where $F_i \subset \partial B_i$ and let the edge between the faces F_i and F_{i+1} be E_i, the angle on the face F_i of the pyramid at the vertex be θ_i and the angle between the upward facing normal to F_i and F_{i+1} at E_i be α_i (where i is taken mod 3 throughout). See Figure 3.

The maximal balls and canonical stratification of U are relatively simple and can be described as follows:

Proposition 2. *The maximal balls in U are all half-spaces of three types which can be described as follows*:

(a) *Half-spaces whose boundary intersect ∂U at exactly one of the three faces F_i. (There are three such maximal balls and the convex hull of their ideal boundary are infinite three-dimensional hyperbolic wedges (or sectors) bounded by intersecting hyperbolic planes. The angle between the bounding planes of each wedge is θ_i.*

(b) *Half-spaces whose boundaries intersect ∂U at exactly one of the edges E_i. For each edge E_i, there is a one-parameter family of such maximal balls; the convex hull of the ideal boundaries of these maximal balls are infinite half hyperbolic planes.*

(c) *Half-spaces whose boundaries intersect ∂U at only the two points $(0, 0, 0)$ and ∞. There is a two-parameter family of such maximal balls; the convex hull of the ideal boundary of these maximal balls is an infinite hyperbolic line.*

The proof of the proposition is easy and is left to the reader.

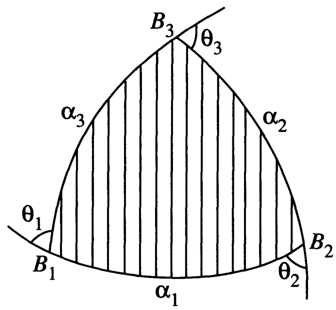

FIGURE 4. THE SPACE OF MAXIMAL BALLS FOR U = THE UNION OF THREE HALF-SPACES

There is a natural metric on the space of maximal balls using the bending measure. Using the metric the space of maximal balls in the above case is isometric to a spherical triangle with edge lengths α_i and exterior angles θ_i. The vertices of the triangle represent the three maximal balls in case (a) of Proposition 2 above where the convex hull of the ideal boundary has dimension three. The edges of the triangle represent the maximal balls in case (b) of Proposition 2 where the convex hull of the ideal boundary has dimension two and finally, the interior points of the triangle represent the maximal balls in case (c) of Proposition 2 where the convex hull of the boundary has dimension one (see Figure 4).

On the other hand, if we look at the intersection of the faces F_1, F_2 and F_3 with the unit sphere centered at 0, we obtain a spherical triangle with edge lengths θ_i and exterior angles α_i. This triangle is dual to the triangle from the space of maximal balls in the following sense:

Proposition 3. *For every convex spherical n-gon with edge lengths e_i and exterior angles β_i, $1 \leq i \leq n$, there exists a dual spherical n-gon with edge lengths β_i and exterior angles e_i.*

Proof. We use the unit sphere centered at the origin as the model for spherical space. For each edge e_i there is a unique plane P_i passing through the origin and the edge. Fixing an orientation, we can take the outward facing normals V_i to the planes P_i at the origin. These intersect the sphere at n points v_i which form the vertices of the dual n-gon. It is easy to see that the dual n-gon has side lengths β_i and exterior angles e_i. q.e.d.

We now look at the pleated image of U in H^4. Since $U \subset S^3$, the pleated image is the boundary of the convex hull of $S^3 - U$ in H^4. This is a singular hypersurface in H^4 consisting of three H^3 wedges W_1, W_2

and W_3 such that the dihedral angles between the bounding planes of each wedge is θ_i, and the angles between the normal vectors to the wedges W_i and W_{i+1} at their intersection is α_i. Thus the pleated image is a singular hyperbolic three-manifold with three codimension-one pleats meeting at a codimension-two singular set with cone angle $\theta_1 + \theta_2 + \theta_3$.

The conformal structure on U can be thought of as being obtained by bending a hyperbolic cone manifold along three totally geodesic hyperplanes intersecting at the singular line. The dihedral angles between the hyperplanes are θ_i, and the bending measures are α_i. Note that the bending measures in this case are determined by the dihedral angles θ_i between the hyperplanes, since the spherical triangular condition must be satisfied and a spherical triangle is completely determined by its exterior angles. In other words, fixing the cone structure and the positions of the hyperplanes in this case completely determines the *admissible* bending weights (compare with [6]).

There are several ways in which the above can be generalized.

First, if we have n totally geodesic hyperplanes intersecting at the singular line of a simply connected hyperbolic cone manifold, bending along the hypersurfaces to obtain a flat conformal structure is equivalent to satisfying a spherical n-gon condition, since U will be conformally equivalent to the union of n half-spaces passing through the origin, and the space of maximal balls will be a spherical n-gon with side lengths given by the bending weights and exterior angles given by the dihedral angles between the hypersurfaces. Thus, if the hyperplanes and their dihedral angles are specified, the deformation space of such deformations is just the deformation space of spherical n-gons with specified exterior angles.

More generally, we can bend along totally geodesic hypersurfaces intersecting at more than one singular line as long as the spherical polygonal conditions are satisfied about all the singular lines.

Second, the above arguments also hold for hyperbolic cone manifolds of dimension $n > 3$ as long as the singular points of the singular set are all of codimension 2 (see [20] for a definition of the codimension of a singular point), and we bend along totally geodesic hypersurfaces intersecting at the singular set. We summarize this in the following lemma.

Lemma 2. *Let M be a simply connected hyperbolic cone manifold of dimension $n \geq 3$ such that the singular points are all of codimension 2. Let $V' = \bigcup_j V'_j$ be the singular set where V'_j are the components of V', and suppose there exists a family $\{F_i\}$ of totally geodesic hypersurfaces in M (possibly with boundary) intersecting at V' such that $M - \{F_i\}$ is hyperbolic* *without singularities. Then bending M along $\{F_i\}$ by $\{\alpha_i\}$ gives a flat conformal structure on M if the bending data $\{\alpha_i\}$ is admissible, i.e,. if, at each component V_j' of V', the spherical polygonal condition relating the weights and the dihedral angles of the faces meeting at V_j' is satisfied.*

Remark. Johnson and Millson showed in [10] that there were obstructions to simultaneously bending along intersecting totally geodesic hypersurfaces for a hyperbolic manifold M of dimension $n \geq 4$. From Lemma 2, we see that this is not possible without first deforming the structure so that the intersection of the two hypersurfaces becomes singular, for this would imply the existence of a nondegenerate spherical polygon with the sum of exterior angles equal to 2π. The author has also been informed by B. Apanasov that Lemma 2 was independently proven in [7].

Finally, if we have n balls in S^n $(n \geq 4)$ such that the intersection of their boundaries is two points, this corresponds to bending a cone hyperbolic manifold with singular points of codimension from 1 to $n-2$ along totally geodesic hypersurfaces that intersect at the singular set. In this case, the space of maximal balls is isometric to a spherical n-simplex. Thus in general a spherical $(n-1)$-dimensional polyhedral condition must be satisfied to bend along hypersurfaces intersecting at the singular set of a hyperbolic cone manifold. (Note that the codimension of the points in the singular set must be between 1 and $n-2$.) However, we do not know of any example of any closed manifold of dimension $n \geq 4$ which admits such general deformations.

5. Bending the cone hyperbolic structures on X

Let X be the manifold defined in §1 and let X_α, $0 < \alpha < \pi/6$, be the hyperbolic cone structures on X with cone angle 12α about the singular set V as in Proposition 1. This pulls back to a cone hyperbolic structure on $\widetilde{X}$ with singular set $\widetilde{V}$. The totally geodesic faces A, B, C, D lift to totally geodesic faces $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ intersecting at $\widetilde{V}$ where $\widetilde{X} - (\widetilde{A} \cup \widetilde{B} \cup \widetilde{C} \cup \widetilde{D})$ is hyperbolic with no singularities. To obtain a flat conformal structure on X with underlying hyperbolic cone structure X_α, we can bend along the totally geodesic faces $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ equivariantly with respect to the covering transformations so that the spherical polygonal condition about $\widetilde{V}$ is satisfied. Clearly, it is only necessary to check this locally for each component of $\widetilde{V}$. Looking at the cross-section of V, we can easily check that the faces A, B, C and D intersect V twelve times in the pattern in Figure 5 (next page).

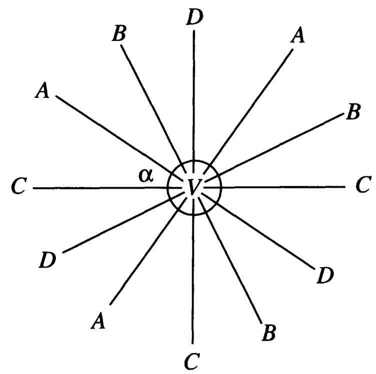

FIGURE 5. THE CROSS-SECTION OF V

If we bend along $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ by a, b, c and d respectively, by Lemma 2 we will get a flat conformal structure (i.e., (a, b, c, d) is *admissible*) if and only if there exists a spherical 12-gon with exterior angles all equal to α and with side lengths $(d, a, b, c, d, b, c, a, d, c, a, b)$ in that order. By a dimension count, it is easily seen that the number of degrees of freedom for such a 12-gon is $4 - 3 = 1$. (The four parameters a, b, c and d gives us the four degrees of freedom, but we require that the polygon closes up at the right angle which is $2+1$ degrees of freedom.) Thus a, b, and c are functions of d and it suffices to show that we obtain a one-parameter family of admissible weights when $a = b = c$ which is the following:

Lemma 3. *For each $0 < \alpha < \pi/6$ there exists a one-parameter family of spherical 12-gons with all the exterior angles equal to α and such that the side lengths s_i $(1 \leq i \leq 12)$ satisfy $s_1 = s_5 = s_9$, $s_i = s_j$ if $i, j \neq 1, 5$ or 9. If we let the two possible lengths be d and a respectively, then a is a function of α and d; i.e., $a = f(\alpha, d)$, and the family is parametrized by $0 < d < d_0$ where d_0 is the length of the sides of the regular spherical triangle with exterior angles 4α.*

Proof. There are three obvious (possibly degenerate) 12-gons which satisfy the above lemma. The first is where $d = a$, so we have the unique regular spherical 12-gon with exterior angle α (this exists since $12\alpha < 2\pi$). The second is where $a = 0$ so that we have a regular spherical triangle with side lengths d_0 and exterior angles 4α, and the third case is where $d = 0$ and we have a 9-gon with the side lengths all equal to a_0, $a_0 = f(\alpha, 0)$ and the exterior angles are $(\alpha, \alpha, 2\alpha, \alpha, \alpha, 2\alpha, \alpha, \alpha, 2\alpha)$.

It is easy to see that for each $0 < d < d_0$, we can find an a such that we get a spherical 12-gon with exterior angles all equal to α and side lengths $(d, a, a, a, d, a, a, a, d, a, a, a)$ (start with the regular triangle of side length d and open it up and expand it in a regular fashion putting in three sides of equal length at each opening. At some point, we can make the exterior angles all equal to α). Clearly, $a = f(d, \alpha)$ is a monotone decreasing function of d for each fixed α, so the spherical 12-gons satisfying the conditions of the lemma are parametrized by the side length d, $0 < d < d_0$. q.e.d.

Proof of Theorem 1. We have shown that for each hyperbolic cone structure X_α on X $(0 < \alpha < \pi/6)$ there is a set of totally geodesic hypersurfaces A, B, C and D in X and a one-parameter family of admissible weights on them parametrized by d, $0 < d < d_0$. The degenerate case where $d = 0$ corresponds to the case where the bending data is 0 on the D faces; similarly, the degenerate case where $d = d_0$ corresponds to the case where the bending data is 0 on the A, B and C faces. Clearly, they are also admissible, so we have a one-parameter family of admissible data, parametrized by d, $0 \leq d \leq d_0$, for each hyperbolic cone structure X_α on X. Thus there is a two-parameter family of flat conformal structures on X, parametrized by (α, d), $0 < \alpha < \pi/6$ and $0 \leq d \leq d_0$.

On the other hand, if we take the regular hyperbolic structure on X, then the triangular faces of the truncated simplices form a totally geodesic hypersurface in X, and we can put any positive weight on this surface and bend along it by this weight. (Since there are no singular points, the spherical polygonal condition is vacuously satisfied.) The flat conformal structures obtained this way obviously have different underlying structure and bending data from those constructed above, hence give different structures. This proves our theorem.

6. Deformations on other manifolds

The deformations we constructed can be applied to many other manifolds; we give two examples in this section.

Example 1. It is well known that the complement of the figure eight know in S^3 admits a complete hyperbolic structure of finite volume, obtained by identifying the faces of two copies of the regular hyperbolic ideal tetrahedrons (with the dihedral angles between the faces equal to $\pi/3$) according to Figure 6 (next page).

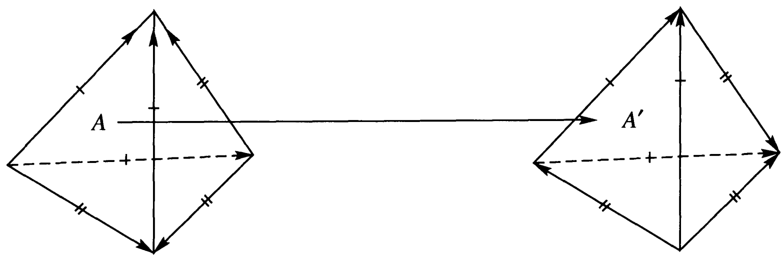

FIGURE 6. IDENTIFICATION PATTERN FOR THE COMPLEMENT OF THE FIGURE EIGHT KNOT

Choosing a horospherical neighborhood of the cusp, we can double the structure about the horosphere (this is essentially a Möbius inversion, see [2], [4] and [12] for more details) to obtain a flat conformal structure on the double of the figure eight knot complement M. This is a closed manifold which we denote by M'. The developing map is a bijection of $\widetilde{M}'$ to a subset of a ball in S^3 with infinitely many smaller balls removed. Changing the horosphere used changes the developing image (conformally) so that there is a one-parameter family of flat conformal structures on M' from this construction. On the other hand, by expanding the regular tetrahedron so that the vertices are outside the sphere at infinity, we can construct a hyperbolic cone structure on M' as before. We can then attach an admissible bending data on the faces of the tetrahedron (e.g., by taking the weights on all the faces to be equal so that we get a regular spherical polygon at the cross-section of the singular set) to obtain new flat conformal structures on M'.

Example 2 (Gromov-Thurston example). In §3.7 of [9], Gromov and Thurston constructed examples of hyperbolic manifolds V_i of dimension $n \geq 4$ admitting i-isometric involutions fixing some hypersurfaces in V_i which divide V_i into $2i$ isometric sectors meeting at a codimension-two submanifold V' (so that the angle of each sector at V' is π/i). Gluing together $2j$ such sectors, one obtains a cone hyperbolic manifold, say $W_{j,i}$ with singular set V' with cone angle $2\pi j/i$. If $j < i$ we can obtain flat conformal structures on $W_{j,i}$ by bending along the faces of the sectors by admissible weights. Again, this amounts to a spherical polygonal condition, i.e., finding a spherical $2j$-gon with exterior angles all equal to π/i. Here since there are $2j$ independent faces meeting at V', the number of degrees of freedom is $2j-3$, for $j > 1$. For $j = 1$, there is a unique bi-gon with exterior angles $2\pi/i$ (the sides have length π), so there is a unique flat conformal structure. Choosing $j = i/2$, we can form manifolds $W_i = W_{i/2,i}$ which are doubly branch covered by the hyperbolic manifolds V_i and whose deformation spaces of flat conformal structures have dimension greater than or equal to $i - 3$. This proves Corollary 1.

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