

PUKANSZKY'S CONDITION AND SYMPLECTIC INDUCTION

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Abstract

Pukanszky's condition is a condition used in obtaining representations from coadjoint orbits. In order to obtain more geometric insight into this condition, we relate it to symplectic induction. It turns out to be equivalent to the condition that the orbit in question is a symplectic subbundle of a modified cotangent bundle.

1. Introduction

One of the original goals of geometric quantization was to obtain a general method of constructing (irreducible) representations of Lie groups out of their coadjoint orbits. The idea was to generalize the Borel-Weil-Bott theorem for compact groups and Kirillov's results for nilpotent groups. Since then geometric quantization has led a somewhat dual life. On the one hand, in representation theory where it is called the orbit method (see [8] for a relatively recent review). On the other hand, in physics where it serves as a procedure that starts with a symplectic manifold (a classical theory) and creates a Hilbert space and a representation of the Poisson algebra as operators on it (the quantum theory).

Recent results in quantum reduction theory [5] allow us to show rigorously in some particular cases that geometric quantization intertwines the procedures of symplectic induction and unitary induction. Since the latter is one of the ingredients in the orbit method, this gives a geometrical insight into the "classical" part of the orbit method. In particular, it allows us to give a geometrical interpretation of Pukanszky's condition on a polarization which is completely different from the well-known interpretation that says that the coadjoint orbit contains an affine plane. In fact, we prove (Proposition 3.9) that Pukanszky's condition is equivalent to the statement that the coadjoint orbit in question is, in a noncanonical way, symplectomorphic to a symplectic subbundle of a modified cotangent bundle (where modified means that the canonical symplectic form on

the cotangent bundle is modified by adding a closed 2-form on the base space). For real polarizations these results have also been obtained with completely different methods in [12]. Again for real polarizations we obtain as a corollary that Pukanszky's condition links the two dual lives of geometric quantization.

This paper is organized as follows. In §2.1 we recall briefly the basics of symplectic induction, and in §2.2 we show (heuristically) that geometric quantization intertwines symplectic induction and unitary induction. Then in §2.3 we prove that unitary induction from a one-dimensional unitary representation of a subgroup is equivalent to geometric quantization of an induced symplectic manifold. In §3 we use this induced symplectic manifold to prove the above-mentioned interpretation of Pukanszky's condition. In §4 we give an example of this interpretation that is particularly interesting for physics: the fact that the symplectomorphism of the orbit with modified cotangent bundle is not canonical translates as the fact that the position of a photon has no intrinsic meaning, i.e., depends heavily upon the observer. Finally in §5, an appendix, we collect some notation and conventions used throughout this paper.

2. Symplectic induction and induced representations

2.1. Symplectic induction. Let (M, ω) be a symplectic manifold and let H be a closed Lie subgroup of a connected Lie group G . Suppose H acts smoothly on M by symplectomorphisms and admits an equivariant momentum map $J_M: M \rightarrow \mathfrak{h}^*$, where \mathfrak{h} denotes the Lie algebra of H . Symplectic induction ([9], [13], [25]) then constructs in a canonical way a symplectic manifold $(M_{\text{ind}}, \omega_{\text{ind}})$ on which G acts smoothly by symplectomorphisms with an equivariant momentum map $J_{\text{ind}}: M_{\text{ind}} \rightarrow \mathfrak{g}^*$, where \mathfrak{g} is the Lie algebra of G .

To construct M_{ind} one proceeds as follows. The group H acts on G by $h: g \mapsto R_{h^{-1}}g \equiv gh^{-1}$ and we denote by Φ_{T^*G} the canonical lift of this action to T^*G , equipped with its canonical symplectic form $d\vartheta_G$. We identify T^*G with $G \times \mathfrak{g}^*$ by identifying \mathfrak{g}^* with the left-invariant 1-forms on G . In this trivialization the action of H on T^*G is given by

$$\Phi_{T^*G}(h)(g, \mu) = (gh^{-1}, \text{Coad}_G(h)\mu),$$

where we have added the subscript G to stress that it concerns the coadjoint action with respect to the group G . This action admits a canonically

defined equivariant momentum map $J_{T^*G}: T^*G \rightarrow \mathfrak{h}^*$ given by

$$J_{T^*G}(g, \mu) = -i_{\mathfrak{h}}^* \mu.$$

We denote by Φ_M the action of H on M , and we construct an action $\Phi_{\tilde{M}}$ of H on $\tilde{M} = M \times T^*G$ by $\Phi_{\tilde{M}} = \Phi_M \times \Phi_{T^*G}$, i.e.,

$$\Phi_{\tilde{M}}(h)(m, g, \mu) = (\Phi_M(h)(m), gh^{-1}, \text{Coad}_G(h)\mu).$$

This action is symplectic for the symplectic form $\tilde{\omega} = \omega + d\vartheta_G$; it is a proper action because H is closed. Moreover, this action admits an equivariant momentum map $J_{\tilde{M}} = J_M + J_{T^*G}$ for which $0 \in \mathfrak{h}^*$ is a regular value. The sought-for induced symplectic manifold $(M_{\text{ind}}, \omega_{\text{ind}})$ is the Marsden-Weinstein reduced symplectic manifold

$$M_{\text{ind}} = J_{\tilde{M}}^{-1}(0)/H.$$

To obtain the hamiltonian action of G on M_{ind} we make the following observations. The group G acts naturally on itself on the left; the canonical lift of this action to T^*G is hamiltonian and given by $g: (\hat{g}, \mu) \mapsto (g\hat{g}, \mu)$. We let G act trivially on M to obtain a hamiltonian action of G on \tilde{M} with the canonical equivariant momentum map $\tilde{J}: \tilde{M} \rightarrow \mathfrak{g}^*$ given by

$$(2.1) \quad \tilde{J}(m, g, \mu) = \text{Coad}_G(g)\mu.$$

This action commutes with the H -action on \tilde{M} and leaves $J_{\tilde{M}}$ invariant; hence it induces a symplectic action of G on M_{ind} . Since \tilde{J} is invariant under the H -action, it descends as an equivariant momentum map for the G -action on M_{ind} which we denote by J_{ind} . This finishes the construction of the induced symplectic manifold. The following proposition describes the relation between $(M_{\text{ind}}, \omega_{\text{ind}})$, (M, ω) , G , and H ; it is a special case of a result of A. Weinstein [24].

Proposition 2.2. *M_{ind} is a fiber bundle over $T^*(G/H)$ with typical fiber M . Moreover, restriction of ω_{ind} to a fiber yields the original symplectic form ω on M .*

Proof. Let α be a connection on the principal H -bundle $G \rightarrow G/H$, i.e., α is a \mathfrak{h} -valued 1-form on G satisfying:

$$\begin{aligned} \forall X \in \mathfrak{h} \forall g \in G : \alpha_g(X) &= X, \\ \forall h \in H \forall Y \in T_g G : (R_{h^{-1}}^* \alpha)_g(Y) &= \text{Ad}_H(h)(\alpha_g(Y)), \end{aligned}$$

where we interpret elements of $\mathfrak{h} \subset \mathfrak{g}$ as left-invariant vector fields on G . Restricting our attention to left-invariant vector fields, we can interpret α_g

as a projection $\alpha_g: \mathfrak{g} \rightarrow \mathfrak{h}$; by dualization we obtain a family of injections $\{\alpha_g^*: \mathfrak{h}^* \rightarrow \mathfrak{g}^* | g \in G\}$ satisfying $i_{\mathfrak{h}}^* \circ \alpha_g^* = \text{id}_{\mathfrak{h}^*}$. Going to cotangent bundles, we consider the canonical projections $\text{pr}: T^*G \equiv G \times \mathfrak{g}^* \rightarrow G$, $\pi: G \rightarrow G/H$, and $\overline{\text{pr}}: T^*(G/H) \rightarrow G/H$. It is easy to verify that $\text{pr}: J_{T^*G}^{-1}(0) = G \times \mathfrak{h}^0 \rightarrow G$ is the pull-back bundle of the bundle $T^*(G/H) \rightarrow G/H$ over the map π , where the projection $J_{T^*G}^{-1}(0) \rightarrow T^*(G/H)$ is just π^{*-1} .

We now note that $J_M^{-1}(0) = \{(m, g, \mu) | J_M(m) = i_{\mathfrak{h}}^* \mu\} \cong M \times G \times \mathfrak{h}^0$, and we define a map $P: J_M^{-1}(0) \rightarrow G \times \mathfrak{h}^0$ by

$$P(m, g, \mu) = (g, \mu - \alpha_g^*(J(m))).$$

The kernel of this map is obviously diffeomorphic to M , and the defining properties of a connection show that it is equivariant for the H -actions. Hence P induces a map $\overline{P}: J_M^{-1}(0)/H = M_{\text{ind}} \rightarrow J_{T^*G}^{-1}(0)/H \cong T^*(G/H)$ whose fiber is diffeomorphic to M . We thus obtain the following commutative diagram:

$$(2.3) \quad \begin{array}{ccccc} J_M^{-1}(0) & \xrightarrow{P} & G \times \mathfrak{h}^0 & \xrightarrow{\text{pr}} & G \\ \downarrow \text{mod } H & & \downarrow \pi^{*-1} & & \downarrow \pi \\ M_{\text{ind}} & \xrightarrow{\overline{P}} & T^*(G/H) & \xrightarrow{\overline{\text{pr}}} & G/H. \end{array}$$

This proves the first assertion of the proposition; the second is left to the reader.

2.2. Geometric quantization and induced representations. To forge the link between symplectic induction and induced representations, we make two additional assumptions. In the first place we assume that H is connected, and in the second place we assume that geometric quantization applied to the quadruple (M, ω, H, J_M) yields a unitary representation U_M of the Lie group H on the Hilbert space \mathcal{H}_M . Of course this requires additional data such as a polarization \mathcal{F}_M on (M, ω) , but we will not specify these explicitly. The aim now is to apply geometric quantization to the quadruple $(M_{\text{ind}}, \omega_{\text{ind}}, G, J_{\text{ind}})$ in order to obtain a unitary representation of G .

We start by applying geometric quantization to the symplectic manifold $(\widetilde{M}, \widetilde{\omega})$. We equip T^*G with the vertical polarization \mathcal{F}_v and define on \widetilde{M} the composite polarization $\widetilde{\mathcal{F}} = \mathcal{F}_M \oplus \mathcal{F}_v$. It is an elementary exercise in geometric quantization to prove that the Hilbert space \mathcal{H} obtained by

quantization of $(\widetilde{M}, \widetilde{\omega})$ with this polarization can be described as

$$\widetilde{\mathcal{H}} = \left\{ \psi : G \rightarrow \mathcal{H}_M \mid \int_G \langle \psi(g), \psi(g) \rangle dg < \infty \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathcal{H}_M , and dg denotes a nowhere vanishing volume form on G . For convenience we will now make the choice that dg is a left-invariant volume form (which is unique up to a nonzero real factor). When one then quantizes the action of G on \widetilde{M} , one obtains the unitary representation \widetilde{U} of G on $\widetilde{\mathcal{H}}$ given by

$$(2.4) \quad (\widetilde{U}(g)\psi)(k) = \psi(g^{-1}k).$$

Note that this nice description is due to our particular choice of the volume dg on G .

The next step is to implement the Marsden-Weinstein reduction from \widetilde{M} to M_{ind} by means of the group H . Although no proof is known, partial results obtained in [5], [6], [7], [9], and [21] all indicate that the following conjecture is true, a conjecture which describes the Hilbert space \mathcal{H}_{ind} obtained by applying geometric quantization to the reduced symplectic manifold $M_{\text{ind}} = J_{\widetilde{M}}^{-1}(0)/H$.

Conjecture 2.5.

$$\mathcal{H}_{\text{ind}} \cong \{ \psi \in \widetilde{\mathcal{H}} \mid \forall h \in H : U_{\widetilde{M}}(h)\psi = \text{Det}(\text{Ad}_H(h))^{-1/2} \cdot \psi \}.$$

In this conjecture $U_{\widetilde{M}}$ is the unitary representation of H on $\widetilde{\mathcal{H}}$ obtained by geometric quantization; note that the adjoint representation is with respect to the reducing group H . Of course this equivalence has to be read with caution because (i) in general one has to enlarge $\widetilde{\mathcal{H}}$ before there are elements satisfying the condition of the right-hand side and (ii) one then has to restrict to elements that are square-integrable with respect to a measure that is not specified in terms of $\widetilde{\mathcal{H}}$.

The representation $U_{\widetilde{M}}$ is readily calculated as being given by

$$(U_{\widetilde{M}}(h)\psi)(g) = \text{Det}(\text{Ad}_G(h))^{-1/2} \cdot U_M(h)\psi(gh).$$

The factor $\text{Det}(\text{Ad}_G(h))^{-1/2}$ is due to the fact that the left-invariant volume form dg on G is not invariant under the right-action of H , but transforms with $\text{Det}(\text{Ad}_G(h))$. Combining this with the conjecture, we find the following description of \mathcal{H}_{ind} :

$$(2.6) \quad \mathcal{H}_{\text{ind}} \cong \{ \psi : G \rightarrow \mathcal{H}_M \mid \forall h \in H : \psi(gh^{-1}) = \gamma(h) \cdot U_M(h)\psi(g) \},$$

where we have defined the function γ on H by

$$(2.7) \quad \gamma(h) = \sqrt{\frac{\text{Det}(\text{Ad}_H(h))}{\text{Det}(\text{Ad}_G(h))}}.$$

The scalar product on \mathcal{H}_{ind} can be described intrinsically by the following procedure (sketched). For $\psi, \chi \in \mathcal{H}_{\text{ind}}$ we construct the volume form $dV = \langle \psi(g), \chi(g) \rangle dg$ on G . We then contract this volume form with the generators of the right-action of H on G to obtain a form α on G . Due to the defining property of \mathcal{H}_{ind} , this form is closed and hence is the pull-back of a volume form $d\bar{V}$ on the quotient G/H . Integration of this volume form over G/H then gives the scalar product $\langle \psi, \chi \rangle_{\text{ind}}$ on \mathcal{H}_{ind} . We can find the usual description directly in terms of the functions ψ and χ if we introduce an auxiliary function ρ on G which is strictly positive and satisfies:

$$\forall h \in H : \rho(gh) = \gamma(h)^2 \cdot \rho(g).$$

We denote by $d\mu$ the volume form on G/H obtained from the volume form $dV = \rho(g) dg$ on G by the procedure described above. With these preparations we have

$$(2.8) \quad \langle \psi, \chi \rangle_{\text{ind}} = \int_{G/H} \frac{\langle \psi(g), \chi(g) \rangle}{\rho(g)} d\mu,$$

where we note that the quotient under the integral sign is a function on G/H , again due to the definition of \mathcal{H}_{ind} . Finally we note that a different choice for the generators of the right-action of H on G changes the scalar product on \mathcal{H}_{ind} by a constant factor.

After the description of \mathcal{H}_{ind} as quantum Hilbert space of $(M_{\text{ind}}, \omega_{\text{ind}})$ we have to determine the representation of G on \mathcal{H}_{ind} associated to the hamiltonian action of G on M_{ind} . With reference to the same partial results as for Conjecture 2.5 and using that J_{ind} is obtained from the (globally) H -invariant momentum map \tilde{J} (formula (2.1)) one “deduces” that this representation is just the restriction of the representation \tilde{U} (formula (2.4)) of G on \mathcal{H} restricted to \mathcal{H}_{ind} . If we compare this representation of G (formulas (2.4), (2.6), and (2.8)) with the standard description of the induced representation of G , induced from the unitary representation U_M of H on \mathcal{H}_M ([8], [16], [23]), then we see that they are the same. In other words, geometric quantization intertwines the constructions “symplectic induction” and “induced representations,” of course modulo the fact that Conjecture 2.5 is still open in the general case.

2.3. A particular case. We now consider the particular example of symplectic induction in which the original symplectic manifold is a single point. This example will play an important role in our interpretation of Pukanszky's condition. Although it might seem to be singular, the constructions all make sense. Since the action of H on $M = \{\text{pt}\}$ is trivial, a momentum map J_M for this action has a single value $\nu_0 \in \mathfrak{h}^*$. The condition that J_M is equivariant is equivalent to the condition that ν_0 is invariant under the coadjoint action of H on \mathfrak{h}^* :

$$J \text{ equivariant} \Leftrightarrow \nu_0 \text{ Coad}_H\text{-invariant.}$$

We continue with the symplectic manifold $(\widetilde{M}, \widetilde{\omega}) = (\{\text{pt}\} \times T^*G, d\vartheta_G)$. The momentum map $J_{\widetilde{M}}: \widetilde{M} \rightarrow \mathfrak{h}^*$ is given by

$$(2.9) \quad J_{\widetilde{M}}(\text{pt}, g, \mu) = \nu_0 + J_{T^*G}(g, \mu) = \nu_0 - i_{\mathfrak{h}}^* \mu.$$

It follows that the constraint set $J_{\widetilde{M}}^{-1}(0)$ is given by

$$(2.10) \quad J_{\widetilde{M}}^{-1}(0) = \{\text{pt}\} \times J_{T^*G}^{-1}(-\nu_0).$$

We thus see that if we drop the (now superfluous) reference to the point pt , we just have to reduce the canonical action of H on T^*G at ν_0 , i.e., $M_{\text{ind}} = J_{T^*G}^{-1}(-\nu_0)/H$.

We now invoke the Sternberg-Satzer-Marsden-Kummer reduction theorem ([20], [18], [1], [15]) to describe this reduced manifold. With α a connection on $G \rightarrow G/H$ as in the proof of Proposition 2.2, we define the 1-form $\alpha_{\nu_0} = \nu_0 \circ \alpha$ on G . Using that α is a connection and that ν_0 is invariant, it is elementary to show the existence of a closed 2-form β on G/H such that $d\alpha_{\nu_0} = \pi^* \beta$. Careful inspection of diagram (2.3) then proves the next proposition.

Proposition 2.11. $\overline{P}: (M_{\text{ind}}, \omega_{\text{ind}}) \rightarrow (T^*(G/H), d\vartheta_{G/H} + \overline{\text{pr}}^* \beta)$ is a symplectomorphism.

We call a cotangent bundle T^*Q in which the canonical symplectic form $d\vartheta_Q$ is modified with the pull-back of a closed 2-form β on Q a *modified cotangent bundle* (note that $d\vartheta_Q + \overline{\text{pr}}^* \beta$ is always symplectic). Thus Proposition 2.11 states that M_{ind} is symplectomorphic to the modified cotangent bundle $T^*(G/H)$. Note however that this symplectomorphism depends upon the choice of the connection α and hence is not canonical in the general case. The induced action of G on $T^*(G/H)$ can be described as the unique action that covers the canonical left-action of G on G/H and that is symplectic with respect to the symplectic form $d\vartheta_{G/H} + \overline{\text{pr}}^* \beta$ (see also [3]).

Next we tackle the question of quantization and we start with $(M, \omega) = (\{\text{pt}\}, 0)$. The Hilbert space \mathcal{H}_M consists of sections of a complex line bundle over a point, i.e.,

$$\mathcal{H}_M = \mathbf{C}.$$

Geometric quantization of the momentum map J_M yields the infinitesimal representation $\tau: \mathfrak{h} \rightarrow \text{End}(\mathcal{H}_M)$ given by

$$\tau(\xi) = i \cdot \nu_0(\xi) \in \mathbf{C} = \text{End}(\mathbf{C}).$$

The assumption that geometric quantization of (M, ω) yields a unitary representation translates in this context as the assumption that this algebra representation τ can be integrated to a group representation $\chi: H \rightarrow U(1) \subset \text{End}(\mathbf{C})$. In other words, we assume that $i \cdot \nu_0$ is the derivative (at the identity) of a character χ of H . By abuse of language we will say that $\nu_0 \in \mathfrak{h}^*$ is the infinitesimal form of the character χ .

It turns out that the assumptions we have made so far allow us to apply the results obtained in [7] and [5], results which tell us that in this particular case Conjecture 2.5 is true. We note in particular that the assumption that ν_0 is the infinitesimal form of a character χ is equivalent to the condition in [7] and [5] that the H -action lifts to a connection-preserving action on the prequantum bundle above $(T^*G, d\vartheta_G)$. We thus have proven the following proposition.

Proposition 2.12. *Let χ be a character of H , a closed and connected Lie subgroup of a connected Lie group G . Denote by $\nu_0 \in \mathfrak{h}^*$ its infinitesimal form and by U the unitary representation of G obtained by induction from χ . Then we have:*

- (i) χ is obtained by geometric quantization of the quadruple $(M, \omega, H, J_M) \equiv (\{\text{pt}\}, 0, H, \nu_0)$, and
- (ii) U is obtained by geometric quantization (using the vertical polarization) of the quadruple $(M_{\text{ind}}, \omega_{\text{ind}}, G, J_{\text{ind}}) \equiv (T^*(G/H), d\vartheta_{G/H} + \overline{\text{pr}}^* \beta, G, J_{\text{ind}})$ which is obtained by symplectic induction from the quadruple $(\{\text{pt}\}, 0, H, \nu_0)$.

Remark 2.13. Without additional hypotheses the above proposition is true for geometric quantization using half-densities; for half-forms quantization additional conditions concerning metilinear structures are necessary [7]. If H is not connected, Proposition 2.11 remains true; Proposition 2.12 also remains true with half-density quantization, provided we add absolute values under the square root sign in (2.7).

3. Pukanszky's condition and the structure of coadjoint orbits

3.1. Polarizations and Pukanszky's condition. For the remainder of this paper we fix a connected Lie group G and an element $\mu_0 \in \mathfrak{g}^*$. To make notation less cumbersome, we denote the coadjoint action of G on \mathfrak{g}^* by a simple dot, i.e., for $g \in G$ and $\mu \in \mathfrak{g}^*$ we have $g \cdot \mu \equiv \text{Coad}_G(g)\mu$. We define $G_{\mu_0} \subset G$ as the isotropy subgroup of μ_0 with Lie algebra $\mathfrak{g}_{\mu_0} \subset \mathfrak{g}$, and we denote by $\mathcal{O}_{\mu_0} \equiv G \cdot \mu_0 \cong G/G_{\mu_0}$ the coadjoint orbit of μ_0 in \mathfrak{g}^* . We denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of \mathfrak{g} with its canonical injection $\mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$ and complex conjugation $\bar{\cdot} : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$; the adjoint action of G is extended by linearity to $\mathfrak{g}^{\mathbb{C}}$.

Given a linear subspace $\mathfrak{a} \subset \mathfrak{g}$ containing \mathfrak{g}_{μ_0} we define the symplectic orthogonal \mathfrak{a}^\perp by

$$\mathfrak{a}^\perp = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{a} : \mu_0([X, Y]) = 0\},$$

and we extend this notion in the obvious way to subspaces of $\mathfrak{g}^{\mathbb{C}}$ containing $\mathfrak{g}_{\mu_0} + i\mathfrak{g}_{\mu_0}$.

Lemma 3.1. $\mathfrak{g}^\perp = \mathfrak{g}_{\mu_0} \subset \mathfrak{a}^\perp$, $(\mathfrak{a}^\perp)^\perp = \mathfrak{a}$, and $\dim \mathfrak{a} + \dim \mathfrak{a}^\perp = \dim \mathfrak{g} + \dim \mathfrak{g}_{\mu_0}$.

Definition 3.2. A *polarization* is a complex Lie subalgebra \mathfrak{h} of $\mathfrak{g}^{\mathbb{C}}$ containing $\mathfrak{g}_{\mu_0} + i\mathfrak{g}_{\mu_0}$ and satisfying:

- (i) \mathfrak{h} is invariant under the Ad_G -action of G_{μ_0} ;
- (ii) $\mathfrak{h}^\perp = \mathfrak{h}$; and
- (iii) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Remark 3.3. Condition 3.2(ii) is usually given as two separate conditions: (ii-a) $\dim_{\mathbb{C}} \mathfrak{h} = \frac{1}{2}(\dim_{\mathbb{R}} \mathfrak{g} + \dim_{\mathbb{R}} \mathfrak{g}_{\mu_0})$ and (ii-b) $\mu_0([\mathfrak{h}, \mathfrak{h}]) = 0$.

Remark 3.4. One can easily show that polarizations as defined above are in 1-1 correspondence with G -invariant polarizations \mathcal{F} on \mathcal{O}_{μ_0} in the sense of geometric quantization, the correspondence being given by $\text{coad}_{\mathfrak{g}}(\mathfrak{h})\mu_0 = \mathcal{F}_{\mu_0} \subset (T_{\mu_0} \mathcal{O}_{\mu_0})^{\mathbb{C}}$. Condition (i) guarantees that \mathcal{F} so defined is indeed well defined, (ii) translates to the fact that \mathcal{F} is Lagrangian, and (iii) states that $\mathcal{F} + \bar{\mathcal{F}}$ is involutive.

Remark 3.5. In the special case $\mathfrak{h} = \bar{\mathfrak{h}}$ one says that \mathfrak{h} is a real polarization; at the other extreme $\mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{g}^{\mathbb{C}}$, one calls \mathfrak{h} a purely complex polarization.

To any polarization \mathfrak{h} we can associate two (real) Lie subalgebras $\mathfrak{d} \subset \mathfrak{e}$ of \mathfrak{g} by the relations $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$ and $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$. These Lie subalgebras

will be fixed throughout the remaining part of this paper. We denote by $D_0 \subset E_0$ the connected Lie subgroups of G whose Lie algebras are \mathfrak{d} , resp. \mathfrak{e} . It follows from conditions (i) and (ii) that \mathfrak{d} and \mathfrak{e} are Lie subalgebras of \mathfrak{g} containing \mathfrak{g}_{μ_0} that are invariant under the Ad_G -action of G_{μ_0} . We deduce that the subsets $D = D_0 \cdot G_{\mu_0} \subset E = E_0 \cdot G_{\mu_0}$ are subgroups of G .

We now collect some elementary facts about these objects (e.g., see [14] or [22]).

Lemma 3.6. (i) $\mathfrak{d}^\perp = \mathfrak{e}$.

(ii) D_0 and D are closed Lie subgroups of G with Lie algebra \mathfrak{d} .

(iii) $\iota_0^* \mu_0 \in \mathfrak{d}^*$ is Coad_D -invariant.

(iv) \mathfrak{e}^0 and $\mu_0 + \mathfrak{e}^0$ are invariant under the Coad_G -action of D .

(v) If E is a Lie subgroup of G then its Lie algebra is \mathfrak{e} .

Lemma 3.7 (Pukanszky’s condition). *The following three conditions are equivalent:*

(i) $\mu_0 + \mathfrak{e}^0 \subset G \cdot \mu_0 \equiv \mathcal{O}_{\mu_0}$.

(ii) $D \cdot \mu_0 = \mu_0 + \mathfrak{e}^0$.

(iii) $D \cdot \mu_0$ is closed in \mathfrak{g}^* .

Proof. Since $\mu_0 + \mathfrak{e}^0$ is invariant under D , it follows that $D \cdot \mu_0 \subset \mu_0 + \mathfrak{e}^0$. From Lemma 3.1 we deduce that $\dim(D \cdot \mu_0) = \dim(\mu_0 + \mathfrak{e}^0)$ and thus we conclude that $D \cdot \mu_0$ is open in $\mu_0 + \mathfrak{e}^0$. Hence we find the implication (iii) \Rightarrow (ii). Since the implications (ii) \Rightarrow (iii) and (ii) \Rightarrow (i) are obvious, we only have to prove (i) \Rightarrow (iii).

Therefore assume $\mu \in \mu_0 + \mathfrak{e}^0$ lies in the closure of $D \cdot \mu_0$. As above we have $D \cdot \mu \subset \mu_0 + \mathfrak{e}^0$ and because by hypothesis $\mu \in \mathcal{O}_{\mu_0}$ we still have $\dim(D \cdot \mu) = \dim(\mu_0 + \mathfrak{e}^0)$. It follows that $D \cdot \mu$ is also open in $\mu_0 + \mathfrak{e}^0$ and thus $D \cdot \mu$ intersects $D \cdot \mu_0$. It then follows immediately from the existence of $g \in G$ with $\mu = g \cdot \mu_0$ that $\mu \in D \cdot \mu_0$.

Remark 3.8. The definition of Pukanszky’s condition as given above is the one used by M. Vergne [22] in the context of solvable Lie groups; there are other versions of this condition oriented more toward Lie groups that may have semisimple subgroups. One by M. Duflo [4] is that $H \cdot \mu_0$ should be closed in $(\mathfrak{g}^C)^*$, and one by B. Kostant [14] is that $E \cdot \mu_0$ should be closed in \mathfrak{g}^* . In the case of a real polarization all these conditions imply the original one of Pukanszky [17].

3.2. The structure of coadjoint orbits. We saw in the previous subsection that D is a closed subgroup of G and that $\nu_0 = \iota_0^* \mu_0 \in \mathfrak{d}^*$ is Coad_D -invariant. We thus can apply symplectic induction from a point as

explained in §2.3 with D as the closed subgroup H . In this case we have $J_{\tilde{M}}^{-1}(0) \cong J_{T^*G}^{-1}(-\nu_0) \cong G \times (\mu_0 + \mathfrak{d}^0)$, while the subset $\mu_0 + \mathfrak{e}^0 \subset \mu_0 + \mathfrak{d}^0$ is also invariant under the D -action. Hence we can enlarge and simplify the commutative diagram (2.3) to:

$$\begin{array}{ccccccc}
 G \times (\mu_0 + \mathfrak{e}^0) & \longrightarrow & G \times (\mu_0 + \mathfrak{d}^0) & \xrightarrow{P} & G \times \mathfrak{d}^0 & \xrightarrow{\text{pr}} & G \\
 \downarrow \text{mod } D & & \downarrow \text{mod } D & & \downarrow \pi^{*-1} & & \downarrow \pi \\
 F & \longrightarrow & M_{\text{ind}} & \xrightarrow{\bar{P}} & T^*(G/D) & \xrightarrow{\bar{\text{pr}}} & G/D
 \end{array}$$

with $F = (G \times (\mu_0 + \mathfrak{e}^0))/D$. Since we apply symplectic induction from a point, the maps P and \bar{P} are diffeomorphisms. Moreover, each D -orbit intersects $\{g\} \times (\mu_0 + \mathfrak{d}^0)$ in a single point and P is affine on it; hence we can identify F as a subbundle of $T^*(G/D)$.

We now restrict the canonical symplectic form $d\vartheta_G$ of $T^*G \cong G \times \mathfrak{g}^*$ to the subspace $G \times (\mu_0 + \mathfrak{e}^0)$. We leave it to the reader to verify that the leaves of the characteristic foliation of this restricted 2-form are exactly the orbits of D_0 . It then follows that F can be identified as a symplectic subbundle of $(T^*(G/D), d\vartheta_{G/D} + \bar{\text{pr}}^* \beta)$; the induced symplectic form ω_F on F being the restriction of $d\vartheta_{G/D} + \bar{\text{pr}}^* \beta$ to $F \subset T^*(G/D)$.

Since the (left) action of G on T^*G obviously preserves $G \times (\mu_0 + \mathfrak{e}^0)$, we obtain an induced symplectic action of G on F ; its equivariant momentum map J_F is obtained from the momentum map \tilde{J} defined in (2.1).

Proposition 3.9. *The following four conditions on the polarization are equivalent:*

- (i) *Pukanszky's condition.*
- (ii) *The momentum map $J_F: F \rightarrow \mathfrak{g}^*$ is onto \mathcal{O}_{μ_0} .*
- (iii) *The symplectic action of G on F is transitive.*
- (iv) *J_F is a symplectomorphism between (F, ω_F) and \mathcal{O}_{μ_0} .*

Proof. From the definitions of F and J_F we deduce that

$$\text{im}(J_F) = \{g \cdot \mu \mid g \in G, \mu \in \mu_0 + \mathfrak{e}^0\}.$$

In particular, $\mu_0 \in \text{im}(J_F)$, and thus $\mathcal{O}_{\mu_0} \subset \text{im}(J_F)$. The equivalence (i) \Leftrightarrow (ii) now follows immediately from 3.7(i). Since the implications (iv) \Rightarrow (iii) \Rightarrow (ii) are obvious, it suffices to show the implication (i) \Rightarrow (iv).

To that end, consider $(g, \mu), (\hat{g}, \hat{\mu}) \in G \times (\mu_0 + \mathfrak{e}^0)$ that have the same image under J_F , i.e., $g \cdot \mu = \hat{g} \cdot \hat{\mu}$. From 3.7(ii) we deduce the existence of $d, \hat{d} \in D$ such that $\mu = d \cdot \mu_0, \hat{\mu} = \hat{d} \cdot \mu_0$, and hence we have $d^{-1}g^{-1}\hat{g}\hat{d} \in G_{\mu_0}$. Since $G_{\mu_0} \subset D$, it follows that $d_0 = g^{-1}\hat{g} \in D$ and thus $(g, \mu) = \Phi_{T^*G}(d_0)(\hat{g}, \hat{\mu})$, i.e., (g, μ) and $(\hat{g}, \hat{\mu})$ lie in the same D -orbit. Together with (i) this shows that J_F maps F bijectively to \mathcal{O}_{μ_0} . It is then standard to show that it is a symplectomorphism, and thus we have shown the implication (i) \Rightarrow (iv).

Remark 3.10. If \mathfrak{h} is a real polarization, the two subalgebras \mathfrak{d} and \mathfrak{e} are the same. In that case Pukanszky’s condition states that \mathcal{O}_{μ_0} is isomorphic to the (full) modified cotangent bundle $T^*(G/D)$. At the other extreme when \mathfrak{h} is purely complex, Pukanszky’s condition is always satisfied and the above proposition reduces to the rather trivial statement that \mathcal{O}_{μ_0} is symplectomorphic to the zero section of the modified cotangent bundle $T^*\mathcal{O}_{\mu_0}$.

Remark 3.11. If G is an exponential group, the orbit method proceeds as follows. One assumes the polarization to be real and such that there exists a global character χ of D with $d\chi = i\iota_{\mathfrak{d}}^*\mu_0$. The representation of G associated to the orbit \mathcal{O}_{μ_0} then is the representation obtained by unitary induction from χ . Pukanszky [17] has shown that this representation is irreducible if and only if the condition that bears his name is satisfied.

However, there is another representation of G we can associate to this orbit, i.e., the one obtained by geometric quantization (using the given real polarization). Combining Propositions 2.12 and 3.9 we see that these two representations of G coincide if Pukanszky’s condition is satisfied. This last result thus provides an even stronger link between geometric quantization and the orbit method.

Without additional assumptions not much more can be said about the geometric implications of Pukanszky’s condition. However, if the subgroup E of G happens to be closed, we have the following proposition.

Proposition 3.12. *If E is a closed subgroup of G , and if Pukanszky’s condition is satisfied, then there exists a commutative diagram*

$$\begin{array}{ccccc}
 T^*(G/D) & \xleftarrow{i} & \mathcal{O}_{\mu_0} & \xrightarrow{\bar{P}_E} & T^*(G/E) \\
 \downarrow & & \downarrow f & & \downarrow \pi \\
 G/D & \xlongequal{\quad} & G/D & \longrightarrow & G/E
 \end{array}$$

with the following properties:

(i) (i, f) is the identification of \mathcal{O}_{μ_0} as a symplectic subbundle of $T^*(G/D)$ according to Proposition 3.9, and

(ii) \overline{P}_E is a fiber bundle whose fibers, together with the restricted symplectic form, are symplectomorphic to the pseudo-Kähler space E/D .

Proof. Define $\rho_0 = i_e^* \mu_0 \in \mathfrak{e}^*$, and denote by \mathcal{O}_{ρ_0} the orbit of ρ_0 in \mathfrak{e}^* under the Coad_E -action. We compute the isotropy subgroup of ρ_0 in E as follows. $e \in E$ lies in the isotropy subgroup of ρ_0 iff $e \cdot \mu_0 - \mu_0 \in \mathfrak{e}^0$. According to Pukanszky's condition this is equivalent to $e \cdot \mu_0 \in D \cdot \mu_0$. Since $G_{\mu_0} \subset D \subset E$ we deduce that the isotropy subgroup is D .

We now consider symplectic induction from the subgroup $E \subset G$ with $M = \mathcal{O}_{\rho_0}$. From Proposition 2.2 we deduce that M_{ind} fibers over $T^*(G/E)$ with symplectic fiber $\mathcal{O}_{\rho_0} \cong E/D$. Since one can show (e.g., [14]) that $\mathcal{O}_{\rho_0} \cong E/D$ admits a pseudo-Kähler structure, it thus only remains to show that M_{ind} is symplectomorphic to \mathcal{O}_{μ_0} and that the diagram containing \overline{P}_E commutes.

To that end we investigate $J_M^{-1}(0)$, which is given by

$$J_M^{-1}(0) = \{(\rho, g, \mu) \in \mathfrak{e}^* \times G \times \mathfrak{g}^* \mid i_e^* \mu = \rho \in \mathcal{O}_{\rho_0}\}$$

(note that J_M is the identity map for coadjoint orbits). Now if $\rho \in \mathcal{O}_{\rho_0}$ then there exists $e \in E : \rho = i_e^*(e \cdot \mu_0)$ and thus we find the condition $\mu - e \cdot \mu_0 \in \mathfrak{e}^0$, or equivalently (using Pukanszky's condition) $\mu = ed \cdot \mu_0$ for some $d \in D$. Since $D \subset E$ we thus find

$$J_M^{-1}(0) = \{(i_e^*(e \cdot \mu_0), g, e \cdot \mu_0) \mid g \in G, e \in E\}.$$

It then follows (with the same techniques as in the proof of 3.9) that $J_{\text{ind}} : M_{\text{ind}} \rightarrow \mathcal{O}_{\mu_0}$ is a symplectomorphism. The map \overline{P}_E now is the composition of J_{ind}^{-1} with the map \overline{P} from Proposition 2.2 for this induction.

Tracing diagram (2.3) for the two different symplectic inductions, one finds that the projection $f : F \cong \mathcal{O}_{\mu_0} \rightarrow G/D$ maps the element $g \cdot \mu_0 \in \mathcal{O}_{\mu_0}$ to $[g]_D \in G/D$ and that $\pi \circ \overline{P}_E$ maps it to $[g]_E \in G/E$, and thus the given diagram is commutative.

4. Pukanszky's condition and localization of massless particles

We tackle here, in purely geometric terms, the question of localization of massless relativistic particles in the light of our interpretation of Pukanszky's condition.

Let $\mathbf{R}^{3,1} = (\mathbf{R}^4, g)$ denote flat space-time whose metric g has the Lorentz signature $(- - - +)$. We also assume for convenience that $\mathbf{R}^{3,1}$ is oriented and time oriented as well. The group G of interest to us is the neutral component of the Poincaré group $\text{Isom}(\mathbf{R}^{3,1})$, i.e., $G = O(3, 1)_0 \otimes \mathbf{R}^{3,1}$. We will denote by $\xi = (\Lambda, \Gamma)$ a typical element of $\mathfrak{g} = o(3, 1) \otimes \mathbf{R}^{3,1}$. Likewise, a point in \mathfrak{g}^* is a pair $\mu = (M, P)$ with $M \in o(3, 1)$ and $P \in \mathbf{R}^{3,1}$ —interpreted as the angular and linear momentum respectively—where the pairing with \mathfrak{g} is given by $\langle \mu, \xi \rangle = -\frac{1}{2} \text{Tr}(M\Lambda) - g(P, \Gamma)$.

According to the point of view espoused in [19], the coadjoint orbit \mathcal{O}_{μ_0} representing the space of motions (or in other words, the classical phase space) of a massless particle with helicity $s \in \mathbf{R} \setminus \{0\}$ is specified by $\mu_0 = (M_0, P_0)$ with

$$(4.1) \quad \star(M_0)P_0 = sP_0, \quad \text{Det } M_0 = 0, \quad \text{and } P_0 \text{ future-pointing,}$$

where the star “ \star ” denotes the standard Hodge anti-involution of the Lorentz Lie algebra $o(3, 1)$ identified with $\Lambda^2 \mathbf{R}^{3,1} \cong \Lambda^2(\mathbf{R}^{3,1})^*$ by $(A \wedge B)V = g(B, V)A - g(A, V)B$. We note that the conditions (4.1) imply

$$(4.2) \quad g(P_0, P_0) = 0 \quad \text{and} \quad M_0 P_0 = 0.$$

The coadjoint action of G on \mathfrak{g}^* is given by

$$\text{Coad}_G(L, C)(M, P) = (LML^{-1} + C \wedge (LP), LP),$$

and an elementary (but tedious) computation shows that the isotropy subgroup is given by $G_{\mu_0} \cong \text{SO}(2) \times \mathbf{R}^3$.

As a next step, we consider the seven-dimensional (real) subalgebra

$$(4.3) \quad \mathfrak{d} = \{(\Lambda, \Gamma) \in \mathfrak{g} \mid \Lambda P_0 = 0\}.$$

Its main interest is that $\mathfrak{h} = \mathfrak{d}^C$ is a real polarization. To prove this, we first recall the fact that for $\Lambda \in o(3, 1)$ the condition $\Lambda P = 0$ implies that there exists $V \in \mathbf{R}^{3,1}$ such that $\Lambda = \star(V \wedge P)$. Using this fact, one can show that for $\xi, \xi' \in \mathfrak{d}$ there exists $Q \in \mathbf{R}^{3,1}$ such that $[\Lambda, \Lambda'] = P_0 \wedge Q$, and hence

$$\begin{aligned} \langle \mu_0, [\xi, \xi'] \rangle &= -\frac{1}{2} \text{Tr}(M_0[\Lambda, \Lambda']) - g(P_0, \Lambda\Gamma' - \Lambda'\Gamma) \\ &= -g(Q, M_0 P_0) \in (\Lambda P_0, \Gamma') - g(\Lambda' P_0, \Gamma) = 0 \end{aligned}$$

because of (4.2) and (4.3). As a notable feature, this polarization satisfies Pukanszky’s condition. To see this, note first that, since \mathfrak{h} is real, we have

$\mathfrak{e} = \mathfrak{d}$, and, since G_{μ_0} is connected, we have $D = D_0$. Integrating the subalgebra \mathfrak{d} yields the closed connected subgroup

$$D = \{(L, C) \in G \mid LP_0 = P_0\}.$$

With these ingredients we now compute

$$\begin{aligned} \mu_0 + \mathfrak{d}^0 &= \{(M, P_0) \in \mathfrak{g}^* \mid \star(M - M_0)P_0 = 0\} \\ &= \{(M, P_0) \in \mathfrak{g}^* \mid \exists C \in \mathbf{R}^{3,1} : M = M_0 + C \wedge P_0\} \subset D \cdot \mu_0, \end{aligned}$$

and thus $\mu_0 + \mathfrak{d}^0 = D \cdot \mu_0$.

We thus may apply Proposition 3.9 to conclude that \mathcal{O}_{μ_0} is symplectomorphic to the (modified) cotangent bundle of the forward light-cone of $\mathbf{R}^{3,1}$:

$$\mathcal{O}_{\mu_0} \cong T^*\mathcal{E} \quad \text{with } \mathcal{E} = G/D,$$

the projection $\pi: G \rightarrow \mathcal{E}$ being given by $P \equiv \pi((L, C)) = LP_0$. The base manifold $\mathcal{E} \cong \mathbf{R}^3 \setminus \{0\}$ is physically interpreted as the space of linear momentum and energy of the massless particle, whereas the typical fiber of the phase space $T^*\mathcal{E} \equiv \mathcal{E} \times \mathbf{R}^3$ may be identified with the configuration space where our massless particle dwells. Such an identification thus assigns to each point of the classical phase space of the massless particle a position in our three-dimensional space, i.e., it becomes “localized.” However, we must emphasize that this localization procedure relies on a specific *noncanonical* choice for the connection α on the principal bundle $G \rightarrow G/D$ used to define the modified symplectic structure $d\vartheta_{G/D} + \overline{\text{pr}}^*\beta$ of $T^*\mathcal{E}$. The fact that there does not exist a preferred G -invariant connection is due to the nonexistence of a reductive splitting $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{s}$ with $[\mathfrak{d}, \mathfrak{s}] \subseteq \mathfrak{s}$.

A straightforward calculation shows that to each future-pointing unit vector $I \in \mathbf{R}^{3,1}$, we can associate a connection $\alpha_I = (\tilde{\Lambda}, \tilde{\Gamma})$ on $G \rightarrow \mathcal{E}$ by

$$\begin{aligned} \tilde{\Lambda} &= L^{-1} dL - \frac{(L^{-1} dLP_0) \wedge (L^{-1} I)}{g(I, LP_0)}, \\ \tilde{\Gamma} &= L^{-1} dC - \frac{g(I, C) \cdot L^{-1} dLP_0 - g(dLP_0, C) \cdot L^{-1} I}{g(I, LP_0)}. \end{aligned}$$

These expressions all make sense because $g(I, LP_0) > 0$ for all $L \in O(3, 1)_0$.

More effort is needed to find out the modified symplectic structure on $T^*\mathcal{E}$, which turns out to be given by

$$(4.4) \quad \omega_I = d(Q dP) + s \cdot \overline{\text{pr}}^* \left(\frac{\text{Vol}(I, P)}{g(I, P)^3} \right),$$

where $P \in \mathcal{E}$ and $\overline{\text{pr}}$ denotes the projection $T^*\mathcal{E} \rightarrow \mathcal{E}$. We have denoted by $\text{Vol}(I, P)$ the 2-form on \mathcal{E} obtained by contracting the prescribed volume form Vol of space-time with the vectors I and P . Note also that we have interchanged the traditional roles of the symbols P and Q : P denotes the coordinates in the base manifold and Q denotes the coordinates into the fiber.

By using the g -orthogonal decomposition $P = p + \|p\| \cdot I$, with $p \in I^\perp \setminus \{0\} \cong \mathbf{R}^3 \setminus \{0\}$ and $q \in \mathbf{R}^3$ (our three-dimensional space), i.e., in a Lorentz frame adapted to the “observer” I , we get

$$\omega_I = -d(q dp) - s \cdot \frac{\text{vol}(p)}{\|p\|^3},$$

where vol stands for the canonical volume element of \mathbf{R}^3 . Following a completely different route, we thus recover the symplectic structure which is derived in [19] by means of another localization procedure.

We finish this discussion by noting that any connection α will provide us with an identification of \mathcal{E}_{μ_0} with $T^*\mathcal{E}$ equipped with a modified symplectic structure $\omega_\alpha = d\vartheta_{G/D} + \overline{\text{pr}}^* \beta$. However, because of the invariance of $I_3^* \mu_0 \in \mathfrak{d}^*$, there will exist a 1-form ψ on \mathcal{E} such that $\omega_\alpha = \omega_I + d(\overline{\text{pr}}^* \psi)$. This implies that, modulo a redefinition of the canonical 1-form of $T^*\mathcal{E}$ —a “gauge transformation” which reveals the affine structure of our three-dimensional space—the localization procedure we have spelled out in terms of Pukanszky’s condition merely reduces to the choice of an otherwise arbitrary observer I in space-time.

Remark 4.5. For completeness, we recall that the massless coadjoint orbit with $s = 0$ corresponds to the choice of origin $\mu_0 = (0, P_0)$, where P_0 is null and future-pointing. In this case, the polarization is still given by (4.3) and all previous results hold except that our localization is now “canonical” since this orbit is symplectomorphic with $T^*\mathcal{E}$ endowed with its canonical symplectic structure.

Remark 4.6. It is worth mentioning that the position observables we have defined above do not Poisson-commute since for $u, v \in \mathbf{R}^3$ we have:

$$\{q \cdot u, q \cdot v\} = s \cdot \frac{\text{vol}(p, u, v)}{\|p\|^3}.$$

We can easily single out the prequantizable massless orbits \mathcal{O}_{μ_0} as those satisfying $2s \in \mathbb{Z}$, and to these we can apply the geometric quantization procedure using the previously introduced real polarization, which is (of course) the vertical polarization of $(T^*\mathcal{E}, \omega_I)$. In doing so, we can easily quantize the *three* position observables (they preserve the G -invariant polarization) and end up with *noncommuting* position operators in the case of nonzero helicity. In this way we recover results already known to physicists (e.g., [2], [11]).

5. Appendix: Some notations and sign conventions

Notation 5.1. If \mathfrak{a} is any (real) vector space, we denote by \mathfrak{a}^* its dual space. If \mathfrak{a} is a linear subspace of a vector space \mathfrak{g} , we denote the canonical injection by $\iota_{\mathfrak{a}}: \mathfrak{a} \rightarrow \mathfrak{g}$. Dual to the canonical injection we have the projection $\iota_{\mathfrak{a}}^*: \mathfrak{g}^* \rightarrow \mathfrak{a}^*$, and we denote by \mathfrak{a}^0 the annihilator of \mathfrak{a} in $\mathfrak{g}^* : \mathfrak{a}^0 = \ker(\iota_{\mathfrak{a}}^*) = \{\mu \in \mathfrak{g}^* | \forall X \in \mathfrak{a} : \mu(X) = 0\}$.

Sign convention 5.2. Let Φ be a (left) action of a Lie group G on a symplectic manifold (M, ω) , i.e., $\Phi: G \rightarrow \text{Diff}(M)$ is a group homomorphism. For $X \in \mathfrak{g}$ (\mathfrak{g} the Lie algebra of G) we define the fundamental vector field X_M on M as the vector field whose flow is $\Phi(\exp(Xt))$. The map $X \mapsto X_M$ so defined is a Lie algebra anti-homomorphism.

A momentum map (if it exists) is a map $J: M \rightarrow \mathfrak{g}^*$ satisfying $\iota(X_M)\omega + d(J^*X) = 0$ for all $X \in \mathfrak{g}$. It is called an equivariant momentum map if it is equivariant for the given action of G on M and the coadjoint action of G on \mathfrak{g}^* .

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