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MODULI OF RANK-2 VECTOR BUNDLES, THETA DIVISORS, AND THE GEOMETRY OF CURVES IN PROJECTIVE SPACE

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0. Introduction

Let C be a fixed curve (smooth, irreducible, projective) of genus g defined over the complex numbers, let ω_C denote the canonical line bundle of one-forms on C, and let L be a fixed line bundle. If E is a vector bundle of rank n on C, define $\det(E) = \bigwedge^n E$, $\deg(E) = \deg(\det(E))$, and $\mu(E) = \deg(E)/n$. We say E is stable (resp. semistable) if for all quotient vector bundles $E \to F \to 0$, $\mu(E) < \mu(F)$ (resp. $\mu(E) \leq \mu(F)$).

Let $\mathscr{M}_{n,L}$ be the moduli space of semistable vector bundles on C of rank n and determinant L. Narasimhan and Seshadri [12] showed that $\mathscr{M}_{n,L}$ can be given the structure of a projective variety, with an open subset $\mathscr{M}_{n,L}^{\text{stab}} \subset \mathscr{M}_{n,L}$ corresponding to the stable bundles, and the complement corresponding to certain equivalence classes of semistable bundles. More recently, Drezet and Narasimhan [5] have shown that $\operatorname{Pic}(\mathscr{M}_{n,L}) \cong \mathbf{Z}$ and further, that there are geometrically defined Cartier divisors $\Theta_{n,L} \subset \mathscr{M}_{n,L}$ generating $\operatorname{Pic}(\mathscr{M}_{n,L})$. For example, when $\deg(L) = n(g-1)$, then $\chi(E) = 0$ for all $E \in \mathscr{M}_{n,L}$ and $\Theta_{n,L} = \{E \in \operatorname{Pic}(\mathscr{M}_{n,L}) | h^0(E) > 0\}$ is naturally such a divisor.

There has been considerable interest recently in the space of sections $H^0(\mathscr{M}_{n,L}, \mathscr{O}(k\Theta))$. For example, these spaces arise in the theory of geometric quantization, and have been used by Witten to recover the Jones polynomials [13].

The primary goal of this paper is to use the (rational) "extension maps" $\phi_L \colon \mathbf{P}_L := \mathbf{P}(\operatorname{Ext}^1(L, \mathscr{O})^*) \to \mathscr{M}_{2,L}$ to study $\mathscr{M}_{2,*}$. The idea of using extension classes to study vector bundles dates back to the famous theorem of Grothendieck that all vector bundles on $\mathbf{P}^1$ split as a direct sum of line bundles. Atiyah [2] used extensions to analyze vector bundles on an elliptic curve, and Newstead [9] used them to analyze $\mathscr{M}_{2,*}$ in the case $g = 2$.

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Finally, if $\deg(L) = 2g-1$, then the birationality of the extension map is the standard proof that $\mathcal{M}_{2,L}$ is a rational variety if $\deg(L)$ is odd.

The first observation about ϕ_L is that it is not everywhere defined. Indeed, given an extension class $(*)$: $0 \to \mathcal{O} \to E \to L \to 0$, then E may have a quotient line bundle of any positive degree. However, if $E \to M \to 0$ is such a quotient and $\deg(M) < \deg(L)$, then M must have a section. In particular, the "maximally unstable" such E are those with quotient line bundles $M \cong \mathcal{O}(p)$, $p \in C$. It follows from the definitions that if $(*)$ gives rise to such an E, then in fact $(*) \in C$, where C is embedded in $\mathbf{P}_L$ via the natural map. More generally, $(*)$ lies in the linear span $\overline{D}$ of D in $\mathbf{P}_L$ if and only if $(*)$ gives rise to an E with quotient isomorphic to $\mathcal{O}(D)$. This observation suggests the following theorem, whose proof will occupy the first three sections.

Theorem 1. *A natural sequence σ of blow-ups along smooth centers (commencing with the blow-up of $\mathbf{P}_L$ along C) resolves the extension map ϕ_L into a morphism $\Phi_L: \widetilde{P}_L \to \mathcal{M}_{2,L}$.*

$$\begin{tikzcd}
\widetilde{P}_L \arrow[d, dotted] \arrow[dd, bend right, "\sigma"'] \arrow[ddr, "\Phi_L"] & \\
bl_1(\mathbf{P}_L) \arrow[d] & \\
\mathbf{P}_L \arrow[r, "\phi_L"] & \mathcal{M}_{2,L}
\end{tikzcd}$$

Further, we can number the exceptional divisors as $E_0, \cdots, E_{k(L)-1}$ with the following properties:

(1) *E_k dominates the secant variety* $\mathrm{Sec}^k(C) := \bigcup_{\deg(D)=k+1}(\overline{D})$.

(2) *If $x \in \overline{D}$ is not in the span of any proper subdivisor of D, then there is a natural isomorphism $\sigma^{-1}(x) \cong \widetilde{P}_{L(-2D)}$, and when restricted to $\sigma^{-1}(x)$, Φ_L coincides with the map*

$$\Phi_{L(-2D)} \otimes \mathcal{O}(D): \widetilde{P}_{L(-2D)} \to \mathcal{M}_{2,L(-2D)} \to \mathcal{M}_{2,L}.$$

(3) *There is a Poincaré vector bundle $\widetilde{\mathcal{E}}_L$ on $C \times \widetilde{P}_L$ realizing the map Φ_L.*

Parts (1) and (2) of Theorem 1 can be thought of as an analog of the classical theory of complete quadrics (see [4] for details).

The main application of Theorem 1 states that, at least in rank 2, the spaces $H^0(\mathcal{M}_{2,L}, \mathcal{O}(k\Theta))$ have a particularly nice description.

Theorem 2. *Let $\mathcal{I}_C$ be the ideal sheaf of $C \subset \mathbf{P}_L$.*

(a) *If* $\deg(L) = 2g$, *then there is a natural identification*:

$$H^0(\mathcal{M}_{2,L}, \mathcal{O}(k\Theta)) \cong H^0(\mathbf{P}_L, \mathcal{O}(kg) \otimes \mathcal{I}_C^{k(g-1)}).$$

(b) *If* $\deg(L) = 2g-1$, *then there is a natural identification*:

$$H^0(\mathcal{M}_{2,L}, \mathcal{O}(k\Theta)) \cong H^0(\mathbf{P}_L, \mathcal{O}(k(2g-1)) \otimes \mathcal{I}_C^{k(2g-3)}).$$

As additional applications, we will give a new and extremely simple proof that $\operatorname{Pic}(\mathcal{M}_{2,L}) \cong \mathbf{Z}$ if $\deg(L)$ is odd; we will similarly get a simple proof that the class of Θ_{ω_C} in $\operatorname{Pic}(\mathcal{M}_{2,\omega_C})$ is irreducible, and we will calculate the canonical divisors on $\mathcal{M}_{2,*}$. All these results are already known ([10], [3]), but the following additional applications seem to be new.

Proposition 4.9. *There are (singular) rational hypersurfaces in* $\mathbf{P}^{3g-2}$ *dominating* $\mathcal{M}_{2,L}$ *with degree* 2 *if* $\deg(L)$ *is even.*

Proposition 4.10. *If* C *is not hyperelliptic, then* $\Theta \subset \mathcal{M}_{2,L}$ *is birationally very ample for odd* $\deg(L)$.

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Notation. (1) If V is a vector space, $\mathbf{P}(V)$ stands for the projective space of one-dimensional *quotients* of V.

(2) E, F will stand for vector bundles, and L, M for line bundles. If $\alpha: L \to F$, then $Z(\alpha)$ is the (scheme-theoretic) zero locus of α. If $Z(\alpha) = \varnothing$, we write $\alpha: L \hookrightarrow F$.

(3) $(*)$ and $(**)$ will stand for short exact sequences, or for the extension class which they represent.

(4) If a morphism $C \to \mathbf{P}^r$ is understood, then $\overline{D}$ stands for the linear span of D, and $U_D \subset \overline{D}$ stands for the open complement of the spans of all proper subdivisors of D

(5) C_k stands for the kth symmetric power of C.

1. Secant bundles, relative secant bundles, and the natural maps among them

Definition. We say that M *separates* k *points* if $M \in \operatorname{Pic}^d(C)$ has the property that $h^0(C, M) = h^0(C, M(-D)) + k$ for all $D \in C_k$.

Thus M separates one point if and only if M is base point free, and M separates two points if and only if M is very ample.

Let $\mathcal{D}_{k+1} = C \times C_k \subset C \times C_{k+1}$ be the "universal" divisor, embedded via $(p, D) \mapsto (p, p+D)$, and let $\pi_C: C \times C_{k+1} \to C$ and $\pi_{k+1}: C \times C_{k+1} \to C_{k+1}$ be the projections. If M separates $k+1$ points, then the sequence of sheaves on $C \times C_{k+1}$,

$$0 \to \pi_C^* M \otimes \mathscr{O}(-\mathscr{D}_{k+1}) \to \pi_C^* M \to \pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{k+1}} \to 0,$$

remains exact when pushed down to C_{k+1}. Following Schwarzenberger [11] we define:

Definition. The secant bundle (with respect to M) of k-planes over C_{k+1} is $B^k(M) := \mathbf{P}(\pi_{Ck+1_*}(\pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{k+1}}))$, and the natural map to $\mathbf{P}(H^0(C, M))$ is

$$\beta_k \colon B^k(M) \to \mathbf{P}(\pi_{C_{k+1_*}}(\pi_C^* M)) = \mathbf{P}(H^0(C, M)) \times C_{k+1} \to \mathbf{P}(H^0(C, M)).$$

The image of $B^k(M)$ in $\mathbf{P}(H^0(C, M))$ is the usual secant variety $\operatorname{Sec}^k(C)$ of k-planes through $k+1$ points of C.

Note. $B^k(M)$ is well defined for any M, but if M does not separate $k+1$ points, then β_k is only a rational map.

Similarly, let π_C, $\pi_{C_{m+1}}$, $\pi_{C_{k-m}}$ be the projections of $C \times C_{m+1} \times C_{k-m}$ to the various factors for $m < k$, and let π^C, $\pi^{C_{m+1}}$, $\pi^{C_{k-m}}$ be the projections to the complement of the indicated factors, so for example, $\pi^C \colon C \times C_{m+1} \times C_{k-m} \to C_{m+1} \times C_{k-m}$. Finally, define the addition map $r_{k+1} \colon C_{m+1} \times C_{k-m} \to C_{k+1}$.

Abusing notation, we define divisors $\mathscr{D}_{m+1} = (\pi^{C_{k-m}})^{-1}(\mathscr{D}_{m+1})$, $\mathscr{D}_{k-m} = (\pi^{C_{m+1}})^{-1}(\mathscr{D}_{k-m})$, and $\mathscr{D}_{k+1} = (1, r_{k+1})^{-1}(\mathscr{D}_{k+1})$. Then it is easy to check that as divisors, $\mathscr{D}_{m+1} + \mathscr{D}_{k-m} = \mathscr{D}_{k+1}$, and $\mathscr{D}_{m+1} \cap \mathscr{D}_{k-m}$ is transverse, so we get an exact sequence of sheaves on $C \times C_{m+1} \times C_{k-m}$:

$$0 \to \pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{k-m}}(-\mathscr{D}_{m+1}) \to \pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{k+1}} \to \pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{m+1}} \to 0.$$

This sequence remains exact when pushed down to $C_{m+1} \times C_{k-m}$ *regardless of* M (because the map π^C is finite on $\mathscr{D}_{k-m}$), and we define:

Definition. The relative secant bundle (with respect to M) over $C_{m+1} \times C_{k-m}$ is $\mathbf{P}(\pi)_*^C(\pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{m+1}})) = B^m(M) \times C_{k-m}$, and the natural map to $B^k(M)$ is

$$\alpha_{m,k} \colon B^m(M) \times C_{k-m} \to \mathbf{P}(\pi_*^C(\pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{k+1}})) \to B^k(M),$$

where the first map is the map on projective bundles, and the second is the lift of r_{k+1}.

We further define the relative secant variety of m-planes in $B^k(M)$ to be the image of $B^m(M) \times C_{k-m}$ under the map $\alpha_{m,k}$.

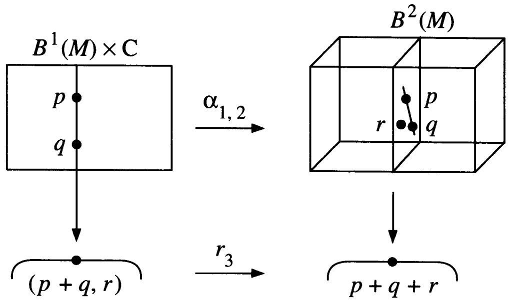

FIGURE 1

It is probably easier to think of this last definition on the level of fibers over the symmetric products. If we fix $E \in C_{m+1}$, $F \in C_{k-m}$, and $D = E + F$, then the fiber of $B^k(M)$ over D is $\mathbf{P}(H^0(C, M_D))$, and likewise the fiber of $B^m(M) \times C_{k-m}$ over (E, F) is $\mathbf{P}(H^0(C, M_E)) \times \{F\}$. The map $\alpha_{m,k}$ on the fiber level is simply the inclusion of projective spaces: $\mathbf{P}(H^0(C, M_E)) \times \{F\} \subset \mathbf{P}(H^0(C, M_D))$ dual to the restriction map $M_D \to M_E$. As an example, Figure 1 shows $\alpha_{1,2}$.

As one can see from this description, the image of $\alpha_{0,k}$ in $B^k(M)$ is the collection of points in each fiber $\mathbf{P}(H^0(C, M_D))$ corresponding to D (with suitable multiplicity). The image of $\alpha_{1,k}$ consists of lines through pairs of these points, etc. Because of this, we call the images of the $\alpha_{m,k}$ the relative secant varieties.

The fact that, for all effective divisors $F \subset E \subset D$, the natural inclusion maps

$$\mathbf{P}(H^0(C, M_F)) \xrightarrow{\phi_{F,E}} \mathbf{P}(H^0(C, M_E)) \xrightarrow{\phi_{E,D}} \mathbf{P}(H^0(C, M_D)) \xrightarrow{\phi_D} \mathbf{P}(H^0(C, M))$$

commute (i.e., $\phi_{E,D} \circ \phi_{F,E} = \phi_{F,D}$ and $\phi_D \circ \phi_{E,D} = \phi_E$) globalizes to the following "compatibility" lemma.

Lemma 1.1 (Compatibility). *For $m < l < k$, the following diagrams commute:*

(a)
$$\begin{tikzcd} B^m(M) \times C_{k-m} \arrow[r, "\alpha_{m,k}"] \arrow[dr, "\pi_{B^m(M)}"'] & B^k(M) \arrow[r, "\beta_k"] & \mathbf{P}(H^0(C, M)) \arrow[dl, "\beta_m"] \\ & B^m(M) & \end{tikzcd}$$

where $\pi_{B^m(M)}$ *is the projection,*

(b)
$$\begin{tikzcd} B^m(M) \times C_{l-m} \times C_{k-l} \arrow[r, "(\alpha_{m,l}, 1)"] \arrow[dr, "(1, r_{k-m})"'] & B^l(M) \times C_{k-l} \arrow[r, "\alpha_{l,k}"] \arrow[d, "\alpha_{m,k}"] & B^k(M) \\ & B^m(M) \times C_{k-m} & \end{tikzcd}$$

Next, suppose that M separates d points. Then, by definition, an effective divisor D of degree d spans a $\mathbf{P}^{d-1}$ in $\mathbf{P}(H^0(C, M))$. But this also means that for effective divisors $E = \sum e_p p$ and $F = \sum f_p p$ with $e+f = \deg(E)+\deg(F) \leq d$, $\overline{E}$ and $\overline{F}$ together span a $\mathbf{P}^{e+f-1-g}$, where $g = \deg(E \cap F)$. In exactly the same way, if E and F are subeffective divisors of an effective divisor D, and $\overline{E}, \overline{F}$ together span a $\mathbf{P}^{e+f-1-g}$ in $\mathbf{P}(H^0(C, M_D))$, then $g = \deg(E \cap F)$ regardless of M.

Let $U^k(M) = B^k(M) - \alpha_{k-1,k}(B^{k-1}(M) \times C)$. That is, $U^k(M)$ is the complement of the "largest" relative secant variety. Then the observations above yield the following "intersection" lemma.

Lemma 1.2 (Intersection). (a) *If* M *separates* $2k+2$ *points, then* $\beta_m|_{U^m(M)}: U^m(M) \to \mathbf{P}(H^0(C, M))$ *is an injective map of sets for all* $m \leq k$, *and*

$$\beta_m(U^m(M)) \cap \beta_l(U^l(M))) = \varnothing \quad \text{if } m < l \leq k.$$

(b) *For all line bundles* M, *the map* $\alpha_{m,k}: U^m(M) \times C_{k-m} \to B^k(M)$ *is injective on sets for all* $m < k$, *and*

$$\alpha_{m,k}(U^m(M) \times C_{k-m})) \cap \alpha_{l,k}(U^l(M) \times C_{k-1})) = \varnothing \quad \text{if } m < l < k.$$

Next, we turn to the differentials of the β_k and $\alpha_{m,k}$ maps. Following the lead of the intersection lemmas, we show that $\alpha_{m,k}$ is an immersion on $U^m(M) \times C_{k-m}$, and, if M separates $2k+2$ points, then β_k is an immersion on $U^k(M)$. When we projectivize the conormal bundles to these restricted α and β maps, the resulting varieties are themselves secant bundles and projective spaces associated to sub-line-bundles of M. Since the constructions in §§2 and 3 will involve blowing-up, which naturally introduces the projectivized conormal bundles, the following "Terracini" lemmas are really the key to understanding the recursive proofs of the next sections.

Lemma 1.3 (Relative Terracini). *Let* $x \in U^m(M)$, *and consider* $\alpha_{m,k}$: $B^m(M) \times C_{k-m} \to B^k(M)$. *Then*:

(a) $d\alpha_{m,k}: \alpha^*_{m,k} T^* B^k(M) \to T^*(B^m(M) \times C_{k-m})$ *is surjective when restricted to* $U^m(M) \times C_{k-m}$,

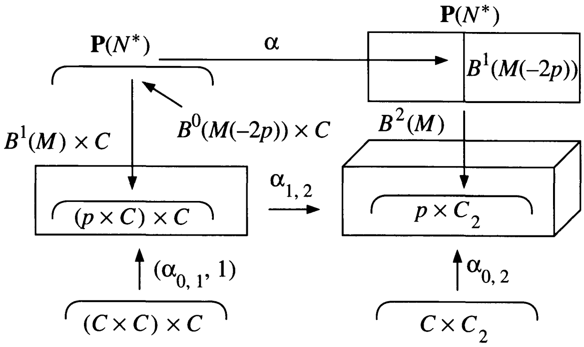

FIGURE 2. DEPICTION OF LEMMA 1.3 FOR $\alpha_{1,2}$

(b) $\mathbf{P}(N^*_{\alpha_{m,k}}(x \times C_{k-m})) \cong B^{k-m-1}(M(-2D_x))$, *where* $N^*_{\alpha_{k,m}}(x \times C_{k-m})$ *is the restriction of the conormal bundle* $N^*_{\alpha_{m,k}} = \ker(d\alpha_{m,k})$ *to* $x \times C_{k-m} \subset U^m(M) \times C_{k-m}$, *and* D_x *is the divisor of degree* $m+1$ *obtained by projecting* $x \in B^m(M)$ *to* C_{m+1}.

From Lemma 1.1(b), *we get an induced map* $\mu_{m,l}$ *of conormal bundles on* $C_{l-m} \times C_{k-l}$:

$$\mu_{m,l}\colon (1, r_{k-m})^* N^*_{\alpha_{m,k}}(x \times C_{k-m}) \to N^*_{(\alpha_{m,l},1)}(x \times C_{l-m} \times C_{k-l}).$$

Then:

(c) $\mu_{m,l}$ *is surjective,*

(d) *the following diagram commutes*:

$$\begin{array}{ccc} \mathbf{P}(N_{(\alpha_{m,l},1)}(x \times C_{l-m} \times C_{k-l})) & \xrightarrow{\tilde{\mu}} & \mathbf{P}(N^*_{\alpha_{m,k}}(x \times C_{k-m})) \\ \Vert & & \Vert \\ B^{l-m-1}(M(-2D_x)) \times C_{k-l} & \xrightarrow{\alpha} & B^{k-m-1}(M(-2D_x)) \end{array}$$

where the vertical identifications are from (b), $\tilde{\mu}$ *is the map derived from* $\mu_{m,l}$, *and* α *is the appropriate* α*-map associated to the line bundle* $M(-2D_x)$.

Proof. From the definition, we saw that $\alpha_{m,k}$ is a composition of two maps $\alpha_{m,k} = \tilde{r} \circ i$, where $i\colon B^m(M) \times C_{k-m} \to r^* B^k(M)$ is the inclusion of projective bundles, and $\tilde{r}\colon r^* B^k(M) \to B^k(M)$ is the lift of the addition map. The proof of the lemma accordingly breaks up into three logical parts:

(i) An analysis of $N_i^* = \ker(di)$,
(ii) an analysis of $\operatorname{coker}(d\tilde{r})$, and
(iii) an analysis of how N_i^* and $\operatorname{coker}(d\tilde{r})$ fit together to produce $N^*_{\alpha_{m,k}} = \ker(d\alpha_{m,k})$.

Part (i). In general, given an exact sequence of vector bundles on $X: 0 \to E' \to E \to E'' \to 0$, if we let $\pi: \mathbf{P}(E'') \to X$ be the projection and $i: \mathbf{P}(E'') \to \mathbf{P}(E)$ be the inclusion, then $N_i^* \cong \pi^* E' \otimes \mathscr{O}_{\mathbf{P}(E'')}(-1)$. In our situation, $X = C_{m+1} \times C_{k-m}$ and we are given an exact sequence of vector bundles on X:

$$0 \to H_{M_{D_{k-m}}(-D_{m+1})} \to H_{M_{D_{k+1}}} \to H_{M_{D_{m+1}}} \to 0,$$

where $H_{M_{D_{m+1}}} = \pi_*^C(\pi_C^* M \otimes \mathscr{O}_{\mathscr{D}_{m+1}})$, and the others are defined similarly.

This notation is meant to be more descriptive. For example, $H_{M_{D_{k-m}}(-D_{m+1})}$ is the bundle with fiber $H^0(C, M(-D_{m+1}) \otimes \mathscr{O}_{D_{k-m}})$ over the point $(D_{m+1}, D_{k-m}) \in C_{m+1} \times C_{k-m}$.

Thus $N_i^* = \pi^* H_{M_{D_{k-m}}(-D_{m+1})} \otimes \mathscr{O}(-1)$. One last remark that will be useful later is the following: $H_{M_{D_{k-m}}(-D_{m+1})}$ is naturally the quotient vector bundle in the exact sequence

$$0 \to H_{M_{D_{m+1}}(-D_{k+1})} \to H_{M_{D_{k+1}}(-D_{m+1})} \to H_{M_{D_{k-m}}(-D_{m+1})} \to 0,$$

where the other two vector bundles are defined in the same manner. So we get the exact sequence

$$(*): 0 \to \pi^* H_{M_{D_{m+1}}(-D_{k+1})} \otimes \mathscr{O}(-1) \to \pi^* H_{M_{D_{k+1}}(-D_{m+1})} \otimes \mathscr{O}(-1) \to N_i^* \to 0.$$

Part (ii). The map $\tilde{r}$ fits into a fiber square:

$$\begin{array}{ccc} \mathbf{P}(H_{M_{D_{k+1}}}) & \xrightarrow{\tilde{r}} & B^k(M) \\ {\scriptstyle\tilde{p}}\downarrow & & {\scriptstyle p}\downarrow \\ C_{m+1} \times C_{k-m} & \xrightarrow{r} & C_{k+1} \end{array}$$

where the vertical maps are the (smooth) projections. Thus, $\operatorname{coker}(d\tilde{r}) = \tilde{p}^* \operatorname{coker}(dr)$, so it suffices to analyze $\operatorname{coker}(dr)$.

But $TC_{k+1} = \pi_{C_{k+1_*}} \mathscr{O}_{\mathscr{D}_{k+1}}(\mathscr{D}_{k+1})$ (see [1, p. 172]), so, in our notation,

$$r^* T^* C_{k+1} = H^*_{\mathscr{O}_{D_{k+1}}(D_{k+1})},$$
$$T^*(C_{m+1} \times C_{k-m}) = H^*_{\mathscr{O}_{D_{m+1}}(D_{m+1})} \oplus H^*_{\mathscr{O}_{D_{k-m}}(D_{k-m})}.$$

Furthermore, one checks that the map $H^*_{\mathscr{O}_{D_{k+1}}(D_{k+1})} \to H^*_{\mathscr{O}_{D_{k-m}}(D_{k-m})}$ induced by dr is the natural one, hence surjective with kernel $H^*_{\mathscr{O}_{D_{m+1}}(D_{k+1})}$. Finally, the cokernel $Q = \operatorname{coker}(dr)$ is the quotient in the sequence

$$(**)\colon 0 \to H^*_{\mathscr{O}_{D_{m+1}}(D_{k+1})} \xrightarrow{\rho^*} H^*_{\mathscr{O}_{D_{m+1}}(D_{m+1})} \to Q \to 0,$$

where ρ^* is dual to the pushdown of the line bundle inclusion $\mathscr{O}_{\mathscr{D}_{m+1}}(\mathscr{D}_{m+1}) \to \mathscr{O}_{\mathscr{D}_{m+1}}(\mathscr{D}_{k+1})$.

Note. Q is not a vector bundle. It is a coherent sheaf supported on the points $(D_{m+1}, D_{k-m}) \in C_{m+1} \times C_{k-m}$, where $D_{m+1} \cap D_{k-m} \neq \varnothing$.

Thus, to answer our original question, $\operatorname{coker}(d\tilde{r}) = \tilde{p}^* Q$ is the quotient in the lift of $(**)$ by $\tilde{p}^*$, which remains exact because $\tilde{p}$ is smooth:

$$0 \to \tilde{p}^* H^*_{\mathscr{O}_{D_{m+1}}(D_{k+1})} \to \tilde{p}^* H^*_{\mathscr{O}_{D_{m+1}}(D_{m+1})} \to \tilde{p}^* Q \to 0.$$

Part (iii). From the commuting diagram

$$\begin{tikzcd} B^m(M) \times C_{k-m} \arrow[r, "i"] \arrow[dr, "\pi"'] & \mathbf{P}(H_{MD_{k+1}}) \arrow[r, "\tilde{r}"] \arrow[d, "\tilde{p}"] & B^k(M) \\ & C_{m+1} \times C_{k-m} & \end{tikzcd}$$

it follows that $(**)$ remains exact after pulling back by $\tilde{p} \circ i = \pi$ (since π is smooth), so we have the following sequence of sheaves on $B^m(M) \times C_{k-m}$:

$$0 \to \alpha^*_{m,k} T^* B^k(M) \to i^* T^* \mathbf{P}(H_{M_{D_{k+1}}}) \to \pi^* Q \to 0.$$

This exact sequence, together with the conormal sequence for i,

$$0 \to N^*_i \to i^* T^* \mathbf{P}(H_{M_{D_{k+1}}}) \to T^*(B^m(M) \times C_{k-m}) \to 0,$$

fit into the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
 & & & & 0 & & & & \ker(\gamma) \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & N_i^* & & = & & N_i^* \\
 & & & & \downarrow & & & & \downarrow \gamma \\
 & 0 & \to \alpha_{m,k}^* T^* B^k(M) & \xrightarrow{i^* d\tilde{r}} & i^* T^* \mathbf{P}(H_{M_{D_{k+1}}}) & & \to & & \pi^* Q \to 0 \\
 & & \| & & \downarrow di & & & & \\
0 \to N_{\alpha_{m,k}}^* & & \to \alpha_{m,k}^* T^* B^k(M) & \xrightarrow{d\alpha_{m,k}} & T^*(B^m(M) \times C_{k-m}) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

An easy diagram chase shows that $\ker(\gamma) \cong N_{\alpha_{m,k}}^*$, and $\operatorname{coker}(\gamma) \cong \operatorname{coker}(d\alpha_{m,k})$. It should be emphasized that these are coherent sheaves but not vector bundles, since $\alpha_{m,k}$ is not globally an immersion.

The natural multiplication maps

$$M_1 \colon H^0(C, M_{D_{m+1}}(-D_{k+1})) \otimes H^0(C, \mathscr{O}_{D_{m+1}}(D_{k+1})) \to H^0(C, M_{D_{m+1}}),$$
$$M_2 \colon H^0(C, M_{D_{k+1}}(-D_{m+1})) \otimes H^0(C, \mathscr{O}_{D_{m+1}}(D_{m+1})) \to H^0(C, M_{D_{m+1}})$$

induce maps of sheaves on $B^m(M) \times C_{k-m} = \mathbf{P}(H_{M_{D_{m+1}}})$:

$$m_1 \colon \pi^* H_{M_{D_{m+1}}}(-D_{k+1}) \otimes \mathscr{O}(-1) \to \pi^* H^*_{\mathscr{O}_{D_{m+1}}(D_{k+1})},$$
$$m_2 \colon \pi^* H_{M_{D_{k+1}}(-D_{m+1})} \otimes \mathscr{O}(-1) \to \pi^* H^*_{\mathscr{O}_{D_{m+1}}(D_{m+1})}.$$

Finally, $(*)$ and $(**)$ fit into the following commuting diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \pi^* H_{M_{D_{m+1}}(-D_{k+1})} \otimes \mathscr{O}(-1) & \longrightarrow & \pi^* H_{M_{D_{k+1}}(-D_{m+1})} \otimes \mathscr{O}(-1) & \longrightarrow & N_i^* & \longrightarrow 0 \\
 & \downarrow m_1 & & \downarrow m_2 & & \downarrow \gamma & \\
0 \longrightarrow & \pi^* H^*_{\mathscr{O}_{D_{m+1}}(D_{k+1})} & \longrightarrow & \pi^* H^*_{\mathscr{O}_{D_{m+1}}(D_{m+1})} & \longrightarrow & \pi^* Q & \longrightarrow 0
\end{array}$$

The fact that the left square commutes follows immediately from the definitions. The commutativity of the right square follows from the explicit identification of $H^0(C, \mathscr{O}_{D_{k+1}}(D_{k+1}))$ with the tangent space to C_{k+1} at D_{k+1} (see [7, Lecture 22]).

The claims of Lemma 1.3 now follow from this last diagram. In the first place, m_1 and m_2 are both surjective when restricted to $U^m(M) \times C_{k-m}$. One can see this by looking at the M_1 and M_2 maps. But if

m_2 is surjective on $U^m(M) \times C_{k-m}$, then γ is also surjective, and since $\operatorname{coker}(\gamma) \cong \operatorname{coker}(d\alpha_{m,k})$, we get (a). Next, m_1 is an isomorphism when restricted to $U^m(M) \times C_{k-m}$ since it is a surjective map of vector bundles of the same rank, so, on this set, $\ker(m_2) \cong \ker(\gamma)$. But m_2 factors as $m_2 = m \circ \rho$ below:

$$\pi^* H_{M_{D_{k+1}}(-D_{m+1})} \otimes \mathscr{O}(-1) \xrightarrow{\rho} \pi^* H_{M_{D_{m+1}}(-D_{m+1})} \otimes \mathscr{O}(-1) \xrightarrow{m} \pi^* H^*_{\mathscr{O}_{D_{m+1}}(D_{m+1})},$$

where ρ is the natural restriction and m is an isomorphism on $U^m(M) \times C_{k-m}$ just as m_1 was. Thus, $\ker(m_2) \cong \ker(\rho)$.

But $\ker(\rho) \cong \pi^* H_{M_{D_{k-m}}(-2D_{m+1})} \otimes \mathscr{O}(-1)$, so, for $x \in U^m(M)$,

$$\mathbf{P}(N^*_{\alpha_{m,k}}(x \times C_{k-m})) \cong \mathbf{P}(\ker(\gamma)) \cong \mathbf{P}(H_{M_{D_{k+1}}(-2D_x)}) \cong B^{k-m-1}(M(-2D_x)).$$

This is part (b) of the lemma.

The proofs of (c) and (d) are more of the same. The point is that if the induced map $\mu_{m,k}$ is pulled back through all the various isomorphisms described above, it becomes the restriction map of bundles on $U^m(M) \times C_{l-m} \times C_{k-l}$:

$$\pi^* H_{M_{D_{k-m}}(-2D_{m+1})} \otimes \mathscr{O}(-1) \to \pi^* H_{M_{D_{l-m}}(-2D_{m+1})} \otimes \mathscr{O}(-1),$$

which gives the desired results when restricted to $x \times C_{l-m} \times C_{k-l}$. This concludes the proof of the lemma. q.e.d.

There is an exact analog of Lemma 1.3 for the map $d\beta_m$:

Lemma 1.4 (Terracini). *If M separates $2m+2$ points, let $x \in U^m(M)$ and consider $\beta_m \colon B^m(M) \to \mathbf{P}(H^0(C, M))$. Then:*

(a) *$d\beta_m \colon \beta_m^* T^* \mathbf{P}(H^0(C, M)) \to T^* B^m(M)$ is surjective,*

(b) *$\mathbf{P}(N^*_{\beta_m}(x)) \cong \mathbf{P}(H^0(C, M(-2D_x)))$, where, as usual, $N^*_{\beta_m}(x) = \ker(d\beta_m(x))$ is the fiber of the conormal sheaf at x, and D_x is the projection of x to C_{m+1}.*

Furthermore, from Lemma 1.1(a), *we get the induced map*

$$\mu_{m,k} \colon \pi^*_{B^m(M)} N^*_{\beta_m}(x) \to N^*_{\alpha_{m,k}}(x \times C_{k-m}).$$

Then:

(c) *$\mu_{m,k}$ is surjective.*

(d) *The following diagram commutes*:

$$\begin{array}{ccc}
\mathbf{P}(N^*_{\alpha_{m,k}}(x \times C_{k-m})) & \xrightarrow{\tilde{\mu}} & \mathbf{P}(N^*_{\beta_n}(x)) \\
\| & & \| \\
B^{k-m-1}(M(-2D_x)) & \xrightarrow{\beta} & \mathbf{P}(H^0(C, M(-2D_x)))
\end{array}$$

where the vertical identifications come from (b) *of Lemmas* 1.3 *and* 1.4, *and* β *is the appropriate* β*-map for* $M(-2D_x)$.

Proof. The proof of this lemma follows exactly the same pattern as the proof of Lemma 1.3, and is left to the reader. In fact, it is easier since, in this case, $\beta_m = \tilde{\pi} \circ i$, where i is an inclusion of projective bundles, and $\tilde{\pi}: \mathbf{P}(H^0(C, M)) \times C_{m+1} \to \mathbf{P}(H^0(C, M))$ is the projection, hence smooth (unlike $\tilde{r}$ of the previous lemma).

Corollary. *If* M *separates* $2m+2$ *points, then* $\mathrm{Sec}^m(C)$ *is smooth away from* $\mathrm{Sec}^{m-1}(C)$.

Proof. In fact, by Lemma 1.2(a), $\beta_m: U^m(M) \to \mathbf{P}(H^0(C, M))$ is injective, with image $\mathrm{Sec}^m(C) - \mathrm{Sec}^{m-1}(C)$. Further, by the same lemma, $\beta_m(B^m(M) - U^m(M)) \subset \mathrm{Sec}^{m-1}(C)$. By Lemma 1.5(a), β_m is an immersion on $U^m(M)$, so $U^m(M) \cong \mathrm{Sec}^m(C) - \mathrm{Sec}^{m-1}(C)$ and since $U^m(M) \subset B^m(M)$ is obviously smooth, the corollary follows. (One should note, though, that $\mathrm{Sec}^m(C)$ is singular at all points of $\mathrm{Sec}^{m-1}(C)$.) q.e.d.

Along the same lines, we have:

Corollary 1.6. *For any* M *and integers* $0 \leq m < k$, $\alpha_{m,k}(B^m(M) \times C_{k-m})$ *is smooth away from* $\alpha_{m-1,k}(B^{m-1}(M) \times C_{k-m+1})$ (*i.e., the* m*th relative secant variety is smooth away from the* $(m-1)$*st*).

Proof. Exactly as above. By Lemmas 1.2(b) and 1.4, $\alpha_{m,k}: U^m(M) \times C_{k-m} \to B^k(M)$ is an embedding, with the desired image.

Warning. Although we have just shown that the relative secant variety is smooth away from the smaller secant varieties, one must keep in mind that the map to C_{k+1} is not smooth.

For example, by the corollary, $\alpha_{0,k}: C \times C_k \to B^k(M)$ is an embedding. But the composition with the projection $C \times C_k \to C_{k+1}$ results in the addition map r_{k+1}, which is not smooth.

2. The blow-up constructions

We start this section with a series of definitions which will be needed in the proof of Theorem 1.

Let X be a projective variety.

Definition. A *chain of* $k+1$ *maps to* X is a sequence of morphisms $\{f_i: X_i \to X\}_{i=0}^k$ from projective varieties X_i with the property that for each $i < j$, the induced morphism $\tilde{f}_j$ in the fibered square below is surjective:

$$\begin{tikzcd} & X_i \arrow[dr, "f_i"] & \\ X_i \times_X X_j \arrow[ur, "\tilde{f}_j"] \arrow[dr, "\tilde{f}_i"'] & & X \\ & X_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

Equivalently, $\{f_i\}$ is a chain of maps is a chain of maps if there are projective varieties $Y_{i,j}$ and morphisms $g_{i,j}\colon Y_{i,j} \to X_i$ and $h_{i,j}\colon Y_{i,j} \to X_j$ for each $i < j$ so that $g_{i,j}$ is surjective, and $f_i \circ g_{i,j} = f_j \circ h_{i,j}$.

In addition, we say that $\{f_i\}$ is a *proper* chain if the $\tilde{f}_i$ above are not surjective.

Examples. (0) $X_0 \subset X_1 \subset \cdots \subset X_k \subset X$ is a chain of maps. It is proper if all the inclusions are proper.

(1) If M separates $k+1$ points, then $\{\beta_m : B^m(M) \to \mathbf{P}(H^0(C, M))\}_{m=0}^{k}$ is a chain of $k+1$ maps because of the compatibility Lemma 1.1(a) and the fact that the projection map is surjective. Moreover, since $\beta_m(B^m(M)) \subset \beta_l(B^l(M))$ is proper for $m < l$, this is a proper chain.

(2) For arbitrary M, $\{\alpha_{m,k}\colon B^m(M) \times C_{k-m} \to B^k(M)\}_{m=0}^{k-1}$ is a chain of k maps, again by the compatibility Lemma 1.1(b), and the fact that the addition map is surjective. It is also a proper chain, as is easily verified.

Notation. *The* $k+1$ *chain* $\mathbf{P}(H^0(C, M))$ will be understood to mean Example (1). Similarly, *the chain* $B^k(M)$ will be understood to mean Example (2). Moreover, if S is any variety, then *the chain* $B^k(M) \times S$ will signify the chain $\{(\alpha_{m,k}, 1)\}_{m=0}^{k-1}$, i.e., the collection of maps that are $\alpha_{m,k}$ on the first factors and the identity on S. Finally, *the m-chain* $B^k(M)$ will signify the chain $B^k(M)$ truncated after the first m maps.

Suppose $\{f_i\colon X_i \to X\}_{i=0}^{k}$ is a proper chain.

(Inductive) **Definition.** If f_0 is injective, we identify X_0 with its image, and we define:

$bl_1(X) :=$ the blow-up of X along X_0,

$bl_1(X_i) :=$ the blow-up of X_i along $f_i^{-1}(X_0)$, and

$bl_1(f_i) :=$ the (unique) lift of f_i to a map $bl_1(X_i) \to bl_1(X)$.

If $bl_n(X)$, $bl_n(X_i)$, and $bl_n(f_i)$ are all defined (for all $i \geq n$), and $bl_n(f_n)$ is injective, then we identify $bl_n(X_n)$ with its image, and define

$bl_{n+1}(X) :=$ the blow-up of $bl_n(X)$ along $bl_n(X_n)$, etc.

Definition. If $\{f_i\colon X_i \to X\}_{i=0}^{k}$ is a proper chain and $bl_{k+1}(X)$ is defined, then we say $\{f_i\}$ is a *chain of* $k+1$ *centers.*

If, in addition, X and $bl_n(X_n)$ are all smooth, then we say that $\{f_i\}$ is a chain of smooth centers (and of course it follows that $bl_{k+1}(X)$ is smooth).

Now, suppose that $\{f_i\colon X_i \to X\}_{i=0}^k$ and $\{g_i\colon Y_i \to Y\}_{i=0}^k$ are two chains of centers.

Definition. We say that a map $\phi\colon X \to Y$ is a *map of chains of centers* if it satisfies:

(0) $\phi^{-1}(Y_0) = X_0$, so $bl_1(\phi)\colon bl_1(X) \to bl_1(Y)$ is defined, and

(n) for all $0 < n \le k$, $bl_n(\phi)^{-1}(bl_n(Y_n)) = bl_n(X_n)$, so $bl_{n+1}(\phi)$ is defined.

We say the map ϕ is an injective map of chains of centers (or just an injective map of centers) if, in addition, $bl_{k+1}(\phi)$ is injective.

Note. If ϕ is an injective map of centers, it does not follow that ϕ is itself injective!

Before we get to the main propositions of the section, we prove a general lemma, which guarantees that, under certain conditions, the exceptional divisor in the blow-up $bl_{k+1}(X)$ has normal crossings.

Lemma 2.1. *Suppose that X, X_j, and $X_{i,j}$ are smooth projective varieties for $0 \le j \le k$ and $0 \le i < j$, and that $\{\phi_j\colon X_j \to X\}_{j=0}^k$ and, for each j, $\{f_{i,j}\colon X_{i,j} \to X_j\}_{i=0}^{j-1}$ are all chains of smooth centers. Suppose finally that each ϕ_j is an injective map of smooth centers.*

Let $U_j = X_j - f_{j-1,j}(X_{j-1,j})$ and $U = X - \phi_k(X_k)$. Then there are natural inclusions $U_j \subset bl_j(X_j)$ and $U \subset bl_{k+1}(X)$. If each exceptional divisor $bl_j(X_j) - U_j$ has j components and normal crossings, then the exceptional divisor $bl_{k+1}(X) - U$ has $k+1$ components and normal crossings.

Proof. First, recall that if $Z \subset Y$ is a proper inclusion of projective varieties and $\varepsilon\colon bl_Z(Y) \to Y$ is the blow-up, then ε is an isomorphism over the open set $Y - Z \subset Y$, so we may consider $Y - Z$ as a subset of $bl_Z(Y)$. By the definition of a chain of centers, the open subsets U_j and U are consistently in the complement of the centers of blowings-up, so we may consider them as open subsets of $bl_j(X_j)$ and $bl_{k+1}(X)$ respectively.

The proof of the normal crossings is by induction on k. If $k = 0$, then $\phi_0\colon X_0 \to X$ is an embedding of smooth projective varieties, and $U \subset bl_1(X)$ is the complement of the (smooth, irreducible) exceptional divisor.

Before we attack the case $k \ge 1$, recall the following facts about blowing-up (which we state here only in the generality we will use).

Fact A (Functoriality). If Y is a smooth projective variety, $Z \subset Y$ a smooth projective subvariety, and $f\colon W \to Y$ a morphism such that

$f^{-1}(Z)$ is smooth, properly contained in W, then the lift of f to $bl_Z(f)$:

$$\begin{array}{ccc} bl_{f^{-1}(Z)}(W) & \xrightarrow{bl_Z(f)} & bl_Z(Y) \\ \downarrow{\scriptstyle \varepsilon_{f^{-1}(Z)}} & & \downarrow{\scriptstyle \varepsilon_Z} \\ W & \xrightarrow{f} & Y \end{array}$$

has the property that $bl_Z(f)^{-1}(E_Z) = E_{f^{-1}(Z)}$, where E_Z and $E_{f^{-1}(Z)}$ are the exceptional divisors.

Proof of Fact A. $bl_Z(f)^{-1}(E_Z) = bl_Z(f)^{-1} \circ \varepsilon_Z^{-1}(Z) = \varepsilon_{f^{-1}(Z)}^{-1} \circ f^{-1}(Z) = E_{f^{-1}(Z)}$.

Corollary. *If* $bl_Z(f)$ (*though not necessarily* f *itself*) *is injective, then* $E_Z \cap bl_{f^{-1}(Z)}(W) = E_{f^{-1}(Z)}$.

Fact B (Transversality). Suppose $W \subset Y$ in Fact A is an embedding of smooth, projective varieties, and $Z \cap W$ is transverse. Then the diagram from Fact A:

$$\begin{array}{ccc} bl_{Z\cap W}(W) & \subset & bl_Z(Y) \\ \downarrow & & \downarrow \\ W & \subset & Y \end{array}$$

is a fiber square.

Proof of Fact B. This is just a local coordinate computation.

Returning to the lemma, suppose $k \geq 1$ and let $V = X - \phi_k(X_{k-1})$. By induction, we may write $bl_k(X) - V = E_0 \cup E_1 \cup \cdots \cup E_{k-1}$, where the E_j are smooth, with transverse intersections. By repeated application of Facts A and B, if we let $bl_k(X_k) - U_k = E_{0,k} \cup \cdots \cup E_{k-1,k}$, then (for a suitable ordering of the E's), $bl_k(X_k) \cap E_j = E_{j,k}$. Hence the intersection is transverse.

Now, we blow up $bl_k(X)$ along $bl_k(X_k)$ to get $bl_{k+1}(X)$. One component of the complement of U is therefore the exceptional divisor in this blow-up, which we call E_k. The others are all derived from the E_j, $0 \leq j < k$. In fact, by Fact B, the preimage of E_j in $bl_{k+1}(X)$ is the blow-up of E_j along $E_{j,k}$, which is smooth and irreducible, and is called $\widetilde{E}_j$.

To sum up, $bl_{k+1}(X) - U = \widetilde{E}_0 \cup \cdots \cup \widetilde{E}_{k-1} \cup E_k$ is a union of smooth, irreducible components. A similar argument shows that the divisors intersect transversely. q.e.d.

The rest of this section is devoted to the following propositions.

Proposition 2.2. *If M is any line bundle, and $0 \leq m < k$, then $\alpha_{m,k}: B^m(M) \times C_{k-m} \to B^k(M)$ is an injective map of chains of smooth centers. (Hence, in particular, $bl_{m+1}(B^k(M))$ is smooth.)*

Proposition 2.3. *If M separates $2k+2$ points, then $\beta_k: B^k(M) \to \mathbf{P}(H^0(C, M))$ is an injective map of chains of smooth centers. (Hence, in particular, $bl_{k+1}(\mathbf{P}(H^0(C, M)))$ is smooth.)*

Let $\widetilde{B}^m(M) = bl_m(B^m(M))$. Recall the definition of $U^m(M)$ and in addition define $U^k = \mathbf{P}(H^0(C, M)) - \beta_k(B^k(M))$ when M separates $2k+2$ points. Then Lemma 2.1 gives the following corollaries to the propositions.

Corollary 2.4. (a) *There is a natural inclusion $U^k(M) \subset \widetilde{B}^k(M)$, and $\widetilde{B}^k(M) - U^k(M)$ is a divisor with k components and normal crossings.*

(b) *If M separates $2k+2$ points, then there is a natural inclusion $U^k \subset bl_{k+1}(\mathbf{P}(H^0(C, M)))$, and $bl_{k+1}(\mathbf{P}(H^0(C, M))) - U^k$ is a divisor with $k+1$ components and normal crossings.*

Proof of the corollary. (a) The statement is vacuously true if $k = 0$. If $k > 0$, we apply Lemma 2.1 to the varieties $B^k(M)$, $B^m(M) \times C_{k-m}$ $(m < k)$, and $B^n(M) \times C_{m-n} \times C_{k-m}$ $(n < m)$, and the chains of smooth centers:

$$\{(\alpha_{n,m}, 1): B^n(M) \times C_{m-n} \times C_{k-m} \to B^m(M) \times C_{k-m}\}_{n=0}^{m-1},$$
$$\{\alpha_{m,k}: B^m(M) \times C_{k-m} \to B^k(M)\}_{m=0}^{k-1}.$$

By induction, we may assume that $\widetilde{B}^m(M) - U^m(M)$ are all divisors with normal crossings for $m < k$, so the lemma applies.

(b) This time, we apply Lemma 2.1 to the varieties $\mathbf{P}(H^0(C, M))$, $B^m(M)$ $(m \leq k)$, and $B^n(M) \times C_{m-n}$ $(n < m)$, and the chains of smooth centers:

$$\{\alpha_{n,m}: B^n(M) \times C_{m-n} \to B^m(M)\}_{n=0}^{m-1},$$
$$\{\beta_m: B^m(M) \to \mathbf{P}(H^0(C, M))\}_{m=0}^{k}.$$

By (a), $\widetilde{B}^m(M) - U^m(M)$ are all divisors with normal crossings, so again Lemma 2.1 applies.

Proof of Proposition 2.2. We need to prove two statements: first, $B^m(M)$ and $B^k(M)$ are chains of m smooth centers, and then, $\alpha_{m,k}$ is an injective map of smooth centers. These follow from the definitions once we show that for all k and $m < k$, $bl_m(\alpha_{m,k}): \widetilde{B}^m(M) \times C_{k-m} \to bl_m(B^k(M))$ is an embedding, and, if we identify $\widetilde{B}^n(M) \times C_{k-n}$ with its image in $bl_n(B^k(M))$, then

$$bl_n(\alpha_{m,k})^{-1}(\widetilde{B}^n(M) \times C_{k-n}) = \widetilde{B}^n(M) \times C_{m-n} \times C_{k-m}.$$

By Corollary 1.6, we already know that $bl_m(\alpha_{m,k})$ is an embedding when restricted to $U^m(M) \times C_{k-m}$. The problem is to understand what is going on over the complement. This is accomplished using the Terracini Lemma 1.3 and induction. Again by Corollary 1.6, $B^k(M)$ is covered by $U^k(M)$, together with the locally closed subvarieties $U^n(M) \times C_{k-n}$, $(n < k)$. If we let $\varepsilon_m : bl_m(B^k(M)) \to B^k(M)$ be the (multiple) blow-up, then we will show that the induced maps

$$\pi \circ \varepsilon_m : \varepsilon_m^{-1}(U^n(M) \times C_{k-n}) \to U^n(M) \times C_{k-n} \to U^n(M)$$

are all smooth, and that, for each $x \in U^n(M)$,

$$(\pi \circ \varepsilon_m)^{-1}(x) \cong bl_{m-n}(B^{k-m-1}(M(-2D_x))).$$

(The notation here is as in Lemma 1.3.)

The description of $(\pi \circ \varepsilon_m)^{-1}(x)$ will not only be an essential ingredient of this proof, but will be used extensively in the next section.

The proposition is proved by induction on m for each fixed k. In order to demonstrate the ideas in the proof, we start by proving the proposition for the first few values of m.

Case $m = 0$. We only need to show that $\alpha_{0,k}$ is an embedding.

Proof. Since, in this case, $U^0(M) = B^0(M)$, this is Corollary 1.6.

Case $m = 1$. We need to show that:

(0) $\alpha_{0,k}$ is an embedding,

(1) $\alpha_{1,k}^{-1}(B^0(M) \times C_k) = B^0(M) \times C \times C_{k-1}$, and

(2) $bl_1(\alpha_{1,k})$ is an embedding.

Proof. (0) has been shown already. (This is case $m = 0$.) (1) is something new, but also follows immediately from §1. First, by Corollary 1.6, $\alpha_{1,k}^{-1}(U^1(M) \times C_{k-1}) \cap (B^0(M) \times C \times C_{k-1}) = \varnothing$, so, at least set-theoretically, (1) holds. In order to get a scheme-theoretic equality, we need to show that the induced map on conormal bundles $d\alpha_{1,k} : \alpha_{1,k}^* N_{\alpha_{0,k}}^* \to N_{(\alpha_{0,1},1)}^*$ is surjective. But this is precisely the content of Lemma 1.3(c).

By Lemma 1.3(d), the exceptional divisor in $bl_1(B^k(M))$, which is just $\mathbf{P}(N_{\alpha_{0,k}}^*)$, is smooth over C, with fibers $B^{k-1}(M(-2p))$ over each $p \in C$. Furthermore, by the same lemma, the map on exceptional divisors induced by $bl_1(\alpha_{1,k})$ is a map of varieties over C, which is just $\alpha_{0,k-1} : B^0(M(-2p)) \times C_{k-1} \to B^{k-1}(M(-2p))$ on the fiber over each $p \in C$.

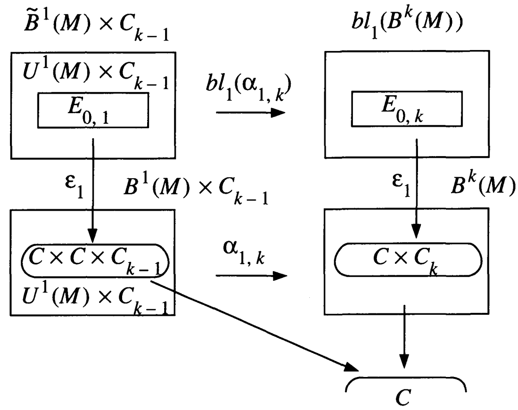

FIGURE 3

Finally, we need to prove (2). As remarked earlier, $bl_1(\alpha_{1,k})$ is an embedding along $U^1(M) \times C_{k-1}$, so we need to focus on points in the exceptional divisor. As in Figure 3, let $E_{0,1}$ and $E_{0,k}$ denote the two exceptional divisors, and let $\tilde{\alpha}_{1,k}$ denote $bl_1(\alpha_{1,k})$. Then we get the following commuting diagram of conormal sequences for the exceptional divisors:

$$\begin{array}{ccccccccc}
0 \to & \tilde{\alpha}^*_{1,k} N^*_{E_{0,k}} & \to & \alpha^*_{1,k} T^*(bl_1(B^k(M)))|_{E_{0,k}} & \to & \tilde{\alpha}^*_{1,k} T^*(E_{0,k}) & \to 0 \\
& \Vert \wr & & \downarrow d\tilde{\alpha}_{1,k} & & \downarrow & \\
0 \to & N^*_{E_{0,1}} & \to & T^*(\widetilde{B}^1(M) \times C_{k-1})|_{E_{0,1}} & \to & T^*(E_{0,1}) & \to 0
\end{array}$$

The isomorphism on the left follows from the surjectivity of the map $d\alpha_{1,k}$ on conormal bundles. The map on the right is surjective by Case $m = 0$ since the map on exceptional divisors is a C-map which is $\alpha_{0,k-1}$ on the fibers. So $\tilde{\alpha}_{1,k}$ is an immersion along the exceptional divisor. We only need to show that $\tilde{\alpha}_{1,k}$ is injective as a map of sets, but this, too, follows from Case $m = 0$.

Case $m = 2$. In addition to the items already proved, we have to show:

(1) $\alpha^{-1}_{2,k}(C \times C_k) = C \times C_2 \times C_{k-2}$,

(2) $bl_1(\alpha_{2,k})^{-1}(\widetilde{B}^1(M) \times C_{k-1}) = \widetilde{B}^1(M) \times C \times C_{k-2}$, and

(3) $bl_2(\alpha_{2,k}) \colon \widetilde{B}^2(M) \times C_{k-2} \to bl_2(B^k(M))$ is an embedding.

Proof. The proof of (1) runs exactly as in the previous case. For (2), we refer to Figure 4.

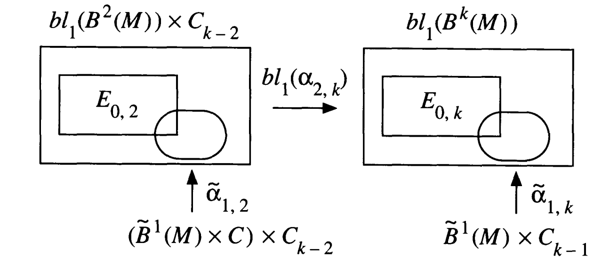

FIGURE 4

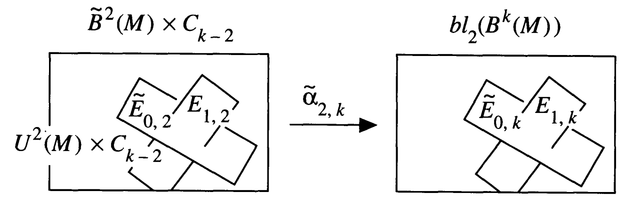

FIGURE 5

On the complement of $E_{0,2}$ and $E_{0,k}$, (2) is easily checked by Lemma 1.3. We just have to show that for all y in $E_{0,k} \cap (\tilde{B}^1(M) \times C_{k-1})$, it follows that $bl_1(\alpha_{2,k})^{-1}(y) \subset \tilde{B}^1(M) \times C \times C_{k-2}$ and if $y = bl_1(\alpha_{2,k})(x)$, then $d(bl_1(\alpha_{2,k}))\colon N^*_{\tilde{\alpha}_{1,k}}(y) \to N^*_{\tilde{\alpha}_{1,2}}(x)$ is surjective.

By Fact A, $E_{0,k} \cap (\tilde{B}^1(M) \times C_{k-1}) = (C \times C) \times C_{k-1}$, the exceptional divisor in $\tilde{B}^1(M) \times C_{k-1}$, and similarly, $E_{0,2} \cap (\tilde{B}^1(M) \times C) \times C_{k-2} = (C \times C \times C) \times C_{k-2}$. These intersections are transverse, so, in fact, $N^*_{\tilde{\alpha}_{1,k}}(y) \cong N^*_{(C\times C\times C_{k-1})/E_{0,k}}(y)$ and similarly for $N^*_{\tilde{\alpha}_{1,2}}(x)$. But now, as before, when we restrict $bl_1(\alpha_{2,k})$ to $E_{0,2}$, the resulting map of C-varieties is $\alpha_{1,k-1}$ on the fibers. By the identification of the intersections and normal bundles, we can reduce the problem to the previous case.

Finally, for (3), consider Figure 5, where $\tilde{E}_{0,2}$ and $\tilde{E}_{0,k}$ are the blown-up divisors $E_{0,2}$ and $E_{0,k}$ from Figure 4, and $E_{1,2}$ and $E_{1,k}$ are the new exceptional divisors over $\tilde{B}^1(M)$. The crucial observation here is that $\tilde{E}_{0,2}$ is a C-variety, with fibers isomorphic to $\tilde{B}^1(M(-2p)) \times C_{k-2}$ and induced map $\tilde{\alpha}_{1,k-1}$ to the fibers of $\tilde{E}_{0,k}$, while $E_{1,2} - \tilde{E}_{0,2}$ is a $U^1(M)$-variety with fibers $B^0(M(-2D_x)) \times C_{k-2}$ and induced map $\alpha_{0,k-2}$

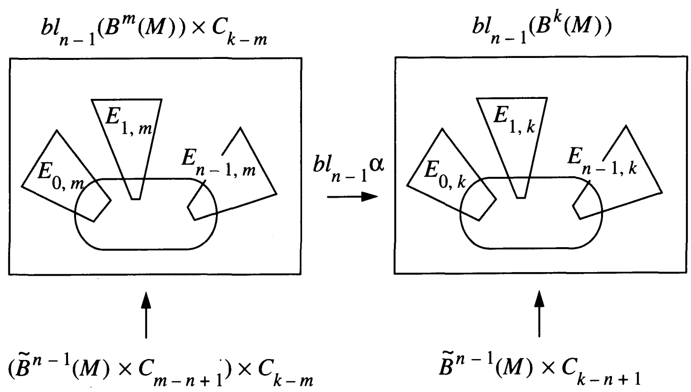

FIGURE 6

to the fibers of $E_{1,k}$ over $U^1(M)$. But once we make these observations, the proof immediately follows from Cases $m = 0$ and $m = 1$.

Case m arbitrary. Let $\tilde{\alpha}_{m,k} = bl_m(\alpha_{m,k})$. We have to show:

(1) $\alpha_{m,k}^{-1}(C \times C_k) = (C \times C_m) \times C_{k-m}$,

.

.

.

(m) $bl_{m-1}(\alpha_{m,k})^{-1}(\tilde{B}^{m-1}(M) \times C_{k-m+1}) = (\tilde{B}^{m-1}(M) \times C) \times C_{k-m}$,
and

$(m+1)$ $\tilde{\alpha}_{m,k}$ is an embedding.

Proof. To prove (n), $1 < n \le m$, we assume by induction that $bl_{n-1}(\alpha_{m,k})$: $bl_{n-1}(B^m(M)) \times C_{k-m} \to bl_{n-1}(B^k(M))$ exists (see Figure 6).

Further, if $E_{i,m}$ and $E_{i,k}$ are the exceptional divisors as pictured, then $bl_{n-1}(\alpha)(E_{i,m} - \bigcup_{j<i} E_{j,m}) \subset E_{i,k} - \bigcup_{j<i} E_{j,k}$, and the induced map is a morphism of $U^i(M)$-varieties which is $bl_{n-i-1}(\alpha_{m-i-1,k-i-1})$ on the fibers.

By repeated use of Facts A and B and induction, we have

$$\begin{aligned}(\tilde{B}^{n-1}(M) \times C_{m-n-1} \times C_{k-m}) \cap E_{i,m} \\ = bl_{n-1}(\alpha)^{-1}((\tilde{B}^{n-1}(M) \times C_{k-n+1}) \cap E_{i,k}),\end{aligned}$$

and because the intersections are transverse, (n) holds. But, in addition, if $\tilde{E}_{i,m}$ is the blown-up exceptional divisor in $bl_n(B^m(M)) \times C_{k-m}$, then $\tilde{E}_{i,m} - \bigcup_{j<i} \tilde{E}_{j,m}$ is a $U^i(M)$-variety with appropriate fibers and induced map.

Finally, $(m+1)$ is proved in the same way, reducing to the exceptional divisors and arguing by induction. This completes the proof of the proposition. q.e.d.

The proof of Proposition 2.3 follows the same pattern by using Lemma 1.4 instead of Lemma 1.3. The only difference is that one has to keep track of the fact that if M separates $2k+2$ points, then $M(-2D_x)$ separates $2k+2-2\deg(D_x)$ points.

As mentioned earlier, the proofs of Propositions 2.2 and 2.3 give useful information about the structure of the embeddings $\tilde{\alpha}_{m,k}$ and $\widetilde{\beta}_m := bl_m(\beta_m)$ which we collect in the following corollary.

Corollary 2.5. (a) *The m smooth exceptional divisors of $bl_m(B^k(M))$, which we denote by $E_{0,k}, \cdots, E_{m,k}$, come equipped with maps $\pi_{i,k}$: $E^0_{i,k} := E_{i,k} - \bigcup_{j<i} E_{j,k} \to U^i(M)$ with the property that*

$$\pi^{-1}_{i,k}(x) \cong bl_{m-i-1}(M(-2D_x))).$$

Further, $\tilde{\alpha}_{m,k}$ induces morphisms $E^0_{i,m} \to E^0_{i,k}$ over $U^i(M)$ which are the appropriate $\tilde{\alpha}$ map on each fiber.

(b) *The $k+1$ smooth exceptional divisors of $bl_{k+1}(\mathbf{P}(H^0(C, M)))$, which we denote by $E_0, \cdots, E_k$, come equipped with maps π_i: $E^0_i \to U^i(M)$ with the property that $\pi_i^{-1}(x) \cong bl_{k-i-1}(\mathbf{P}(H^0(C, M(-2D_x))))$. Furthermore, $\tilde{\beta}_k$ induces morphisms $E^0_{i,k} \to E^0_i$ over $U^i(M)$ which are the appropriate $\tilde{\beta}_{k-i-1}$ maps on the fibers.*

3. Blowing up the extension map

For the rest of this paper, suppose L is a line bundle of nonnegative degree on C and $M = L \otimes \omega_C$. Then we define $\mathbf{P}_L := \mathbf{P}(H^0(C, M))$. Now suppose $\deg(L) = 2k+1$ or $\deg(L) = 2k+2$. Then the work in §2 allows us to construct $\widetilde{P}_L := bl_k(\mathbf{P}_L)$. We will see that this blow-up is "stable" in the sense that we can naturally assign to each point in $\widetilde{P}_L$ a semistable rank 2 vector bundle with determinant L. This assignment will extend the extension map mentioned in the introduction to a morphism $\Phi_L\colon \widetilde{P}_L \to \mathscr{M}_{2,L}$ and will be seen to satisfy the requirements of Theorem 1. But first, we need to rigorously define the extension map ϕ_L.

Recall that $\operatorname{Ext}^1(L, \mathscr{O}_C)$ is defined to be the set of short exact sequences $(*)\colon 0 \to \mathscr{O}_C \to E \to L \to 0$ modulo the equivalence $(*) \sim (**)$ if there is a commuting diagram:

$$\begin{array}{ccccccccc}
(*): 0 & \longrightarrow & \mathscr{O}_C & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
& & \| & & \wr\| & & \| & & \\
(**): 0 & \longrightarrow & \mathscr{O}_C & \longrightarrow & E' & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

Thus there are maps of sets:

$$\{\text{Sequences}\,(*)\,\} \to \operatorname{Ext}^1(L, \mathscr{O}_C) \xrightarrow{f_L} \{\text{rank 2 bundles } E\}.$$

Recall also that:
(1) $\operatorname{Ext}^1(L, \mathscr{O}_C) \cong H^1(C, L^*) \cong H^0(C, M)^*$,
(2) if $k \in \mathbf{C}^*$, then for all $x \in \operatorname{Ext}^1(L, \mathscr{O}_C)$, $f_L(x) = f_L(kx)$,
(3) $f_L(x) \cong \mathscr{O}_C \oplus L$ if and only if $x = 0$.
Putting all this together, we get a well-defined extension map

$$\phi_L: \mathbf{P}_L \to \mathscr{M}_{2,L}.$$

Indeed, if we let π_C and π_L be the two projections on $C \times \mathbf{P}_L$, then we have

Definition-Claim 3.1. There is a natural Poincaré extension on $C \times \mathbf{P}_L$:

$$(*): 0 \to \pi_L^* \mathscr{O}(1) \to \mathscr{E}_L \to \pi_C^* L \to 0$$

with the property that if $x \in \mathbf{P}_L$, then the restriction of $(*)$ to $C \times \{x\}$ yields x.

Proof. The extension space

$$H^1(C \times \mathbf{P}_L, \pi_C^* L^* \otimes \pi_L^* \mathscr{O}(1)) \cong H^1(C, L^*) \otimes H^1(C, L^*)^*$$

has the natural identity element. One checks that the Poincaré property holds for this extension.

Thus, ϕ_L is a rational map with domain:

$$\operatorname{dom}(\phi_L) = \{(*): 0 \to \mathscr{O}_C \to E \to L \to 0 \mid E \text{ is semistable}\}.$$

Observation 0. If $\deg(L) < 0$, then $\operatorname{dom}(\phi_L) = \varnothing$.

Observation 1. If $\deg(L) = 0$, 1, or 2, then $\operatorname{dom}(\phi_L) = \mathbf{P}_L$.

Proof. If $(*): 0 \to \mathscr{O}_C \to E \to L \to 0$ and E is not semistable, then there is a quotient $E \to L' \to 0$ where $\deg(L') < \deg(E)/2$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[dr, "\alpha"'] & E \arrow[r] \arrow[d] & L \arrow[r] \arrow[dl, "\beta"] & 0 \\
& & L' \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

If $\deg(L) = 0$, then $\deg(L') < 0$, so $\alpha = 0$, β is defined, and $\beta = 0$, a contradiction. If $\deg(L) = 1$, then the same reasoning shows that $\deg(L') = 0$. But then α is an isomorphism, and $(*)$ is split, a contradiction. Similarly for $\deg(L) = 2$.

Remark. By definition of $\mathcal{M}_{2,L}$, ϕ_L is the constant map if $\deg(L) = 0$.

If $\deg(L) \geq 2$, then there is the usual linear series morphism $C \to \mathbf{P}(H^0(C, M))$.

Observation 2. If $d = \deg(L) \geq 2$, and $(*)\colon \mathcal{O}_C \to E \to L \to 0$ determines $x \in \mathbf{P}_L$, then E is not semistable (resp. semistable but not stable) if and only if $x \in \overline{D}$ with $\deg(D) < d/2$ (resp. $\deg(D) = d/2$).

Proof. As in Observation 1, we see that E is not semistable if and only if there is a quotient $E \to \mathcal{O}_C(D) \to 0$ with $\deg(D) < d/2$, and $\alpha\colon \mathcal{O}_C \to \mathcal{O}_C(D)$ is not the zero map. But via the identification $\operatorname{Ext}^1(L, \mathcal{O}_C) \cong H^0(C, M)^*$, this exactly means that $x \in \overline{D}$.

Let $k(L)$ be the largest integer less than $d/2$. We have just seen that $\operatorname{Sec}^{k(L)-1}(C) \subset \mathbf{P}_L$ is precisely the locus of extensions which determine bundles that are not semistable. In addition, though, if L is effective, then $k(L)$ has the property that M separates $2k(L)$ points, so, as mentioned above, we can define $\widetilde{P}_L = bl_{k(L)}(\mathbf{P}_L)$. In addition, let $U_L = \mathbf{P}_L - \operatorname{Sec}^{k(L)-1}(C)$, so, by Corollary 2.4(b), $\widetilde{P}_L = U_L \cup E_0 \cup \cdots \cup E_{k(L)-1}$, where the E_i are all smooth divisors, intersecting transversely.

Using Corollary 2.5(b), we can construct the map $\Phi_L\colon \widetilde{P}_L \to \mathcal{M}_{2,L}$ as follows:

(0) If $k(L) = 0$ (i.e., $\deg(L) = 1$ or 2), then by Observation 1, ϕ_L is defined on all of $\mathbf{P}_L$, and $\widetilde{P}_L = \mathbf{P}_L$, so let $\Phi_L = \phi_L$.

(k) If $k(L) = k$ and $y \in U_L$, let $\Phi_L(y) = \phi_L(y)$ as in (0). Otherwise, let i be minimal so that $y \in E_i$. Then, by definition, $y \in E_i^0$, so $y \in bl_{k-i-1}(\mathbf{P}_{L(-2D_x)})$ for a unique $x \in U^i(M)$. (Notation and result are from

Corollary 2.5(b).) But $k - i - 1 = k(L(-2D_x))$, so $bl_{k-i-1}(\mathbf{P}_{L(-2D_x)}) = \widetilde{P}_{L(-2D_x)}$, and, by induction, we let $\Phi_L(y) = \Phi_{L(-2D_x)}(y) \otimes \mathscr{O}_C(D_x)$.

This map is well defined everywhere on $\widetilde{P}_L$ and satisfies (2) of Theorem 1 by definition. Thus, to complete the proof of Theorem 1, we only need to prove:

Proposition 3.2. *There is a rank-2 vector bundle $\widetilde{\mathscr{E}}_L$ on $C \times \widetilde{P}_L$ with the property that for each $y \in \widetilde{P}_L$, $\widetilde{\mathscr{E}}_L|_{C\times\{y\}} \cong \Phi_L(y)$. In particular, by the universal property of $\mathscr{M}_{2,L}$, Φ_L is a morphism.*

As usual, we start the proof with a construction living on the secant bundles. Recall the universal divisor $C \times C_k = \mathscr{D}_{k+1} \subset C \times C_{k+1}$. Let $A_k = \widetilde{B}^k(M) - U^k(M) = \bigcup E_{i,k}$ and let $\tilde{\varepsilon}\colon \widetilde{B}^k(M) \to B^k(M)$ be the (multiple) blow-down.

Let π_C, $\pi_{\widetilde{B}^k(M)}$, and ρ_{k+1} be the natural maps in the following diagram:

$$\begin{tikzcd} C \times \widetilde{B}^k(M) \arrow[rr, "\rho_{k+1}"] \arrow[d, "\pi_C"'] \arrow[dr, "\pi_{\widetilde{B}^k(M)}"] & & C \times C_{k+1} \\ C & \widetilde{B}^k(M) & \end{tikzcd}$$

(**Note.** ρ_{k+1} is not, in general, a smooth map.)

Finally, let $\mathscr{O}_{\widetilde{B}^k(M)}(1) = \tilde{\varepsilon}^*\mathscr{O}_{B^k(M)}(1)$.

Definition. We define the following line bundle on $C \times \widetilde{B}^k(M)$:

$$\mathscr{L}_L^k := \pi^*_{\widetilde{B}^k(M)}(\mathscr{O}(1) \otimes \mathscr{O}(-A_k)) \otimes \rho^*_{k+1}\mathscr{O}(\mathscr{D}_{k+1}).$$

Example. The fiber Γ of $\widetilde{B}^2(M)$ over $p + q + r$ is isomorphic to $\mathbf{P}^2$ blown up at three points. Let E_p, E_q, and E_r denote the three exceptional divisors (see Figure 7).

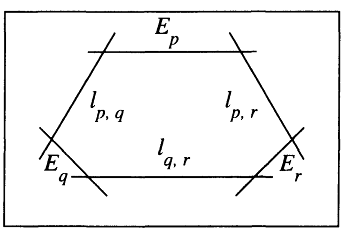

FIGURE 7

If we restrict $\mathscr{L}_L^2$ to Γ, then we get

$$\begin{aligned}\mathscr{L}_L^2|_{C\times\Gamma} &\cong \pi_C^*\mathscr{O}(p+q+r)\\ &\quad\otimes \pi_\Gamma^*(\mathscr{O}(1)\otimes\mathscr{O}(-(E_p+E_q+E_r)-(l_{p,q}+l_{p,r}+l_{q,r}))).\end{aligned}$$

Now recall from Corollary 2.5(a) that $E_{i,k}^0\subset\widetilde{B}^k(M)$ are fibered over $U^i(M)$ with fibers $\widetilde{B}^{k-i-1}(M(-2D_x))$ over each $x\in U^i(M)$. Consider

$$\widetilde{B}^k(M)-U^k(M)=\bigcup_{i=0}^{k-1}E_{i,k}^0=\bigcup_{i=0}^{k-1}\bigcup_{x\in U^i(M)}\widetilde{B}^{k-i-1}(M(-2D_x)),$$

where all the unions are disjoint.

Lemma 3.3. $\mathscr{L}_L^k|_{C\times\widetilde{B}^{k-i-1}(M(-2D_x))}\cong\mathscr{L}_{L(-2D_x)}^{k-i-1}\otimes\pi_C^*\mathscr{O}(D_x)$.

Remarks. (1) In the example above, we identify E_p with the fiber $\Gamma_{q,r}$ of $\widetilde{B}^1(M(-2p))$ over $p+q$. Then

$$\begin{aligned}\mathscr{L}_L^2|_{C\times E_p} &\cong \pi_C^*\mathscr{O}(p+q_{p+q+r})\otimes\pi_{E_p}^*(\mathscr{O}_{E_p}\otimes\mathscr{O}_{E_p}(-E_p-l_{p,q}-l_{p,r}))\\ &\cong \pi_C^*(\mathscr{O}(p)\otimes\mathscr{O}(q+r))\otimes\pi_{\Gamma_{q,r}}^*(\mathscr{O}(1)\otimes\mathscr{O}(-q-r))\\ &\cong \mathscr{L}_{L(-2p)}^1|_{C\times\Gamma_{q,r}}\otimes\pi_C^*\mathscr{O}(p).\end{aligned}$$

(2) This lemma tells us that $\mathscr{L}_L^k$ has the same sort of recursive property which we want $\widetilde{\mathscr{E}}_L$ to have. (Compare the lemma with the definition of Φ_L.)

Proof of the lemma. The proof works exactly as the example did. Let $\widetilde{B}=\widetilde{B}^{k-i-1}(M(-2D_x))$. Then we observe that $\pi_C^*\mathscr{O}_C|_{C\times\widetilde{B}}\cong\pi_C^*\mathscr{O}_C$. Since $\widetilde{B}=\widetilde{B}^{k-i-1}(M(-2D_x))$, we observe that $\pi_C^*\mathscr{O}_C|_{C\times\widetilde{B}}\cong\pi_C^*\mathscr{O}_C$. Further, from the inclusions $C\times\{x\}\times C_{k-i}\subset C\times U^i(M)\times C_{k-i}\subset C\times B^k(M)$, it follows that $\rho_{k+1}^*\mathscr{O}(\mathscr{D}_{k+1})|_{C\times\widetilde{B}}\cong\rho_{k-i}^*\mathscr{O}(\mathscr{D}_{k-i})\otimes\pi_C^*\mathscr{O}(D_x)$. Next, because $\mathscr{O}_{B^k(M)}(1)|_{\{x\}\times C_{k-i}}$ is trivial by Lemma 1.1(a), $\mathscr{O}_{\widetilde{B}^k(M)}(1)|_{\widetilde{B}}$ is trivial, and finally, by the proof of Corollary 2.4(a), we get

$$\mathscr{O}_{\widetilde{B}^k(M)}(-E_{j,k})|_{\widetilde{B}}\cong\begin{cases}\mathscr{O}_{\widetilde{B}} & \text{if } j<i,\\ \mathscr{O}_{\widetilde{B}}(1) & \text{if } j=i,\\ \mathscr{O}_{\widetilde{B}}(-E_{j-i-1,k-i-1}) & \text{if } j>i.\end{cases}$$

From all this, it follows that

$$\begin{aligned}\mathscr{L}_L^k|_{C\times\widetilde{B}} &\cong \pi_{\widetilde{B}}^*(\mathscr{O}(1)\otimes\mathscr{O}(-A_{k-i-1}))\otimes\rho_{k-i}^*(\mathscr{O}(\mathscr{D}_{k-i}))\otimes\pi_C^*(\mathscr{O}(D_x))\\ &\cong \mathscr{L}_{L(-2D_x)}^{k-i-1}\otimes\pi_C^*(\mathscr{O}(D_x)),\end{aligned}$$

as desired. q.e.d.

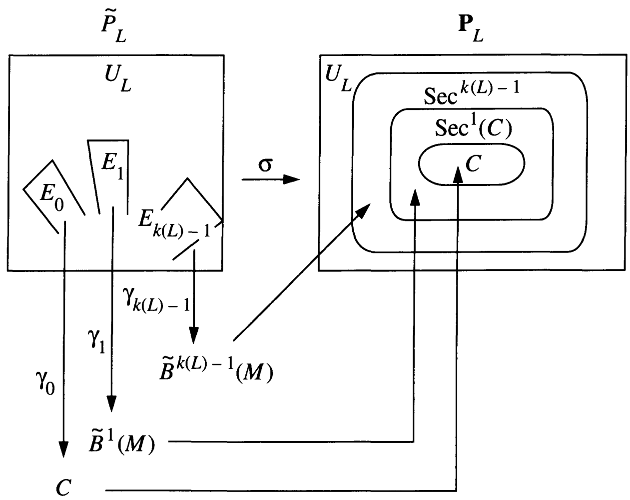

FIGURE 8

Now, recall that $U_L = \widetilde{P}_L - (\bigcup_{m=0}^{k(L)-1} E_m)$. If $\sigma: \widetilde{P}_L \to \mathbf{P}_L$ is the multiple blow-down, then σ factors through $\widetilde{B}^m(M)$ when restricted to E_m. Let $\gamma_m: E_m \to \widetilde{B}^m(M)$ be the induced maps shown in Figure 8.

Abusing notation, we will let $\mathscr{L}_L^m$ denote not only the line bundle on $C \times \widetilde{B}^m(M)$ defined earlier, but also its pullback to the smooth divisors $C \times E_m$. Thus, any surjective map from a vector bundle $\mathscr{E}$ of rank n on $C \times \widetilde{P}_L$ to $\mathscr{L}_L^m$ yields a kernel which is also locally free of rank n.

This observation motivates the following inductive definitions and claims which start with the Poincaré bundle from Definition 3.1 and end by proving Proposition 3.2.

Definition-Claim 3.4. (0) Let $\mathscr{E}_L^0 = (1, \sigma)^*\mathscr{E}_L$. Then

$$\mathscr{E}_L^0|_{C\times\{x\}} \cong \Phi_L(x) \quad \text{for all } x \in \widetilde{P}_L - \left(\bigcup_{m=0}^{k(L)-1} E_m\right).$$

(k) If $0 < k \le k(L)$, there is a natural surjective map of sheaves on $C \times \widetilde{P}_L$ from $\mathscr{E}_L^{k-1} \to \mathscr{L}_L^{k-1}$. Let $\mathscr{E}_L^k$ be the kernel of this map. Then

$$\mathscr{E}_L^k|_{C\times\{x\}} \cong \Phi_L(x) \quad \text{for all } x \in \widetilde{P}_L - \left(\bigcup_{m=k}^{k(L)-1} E_m\right).$$

Note that this sequence of claims and definitions parallels the sequence of claims and definitions implicit in Proposition 2.3. Just as in that case, we rely on a fundamental lemma, analogous to the Terracini lemmas of §1.

Suppose $x \in \operatorname{Sec}^m(C) - \operatorname{Sec}^{m-1}(C)$, $m < k(L)$. Then, by Terracini, we may regard $x \in U^m(M)$, and $\overline{2D_x} \subset \mathbf{P}_L$ is the (projective) tangent plane to $\operatorname{Sec}^m(C)$ at x. By the proof of Observation 2, there is a natural lift λ of bundles on $C \times \{x\}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O} \arrow[r] \arrow[d] & E_x \arrow[r] \arrow[dl, "\lambda"] & L \arrow[r] & 0 \\ & \mathscr{O}(D_x) & & & \end{tikzcd}$$

Let $H \subset \mathbf{P}_L$ be a plane of codimension $2m+1$, meeting $\operatorname{Sec}^m(C)$ transversely at x. Let $\varepsilon\colon \widetilde{H} \to H$ be the blow-up at x, and let $E_H \subset \widetilde{H}$ be the exceptional divisor. Let $\mathscr{E}_H = \mathscr{E}_{\mathbf{P}_L}|_{C\times H}$ and let $\mathscr{E}^0_{\widetilde{H}}$ be the pullback to $C \times \widetilde{H}$. Then we construct a new vector bundle $\mathscr{E}^1_{\widetilde{H}}$ via:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}^1_{\widetilde{H}} \arrow[r] & \mathscr{E}^0_{\widetilde{H}} \arrow[r] \arrow[d, "\rho"] & (1,\varepsilon)^*\mathscr{O}_{C\times\{x\}}(D_x) \arrow[r] & 0 \\ & & (1,\varepsilon)^*E_x \arrow[ur, "(1,\varepsilon)^*\lambda"'] & & \end{tikzcd}$$

where ρ is the restriction map to E_H.

Lemma 3.5. *The natural identification* $E_H = \mathbf{P}_{L(-2D_x)}$, *obtained by projecting, identifies* $\mathscr{E}^1_{\widetilde{H}}|_{C\times EH}$ *with* $\mathscr{E}_{L(-2D_x)} \otimes \pi_C^*\mathscr{O}(D_x)$.

Proof. We construct a new vector bundle $\mathscr{F}^0_{\widetilde{H}}$ on $C \times \widetilde{H}$ by pushing forward the extension which gives $\mathscr{E}^0_{\widetilde{H}}$:

$$\begin{tikzcd} 0 \arrow[r] & \pi_{\widetilde{H}}^*(1) \arrow[r] \arrow[d] & \mathscr{E}^0_{\widetilde{H}} \arrow[r] \arrow[d] & \pi_C^*L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \pi_C^*\mathscr{O}(D_x) \otimes \pi_{\widetilde{H}}^*\mathscr{O}(1) \arrow[r] & \mathscr{F}^0_{\widetilde{H}} \arrow[r] & \pi_C^*L \arrow[r] & 0 \end{tikzcd}$$

The extension giving $\mathscr{F}^0_{\widetilde{H}}$ splits when restricted to $C \times E_H$, and we form $\mathscr{F}^1_{\widetilde{H}}$ as the kernel

$$0 \to \mathscr{F}^1_{\widetilde{H}} \to \mathscr{F}^0_{\widetilde{H}} \to (1,\varepsilon)^*\mathscr{O}_{C\times\{x\}}(D_x) \to 0.$$

But $\mathscr{F}^1_{\widetilde{H}}$ can also be thought of as the pullback of extensions, dividing by the equation for $C \times E_H$:

$$\begin{array}{ccccccc}
(**): 0 \to & \pi_C^*\mathscr{O}(D_x) \otimes \pi_{\widetilde{H}}^*(\mathscr{O}(1) \otimes \mathscr{O}(-E_H)) & \to & \mathscr{F}_{\widetilde{H}}^1 & \to & \pi_C^* L & \to 0 \\
& \downarrow & & \downarrow & & \| & \\
0 \to & \pi_C^*\mathscr{O}(D_x) \otimes \pi_{\widetilde{H}}^*\mathscr{O}(1) & \to & \mathscr{F}_{\widetilde{H}}^0 & \to & \pi_C^* L & \to 0
\end{array}$$

Now, $\mathscr{E}_{\widetilde{H}}^0$, $\mathscr{E}_{\widetilde{H}}^1$, and $\mathscr{F}_{\widetilde{H}}^1$ all fit into the following diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
0 \to & \mathscr{E}_{\widetilde{H}}^1 & \to & \mathscr{E}_{\widetilde{H}}^0 & \to & (1, \varepsilon)^*\mathscr{O}(D_x) & \to 0 \\
& \downarrow & & \downarrow & & \| & \\
0 \to & \mathscr{F}_{\widetilde{H}}^1 & \to & \mathscr{F}_{\widetilde{H}}^0 & \to & (1, \varepsilon)^*\mathscr{O}(D_x) & \to 0 \\
& \downarrow & & \downarrow & & & \\
& \pi_{\widetilde{H}}^*\mathscr{O}(1)|_{C_x \times \widetilde{H}} & = & \pi_{\widetilde{H}}^*\mathscr{O}(1)|_{D_x \times \widetilde{H}} & & & \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

from which it follows that $\mathscr{E}_{\widetilde{H}}^1|_{C \times E_H}$ fits into the following pullback of extensions:

$$\begin{array}{ccccccc}
(**): 0 \to & \pi_C^*\mathscr{O}(D_x) \otimes \pi_{E_H}^*\mathscr{O}(1) & \to & \mathscr{E}_{\widetilde{H}}^1|_{C \times E_H} & \to & \pi_C^* L(-D_x) & \to 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \to & \pi_C^*\mathscr{O}(D_x) \otimes \pi_{E_H}^*\mathscr{O}(1) & \to & \mathscr{F}_{\widetilde{H}}^1|_{C \times E_H} & \to & \pi_C^* L & \to 0
\end{array}$$

But this extension is the twist by $\pi_C^*\mathscr{O}(D_x)$ of the extension giving $\mathscr{E}_{L(-2D_x)}$, as is easily checked, and the lemma is proved. q.e.d.

Now, we prove Claim 3.4 by induction on k. As in the proof of Proposition 2.2, we will demonstrate the first few cases first to illustrate the general proof.

Case $k = 0$. This is covered by Observation 2 and Definition 3.1.

Case $k = 1$. We need to prove:

(1a) There is a natural surjective map $\mathscr{E}_L^0 \to \mathscr{L}_L^0$.

(1b) The kernel $\mathscr{E}_L^1$ satisfies $\mathscr{E}_L^1|_{C\times\{x\}} \cong \Phi_L(x)$ for all $x \in E_0 - \bigcup_{m>0} E_m$.

Proof. (1a) The surjective map is the pullback from $C \times \mathbf{P}_L$ of the lift Λ_0 below:

$$\begin{tikzcd}
(*): 0 \arrow[r] & \pi_{B^0(M)}^*\mathscr{O}(1) \arrow[r] \arrow[d, "\Delta"] & \mathscr{E}_L|_{C\times B^0(M)} \arrow[r] \arrow[dl, "\Lambda_0"] & \pi_C^* L \arrow[r] & 0 \\
& \mathscr{L}_L^0 & & &
\end{tikzcd}$$

From the exact sequence

$$\begin{aligned} H^0(\Delta, \mathscr{O}_\Delta) &\to H^1(C \times B^0(M), \pi_C^* L^* \otimes \pi_{B^0(M)}^*\mathscr{O}(1)) \\ &\to H^1(C \times B^0(M), \pi_C^* L^* \otimes \mathscr{L}_L^0) \end{aligned}$$

and the fact that $(*)$ is by definition the image of the first map, it follows that Λ_0 exists (and is easily seen to be unique), and moreover, $\lambda_0|_{C\times\{x\}}: E_x \to \mathscr{O}(D_x)$ is the map λ in the remarks leading up to Lemma 3.5, hence Λ_0 is surjective, since each λ is. It is important to remark that since Λ_0 is defined on $C \times \mathbf{P}_L$, it follows that $\mathscr{E}_L^1$ is the pullback of a bundle on $C \times bl_1(\mathbf{P}_L)$ defined in the same way. Abusing notation, we will refer to that bundle, too, as $\mathscr{E}_L^1$.

(1b) Pick a hyperplane H transverse to C at p, and apply Lemma 3.5. The restriction of the exact sequence of sheaves on $C \times bl_1(\mathbf{P}_L)$,

$$0 \to \mathscr{E}_L^1 \to \mathscr{E}_L^0 \to \mathscr{L}_L^0 \to 0,$$

remains exact when restricted to $C \times \widetilde{H}$, so $\mathscr{E}_L^1|_{C\times\varepsilon^{-1}(p)} \cong \mathscr{E}_{\widetilde{H}}^1|_{C\times E_H}$ is naturally identified with $\mathscr{E}_{L(-2p)}^0 \otimes \pi_C^*\mathscr{O}(p)$ by the lemma.

By the proof of Proposition 2.3, we may regard $U_{L(-2p)} \subset \mathbf{P}_{L(-2p)} = \varepsilon^{-1}(p)$ as a subset of $E_0 - \bigcup_{m>0} E_m$. Indeed, it follows from the proof that $E_0 - \bigcup_{m>0} E_m = \bigcup_{p\in C} U_{L(-2p)}$. But now, Case (1b) follows from the case $k = 0$ since $\mathscr{E}_L^1|_{C\times U_{L(-2p)}} \cong \mathscr{E}_{L(-2p)}^0|_{C\times U_{L(-2p)}} \otimes \pi_C^*\mathscr{O}(p)$.

Case $k = 2$. Again, we need to show:

(2a) There is a natural surjective map $\mathscr{E}_L^1 \to \mathscr{L}_L^1$.

(2b) The kernel $\mathscr{E}_L^2$ satisfies $\mathscr{E}_L^2|_{C\times\{y\}} \cong \Phi(y)$ for all $y \in E_1 - \bigcup_{m>1} E_m$.

Proof of (2a). We define the lift Λ_1 on $C \times \widetilde{B}^1(M)$ as before:

$$\begin{tikzcd}
(*): 0 \arrow[r] & \pi^*_{\widetilde{B}^1(M)}\mathscr{O}(1) \arrow[r] \arrow[d, "\mathscr{D}_1"] & \mathscr{E}^0_L|_{C\times\widetilde{B}^1(M)} \arrow[r] \arrow[dl, "\Lambda_1"] & \pi^*_C L \arrow[r] & 0 \\
& \mathscr{L}^1_L(A_1) & & &
\end{tikzcd}$$

For $x \in U^1(M)$, the restriction $\Lambda_1|_{C\times\{x\}}: E_x \to \mathscr{O}(D_x)$ is surjective. But if $x \in A_1 \cong C \times C$, say $x = (p, q)$, then $\Lambda_1|_{C\times\{x\}}: E_x \to \mathscr{O}(p+q)$ factors through $\mathscr{O}(p)$, and so Λ_1 fails to be surjective at points of the form $(q, p, q) \in C \times A_1$. However, Λ_1 lifts to a map $\widetilde{\Lambda}_1$:

$$\begin{tikzcd}
\mathscr{E}^1_L|_{C\times\widetilde{B}^1(M)} \arrow[r, "\widetilde{\Lambda}_1"] \arrow[d, "\psi_1"] & \mathscr{L}^1_L \arrow[d, "C\times A_1"] \\
\mathscr{E}^0_L|_{C\times\widetilde{B}^1(M)} \arrow[r, "\Lambda_1"] & \mathscr{L}^1_L(A_1)
\end{tikzcd}$$

To see this, we only need to show that $\Lambda_1 \circ \psi_1$ is zero, when restricted to $C \times A_1$. But $\rho_2^{-1}(\mathscr{D}_1) \cap (C \times A_1) = \Delta_{1,2} \cup \Delta_{1,3}$ where $\Delta_{i,j}$ is the i, j diagonal in $C \times C \times C$, and $\Lambda_1|_{C\times A_1}$ factors into $(\Lambda_0, 1)$ followed by multiplication by $C \times \Delta_{1,3}$. By construction of $\mathscr{E}^1_L$, it follows that $\Lambda_1|_{C\times A_1} \circ (\Lambda_0, 1) = 0$.

Finally, we need to show that $\widetilde{\Lambda}_1$ is surjective. We only need to consider the restriction to $C \times A_1$. But, by Lemma 3.3, if we consider the natural projection $\varepsilon_1: A_1 \to C$, then $\mathscr{L}^1_L|_{C\times\varepsilon_1^{-1}(p)} \cong \mathscr{L}^0_{L(-2p)} \otimes \pi^*_C\mathscr{O}(p)$, and by Case $k = 1$, $\mathscr{E}^1_L|_{C\times\varepsilon_1^{-1}(p)} \cong \mathscr{E}^0_{L(-2p)} \otimes \pi^*_C\mathscr{O}(p)$. Finally, the restriction of $\widetilde{\Lambda}_1$ to $C \times \varepsilon_1^{-1}(p)$ is just Λ_0, and hence is surjective by Case $k = 1$.

(2b) If $x \in \mathrm{Sec}^1(C) - C$, apply the reasoning of Case $k = 1$ to a plane transverse to $\mathrm{Sec}^1(C)$ at x to conclude that if $\varepsilon_2: bl_2(\mathbf{P}_L) \to \mathbf{P}_L$, then $\mathscr{E}^0_L|_{C\times\varepsilon_2^{-1}(x)} \cong \mathscr{E}^0_{L(-2D_x)} \otimes \mathscr{O}(D_x)$. So we only have to worry about $y \in (E_0 \cap E_1) - \bigcup_{m>1} E_m$.

For this, we observe that by (2a), $\mathscr{E}^2_L|_{C\times\varepsilon_2^{-1}(p)} \cong \mathscr{E}^1_{L(-2p)} \otimes \pi^*_C\mathscr{O}(p)$, so the result follows by Case $k = 1$ applied to the fibers.

General Case. We prove the two statements:

(a) There is a natural surjection $\widetilde{\Lambda}_{k-1}: \mathscr{E}^{k-1}_L \to \mathscr{L}^{k-1}_L$.

(b) The kernel $\mathscr{E}^k_L$ restricts to $\Phi(y)$ for each $y \in E_{k-1} - \bigcup_{m\geq k} E_m$.

Proof. (a) We first remark that $\mathscr{E}_L^{k-1}$ is pulled back from the bundle defined the same way on $bl_{k-1}(\mathbf{P}_L)$. Then we construct the lift:

$$\begin{tikzcd} (*)\colon 0 \arrow[r] & \pi^*_{\widetilde{B}^{k-1}(M)}\mathscr{O}(1) \arrow[r] \arrow[d, "\mathscr{D}_{k-1}"] & \mathscr{E}_L^0|_{C\times\widetilde{B}^{k-1}(M)} \arrow[r] \arrow[dl, "\Lambda_{k-1}"] & \pi_C^* L \arrow[r] & 0 \\ & \mathscr{L}_L^{k-1}(A_{k-1}) & & & \end{tikzcd}$$

Then, exactly as in Case 2, we lift Λ_{k-1} to $\widetilde{\Lambda}_{k-1}$:

$$\begin{tikzcd} \mathscr{E}_L^{k-1}|_{C\times\widetilde{B}^{k-1}(M)} \arrow[r, "\widetilde{\Lambda}_{k-1}"] \arrow[d, "\psi_{k-1}"] & \mathscr{L}_L^{k-1} \arrow[d, "C\times A_{k-1}"] \\ \mathscr{E}_L^0|_{C\times\widetilde{B}^{k-1}(M)} \arrow[r, "\Lambda_{k-1}"] & \mathscr{L}_L^{k-1}(A_{k-1}) \end{tikzcd}$$

This time, A_{k-1} consists of $k-1$ components, but ψ_{k-1} is a composition of $k-1$ maps, and each of the factors in the composition $\psi_{k-1}\circ\Lambda_{k-1}$ vanishes on the corresponding component of A_{k-1}.

Finally, $\widetilde{\Lambda}_{k-1}$ is directly seen to be surjective on $C\times U^{k-1}(M)$, while on $C\times A_{k-1}$, one reasons by induction. We conclude in particular that if $x\in U^j(M)$ $(j<k-1)$, then the newly constructed kernel satisfies:

$$(*) \qquad \mathscr{E}_L^k|_{\varepsilon_k^{-1}(x)} \cong \mathscr{E}^{k-j-1}_{L(-2D_x)} \otimes \pi_C^*\mathscr{O}(D_x).$$

(b) If $x\in \operatorname{Sec}^{k-1}(C)-\operatorname{Sec}^{k-2}(C)$, on applies Lemma 3.5 to a transverse plane to conclude that $\mathscr{E}_L^k|_{C\times\varepsilon_k^{-1}(x)} \cong \mathscr{E}^0_{L(-2D_x)}\otimes\pi_C^*\mathscr{O}(D_x)$. Thus, $\mathscr{E}_L^k$ agrees with Φ_L on $C\times\{y\}$ for all $y\in E^0_{k-1}-\bigcup_{m\geq k}E_m$. But by $(*)$ above and induction, $\mathscr{E}_L^k$ agrees with Φ_L on $C\times\{y\}$ for all $y\in(\bigcup_{j<k-1}E_j)-(\bigcup_{m>k-1}E_m)$, which concludes the proof.

4. Fibers of the extension map and applications

We start this section by proving that the fibers of the extension map Φ_L are connected. By the projection formula, this will imply that as soon as $\deg(L)\geq 2g-1$, there is a canonical identification $H^0(\mathscr{M}_{2,L}, \mathscr{O}(k\Theta)) \cong H^0(\widetilde{P}_L, \Phi_L^*\mathscr{O}(k\Theta))$. We will then use Theorem 1 to calculate $\Phi_L^*\mathscr{O}(k\Theta)$ from which we will obtain Theorem 2 and the additional results mentioned in the introduction.

Lemma 4.1 (stable fibers). *Suppose* $E \in \mathscr{M}^{\mathrm{stab}}_{2,L}$. *Then there is a natural rational map* $e\colon \mathbf{P}(H^0(C, E)^*) \to \mathbf{P}_L$ *satisfying the following*:

(a) *The domain* $\operatorname{dom}(e) \supseteq \{\alpha \in H^0(C, E) | \deg(Z(\alpha)) \leq 1\}$.

(b) *The map* e *is injective on* $U = \{\alpha\colon \mathscr{O}_C \hookrightarrow E\}$, *and the image* $e(U) = \phi_L^{-1}(E)$.

(c) *Suppose there is an* $\alpha\colon \mathscr{O}_C \hookrightarrow E$. *For each* $D \subset C$, *if there is an* $\alpha \in H^0(C, E)$ *with* $Z(\alpha) = D$, *then* $\overline{D} \cap \overline{\phi_L^{-1}(E)} \neq \varnothing$.

Proof. We start by defining e. By linearity of $\otimes$ and the functor H^1, we get a linear map

$$\lambda\colon H^0(C, E) \to \operatorname{Hom}(H^1(C, L^*), H^1(C, L^* \otimes E)).$$

If $\alpha \in U$, then there is some short exact sequence $(*)\colon 0 \to \mathscr{O}_C \to E \to L \to 0$, so the long exact sequence on the cohomology of $(*) \otimes L^*$ yields

$$\cdots \to H^0(C, \mathscr{O}) \xrightarrow{\delta} H^1(C, L^*) \xrightarrow{\lambda(\alpha)} H^1(C, L^* \otimes E) \to \cdots.$$

Since E is stable, $\delta \neq 0$, so $\dim(\ker(\lambda(\alpha))) = 1$.

On the other hand, suppose $\alpha \notin U$. Let $D = Z(\alpha)$, and let $\alpha'\colon \mathscr{O}_C(D) \hookrightarrow E$ be the induced map. Then $\lambda(\alpha)$ factors:

$$\lambda(\alpha)\colon H^1(C, L^*) \to H^1(C, L^*(D)) \xrightarrow{\lambda(\alpha')} H^1(C, L^* \otimes E).$$

But from the long exact sequences on cohomology associated to $0 \to \mathscr{O}_C(D) \to E \to L(-D) \to 0$ and $0 \to \mathscr{O}_C \to \mathscr{O}_C(D) \to \mathscr{O}_D(D) \to 0$, one immediately checks that $\lambda(\alpha')$ is injective and so, by Serre duality, $\ker(\lambda(\alpha)) \cong H^0(C, M \otimes \mathscr{O}_D)^*$. Notice that if $\deg(D) = 1$, then this is one-dimensional.

Let $X = \{\psi \in \mathbf{P}(\operatorname{Hom}(H^1(C, L^*), H^1(C, L^* \otimes E))^*) | \operatorname{rank}(\psi) < r\}$, where $r = h^1(C, L^*)$. Then the natural rational map

$$k\colon X \to \mathbf{P}(H^1(C, L^*)^*) = \mathbf{P}_L, \quad \psi \to \ker(\psi),$$

is defined on $\{\psi \in X | \operatorname{rank}(\psi) = r - 1\}$. Further, there is a universal blow-up $\widetilde{X}$ of X resolving k. Set-theoretically, $\widetilde{X} = \{(\psi, \Lambda) | \dim(\Lambda) = 1, \Lambda \subset \ker(\psi)\}$, and the induced morphism $\tilde{k}\colon \widetilde{X} \to \mathbf{P}_L$ maps (ψ, λ) to Λ.

Now, by the analysis above and a dimension count, λ is injective and defines a morphism $\overline{\lambda}\colon \mathbf{P}(H^0(C, E)^*) \to X$. Let $e = k \circ \overline{\lambda}$. From the description of k, we get (a).

Suppose $\alpha, \alpha'\colon \mathscr{O}_C \hookrightarrow E$ and $e(\alpha) = e(\alpha')$. Then by definition, there are short exact sequences $(*)$ and $(**)$ containing α and α' and defining the same extension class $e(\alpha) \in \mathbf{P}_L$. But since E is stable, $\mathrm{Aut}(E) = \mathbf{C}^*$, and it follows that $\alpha = \alpha' \in \mathbf{P}(H^0(C, E)^*)$. This gives the first part of (b). For the second part, if $x \in \phi_L^{-1}(E)$, then x comes from some exact sequence $(*)$: $0 \to \mathscr{O}_C \xrightarrow{\alpha} E \to L \to 0$, and, by definition, $e(\alpha) = x$.

Finally, for (c), we consider the strict transform $\widetilde{Y}$ of the image of $\overline{\lambda}$ in $\widetilde{X}$. This is an irreducible variety mapping to $\mathbf{P}_L$. Further, if $\alpha \in H^0(C, E)$ with $Z(\alpha) = D$, then $\ker(\overline{\lambda}(\alpha)) \cong H^0(C, M \otimes \mathscr{O}_D)^*$, so there is a $y \in \widetilde{Y}$ lying over $\overline{\lambda}(\alpha)$, and y gets sent to a point in $\overline{D}$. This proves (c). q.e.d.

As an immediate corollary, we obtain:

Corollary 4.2. *If* $\deg(L) = 1$, *then* ϕ_L *is injective.*

Proof. If $\det(E) \cong L$ and $h^0(C, E) > 1$, then there is an $\alpha \in H^0(C, E)$ with $Z(\alpha) \neq \varnothing$ and E is not stable, So $h^0(C, E) = 1$ for all stable E, and the corollary follows from the lemma. q.e.d.

In contrast to the previous lemma, we have:

Lemma 4.3 (Semistable fibers). *Suppose that* $\deg(L) \geq 2$, *and* $[E] \in \mathscr{M}_{2,L}$ *does not represent a stable point, so that there is an extension* $0 \to L_1 \to E \to L_2 \to 0$ *with* $\deg(L_1) = \deg(L_2)$ *and* $L_1 \otimes L_2 \cong L$. *Then there are three possibilities for the fiber* $\phi_L^{-1}([E])$. *Either:*

(a) $H^0(C, L_1) = H^0(C, L_2) = 0$, *in which case* $\phi_L^{-1}([E]) = \varnothing$, *or*

(b) $H^0(C, L_i) \neq 0$ *for exactly one of* $i = 1, 2$, *in which case if* $L_i \cong \mathscr{O}_C(D_0)$, *then* $\phi_L^{-1}([E]) = \bigcup_{D \in |D_0|}\{U_D\}$, *or*

(c) $H^0(C, L_i) \neq 0$ *for both* $i = 1, 2$, *in which case if* $L_i \cong \mathscr{O}_C(D_i)$, *then* $\phi_L^{-1}([E]) = \bigcup_{i=1,2}\bigcup_{D \in |D_i|}\{U_D\}$.

Proof. Recall from [12] that $[E'] = [E]$ if and only if E' has either L_1 or L_2 as a quotient line bundle. Now, suppose $\phi_L^{-1}([E]) \neq \varnothing$. Then there must be some E' with $[E'] = [E]$ and some map $\alpha: \mathscr{O}_C \hookrightarrow E'$. So α induces a nonzero element of $H^0(C, L_i)$ for some i. This gives (a).

Next, by symmetry we may suppose that $H^0(C, L_1) \neq 0$ but $H^0(C, L_2) = 0$. Then $\{(*): 0 \to \mathscr{O}_C \to E' \to L \to 0\} \in \phi_L^{-1}([E])$ implies that E' has L_1 as a quotient, and (b) follows from Observation 2. Similarly, we get (c).

Corollary 4.4. *If* $\deg(L) = 2$, *then* ϕ_L *is injective.*

Proof. By the argument in Corollary 4.2, ϕ_L is injective at the stable points of $\mathbf{P}_L$. On the other hand, if E is not stable, then, by Observation 2, $x \in \phi_L^{-1}([E])$ corresponds to a point in the image of $C \to \mathbf{P}(H^0(C, L \otimes \omega_C))$, and there is an exact sequence $(*)$: $0 \to L(-p) \to E' \to \mathscr{O}_C(p) \to 0$.

Now, by Lemma 4.3, there are two possibilities. Either $H^0(C, L(-p)) = 0$ and (by (b)), $\phi_L^{-1}([E]) = \overline{p}$ or (by (c)), $L(-p) \cong \mathscr{O}_C(q)$ and $\phi_L^{-1}([E]) = \overline{p} \cup \overline{q}$. But in that case, p and q have the same image in $\mathbf{P}_L$. q.e.d.

We use Theorem 1 to prove:

Proposition 4.5. *If* $\deg(L) \geq 2g - 1$, *then* $\Phi_{L_*}\mathscr{O}_{\widetilde{P}_L} \cong \mathscr{O}_{\mathscr{M}_{2,L}}$.

Proof. Since $\mathscr{M}_{2,*}$ is always normal (see [5]), it suffices by Zariski's main theorem to show that Φ_L is onto and that the fibers $\Phi_L^{-1}([E])$ are connected for all $[E] \in \mathscr{M}_{2,L}$.

Surjectivity is an easy consequence of the theorem as follows. If $\deg(L) \geq 2g-1$, then by Riemann-Roch, $H^0(C, E) \neq 0$ for all $E \in \mathscr{M}_{2,L}$. Now, if there is an $\alpha: \mathscr{O}_C \hookrightarrow E$, then, as we have seen, $E \in \operatorname{im}(\phi_L)$. Otherwise, suppose $\alpha \in H^0(C, E)$ and α factors through $\alpha': \mathscr{O}_C(D) \hookrightarrow E$, $\deg(D) = k + 1$. Then $[E] \in \operatorname{im}(\phi_{L(-2D)} \otimes \mathscr{O}(D))$, so $[E]$ is in the image of Φ_L restricted to the fiber of E_k over any $x \in U_D$.

The same argument shows that if $\deg(L) < 2g - 1$, then $[E] \in \operatorname{im}(\Phi_L)$ if and only if $H^0(C, E) \neq 0$ (or, if E is not stable, if and only if $H^0(C, E') \neq 0$ for some $E' \sim E$).

We will prove that $\Phi_L^{-1}([E])$ is connected for all L by induction on $\deg(L)$. First note that we have already shown in Corollaries 4.2 and 4.4 that if $\deg(L) = 1$ or 2, then $\Phi_L = \phi_L$ and ϕ_L is injective, so of course the fibers are connected.

If $[E] \in \mathscr{M}_{2,L}$ with $\deg(L) > 2$, then we should think of $\Phi_L^{-1}([E])$ as a union of various "pieces." The first piece (which may be empty!) is the fiber $\phi_L^{-1}([E])$, which we have already described in the two lemmas.

To define the other pieces, suppose $\alpha \in H^0(C, E)$ and $Z(\alpha) = D$. Then, by the surjectivity argument, $(\Phi_{L(-2D')} \otimes \mathscr{O}(D'))^{-1}([E]) \neq \varnothing$ for all $D' \subset D$, so, by Theorem 1(2) and induction, $\Phi_L^{-1}([E]) \cap \sigma^{-1}(U_{D'})$ is a connected fiber bundle over $U_{D'}$ for all $D' \subset D$. But U_D is dense in $\overline{D}$, so $\Phi_L^{-1}([E]) \cap \sigma^{-1}(\overline{D})$ is connected, as well.

Thus, the decomposition $\Phi_L^{-1}([E]) = \phi_L^{-1}([E]) \cup \bigcup_{\mathscr{O}_C(D) \hookrightarrow E}(\Phi_L^{-1}([E]) \cap \sigma^{-1}(\overline{D}))$ consists of $\phi_L^{-1}([E])$ and a collection of connected sets.

We finish the proof by considering various cases.

Case 1. E is stable and $\phi_L^{-1}(E) \neq \varnothing$.

By Lemma 4.1(b), $\phi_L^{-1}(E) = e(\{\alpha: \mathscr{O}_C \hookrightarrow E\})$ so $\phi_L^{-1}(E)$ is itself connected. Moreover, by (c) of the same lemma, if $Z(\alpha) = D$, then there is an $x \in \overline{D} \cap \overline{\phi_L^{-1}(E)}$ (closure in $\mathbf{P}_L$). But this means that $\overline{\phi_L^{-1}(E)}$ (closure in $\widetilde{P}_L$) meets every $\Phi_L^{-1}(E) \cap \sigma^{-1}(\overline{D})$. Thus $\Phi_L^{-1}(E)$ is connected.

Case 2. E is stable and $\phi_L^{-1}(E) = \varnothing$.

In other words, for all $\alpha \in H^0(C, E)$, $Z(\alpha) \neq \varnothing$. But in this case, it is easy to see that there must be a (maximal) line bundle L with the property that every section α factors through L. Now unless $H^0(C, L) = 0$, in which case $\Phi_L^{-1}(E) = \varnothing$, we can write $L = \mathscr{O}_C(D_0)$, and then

$$\Phi_L^{-1}(E) = \bigcup_{D' \in |D_0|} (\Phi_L^{-1}(E) \cap \sigma^{-1}(\overline{D}')) \cup \bigcup_{\mathscr{O}_C(D) \hookrightarrow E} (\Phi_L^{-1}([E]) \cap \sigma^{-1}(\overline{D})).$$

But, just as in Case 1, the first term is connected (it maps to the connected base $|D_0|$ with connected fibers) and meets all the components of the rest.

Suppose $[E] \in \mathscr{M}_{2,L}$ and E is not stable. Then there is an exact sequence $(*)$: $0 \to L_1 \to E \to L_2 \to 0$ with $\deg(L_1) = \deg(L_2)$ and $L_1 \otimes L_2 \cong L$. Thus by the surjectivity argument, $\Phi_L^{-1}([E]) \neq \varnothing$ if and only if there is an $E' \sim E$, and $\mathscr{O}_C(D) \hookrightarrow E'$ for some $D \subset C$. But then either $H^0(C, L_1) \neq 0$ or $H^0(C, L_2) \neq 0$, so by Lemma 4.3(a), we have $\phi_L^{-1}([E]) \neq \varnothing$. This leaves only:

Case 3. $[E]$ is not stable and $\phi_L^{-1}([E]) \neq \varnothing$.

First, we observe that if Lemma 4.3(b) applies, then as in Case 1, $\phi_L^{-1}([E])$ is connected and, for all $\alpha' : \mathscr{O}_C(D') \hookrightarrow E'$ and $E' \sim E$, $\overline{\phi_L^{-1}([E])} \cap \sigma^{-1}(\overline{D}') \neq \varnothing$, so $\Phi_L^{-1}([E])$ is connected.

Next, if Lemma 4.3(c) applies, then either $\phi_L^{-1}([E]) = (\bigcup_{D \in |D_1|} U_D) \cup (\bigcup_{D' \in |D_2|} U_{D'})$ has one connected component, in which case we reason as before, or else there are two connected components. But since $L \cong \mathscr{O}_C(D_1 + D_2)$ it follows that $\overline{D}_1 \cap \overline{D}_2 \neq \varnothing$, so if $\overline{U_D \cap U_{D'}} = \varnothing$, there must be a nonempty $D_0 \subset D \cap D'$. But then $\overline{\phi_L^{-1}([E])}$ is connected through $\Phi_L^{-1}([E]) \cap \sigma^{-1}(\overline{D}_0)$, so, once again, the connectedness of all of $\Phi_L^{-1}([E])$ follows. q.e.d.

For the rest of this section, we will be particularly interested in three degrees for L:

$\deg(L) = 2g - 1$. In this case, Φ_L is surjective by Theorem 1. Since $\dim(\mathbf{P}_L) = 3g-3 = \dim(\mathscr{M}_{2,L})$, the map Φ_L is birational. Also remark by Theorem 1 that there are g exceptional divisors in $\widetilde{P}_L$, which we number $E_0, \cdots, E_{g-1}$.

$\deg(L) = 2g$. In this case Φ_L is again surjective, but this time the generic fiber is of dimension one, and again there are g exceptional divisors.

$\deg(L) = 2g-2$. Here, Φ_L maps birationally to a subvariety of codimension one, and there are $g-1$ exceptional divisors.

Using the Poincaré extension from Definition 3.1, we define two hypersurfaces in $\mathbf{P}_L$.

Definition-Claim 4.6. The variety

$$\Gamma_L := \{(*): 0 \to \mathscr{O} \to E \to L \to 0 |\ h^0(C, E) > 1\} \subset \mathbf{P}_L$$

is a hypersurface of degree g if either $\deg(L) = 2g-1$ or $L = \omega_C$.

Proof. Pushing down the Poincaré extension to $\mathbf{P}_L$, we get

$$\begin{aligned} 0 \to \mathscr{O}_{\mathbf{P}_L}(1) \to \pi_{L*}\mathscr{E}_L \to H^0(C, L) \otimes \mathscr{O}_{\mathbf{P}_L} \xrightarrow{\lambda} H^1(C, \mathscr{O}) \otimes \mathscr{O}(1) \\ \to R^1\pi_{L*}\mathscr{E}_L \to H^1(C, L) \otimes \mathscr{O}_{\mathbf{P}_L} \to 0. \end{aligned}$$

If $\deg(L) = 2g-1$ or $L = \omega_C$, then the map λ is a $g \times g$ matrix of linear forms on $\mathbf{P}_L$ which degenerates exactly along Γ_L, so $\Gamma_L \subset \mathbf{P}_L$ is the hypersurface of degree g defined by $\det(\lambda)$.

Example. If $g = 2$, then $\omega_C^{\otimes 2}$ maps C to a conic in $\mathbf{P}^2$. On the other hand, if $\deg(L) = 3$, then $\omega_C \otimes L$ embeds C as a curve of degree 5 in $\mathbf{P}^3$ and the image is contained in a unique quadric surface.

As a first application of Theorem 1, we obtain a special case of the theorem of [5] mentioned in the Introduction. The result in this form is also proved, for example, in Ramanan [10].

Application 1. $\mathrm{Pic}(\mathscr{M}_{2,L}) \cong \mathbf{Z}$ if $\deg(L)$ is odd.

Proof. Since $\Phi_L: \widetilde{P}_L \to \mathscr{M}_{2,L}$ is birational for $\deg(L) = 2g-1$, it follows that the map on Chow groups $\Phi_*: CH^1(\widetilde{P}_L) \to CH^1(\mathscr{M}_{2,L})$ is surjective. Further, by Theorem 1(1), $\mathrm{Pic}(\widetilde{P}_L) \cong \mathbf{Z}H \oplus (\bigoplus_{k=0}^{g-1} \mathbf{Z}E_k)$, where H is the pullback of the hyperplane class from $\mathbf{P}_L$. Now,

(i) $\Phi_{L*}E_k = 0$ if $k > 0$.

By Theorem 1(2), E_k dominates $\mathrm{Sec}^k(C)$ and is a fiber bundle over $U_D \subset \overline{D}$ for each D of degree $k+1$. Thus, the fibers of the induced map $\Phi_L: E_k \to \mathscr{M}_{2,L}$ have dimension at least k, so, if $k > 0$, then the codimension of the image is at least 2, and we get (i).

(ii) Let $\widetilde{\Gamma}_L$ be the strict transform of Γ_L in $\widetilde{P}_L$. Then $\widetilde{\Gamma}_L \sim gH - (g-1)E_0 - \cdots - E_{g-1}$ and $\Phi_{L*}\widetilde{\Gamma}_L = 0$.

The second part is as in (i) since by Lemma 4.1 and the definition of $\widetilde{\Gamma}_L$, the fibers of the map to $\mathscr{M}_{2,L}$ are positive dimensional.

Let $D \subset C$ be a general divisor of degree $k+1 \leq g$. Then because of Theorem 1(4), for each $x \in U_D$, there is an equality:

$$\widetilde{\Gamma}_L \cap \sigma^{-1}(x) = \{(*)\colon 0 \to \mathscr{O}(D) \to E \to L(-D) \to 0 \mid h^0(C, E) > 1\}.$$

Since $h^0(C, L(-D)) = g - k - 1 = h^1(C, \mathscr{O}(D))$ (remember, D was chosen to be general), we see that $\mathscr{O}_{E_k}(\widetilde{\Gamma}_L) \cong \mathscr{O}_{E_k}(g-k-1) \otimes$(degree 0 over $\mathrm{Sec}^k(C)$), which gives the formula for $\widetilde{\Gamma}_L$.

Now, let V be the span of $\widetilde{\Gamma}_L$ and the E_k $k > 0$, in $CH^1(\widetilde{P}_L)$. Then $V \subset \ker(\Phi_{L*})$. But

$$\mathrm{Pic}(\widetilde{P}_L)/V \cong \mathbf{Z}H \oplus \mathbf{Z}E_0/gH - (g-1)E_0 \cong \mathbf{Z},$$

so $\mathrm{Pic}(\mathscr{M}_{2,L})$ is a quotient of $\mathbf{Z}$. But $\mathscr{M}_{2,L}$ is a projective variety, so $V = \ker(\Phi_{L*})$ and $\mathrm{Pic}(\mathscr{M}_{2,L}) \cong \mathbf{Z}$.

Notes. (1) The group $\mathrm{Pic}(\mathscr{M}_{2,L})$ is not generated by $\Phi_L * H$. In fact, H pushes down to $(g-1)$ times the generator. Two divisors that do push down to generators are $(g-1)H - (g-2)E_0$ and $(2g-1)H - (2g-3)E_0$.

(2) An argument of Beauville's [3] concludes from $\mathrm{Pic}(\mathscr{M}_{2,L}) \cong \mathbf{Z}$ for $\deg(L)$ odd that $\mathrm{Pic}(\mathscr{M}_{2,L(-p)}) \cong \mathbf{Z}$. It should be possible to show this directly by a close analysis of the map Φ_{L*} for $\deg(L) = 2g$.

Definition. If L is a line bundle of degree $2g-2$, define $\Theta_L \subset \mathscr{M}_{2,L}$ to be the image $\Phi_L(\widetilde{P}_L)$ with the reduced scheme structure.

As a second application of the extension map, we obtain a result first shown in [3].

Application 2. $\mathrm{Pic}(\mathscr{M}_{2,\omega_C})$ is generated by (a translate of) Θ_{ω_C}.

Proof. It suffices to show that the class of Θ_{ω_C} is irreducible in $\mathrm{Pic}(\mathscr{M}_{2,\omega_C})$. We will do even better. We will show that $\mathrm{Pic}(\Theta_{\omega_C}) \cong \mathbf{Z}$ and is generated by $\mathscr{O}_{\Theta_{\omega_C}}(\Theta_{\omega_C})$.

Step 1. $\mathrm{Pic}(\Theta_{\omega_C}) \cong \mathbf{Z}$. Since Θ_{ω_C} is normal (see Laszlo [6]), it suffices to show that $CH^1(\Theta_{\omega_C}) \cong \mathbf{Z}$. But exactly as in the previous application, we see that:

(i) $\Phi_{\omega_{C*}} E_k = 0$ for all $k > 0$,

(ii) $\Phi_{\omega_{C*}}(\widetilde{\Gamma}_{\omega_C}) = \Phi_{\omega_{C*}}(gH - (g-1)E_0) = 0$.

Step 2. $\Phi^*_{\omega_C}\mathscr{O}_{\Theta_{\omega_C}}(\Theta_{\omega_C}) \cong \mathscr{O}((g-1)H - (g-2)E_0 - \cdots - E_{g-2})$. Let $\tau \in \mathrm{Pic}^0(C)$ be a two-torsion point. Then $\mathscr{O}_{\Theta_{\omega_C}}(\Theta_{\omega_C \otimes \tau}) \cong \mathscr{O}_{\Theta_{\omega_C}}$. On the other hand, we can explicitly describe

$$\phi^{-1}_{\omega_C}(\Theta_{\omega_C \otimes \tau}) = \{(*)\colon 0 \to \tau \to E \otimes \tau \to \omega_C \otimes \tau \to 0\},$$

and as before we get the expression for the pullback.

But now, as observed in the Note above, the image of $(g-1)H-(g-2)E_0$ generates the quotient of $\mathbf{Z}H \oplus \mathbf{Z}E_0$ by $gH-(g-1)E_0$, and we get the result.

Notation. Since $\mathscr{M}_{2,*}$ always has Picard group isomorphic to $\mathbf{Z}$, we will let Θ stand for the ample generator. We have just seen that there is a canonical choice for Θ in the case $\mathscr{M}_{2,\omega_C}$. There is no such obvious canonical generator for $\deg(L)$ odd. However, we do have:

Definition. If $\deg(L)=2g-1$, let

$$\Xi_L = \{E \in \mathscr{M}_{2,L} | \text{there is an } s \in H^0(C, E) \text{ with } Z(s) \neq \varnothing\}.$$

Application 3. $\Xi_L \sim g\Theta$.

Proof. On the Chow group level, $\Xi_L = \Phi_{L*}(E_0)$. But using Application 1, we see that the image of E_0 is g times the generator of $\mathbf{Z}H \oplus \mathbf{Z}E_0$ modulo $gH-(g-1)E_0$. q.e.d.

As a final illustration of these methods, we show:

Application 4. If $\deg(L)=2g$, then the image of $\Phi_{L*}: CH^1(\widetilde{P}_L) \to CH^0(\mathscr{M}_{2,L})$ is generated by twice the class $[\mathscr{M}_{2,L}]$.

Proof. Just as before,

(i) $\Phi_{L*}(E_k)=0$ for $k>0$, and
(ii) $\Phi_{L*}(gH-(g-1)E_0)=0$.

To see (ii), we show that $\Phi_L^*(\Theta) = gH-(g-1)E_0 - \cdots$. For this, we may as well assume that $L=\omega_C(2p)$, in which case we can represent Θ as the translate $\Theta_{\omega_C(2p)}$ of Θ_{ω_C} by $\otimes\mathscr{O}(p)$. In this case,

$$\phi_L^{-1}(\Theta_{\omega_C(2p)}) = \{(*): 0 \to \mathscr{O}(-p) \to E(-p) \to \omega_C(p) \to 0\}$$

is a hypersurface of degree g, and we finish as before. q.e.d.

Thus, the image of Φ_{L*} in $CH^0(\mathscr{M}_{2,L})$ is isomorphic to $\mathbf{Z}$. But his time, Φ_{L*} is not surjective. Indeed, if $E \in \mathscr{M}_{2,L}$ is general, then the preimage $\Phi_L^{-1}(E) \cong \mathbf{P}^1$ and meets E_0 in exactly $\deg(E)=2g$ points. So $\Phi_L: E_0 \to \mathscr{M}_{2,L}$ is generically finite of degree $2g$. Thus $\Phi_{L*}(E_0) = 2g[\mathscr{M}_{2,L}]$, and it follows immediately that the image $\operatorname{im}(\Phi_{L*}) = 2[\mathscr{M}_{2,L}]\mathbf{Z}$.

Further, the class $\Phi_{L*}((g-1)H-(g-2)E_0) = 2[\mathscr{M}_{2,L}]$. We will see that $(g-1)H-(g-2)E_0$ is effective, so there are rational hypersurfaces in $\mathbf{P}^{3g-2}$ mapping to $\mathscr{M}_{2,L}$ with generic degree 2.

Proposition 4.7. (a) *If* $\deg(L)=2g-2$, *then* $\Phi_L^*(\Theta) \cong (g-1)H-(g-2)E_0 - \cdots - E_{g-2}$.

(b) *If* $\deg(L)=2g$, *then* $\Phi_L^*(\Theta) \cong gH-(g-1)E_0 - \cdots - E_{g-1}$.

(c) *If* $\deg(L)=2g-1$, *then* $\Phi_L^*(\Theta) \cong (2g-1)H-(2g-3)E_0 - \cdots - E_{g-1}$.

Proof. We have already proved (a) in Application 2 and (b) in Application 4. In both cases, we used the fact that Θ has a geometric realization. We will similarly use the geometric realization of Ξ_L to prove (c).

From the expressions derived in the Applications: $\Xi_L \sim g\Theta$ and $\widetilde{\Gamma}_L \sim gH - (g-1)E_0 - (g-2)E_1 \cdots$, it follows that we may write

$$\Phi_L^*(\Xi_L) \sim n_\Gamma \widetilde{\Gamma}_L + \sum n_k E_k,$$

where n_Γ and n_k are integers.

Claim. $n_\Gamma = 2g-1$, and $n_k = k+1$.

Proof. Since $\Xi_L = \Phi_{L*}(E_0)$ and Φ_L is birational to its image when restricted to E_0, we get $n_0 = 1$ for free. In order to calculate n_Γ and the other n_k, we need to understand how E_0 intersects a general fiber of Φ_L restricted to the corresponding divisors.

If $E \in \Phi_L(\widetilde{\Gamma}_L)$ is a general point, then $\Phi^{-1}(E) \cong \mathbf{P}^1$, and $n_{\widetilde{\Gamma}_L} = \deg(E_0 \cap \Phi_L^{-1}(E)) = \deg(E) = 2g-1$. On the other hand, if $E \in \Phi_L(E_k)$ is general, then there is a unique $D \in C_{k+1}$ with a unique $\mathscr{O}(D) \hookrightarrow E$, and the preimage of E in E_k is a connected set containing an open subset isomorphic to U_D and connected but mutually disjoint divisors over each $p \in D \subset C \subset \mathbf{P}_L$. These $k+1$ divisors in $\Phi_L^{-1}(E)$ comprise $E_0 \cap \Phi_L^{-1}(E)$. By generic smoothness, then, $n_k = \deg(D) = k+1$.

Finally, substituting for Ξ_L and $\widetilde{\Gamma}_L$, we get the theorem. q.e.d.

As a quick corollary, we recover the formulas for the canonical line bundles (see, for example [3] or [5]):

Corollary 4.8. *The canonical divisors* $K_{\mathscr{M}_{2,*}}$ *are*:

(a) $K_{\mathscr{M}_{2,L}} \sim -2\Theta$ *if* $\deg(L)$ *is odd.*

(b) $K_{\mathscr{M}_{2,L}} \sim -4\Theta$ *if* $\deg(L)$ *is even.*

Proof. Compute the canonical line bundle in two different ways using the maps $\sigma\colon \widetilde{P}_L \to \mathbf{P}_L$ and $\Phi_L\colon \widetilde{P}_L \to \mathscr{M}_{2,L}$ for line bundles for degree $2g-2$ (odd case) and ω_C (even case). q.e.d.

Suppose $S \subset \mathbf{P}_L$ is a hypersurface of degree d. Then S pulls back to a section of $\mathscr{O}_{\widetilde{P}_L}(dH - a_k E_k)$ if and only if S contains $\operatorname{Sec}^k(C)$ and the tangent cone to S at all points of $\operatorname{Sec}^k(C)$ had degree (at least) a_k (i.e., if and only if $S \in H^0(\mathbf{P}_L, \mathscr{O}(d) \otimes \mathscr{I}^{a_k}_{\operatorname{Sec}^k(C)})$. But suppose S vanishes along C and the tangent cone has order a_0 at all points of C. Since every secant line in $\operatorname{Sec}^1(C)$ meets C at two points, it follows immediately that as soon as $2a_0 > d$, then S contains $\operatorname{Sec}^1(C)$, and, taking derivatives, it follows that in fact, if $a_1 = 2a_0 - d$, then $H^0(\widetilde{P}_L, \mathscr{O}(dH - a_0E_0 - a_1E_1)) = H^0(\widetilde{P}_L, \mathscr{O}(dH - a_0E_0))$. Similarly, a 3-secant $\mathbf{P}^2$ must be contained in S as soon as $3a_1 > d$, which happens as soon as $3a_0 > 2d$. More generally, one sees that $H^0(\widetilde{P}_L, \mathscr{O}(dH - a_0E_0 - \cdots - a_kE_k)) = H^0(\widetilde{P}_L\mathscr{O}(dH - a_0E_0))$ if $a_k = (k+1)a_0 - kd$.

Now let us take another look at the pullbacks of Θ in Proposition 4.7. In all three cases, we find that $a_k = d - (k+1)(d - a_0) = (k+1)a_0 - kd$. As we just saw, this means that the higher a_k are extraneous and in each case $H^0(\widetilde{P}_L, \Phi_L^*(\Theta)) = H^0(\widetilde{P}_L, dH - a_0E_0)$, where $dH - a_0E_0$ are the first two terms in the expansion of $\Phi_L^*(\Theta)$.

Putting this all together, by Proposition 4.5 and the projection formula, we have $H^0(\mathscr{M}_{2,L}, \mathscr{O}(k\Theta)) \cong H^0(\widetilde{P}_L\Phi_L^*(\mathscr{O}(k\Theta)))$. By Proposition 4.7 and the remarks above, we get Theorem 2. As an added bonus, we even see that

$$H^0(\Theta_{\omega_C}, \mathscr{O}_{\Theta_{\omega_C}})) \cong H^0(\mathbf{P}_{\omega_C}, \mathscr{O}(k(g-1)) \otimes \mathscr{I}_C^{k(g-2)}).$$

Suppose $\deg(L) = 2g$, $M = L \otimes \omega_C$, and $C \to \mathbf{P}_L = \mathbf{P}(H^0(C, M))$ is the embedding by the complete linear series. Let $\pi: \mathbf{P}_L \to \mathbf{P}_{L(-p)}$ be the projection from p, so $\pi(C) \to \mathbf{P}_{L(-p)}$ is also embedded by the complete linear series. Now, if $S \in H^0(\mathbf{P}_{L(-p)}, \mathscr{O}(d) \otimes \mathscr{I}_{\pi(C)}^{a_0})$, then $\pi^*S \in H^0(\mathbf{P}_L, \mathscr{O}(d) \otimes \mathscr{I}_C^{a_0})$.

Recall that since $\deg(L(-p)) = 2g - 1$, we saw in Application 1 that we can regard $\widetilde{\Gamma}_L \in H^0(\mathbf{P}_{L(-p)}, \mathscr{O}(g) \otimes \mathscr{I}_{\pi(C)}^{g-1})$. Then the partial derivatives of $\widetilde{\Gamma}_L$ give sections of $H^0(\mathbf{P}_{L(-p)}, \mathscr{O}(g-1) \otimes \mathscr{I}_{\pi(C)}^{g-2})$, and therefore, by pulling back, we see that in particular, $H^0(\mathbf{P}_L, \mathscr{O}(g-1) \otimes \mathscr{I}_C^{g-2}) \neq \varnothing$. This, together with Application 4, gives:

Proposition 4.9. *There exist rational singular hypersurfaces $S \in \mathbf{P}^{3g-2}$ and dominant maps of degree 2 from S to $\mathscr{M}_{2,L}$ if* $\deg(L)$ *is even.*

Note. One sees that these hypersurfaces are rational by projecting from a general point $q \in C$.

Finally, suppose $\deg(L) = 2g - 1$, and consider $\pi_p: \mathbf{P}_L \to \mathbf{P}_{L(-p)}$ for any $p \in C$. Via the map $\Phi_{L(-p)}: \widetilde{P}_{L(-p)} \to \mathscr{M}_{2,L(-p)}$ and Application 3, one readily sees that $\Phi_{L(-p)}^*\mathscr{O}(\Theta_{L(-p)}) \cong \mathscr{O}((g-1)H - (g-2)E_0 - \cdots)$. Now, a theorem of Beauville (Theorem II(b) of [3]) states that if C is not hyperelliptic, then the line bundle $\mathscr{O}(\Theta_{L(-p)})$ determines a map of degree one from $\mathscr{M}_{2,L(-p)}$ to its image. Thus, on projective space $\mathbf{P}_{L(-p)}$, the linear series $H^0(\mathbf{P}_{L(-p)}, \mathscr{O}(g-1) \otimes \mathscr{I}_C^{g-2}) \subset H^0(\mathbf{P}_{L(-p)}, \mathscr{O}(g-1))$ determines a map birational to its image. Now, if $S \in H^0(\mathbf{P}_{L(-p)}, \mathscr{O}(g-1)\otimes\mathscr{I}_C^{g-2})$, then $(\pi_p^* S)(\widetilde{\Gamma}_L) \in H^0(\mathbf{P}_L, \mathscr{O}(2g-1)\otimes\mathscr{I}_C^{2g-3})$, and one checks that via all the possible projections π_p, this gives enough sections of $H^0(\widetilde{P}_L, \Phi_L^*\mathscr{O}(\Theta_L))$ to get:

Proposition 4.10. *The line bundle* $\mathscr{O}(\Theta_L)$ *determines a map birational to its image if* $\deg(L)$ *is odd and* C *is not hyperelliptic.*

Question. In this case, is $\mathscr{O}(\Theta_L)$ in fact very ample?

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