J. DIFFERENTIAL GEOMETRY
35 (1992) 265-281

C^{α}-COMPACTNESS FOR MANIFOLDS WITH RICCI CURVATURE AND INJECTIVITY RADIUS BOUNDED BELOW

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0. Introduction

In this note, we consider the class of Riemannian n-manifolds (M, g) which have a lower bound on the Ricci curvature and on the injectivity radius

$$\mathrm{Ric}_M \geq -\lambda, \qquad \mathrm{inj}_M \geq i_0. \tag{0.1}$$

Our main result is that the C^{α} geometry of the metric g and the $C^{1,\alpha}$ topology of the manifold M are controlled by these bounds. More precisely, we obtain

Theorem 0.1. *Let (M, g) be a compact Riemannian manifold satisfying the bounds*

$$\mathrm{Ric}_M \geq -\lambda, \qquad \mathrm{inj}_M \geq i_0, \quad \mathrm{vol}_M \leq V. \tag{0.2}$$

Then for all $\alpha < 1$ and $Q > 1$, there is a finite atlas of harmonic coordinate charts $F_\nu \colon U_\nu \to \mathbb{R}^n$ for M, having the following properties:

(1) *The domains U_ν are of the form $U_\nu = F_\nu^{-1}(B(r_h))$, $B(r_h)$ a ball in $\mathbb{R}^n$, of radius r_h, satisfying*

$$r_h \geq C(\lambda, i_0, n, \alpha, Q).$$

Further, the domains $F_\nu^{-1}(B(r_h/2))$ cover M.

(2) *The overlaps $F_{\mu\nu} = F_\mu \circ F_\nu^{-1}$ are controlled in the $C^{1,\alpha}$ topology, i.e.,*

$$\|F_{\mu\nu}\|_{C^{1,\alpha}} \leq C(\lambda, i_0, n, \alpha, Q).$$

Received April 16, 1990. This research was partially supported by National Science Foundation Grants.

(3) *There is a bound* $N = N(n, \lambda, i_0, V)$ *on the number of coordinate charts, as well as on the multiplicity* $K = K(n, \lambda, i_0)$ *of their intersections.*

(4) *The metric coefficients* $g_{ij} = g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$ *in the charts* F_ν *are controlled in the* C^α *topology, in the sense that*

$$Q^{-1}\cdot(\delta_{ij}) \leq (g_{ij}) \leq Q\cdot(\delta_{ij}), \quad (\textit{as bilinear forms}),$$
$$r_h^\alpha\cdot\|g_{ij}\|_{C^\alpha} \leq Q - 1.$$

(5) *The square of the distance function* $\rho^2: M \times M \to \mathbb{R}$ *has a* $C^{1,\alpha}$ *bound* $C = C(\lambda, i_0, n, \alpha, Q)$ *for* $\rho < i_0/2$.

There is a C^α compactness theorem naturally associated to Theorem 0.1.

Theorem 0.2. *The space of compact Riemannian n-manifolds* (M, g) *such that*

$$\text{(0.3)} \qquad \mathrm{Ric}_M \geq -\lambda, \quad \mathrm{inj}_M \geq i_0, \quad \mathrm{vol}_M \leq V$$

is precompact in the C^α *topology for any* $\alpha < 1$. *More precisely, given any sequence of n-manifolds* $\{(M_i, g_i)\}$ *satisfying the bounds* (0.3), *and given any fixed* $\alpha < 1$, *there is a compact manifold* M, *and diffeomorphisms* $f_j: \to M_j$, *for a subsequence* $\{j\}$ *of* $\{i\}$, *such that the metrics* $f_j^* g_j$ *converge, in the* $C^{\alpha'}$ *topology for* $\alpha' < \alpha$, *to a Riemannian manifold* (M, g) *with* C^α *metric* g. *The manifold* (M, g) *admits a* $C^{1,\alpha}$ *harmonic coordinate atlas satisfying* (1)–(5) *above.*

In particular, there are only finitely many diffeomorphism types of manifolds satisfying (0.3).

We wish to emphasize that the passage from a theorem like 0.1 to a theorem like 0.2 is *immediate*, given the results of the second author's thesis. (This point seems to have escaped general notice.)

In [3, §4], one considers the collection of n-dimensional manifolds MM^n, admitting a coordinate covering by N coordinate balls $F_\nu^{-1}(B(r))$ (with a given numbering), such that the coordinate changes $F_\mu \circ F_\nu^{-1}$ are uniformly C^2-bounded, and such that the collection $\{F_\nu^{-1}(B(r/2))\}$ also covers. (These assumptions correspond to (1)–(3) of Theorem 0.1.) It is then shown (cf. also [5]) that given an infinite sequence M_i of such manifolds, there is a subsequence M_{i_j} and diffeomorphisms $f_{st}: M_{i_s} \to M_{i_t}$ such that, as $i_s, i_t \to \infty$, the maps $F_{\nu,i_s} \circ f_{st} \circ F_{\nu,i_t}^{-1}|_{B(r/2)}$ converge to the *inclusion* $B(r/2) \subset B(r)$ in the C^1 topology. Exactly the same argument shows that for $k \geq 1$, one actually obtains $C^{k,\alpha'}$ convergence to the inclusion, if we are given instead of $C^{k,\alpha}$ bounds on the $F_\mu \circ F_\nu^{-1}$ $(\alpha' < \alpha)$.

Now assume in addition that the metric satisfies the bounds

$$Q^{-1} \cdot (\delta_{ij}) \leq (g_{ij}) \leq Q \cdot (\delta_{ij}), \text{ (as bilinear forms)}$$
$$r^{m+\alpha} \cdot \|g_{ij}\|_{C^{m+\alpha}} \leq Q - 1,$$

where typically $m = k - 1$, in the given coordinate systems. (This corresponds to (4) of Theorem 0.1.) By what was stated above, one can regard the sequence of metrics as living on a *fixed* manifold M (by pulling back by diffeomorphisms), and satisfying the bounds above (with a different Q) in a *fixed* system of coordinates. Then the existence of a subsequence for which the metrics converge in the $C^{m,\alpha'}$ topology $(\alpha' < \alpha)$ follows immediately from the Arzela-Ascoli theorem; compare with the proofs of related convergence theorems in [11], [13], [15], [17], [18].

Before proceeding further, we make several remarks.

Remarks. (1) We note that under the bounds (0.1), an elementary packing argument, based on the volume comparison theorem (cf. [13]), shows that the bound $\operatorname{vol} M \leq V$ is equivalent to a diameter bound $\operatorname{diam}_M \leq D$.

(2) Versions of both Theorems 0.1 and 0.2 hold for Riemannian manifolds with boundary, e.g., for bounded domains, as well as for complete, noncompact but pointed Riemannian manifolds, provided one restricts attention to compact subsets. For instance, let (Ω_i, g_i) be a sequence of smooth domains in complete manifolds (M_i, g_i), and fix an arbitrary $\varepsilon > 0$. Suppose one has the bounds

$$\operatorname{Ric}_{M_i} \geq -\lambda, \quad \operatorname{inj}_{g_i}(x) \geq i_0, \quad \operatorname{vol} \Omega_i \leq V,$$

for any $x \in \Omega_i$. Then there are smooth domains $N_i \subset \Omega_i$, satisfying

$$\varepsilon/2 \leq \operatorname{dist}(z, \partial\Omega_i) \leq \varepsilon \quad \forall z \in \partial N_i,$$

such that the sequence of open manifolds $\{(N_i, g_i)\}$ is precompact in the C^α topology, as above. For the construction of such domains, we refer to [6], in conjunction with Theorem 0.3 below.

(3) Actually, we will prove Theorems 0.1 and 0.2 in the Sobolev spaces $L^{1,p}$, with $n < p < \infty$, in place of C^{α}, and $L^{2,p}$ in place of $C^{1,\alpha}$. By the Sobolev embedding theorem

$$L^{k,p} \subset C^{k-1,\alpha}, \qquad \alpha = 1 - n/p. \tag{0.4}$$

This in fact gives one stronger results. Here the notion of convergence in a given topology is as in the previous paragraph. For instance a sequence of smooth Riemannian metrics $\{g_i\}$ on M converges in the (strong) $L^{1,p}$ topology, or the C^{α} topology, if there is a sequence of diffeomorphisms f_i such that the metrics $\{f_i^* g_i\}$ converge, in the (strong) $L^{1,p}$ or C^{α} topology with respect to a given atlas for M, to a limit metric g on M.

As we shall see shortly, the main point needed to establish Theorem 0.1 is to find a lower bound on the radius r_h for which one has the metric bounds (4). This leads to the following definition.

Definition. Let (M, g) be an n-dimensional Riemannian manifold. Given $p \in (n, \infty)$ and $Q > 1$, the $L^{1,p}$ *harmonic radius* of (M, g) is the largest number $r_H = r_H(p, Q)$ such that on any geodesic ball $B = B_x(r_H)$ of radius r_H in (M, g), there is a harmonic coordinate chart $U = \{u_i\}_1^n : B \to \mathbb{R}^n$, such that the metric tensor is $L^{1,p}$ controlled in these coordinates, i.e., if $g_{ij} \equiv g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$, then

$$\begin{gathered} Q^{-1} \cdot (\delta_{ij}) \leq (g_{ij}) \leq Q \cdot (\delta_{ij}), \text{ (as bilinear forms)}, \\ (r_H)^{1-n/p} \|\partial g_{ij}\|_{L^p} \leq Q - 1. \end{gathered} \tag{0.5}$$

One should note that the condition (0.5) is scale-invariant, so that the harmonic radius scales as the distance function under rescalings of the metric. Of course, the harmonic radius is strictly positive for any fixed compact, smooth Riemannian manifold (cf. also [14], [8]).

From the definition, the $L^{1,p}$ harmonic radius also controls the analogously defined C^{α} harmonic radius for α as in (0.4). We remark also that, in the same way, one may speak of the $L^{1,p}$ harmonic radius as a function of $x \in M$, $r_H(x)$ being the radius of the largest geodesic ball about $x \in M$ on which one has harmonic coordinates satisfying (0.5).

The $C^{1,\alpha}$ or $L^{2,p}$ harmonic radius was studied in [1], and shown to have a lower bound in terms of the geometric quantities $|\mathrm{Ric}_M| \leq \Lambda$, $\mathrm{inj}_M \geq i_0$. Analogously, we have the following result, which we phrase purely locally. First, for the arbitrary Riemannian manifold, define

$$s_M(x) = \sup_r\{\min(r, \inf\{\mathrm{inj}_M(y) : y \in B_x(r)\})\}. \tag{0.6}$$

Note that if M is a closed manifold, then $s_M(x) = \mathrm{inj}_M$ for all $x \in M$.

Theorem 0.3. *Let (M, g) be a Riemannian manifold satisfying the bound $\mathrm{Ric}_M \geq -\lambda^2$ for some $\lambda \geq 0$. Then there are constants c_1 and c_2, depending only on Q, n, p, such that if*

$$r_H(x) \leq c_1 \cdot \lambda^{-1},$$

then

$$r_H(x) \geq c_2 \cdot s_M(x). \tag{0.7}$$

Let us indicate how Theorem 0.3 implies Theorem 0.1(1)–(4) (and thus also Theorem 0.2). Clearly, Theorem 0.3 implies parts (1) and (4), with the r_h of (1) replaced by $r_H/\sqrt{Q}$, and with charts F_ν given by harmonic coordinates satisfying (0.5). Also, the bounds (0.2), together with well-known packing results [13], imply that one may choose a finite subatlas with a uniform upper bound on N and M, which gives (3).

To establish (2), we need to obtain $C^{1,\alpha}$ (or better $L^{2,p}$) estimates for harmonic functions; in fact, we will obtain $L^{2,q}$ estimates for any $q < \infty$. Recall that in an arbitrary local coordinate system $\{x_i\}$, the Laplace operator Δ is given by

$$\begin{aligned} \Delta &= \frac{1}{g}\sum \frac{\partial}{\partial x_i}\left(g \cdot g^{ij}\frac{\partial}{\partial x_j}\right) \\ &= \sum g^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum \frac{1}{g}\frac{\partial(g \cdot g^{ij})}{\partial x_i} \cdot \frac{\partial}{\partial x_j}, \end{aligned} \tag{0.8}$$

where $g = (\det g_{ij})^{1/2}$. In harmonic coordinates $\{x_i\}$, the coefficient of the first-order term vanishes, i.e., one has the simpler expression

$$\Delta = \sum g^{ij}\frac{\partial^2}{\partial x_i \cdot \partial x_j}, \tag{0.9}$$

since $\Delta x_k = 0$ and $\partial^2 x_k/\partial x_i \partial x_j = 0$, while $\partial x_k/\partial x_j = \delta_{kj}$.

Thus, let f be a smooth function defined on a ball B of radius r_H in (M, g). Since the coefficient g^{ij} in (0.9) is controlled in the C^α topology (for a fixed $\alpha \in (0, 1)$), the L^p theory for elliptic operators of the form (0.9) (cf. [10, p. 235]) gives the bound

$$\|f\|_{L^{2,q}(B')} \leq C(r_H, q, B')[\|\Delta f\|_{L^q} + \|f\|_{L^2}] \tag{0.10}$$

for any $B' \subset\subset B$ and any $q < \infty$. Recall that by the Sobolev embedding theorem $L^{2,q} \subset C^{1,\beta}$, where $\beta = 1 - n/q$, so that one also has a $C^{1,\beta}$ estimate on f for any β.

Remark. The estimate (0.10) is one of the central estimates for this paper. We note that (0.10) does not hold in general for elliptic operators of the form (0.8); one needs a priori L^q bounds for the first-order coefficient to obtain (0.10) in general. Of course, we do not have such bounds from the hypotheses on (M, g) or the harmonic radius. This is the basic reason why harmonic coordinates are used here, and not some other coordinate systems, e.g., distance coordinates. The reader may also wonder why we do not use the simpler Schauder theory to obtain $C^{1,\alpha}$ bounds for f above and prove Theorems 0.1–0.3 in the C^α-category. For the answer, see the Remark following the proof of Proposition 1.1.

We now return to consideration of the control of the overlap maps of harmonic coordinate charts. Let F_i, $i = 1, 2$, be harmonic coordinate charts on $B_i(r_H/2)$ with $B_1(r_H/2) \cap B_2(r_H/2) \neq \emptyset$. By definition, we have C^α control of the metric in the F_i coordinates on $B_i(r_H/2)$. If v is a harmonic coordinate function of, say, F_2, we see that by (0.10) we have $C^{1,\beta}$ control of $v|_{B_1(r_H/2)}$ for any $\beta < 1$. This immediately implies that the overlap maps $F_2 \circ F_1^{-1}$ are controlled in the $C^{1,\beta}$ topology by the size r_H, given in the bounds (0.5). The same reasoning shows that in fact the overlap maps are bounded in the $L^{2,p}$ topology.

Thus, we have shown that Theorem 0.3 implies Theorem 0.1 and thus also Theorem 0.2, apart from (5), (which will be proved separately).

We would like to thank Louis Nirenberg and Michael Taylor for helpful discussions concerning P.D.E. topics.

1. Proof of the theorems

As indicated above, Theorems 0.1 and 0.2 follow from Theorem 0.3. To prove Theorem 0.3, we need to show that the $L^{1,p}$ harmonic radius

$r_H(x)$, if less than a sufficiently small constant, depending on the Ricci curvature, has a lower bound depending only on s_M for any $p \in (n, \infty)$. More precisely, if

$$r_H(x) \le c_1 \cdot \lambda^{-1}, \quad \text{then } r_H(x) \ge c_2 \cdot s_M(x). \tag{1.1}$$

Note that (1.1) is scale invariant. Here, we fix the quantities p and Q.

The basic structure of the argument is as in [1], although there are significant differences in detail; in particular, we do not use the equation for the Ricci curvature in harmonic coordinates, although it is also possible to obtain a proof along these lines.

To establish the lower bound on r_H, we argue by contradiction. By scale invariance, it suffices to consider the case $\lambda = 1$. Thus, if the theorem were false, then there would exist a sequence of Riemannian n-manifolds (M_i, g_i) with $\mathrm{Ric}_{M_i} \ge -1$, such that, for some $x_i \in M_i$,

$$\lim_{i\to\infty} r_H(x_i) = 0, \qquad \lim_{i\to\infty} \frac{r_H(x_i)}{s_{M_i}(x_i)} = 0. \tag{1.2}$$

For a fixed i, we would like to choose x_i so that $r_H(x_i)/s_{M_i}(x_i)$ is as small as possible. But for M_i open, no such point needs to exist. To remedy this, we consider instead the sequence (B_i, x_i, g_i), where $B_i = B_{x_i}(s_{M_i}(x_i)) \subset M_i$. Note that, for $y \in B_i$,

$$s_{B_i}(y) = \tfrac{1}{2}\,\mathrm{dist}(y, \partial B_i).$$

In particular,

$$s_{B_i}(x_i) = \tfrac{1}{2} s_{M_i}(x_i),$$

and, as $y \to \partial B_i$, $s_{B_i}(y) \to 0$. Thus, if we continue to define r_H as above, i.e., with respect to M_i, then there exists $y_i \in B_i$ such that the ratio $r_H(y_i)/s_{M_i}(y_i)$ is minimal for points in B_i, and as $i \to \infty$,

$$r_H(y_i) \to 0, \tag{1.3}$$

and

$$\frac{r_H(y_i)}{s_{B_i}(y_i)} \to 0. \tag{1.4}$$

Put $r_i = r_H(y_i)$ and rescale the metrics g_i by r_i^{-2}, i.e., define metrics $h_i = r_i^{-2} \cdot g_i$. Thus, the harmonic radius of (B_i, h_i) at y_i is 1, while

$$(1.5)\qquad \begin{aligned} &\text{(i)} \quad \mathrm{Ric}_{(B_i, h_i)} \geq -r_i^2 \to 0, \\ &\text{(ii)} \quad \mathrm{inj}_{(B_i, h_i)}(y_i) \to \infty, \\ &\text{(iii)} \quad \mathrm{dist}_{h_i}(y_i, \partial B_i) \to \infty, \\ &\text{(iv)} \quad r_H(y, h_i) \geq \frac{s_{B_i}(y, h_i)}{s_{B_i}(y_i, h_i)} = \frac{\mathrm{dist}(y, \partial B_i)}{\mathrm{dist}(y_i, \partial B_i)}. \end{aligned}$$

In particular, if we set $u_i = s_{B_i}(y_i)/r_H(y_i)$, then $u_i \to \infty$ and

$$r_H(y, h_i) \geq \tfrac{1}{2}$$

for all $y \in B_{y_i}(u_i/2, h_i)$. Thus, given $R < \infty$, we have $r_H(y, h_i) \geq \frac{1}{2}$ on $B_{y_i}(R)$, provided i is sufficiently large. As shown in §0 (cf. Remark (2) of §2 and the discussion following the statement of Theorem 0.3), we may apply Theorem 0.2 to the domains $B_{y_i}(R_j) \subset (B_i, h_i, y_i)$, $R_j \to \infty$, and take a diagonal subsequence to conclude that a subsequence of the pointed manifolds (B_i, h_i, y_i) converges in the $C^{\alpha'}$ topology, $\alpha' < \alpha$, to a complete (noncompact) C^α Riemannian manifold (N, h, y), with $y = \lim y_i$. Here the convergence is uniform on compact subsets of (N, h, y). Note also that since $\{(B_i, h_i)\}$ is bounded in $L^{1,p}$, it contains a weakly convergent subsequence, so that we may also assume that (N, h) is an $L^{1,p}$ Riemannian manifold.

We emphasize that the convergence $(B_i, h_i) \to (N, h)$ is not, a priori, in the (strong) $L^{1,p}$ or C^α topologies, but only in a weaker topology. An appeal to some additional hypotheses on the metrics (in our case (1.5)(i), (ii) above) is crucial in order to obtain an improvement in the convergence. This improvement, as well as the remainder of the proof of Theorem 0.3, is contained in the following results.

Proposition 1.1. *Let (M_i, g_i) be a sequence of Riemannian manifolds which converge strongly in the $L^{1,p}$ topology to a limit $L^{1,p}$ Riemannian manifold (M, g). Then*

$$(1.6)\qquad r_H(M) = \lim_{i\to\infty} r_H(M_i),$$

i.e., the $L^{1,p}$ harmonic radius is continuous in the strong $L^{1,p}$ topology. The same is true pointwise, i.e., for the harmonic radius at any sequence $\{z_i\} \to z \in M$.

Proposition 1.2. *Let (M_i, x_i, g_i) be a sequence of manifolds satisfying the bounds* (1.5). *Then a subsequence converges in the strong $L^{1,p}$ topology for any $p < \infty$ to a limit $L^{1,p}$ Riemannian manifold (N, g_0).*

Proposition 1.3. *Any $L^{1,p}$ limit (N, g_0) of a sequence of Riemannian manifolds (M_i, x_i, g_i) satisfying* (1.5) *is isometric to $\mathbb{R}^n$, with the canonical flat metric.*

To see how the propositions above imply Theorem 0.3, first note that Proposition 1.2 implies that the convergence $(M_i, h_i) \to (N, h)$ is not just weakly $L^{1,p}$ but actually in the strong $L^{1,p}$ topology. Proposition 1.3 implies that the limit (N, h) is $\mathbb{R}^n$. It is obvious that $\mathbb{R}^n$ has harmonic radius ∞. However, by construction, the harmonic radius of (M_i, h_i) at x_i is 1, so that Proposition 1.1 gives the required contradiction.

Thus, it remains to prove the propositions above.

Proof of Proposition 1.1. Note that on an $L^{1,p}$ Riemannian manifold the Laplace operator is well defined as in (0.9), so that one may speak of harmonic functions on M, which are then at least in $L^{2,p}$. Similarly, the concept of $L^{1,p}$ harmonic radius is well defined on M. We first prove the less significant inequality

$$r_H(M) \geq \overline{\lim_{i\to\infty}}\, r_H(M_i). \tag{1.7}$$

Thus, let U_i be harmonic coordinate charts satisfying the bounds (0.5) on $B_i = B_{x_i}(r_i) \subset (M_i, g_i)$, where $r_i = r_H(M_i)$; we may suppose that $\overline{\lim}\, r_i > 0$. Since the metrics $g_i \to g$ in the strong $L^{1,p}$ topology, the charts U_i converge in the $L^{2,p}$ topology to a limiting map $U\colon B \to \mathbb{R}^n$, where $B = B_x(r) \subset (M, g)$, $r = \overline{\lim}\, r_i$. Since the bounds (0.5) are clearly preserved under strong $L^{1,p}$ convergence, this gives (1.7).

To obtain the converse

$$r_H(M) \leq \underline{\lim_{i\to\infty}}\, r_H(M_i), \tag{1.8}$$

suppose $r \leq r_H(M)$ is finite and let $\{x_k\}$ be harmonic coordinates on $B = B(r) \subset (M, g)$ satisfying (0.5). Via diffeomorphisms, we may view the metrics g_i as metrics on B, for i sufficiently large. Let Δ_i be the Laplace operator of g_i and write Δ_i in the coordinates $\{x_k\}$ on B,

$$\Delta_i = \frac{1}{g_i} \sum \frac{\partial}{\partial_k} \left(g_i \cdot g_i^{kl} \frac{\partial}{\partial x_l} \right). \tag{1.9}$$

Let $\{y_k\} = \{y_k(i)\}$ be solutions to the Dirichlet problem for Δ_i on B, with boundary values x_k, and set $w_k = x_k - y_k$. Thus,

$$\Delta_i w_k = \Delta_i x_k, \qquad w_k|_{\partial B} = 0.$$

By the L^p estimates (0.10), one has the estimate

$$\|w_k\|_{L^{2,p}(B')} \leq C(B')\cdot\|\Delta_i(x_k)\|_{L^p},$$

where $B' \subset\subset B$, since w_k has zero boundary values. Now since $g_i \to g$ in the strong $L^{1,p}$ topology, the coefficients of Δ_i converge in the strong $L^{1,p}$ topology to those of Δ on B. By definition, we have $\Delta(x_k) = 0$, so that by examining the coefficients of (1.9), we have

$$\|\Delta_i(x_k)\|_{L^p} \to 0 \quad \text{as } i \to \infty. \tag{1.10}$$

Thus, the harmonic coordinates $\{y_k\} = \{y_k(i)\}$ converge in the strong $L^{2,p}$ topology to the harmonic coordinates $\{x_k\}$, uniformly on compact subsets of B. Since the bounds (0.5) are continuous in the strong $L^{1,p}$ topology, they are satisfied for the charts $\{y_k(i)\}$ on arbitrary compact subsets $B' \subset B$, with constants $Q_i \to Q$ as $i \to \infty$. This then establishes (1.8).

Remark. (i) It is not clear if the C^α harmonic radius is continuous in the C^α topology, since one does not obtain the estimate (1.10) in this case. This is another reason why we work with the Sobolev spaces $L^{1,p}$ in place of the C^α Hölder spaces, besides the fact that $L^{1,p} \subset C^\alpha$.

(ii) The same proof shows that the $L^{1,p}$ harmonic radius is continuous in the strong $L^{1,p}$ topology for manifolds with boundary.

Before beginning with the proof of Proposition 1.2, we will need the following simple, but basic, result.

Lemma 1.4. *Let M be a Riemannian manifold with $\mathrm{inj}_M \geq i_0$ and $\mathrm{Ric}_M \geq -\lambda^2$, $\lambda \geq 0$. Let $\rho = \rho_x \equiv \mathrm{dist}(x, \cdot)$, be a distance function from $x \in M$. Then one has the estimate*

$$|\Delta\rho| \leq (n-1)\lambda\cdot\coth\lambda\rho, \tag{1.11}$$

provided $\rho < i_0/2$.

Proof. A well-known version of the Bishop comparison theorem implies that

$$\Delta\rho \le (n-1)\lambda\cdot\coth\lambda\rho, \tag{1.12}$$

provided $\rho \le i_0$. Given x fixed, let p be any point with $t = \operatorname{dist}(x, p) \le i_0/2$; let γ be the geodesic with $\gamma(0) = x$ and $\gamma(t) = p$; then set $p_1 = \gamma(2t)$. Thus (1.12) holds for $\rho = \rho_x$ and for $\rho_1 = \rho_{p_1}$ on $B_p(i_0/2)$. On the other hand, the function $\sigma = \rho + \rho_1 - 2t\colon M \to \mathbb{R}$ is nonnegative by the triangle inequality, and achieves its minimal value, 0, along the line segment γ between p and p_1. Hence, we have

$$\Delta\sigma = \Delta(\rho + \rho_1)|_\gamma \ge 0,$$

i.e., $\Delta\rho \ge \Delta\rho_1 \ge -(n-1)\lambda\cdot\coth\lambda\rho$, which establishes (1.11).

Remark. We note that the proof of Lemma 1.4 gives the following somewhat more precise bound. If $\gamma\colon [0, l] \to M$ is a minimal geodesic, parametrized by arclength, with $x = \gamma(0)$, and $\operatorname{Ric}_M \ge -\lambda$, then one has the bound

$$-(n-1)\lambda\cdot\coth\lambda(l-\rho) \le \Delta\rho|_\gamma \le (n-1)\lambda\cdot\coth\lambda\rho,$$

provided $\rho \le l$.

Estimates of this sort, or as in (1.11), are implicit in the proof of the Splitting Theorem. They probably go back quite far, for instance at least to [4].

Proof of Proposition 1.2. Recall now that by assumption one has the estimates (1.5) on the sequence of manifolds (M_i, g_i). For any tangent vector $v_i \in T_{x_i}M_i$, let γ_i be the geodesic in M_i with $\gamma_i(0) = x_i$ and $\gamma_i'(0) = v_i$. Set $y_i = \gamma_i(-s_i)$, where $s_i = \frac{1}{2} i_0 \cdot r_i \to \infty$ as $i \to \infty$. Then the distance functions $\rho_i = \operatorname{dist}(y_i, \cdot) - s_i$ are smooth on $B_{x_i}(s_i/2)$, and by Lemma 1.4, with (1.5), one obtains the estimate

$$|\Delta_i \rho_i| \to 0 \quad \text{as } i \to \infty \tag{1.13}$$

on $B_{x_i}(s_i/2)$.

On the other hand, on each ball $B = B_i \subset M_i$ of bounded distance to x_i and of sufficiently small but fixed radius (cf. (1.5)(iv)), one has harmonic coordinates $\{u_k\} = \{u_k(i)\}$, with $L^{1,p}$ control, and for which the Laplace operator has the form

$$\Delta_i = \sum g_i^{kl} \frac{\partial^2}{\partial u_k \partial u_l}. \tag{i.14}$$

The (0,10) estimate then gives the bound

$$\|\rho_i\|_{L^{2,q}} \le C(q, B')[\|\Delta_i \rho_i\|_{L^q} + \|\rho_i\|_{L^2}] \tag{1.15}$$

on $B' \subset\subset B$ for any q $(\ge p)$. (Here we are abusing notation slightly, namely, ρ_i is actually $\rho_i \circ F_i^{-1} \colon \mathbb{R}^n \to \mathbb{R}$, where F_i is the harmonic coordinate chart $\{u_k(i)\}$. Thus, (1.14), (1.15) are defined for functions defined on domains in $\mathbb{R}^n$. This is also implicit in what follows.)

In particular, $\{\rho_i\}$ is bounded in $L^{2,q} = L^{2,q}(\mathbb{R}^n)$ for any q, and thus by the compactness of the embedding $L^{2,q} \subset L^{1,q}$, $\{\rho_i\}$ has a subsequence converging strongly in $L^{1,q}$ and weakly in $L^{2,q}$ to $\rho \in L^{2,q}$, where $\rho \circ F$ is a distance function (or more precisely a Busemann function) on (N, g_0).

We claim that in fact $\{\rho_i\}$ has a subsequence converging strongly in $L^{2,q}$. To see this, we apply the estimate (1.15) to $\rho - \rho_i$. Clearly, $\|\rho - \rho_i\|_{L^2} \to 0$. To show that $\|\Delta_i(\rho - \rho_i)\|_{L^q} \to 0$, we have $|\Delta_i \rho_i| \to 0$ by (1.13) and also $\Delta_i \rho \to \Delta \rho$ in L^q since $\rho \in L^{2,q}$, and the coefficients in (1.14) converge in $C^{\alpha'}$ topology. Thus we need to show that $\Delta \rho = 0$ in L^q. Letting $f \in C_0^\infty(B)$, we compute

$$\int f \cdot \Delta \rho \, dV = \int \Delta f \cdot \rho \, dV = \lim_{i \to \infty} \int \Delta_i f \cdot \rho_i \, dV_i = \lim_{i \to \infty} \int f \cdot \Delta_i \rho_i \, dV_i = 0,$$

which establishes the claim.

We are now in a position to verify that $g_i \to g$ strongly in the $L^{1,q}$ topology for any q. Namely, fix i for the moment and consider the distance functions ρ $(= \rho_i)$ constructed above. Then we have

$$|\nabla \rho|^2 = \sum g^{kl} \rho_k \rho_l = 1,$$

where $\rho_k = \partial \rho / \partial u_k$, and the $\{u_k\}$ are harmonic coordinates on $B \subset (M_i, g_i)$. Choose for instance an orthonormal basis e_μ of $T_{z_i} M_i$, where z_i is the center point of B, and consider the $n(n+1)/2$ vectors e_μ, $e_\mu + e_\nu$, $\mu, \nu = 1, \cdots, n$. We let ρ^m, $1 \le m \le n(n+1)/2$, be the associated distance functions described above, so that one has a system of $n(n+1)/2$ equations

$$\sum_{k,l} g^{kl} \rho_k^m \rho^m = 1 \tag{1.16}$$

on B. We view this as a system of linear equations with g^{kl} as unknowns and $\rho_k^m \rho_l^m$ as coefficients. One may algebraically solve this system for g^{kl} provided the determinant of the coefficients is nonzero. Clearly, by choosing Q sufficiently close to 1, g^{kl} is arbitrarily close, in the $C^{\alpha'}$ topology, to δ^{kl}. By the estimate (1.15) and the arguments above, each $\rho^m = \rho^m(i)$ is close in the $C^{1,\alpha'}$ topology to a limit distance function ρ on $B \subset (N, h)$. By choosing a sufficiently small ball $B' \subset B$ (depending on (N, h)), we see that all $\rho^m(i)$ are close, in the $C^{1,\alpha'}$ topology, to the correspondingly defined Euclidean distance functions on B'. One may easily check that the matrix $\rho_k^m \rho_l^m$ is nonsingular in $\mathbb{R}^n$ and thus, by continuity of (1.16), it is nonsingular on B' for i sufficiently large.

Thus, the metric coefficients g_{kl} on (M_i, g_i), in the coordinates $\{u_k\}$, are rational expressions in $\{\rho_l^m\}$. It has been shown that the $\{\rho_l^m\}$ converge strongly in the $L^{1,q}$ topology for any q to limit $L^{1,q}$ functions; hence the same is true of $\{g_{kl}\} = \{g_{kl}(i)\}$.

Proof of Proposition 1.3. To prove that the limit (N, g_0) is isometric to $\mathbb{R}^n$, we return to the family of distance functions $\rho = \rho(i)$ constructed above in the beginning of Proposition 1.2. By the Bochner-Weitzenbock formula,

$$0 = \Delta |d\rho|^2 = |D^2\rho|^2 + \langle d\Delta\rho, d\rho\rangle + \mathrm{Ric}(\nabla\rho, \nabla\rho).$$

Integrating over balls B of fixed size in (M_i, g_i), one obtains

$$\int_B |D^2\rho|^2 \le \lambda \cdot \mathrm{vol}\, B + \int_B (\Delta\rho)^2 + \int_{\partial B} |\Delta\rho|,$$

where λ is a lower bound for the Ricci curvature. By (1.5) and (1.13), it follows that

$$\int_B |D^2\rho_i|^2 \to 0 \quad \text{as } i \to \infty,$$

so that $D^2\rho_i$ converges strongly to 0 in L^2. Since previously we have shown that $\{\rho_i\}$ strongly (sub)converges in the $L^{2,q}$ topology for any q, it follows that $D^2\rho_i$ converges to 0 strongly in L^q.

Now $g_0 \in C^\alpha$ for any α since the metric g_0 is in $L^{1,q}$, and h has a well-defined Levi-Civita connection (Christoffel symbol) D in L^q. In particular, for the (Busemann) function $\rho = \lim \rho_i$, $\nabla\rho$ is in $L^{1,q} \cap C^\alpha$ and $D\nabla\rho = 0$.

It follows in particular that ρ is harmonic (we already knew this from the proof of Proposition 1.2) and thus, by the Schauder estimates for instance, that $\rho \in C^{2,\alpha}$. The above arguments then show that $g_0 \in$

$C^{1,\alpha}$, and repeating this argument by the standard bootstrap shows $g_0 \in C^\infty$. Finally, since $D\nabla\rho = 0$ for any ρ, it follows that the metric splits isometrically in every direction, i.e., g_0 is the canonical flat metric on $\mathbb{R}^n$. q.e.d.

This completes the proof of Theorem 0.3 and thus the proof of Theorems 0.1 and 0.2(1)–(4).

For the proof of (5), we recall from Lemma 1.4 that one has L^∞ bounds on $\Delta\rho^2$. Thus, if the $L^{1,p}$ harmonic radius is bounded below, then by (1.14) one obtains $L^{2,q}$ bounds on ρ^2 for any q. In particular, this gives $C^{1,\beta}$ bounds on ρ^2 for any $\beta < 1$, provided $\rho < \operatorname{inj} M/2$.

Remarks. (1) As in the definition of $L^{1,p}$ harmonic radius, one could define for instance an $L^{1,p}$ normal radius, i.e., the largest radius on which the metric, in some normal coordinate chart, is $L^{1,p}$ bounded. However, such a concept does not behave well under limits. For instance, it is not continuous in C^α, or even $C^{1,\alpha}$, topology for the same reason that the injectivity radius itself is not continuous in the $C^{1,\alpha}$ topology. Specific examples may be found by smoothing the vertex of a sequence of Euclidean cones converging to $\mathbb{R}^n$.

(2) We point out that there do exist examples of manifolds satisfying the bounds

$$\mathrm{Ric}_M \geq -\lambda^2, \qquad \mathrm{inj}_M \geq i_0,$$

but without a uniform upper bound on the Ricci curvature; we learned of this in [12], [19]. (In particular, Theorem 0.3 does not follow from [1].) Very briefly, these examples may be described as follows. Consider a (solid) cone C in $\mathbb{R}^n$ whose cross-section is some $(n-1)$-dimensional submanifold with smooth, nonempty boundary in the unit sphere $S^{n-1}(1)$. Let C_ε denote C, truncated at distance ε from the origin. The ($(n-1)$-dimensional) examples are then obtained by smoothing the seam or crease in ∂C_ε.

2. C^α harmonic radius and volume

We have shown in the previous section that the C^α or $L^{1,p}$ harmonic radius is bounded below by the geometric bounds

$$\mathrm{Ric}_M \geq -\lambda^2, \qquad \mathrm{inj}_M \geq i_0.$$

This is no longer true if one replaces the lower bound on the injectivity radius by a lower bound on the volume vol_M of M. In fact, one does not even have a bound on the contractibility radius under the (stronger) bounds $|\mathrm{Ric}| \leq \Lambda$, $\mathrm{vol} \geq v$ (cf. [2]). In terms of volume, the best one could hope for is expressed by the following test question.

Question 1. If $B = B_0(1)$ is an n-dimensional geodesic ball of radius 1, does there exist $\varepsilon = \varepsilon(n)$ such that if

$$\mathrm{Ric}_B \geq 0 \quad \text{and} \quad \mathrm{vol}\, B \geq (1-\varepsilon)\omega_n, \tag{2.1}$$

then the C^α or $L^{1,p}$ harmonic radius of B at 0 is bounded below by $\delta = \delta(\varepsilon) > 0$?

Here ω_n is the volume of the unit ball in $\mathbb{R}^n$. Note that the hypotheses (2.1) are scale invariant and imply that B is close, in the Hausdorff topology, to the unit ball $B \subset \mathbb{R}^n$. If true, the question would imply that B is close, in the C^α topology, to the Euclidean ball. It was shown in [1] that one obtains a lower bound on the $C^{1,\alpha}$ harmonic radius under the stronger bounds $C \geq \mathrm{Ric}_B \geq 0$, $\mathrm{vol}\, B \geq (1-\varepsilon)\cdot\omega_n$, where $\varepsilon = \varepsilon(C, n)$.

Now the fact that the hypotheses (2.1) are scale invariant leads naturally to a global analogue of this question, namely,

Question 2. Let M be a complete, noncompact Riemannian n-manifold. Does there exist $\varepsilon = \varepsilon(n)$ such that if the volume $v(r)$ of geodesic r-balls satisfies $v(r) \geq (1-\varepsilon)\omega_n r^n$, then does M admit a global harmonic coordinate system $U: M \to \mathbb{R}^n$ with C^0 bounds, i.e.,

$$Q^{-1}\cdot(\delta_{ij}) \leq (g_{ij}) \leq Q\cdot\delta_{ij}, \quad \text{(as bilinear forms)},$$

where $Q = Q(\varepsilon) \to 1$ as $\varepsilon \to 0$?

We do not include the C^α bounds as in (0.5), since these would require that M is flat by their scale-invariance.

In fact, the answer to Question 2 is negative. From this and a scaling argument, one sees that the answer to Question 1 is negative as well.

Proposition 2.1. *Let g be a complete metric of nonnegative curvature on $\mathbb{R}^2$. Then $(\mathbb{R}^2, g)$ admits no global harmonic coordinate system with C^0 bounds, unless g is flat.*

Remark. Note this implies that the C^0 harmonic radius of $(\mathbb{R}^2, g)$ is finite, assuming g is not flat, regardless of the volume growth of $(\mathbb{R}^2, g)$.

Proof. A global harmonic coordinate system with C^0 bounds is a pair of harmonic functions with, in particular, bounded gradients. Thus, let h

be a harmonic function on $(\mathbb{R}^2, g)$ with bounded gradient ∇h. By the Bochner-Weitzenbock formula,

$$\frac{1}{2}\Delta|dh|^2 = |D^2 h|^2 + \mathrm{Ric}(\nabla h, \nabla h) \geq 0. \tag{2.2}$$

Hence, $|dh|^2$ is a globally defined bounded subharmonic function on $(\mathbb{R}^2, g)$. On the other hand, it is well known that any complete metric of nonnegative Gauss curvature on $\mathbb{R}^2$ is parabolic, i.e., is conformally equivalent to $\mathbb{C}$ (cf. [7] for a proof which only requires that the volume growth of geodesic balls is bounded by $c \cdot r^2$). Now, by definition, a parabolic surface admits no nonconstant bounded subharmonic functions, so that we must have $|dh|^2 = \mathrm{const}$. By (2.2), it then follows that $\mathrm{Ric}\,\nabla h = D\nabla h = 0$, so that $(\mathbb{R}^2, g)$ splits isometrically in the direction ∇h and thus g is flat. q.e.d.

The argument of Proposition 2.1 uses rather strongly the fact that M is two-dimensional. For higher dimensions, we use the following result of A. Kasue [16]. Let M be a complete Riemannian manifold with sectional curvature K_M satisfying $0 \leq K_M \leq C/r^2$, for some constant $C < \infty$, where r is the distance from some point in M. If h is a harmonic function on M with uniformly bounded gradient ∇h, then ∇h is parallel on M, and gives an isometric splitting $M = M' \times \mathbb{R}$.

This result immediately gives the following proposition.

Proposition 2.2. *There exist metrics of nonnegative sectional curvature on $\mathbb{R}^n$, arbitrarily close to the Euclidean metric, in the smooth topology on compact subsets, which admit no global harmonic coordinate systems with C^0 bounds.*

Clearly, these examples may have $v(r) \geq \omega_n(1-\varepsilon)\cdot r^n$ for any given $\varepsilon > 0$. By scaling, one may arrange that the $L^{1,p}$ harmonic radius is any prescribed value in $\mathbb{R}^+$ and thus converges to any value in $\mathbb{R}^+ \cup \{0\} \cup \{\infty\}$ for a family of such metrics converging to $\mathbb{R}^n$.

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