

# CONFORMAL FIELD THEORY AND THE COHOMOLOGY OF THE MODULI SPACE OF STABLE BUNDLES

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## 1. Introduction

Let  $\Sigma_g$  be a compact Riemann surface of genus  $g \geq 2$ , and let  $\Lambda$  be a line bundle over  $\Sigma_g$  of degree 1. Then the moduli space of rank-2 stable bundles  $V$  over  $\Sigma_g$  such that  $\Lambda^2 V \cong \Lambda$  was shown by Seshadri [17] to be a nonsingular projective variety  $\mathcal{N}_g$ . Its rational cohomology ring  $H^*(\mathcal{N}_g)$  is exceedingly rich and despite years of study has never quite been computed in full. The object of this paper is to give an essentially complete characterization of this ring, or at least to reduce the problem to a matter of linear algebra. In particular, we find a proof of Newstead's conjecture that  $p_1^g(\mathcal{N}_g) = 0$ . We also obtain a formula for the volume of  $\mathcal{N}_g$ , which can be regarded as a twisted version of the formula for the degree 0 moduli space recently announced by Witten.

The approach which we shall take is not from algebraic geometry but from mathematical physics: it relies on the  $SU(2)$  Wess-Zumino-Witten model, which is a functor  $Z_k$  associating a finite-dimensional vector space to each Riemann surface with marked points. The relationship with the moduli space is that when the "level"  $k$  of the functor is even, the vector space associated to  $\Sigma_g$  with no marked points can be identified with  $H^0(\mathcal{N}_g; L^{k/2})$ , where  $L$  is a fixed line bundle over  $\mathcal{N}_g$ . Now the work of Verlinde provides us with a means of calculating the dimension of any vector space arising from our functor, and in particular  $\dim H^0(\mathcal{N}_g; L^{k/2})$ , which we shall denote  $D(g, k)$ . On the other hand Newstead [13] found explicit generators for  $H^*(\mathcal{N}_g)$ , and we can also express  $D(g, k)$  in terms of them using a Riemann-Roch theorem. Equating the two formulas enables us to evaluate any monomial in the generators on the fundamental class of  $\mathcal{N}_g$ , and by Poincaré duality this is sufficient, at least in principle, to determine the ring structure of  $H^*(\mathcal{N}_g)$ .

In the discussion above one important point has been skated over. The  $SU(2)$  WZW model is of course associated to bundles of degree 0, not degree 1, so in order to make use of Verlinde's work it is necessary to formulate a "twisted" version of the field theory. This is carried out in §2, but the crucial properties of the twisted theory, analogous to those which make the ordinary theory a modular functor, are not proved in this paper. Rather, we will confine ourselves to exploring the consequences of these claims, and hope to return to justify them in a later paper.

An outline of the remaining sections goes as follows. In §3 we review those parts of Verlinde's work we shall need, show how they must be modified in the twisted case, and work out some explicit formulas for  $D(g, k)$ . In §4 we study the cohomology of  $\mathcal{N}_g$ , which we regard throughout as a space of representations via the theorem of Narasimhan and Seshadri. We define Newstead's generators  $\alpha$ ,  $\beta$ , and  $\psi_i$  of  $H^*(\mathcal{N}_g)$  and, using Poincaré duality, show that the entire ring structure is determined by those pairings of the form  $(\alpha^m \beta^n \gamma^p)[\mathcal{N}_g]$ , where  $\gamma = 2 \sum \psi_i \psi_{i+g}$ . A further application of Poincaré duality enables us to eliminate  $\gamma$  recursively from the pairings, so that we need only evaluate  $(\alpha^m \beta^m)[\mathcal{N}_g]$ . This is calculated in §5 using the results of §3 and the Hirzebruch-Riemann-Roch theorem. Finally we extract Newstead's conjecture and the volume formula from our results.

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## 2. The WZW model

We begin with a speedy review of the  $SU(2)$  Wess-Zumino-Witten model and its gluing rules. The reader is referred to [5], [6], or [16] for a more leisurely account. We must first fix a positive integer  $k$ , called the *level*. Let  $\Sigma_g$  be a compact connected Riemann surface of genus  $g$ , with marked points  $x_1, \dots, x_p \in \Sigma_g$ . We choose a *labelling* of the marked points, that is, we associate to each one an irreducible representation  $V$  of  $SL(2, \mathbb{C}) = SU(2)_{\mathbb{C}}$  with  $\dim_{\mathbb{C}} V \leq k+1$ . Such representations are determined up to isomorphism by their dimension: we will write  $V_n$  for the representation of dimension  $n+1$ , and denote by  $(\Sigma_g; x_i, n_i)$  the Riemann surface with points  $x_i$  labelled by  $V_{n_i}$ . To each such labelled Riemann surface we wish to associate a complex vector space  $Z_k(\Sigma_g; x_i, n_i)$ .

Let  $P$  be a smooth principal  $SL(2, \mathbb{C})$  bundle over  $\Sigma_g$ ; then  $P$  is unique up to isomorphism, and the set  $\mathcal{A}$  of compatible holomorphic

structures on the associated rank-2 vector bundle  $U$  is an affine space modelled on  $\Omega_{\mathbb{C}}^{0,1}(\Sigma_g; \text{ad } P)$ . There exists a line bundle  $\mathcal{L} \rightarrow \mathcal{A}$  whose fiber at any holomorphic structure is the determinant line of the associated  $\bar{\partial}$ -operator on  $U$ . It is easy to see that the usual action of the gauge group  $\mathcal{G} = \Gamma(\text{Ad } P)$  on  $\mathcal{A}$  lifts to  $\mathcal{L}$ , so we have an action of  $\mathcal{G}$  on the tensor power  $\mathcal{L}^k$ . Next, at each marked point  $x_i$ , choose an identification of the fiber  $(\text{Ad } P)_{x_i}$  with  $\text{SL}(2, \mathbb{C})$ . (Actually,  $\text{Ad } P$  always has a global trivialization in the present case, so we may as well just use a restriction of that.) Then  $\mathcal{G}$  also acts on each  $V_{n_i}$  via evaluation at the point  $x_i$ . We may now define our vector space at level  $k$  as a space of  $\mathcal{G}$ -equivariant sections:

$$Z_k(\Sigma_g; x_i, n_i) = \Gamma_{\mathcal{G}} \left( \mathcal{L}^k \otimes \bigotimes_i V_{n_i} \right).$$

If the genus  $g$  is at least 2, then the set  $\mathcal{A}_s$  of semistable holomorphic structures has complement of codimension at least 2 (see p. 569 of [1]), so we may regard  $Z_k(\Sigma_g; x_i, n_i)$  as the space of holomorphic sections of the twisted quotient

$$\mathcal{L}^k|_{\mathcal{A}_s} \otimes_{\mathcal{G}} \bigotimes_i V_{n_i},$$

which is a vector bundle over the moduli space  $\mathcal{A}_s/\mathcal{G}$  of semistable  $\text{SL}(2, \mathbb{C})$  bundles over  $\Sigma_g$ . However, some care is needed as the moduli space has singularities.

In fact, all we will really want to know about  $Z_k(\Sigma_g; x_i, n_i)$  is its dimension, which we shall denote  $z_k(\Sigma_g; x_i, n_i)$ . Now it is well known to physicists that  $Z_k$  is an example of a *modular functor*. In particular, it satisfies the two propositions below, which have important implications for the computation of  $z_k(\Sigma_g; x_i, n_i)$ .

**(1) Proposition.** *The value of  $z_k(\Sigma_g; x_i, n_i)$  is independent of the complex structure on  $\Sigma_g$ .*

Consequently, it is independent of the positions of the  $x_i$  and of the trivializations of the fibers over them, because any two choices can be exchanged by a suitable smooth map. Thus we will be justified in henceforth writing simply  $z_k(\Sigma_g; n_i)$  for  $z_k(\Sigma_g; x_i, n_i)$ .

The second proposition gives two *gluing axioms*, which describe the behavior of  $z_k$  under (i) the addition of a handle and (ii) connect-sum. In light of (1), we need not worry about the choice of complex structure.

- (2) Proposition.** (i)  $z_k(\Sigma_{g+1}; n_i) = \sum_{n=0}^k z_k(\Sigma_g; n, n, n_i)$ .  
(ii)  $z_k(\Sigma_{g+g'}; n_i, n_j) = \sum_{n=0}^k z_k(\Sigma_g; n, n_i) \cdot z_k(\Sigma_{g'}; n, n_j)$ .

Proofs are essentially contained in [19].

The definition of  $Z_k$  given above is well known. However, as it stands the definition involves  $SL(2, \mathbb{C})$  bundles over  $\Sigma_g$ , which are topologically trivial. Since we are interested in nontrivial bundles over  $\Sigma_g$ , we shall need to modify and extend the definition slightly. Fortunately, there is a straightforward way to do this. We consider  $(\Sigma_g; x_i, n_i)$  as before. Let  $\widehat{P}$  be a smooth principal  $PSL(2, \mathbb{C})$  bundle over  $\Sigma_g$  with  $w_2(\widehat{P}) \neq 0$ ; again,  $\widehat{P}$  is unique up to isomorphism. The associated  $\mathbb{C}P^1$  bundle lifts to a rank 2 vector bundle  $\widehat{U}$  of degree 1. Let  $\widehat{\mathcal{A}}$  denote the set of compatible holomorphic structures on  $\widehat{U}$  which induce a fixed holomorphic structure  $\omega$  on  $\Lambda^2 \widehat{U}$ . (We did not need to make this explicit in the untwisted case, because  $\Lambda^2 U$  is canonically trivial.) As before we let  $\widehat{\mathcal{L}}$  be the determinant line bundle. However, we need to be careful in our definition of the gauge group; we let  $\widehat{\mathcal{G}} = \Gamma(\widetilde{Ad\widehat{P}})$ , where  $\widetilde{Ad\widehat{P}}$  is the adjoint bundle with fiber  $SL(2, \mathbb{C})$ , *not*  $PSL(2, \mathbb{C})$ . As in the untwisted case, we choose identifications of  $(\widetilde{Ad\widehat{P}})_{x_i}$  with  $SL(2, \mathbb{C})$ , and define

$$\widehat{Z}_k(\Sigma_g; x_i, n_i) = \Gamma_{\widehat{\mathcal{G}}} \left( \widehat{\mathcal{L}}^k \otimes \bigotimes_i V_{n_i} \right).$$

Again, we write  $\hat{z}_k(\Sigma_g; x_i, n_i) = \dim \widehat{Z}_k(\Sigma_g; x_i, n_i)$ . In analogy with the properties of  $z_k$  discussed above, we propose the following.

**(3) Claim.** *The value of  $\hat{z}_k(\Sigma_g; x_i, n_i)$  is independent of the complex structure on  $\Sigma_g$ .*

Consequently, it is again independent of the positions of the  $x_i$  and the trivializations of the fibers, as well as the choice of the holomorphic structure  $\omega$  on  $\Lambda^2 \widehat{U}$ . As in the untwisted case, we shall henceforth write simply  $\hat{z}_k(\Sigma_g; n_i)$  for  $\hat{z}_k(\Sigma_g; x_i, n_i)$ .

**(4) Claim.** (i)  $\hat{z}_k(\Sigma_{g+1}; n_i) = \sum_{n=0}^k \hat{z}_k(\Sigma_g; n, n, n_i)$ .

(ii)  $\hat{z}_k(\Sigma_{g+g'}; n_i, n_j) = \sum_{n=0}^k \hat{z}_k(\Sigma_g; n, n_i) \cdot z_k(\Sigma_{g'}; n, n_j)$ .

Despite their crucial importance for us, we will not attempt to prove these claims here. However, it appears that the proofs of the analogous statements given by Tsuchiya et al. [19] could be adapted to this case. To prove part (ii) of (2), for example, they define a sheaf  $\mathcal{S}$  over a local universal family of curves of genus  $g + g'$  near a curve  $\Sigma_g \vee \Sigma_{g'}$  with an ordinary double crossing. The sheaf is so constructed that its stalk over the generic smooth curve is

$$\mathcal{S}|_{\Sigma_{g+g'}} = Z_k(\Sigma_{g+g'}; n_i, n_j),$$

but over the singular curve is

$$\mathcal{S}|_{\Sigma_g \vee \Sigma_{g'}} = \bigoplus_{n=0}^k Z_k(\Sigma_g; n, n_i) \otimes Z_k(\Sigma_{g'}; n, n_j).$$

The desired result is then obtained by showing that  $\mathcal{S}$  is locally free. The key step in adapting such a proof to the twisted case would be to note that there exists a principal  $\text{PSI}(2, \mathbb{C})$  bundle over the local universal family whose restriction to the generic curve is twisted, but whose restriction to the singular curve  $\Sigma_g \vee \Sigma_{g'}$  is twisted over  $\Sigma_g$ , but untwisted over  $\Sigma_{g'}$ . Of course, similar reasoning would suggest that

$$(5) \quad z_k(\Sigma_{g+g'}; n_i, n_j) = \sum_{n=0}^k \hat{z}_k(\Sigma_g; n, n_i) \cdot \hat{z}_k(\Sigma_{g'}; n, n_j),$$

and we shall see in the next section that this is indeed the case.

### 3. Some calculations

In one respect the twisted theory described above is simpler than the familiar untwisted one: if the genus  $g$  is at least 2, then not only does the set  $\widehat{\mathcal{A}}_g$  have complement of codimension at least 2, as before, but now the moduli space  $\mathcal{A}_g/\widehat{\mathcal{G}}$  is known [17] to be a compact complex manifold  $\mathcal{N}_g$ . The section  $-I \in \widehat{\mathcal{G}}$  now acts nontrivially on  $\widehat{\mathcal{L}}$  as well as the  $V_{n_i}$ , so that  $\widehat{\mathcal{L}}^k \otimes \bigotimes V_{n_i}$  descends to  $\mathcal{N}_g$  if and only if  $k + \sum n_i$  is even. When this is the case, we need have no qualms about identifying  $\widehat{Z}_k(\Sigma_g; n_i)$  with the space of holomorphic sections of a vector bundle over  $\mathcal{N}_g$ . In particular, let  $L$  denote the line bundle over  $\mathcal{N}_g$  such that  $\widehat{\mathcal{L}}^2$  descends to  $L$ . Then if there are no marked points at all, we have for even  $k$

$$\widehat{Z}_k(\Sigma_g) = H^0(\mathcal{N}_g; L^{k/2}).$$

This is the link which will enable us to use  $\widehat{Z}_k$  to study the cohomology of  $\mathcal{N}_g$ . Hence our object in this section will be to give an explicit formula for  $\hat{z}_k(\Sigma_g)$ .

From (4) we deduce

$$\hat{z}_k(\Sigma_g) = \sum_n \hat{z}_k(\Sigma_{g-1}; n, n) = \sum_{m,n} z_k(\Sigma_{g-1}; m, n) \hat{z}_k(\Sigma_0; m, n),$$

where  $\Sigma_0$  is of course just the 2-sphere. Thus, provided we understand  $\hat{z}_k(\Sigma_0; m, n)$ , we can reduce the computation of  $\hat{z}_k$  to the computation of  $z_k$ , which is well understood and which we shall now describe.

Following Verlinde [20], we define the coefficients

$$N_{m_1, m_2, m_3} = z_k(\Sigma_0; m_1, m_2, m_3),$$

which are symmetric in all three indices. Define the  $(k+1) \times (k+1)$  symmetric matrices  $N_m$  by

$$(N_m)_{i,j} = N_{m,i,j}.$$

Both here and in the future we index the rows and columns starting with 0 instead of 1, for convenience. We then obtain the following formula for the torus with two marked points.

**(6) Proposition.**  $z_k(\Sigma_1; m_1, m_2) = \text{tr } N_{m_1} N_{m_2}$ .

*Proof.*

$$\begin{aligned} z_k(\Sigma_1; m_1, m_2) &= \sum_m z_k(\Sigma_0; m, m, m_1, m_2) \\ &= \sum_{m,n} z_k(\Sigma_0; m, n, m_1) \cdot z_k(\Sigma_0; m, n, m_2) \\ &= \sum_{m,n} (N_{m_1})_{m,n} (N_{m_2})_{n,m} = \text{tr } N_{m_1} N_{m_2}. \quad \text{q.e.d.} \end{aligned}$$

Now define another  $(k+1) \times (k+1)$  symmetric matrix  $M$  by  $(M)_{i,j} = \text{tr}(N_i N_j)$ . This enables us to generalize the previous formula to arbitrary genus.

**(7) Proposition.** For  $g \geq 1$ ,  $z_k(\Sigma_g; m_1, m_2) = (M^g)_{m_1, m_2}$ .

*Proof.* The proof is by induction. Proposition (6) is the case  $g = 1$ . Then, if  $z_k(\Sigma_{g-1}; m_1, m_2) = (M^{g-1})_{m_1, m_2}$ , by attaching a handle we obtain

$$\begin{aligned} z_k(\Sigma_g; m_1, m_2) &= \sum_n z_k(\Sigma_{g-1}; m_1, n) \cdot z_k(\Sigma_1; n, m_2) \\ &= \sum_n (M^{g-1})_{m_1, n} (M)_{n, m_2} \\ &= (M^g)_{m_1, m_2}. \end{aligned}$$

**(8) Corollary.**  $z_k(\Sigma_g) = \text{tr}(M^{g-1})$ .

*Proof.*

$$\begin{aligned} z_k(\Sigma_g) &= \sum z_k(\Sigma_{g-1}; m, m) \\ &= \sum (M^{g-1})_{m,m} = \text{tr } M^{g-1}. \quad \text{q.e.d.} \end{aligned}$$

It is worth mentioning that the coefficients  $N_{m_1, m_2, m_3}$  can actually be calculated explicitly:

(9) **Proposition.**

$$N_{m_1, m_2, m_3} = \begin{cases} 1 & \text{if } |m_1 - m_2| \leq m_3 \leq \min(m_1 + m_2, k - m_1 - m_2), \\ & \text{and } m_1 + m_2 + m_3 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The proof closest to our point of view appears in [6]; a fuller discussion can be found in [7]. However, we shall not use the coefficient in this form, but rather make use of the Verlinde conjecture, which states that the matrices  $N_m$  can be simultaneously diagonalized. Let  $S$  be the  $(k + 1) \times (k + 1)$  matrix such that

$$(S)_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right).$$

Note that  $S$  is orthogonal and symmetric, so  $S = S^{-1}$ .

(10) **Proposition.**  $SN_m S = \text{diag}((S)_{n,m}/(S)_{0,m})$ .

Proofs are given in [2], [7], and [21].

(11) **Corollary.**  $SMS = \text{diag}(1/(S)_{0,m}^2)$ .

*Proof.*

$$\begin{aligned} (SMS)_{i,j} &= \sum_{m,n} (S)_{i,m} \text{tr}(N_m N_n) (S)_{n,j} \\ &= \sum_{m,n,p,q} (S)_{i,m} N_{m,p,q} N_{n,p,q} (S)_{n,j} = \sum_p (SN_p^2 S)_{i,j}. \end{aligned}$$

(12) **Corollary.**

$$z_k(\Sigma_g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=1}^{k+1} \frac{1}{(\sin(m\pi/(k+2)))^{2g-2}}.$$

*Proof.* (8) and (11). q.e.d.

In order to find a similar formula for  $\hat{z}_k$ , as we mentioned above, we need to understand  $\hat{z}_k(\Sigma_0; m_1, m_2)$ . This can be computed directly.

(13) **Proposition.**  $\hat{z}_k(\Sigma_0; m_1, m_2) = \delta_k^{m_1+m_2}$ .

*Proof.* In this case the action of  $\hat{\mathcal{G}}$  on  $\hat{\mathcal{A}}$  is well understood [1]. The orbits are indexed by positive integers  $n$ ;  $\hat{\mathcal{A}}_n$  consists of those holomorphic structures which are isomorphic to  $\mathcal{O}(-n) \oplus \mathcal{O}(n+1)$ . The complex codimension of  $\hat{\mathcal{A}}_n$  in  $\hat{\mathcal{A}}$  is  $2n$ . In particular  $\hat{\mathcal{A}}_0$  is a dense open orbit. Hence any equivariant section over  $\hat{\mathcal{A}}_0$  will extend over  $\hat{\mathcal{A}}$  by the

Hartogs theorem. That is,

$$\widehat{Z}_k(\Sigma_0; m_1, m_2) = \Gamma_{\widehat{\mathcal{G}}}(\widehat{\mathcal{L}}^k|_{\widehat{\mathcal{A}}_0} \otimes V_{m_1} \otimes V_{m_2}).$$

However, since  $\widehat{\mathcal{A}}_0$  is a single orbit of  $\widehat{\mathcal{G}}$ , such an equivariant section is determined by its value at a single fixed  $a \in \widehat{\mathcal{A}}_0$ . Conversely, any element of  $\widehat{\mathcal{L}}^k_a \otimes V_{m_1} \otimes V_{m_2}$  which is fixed by  $\widehat{\mathcal{G}}_a$ , the stabilizer of  $a$ , can be moved around by  $\widehat{\mathcal{G}}$  to give an equivariant section. Hence

$$(14) \quad \widehat{Z}_k(\Sigma_0, m_1, m_2) = \Gamma_{\widehat{\mathcal{G}}_a}(\widehat{\mathcal{L}}^k_a \otimes V_{m_1} \otimes V_{m_2}).$$

Now  $\widehat{\mathcal{G}}_a$  consists precisely of the  $\text{ad } P$ -compatible automorphisms of  $\widehat{U}$  which are holomorphic with respect to  $a$ . Since  $a \in \widehat{\mathcal{A}}_0$  this means that  $\widehat{\mathcal{G}}_a$  is isomorphic to the group of holomorphic automorphisms of  $\mathcal{O} \oplus \mathcal{O}(1)$  having determinant 1 on each fiber. If  $x_1, x_2$  are our two marked points, we may choose a basis  $\{s_1, s_2\}$  of  $H^0(\Sigma_0; \mathcal{O}(1))$  such that  $s_1$  vanishes at  $x_1$  and  $s_2$  vanishes at  $x_2$ . Then we can write

$$(15) \quad \widehat{\mathcal{G}}_a = \left\{ \begin{pmatrix} z & 0 \\ y_1 s_1 + y_2 s_2 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times, y_i \in \mathbb{C} \right\},$$

where the rows and columns of the matrix correspond to the decomposition  $\mathcal{O} \oplus \mathcal{O}(1)$ .

Now since  $H^1(\Sigma_0; \mathcal{O} \oplus \mathcal{O}(1)) = 0$ , the fiber of the determinant line bundle  $\mathcal{L}$  at  $a$  is just

$$\mathcal{L}_a^2 = \Lambda^3 H^0(\Sigma_0; \mathcal{O} \oplus \mathcal{O}(1))^*.$$

A straightforward computation shows that, with respect to the presentation in (15), an element of  $\widehat{\mathcal{G}}_a$  acts on  $\mathcal{L}_a$  by multiplication by  $z$ . On the other hand, for  $i = 1, 2$ , the vanishing of  $s_i$  at  $x_i$  implies that, under an appropriate identification of  $(\widehat{\text{Ad}}\widehat{P})_{x_i}$  with  $\text{SL}(2, \mathbb{C})$ ,  $\widehat{\mathcal{G}}_a$  acts on  $V_{n_i}$  via the homomorphism  $\widehat{\mathcal{G}}_a \rightarrow \text{SL}(2, \mathbb{C})$  given by

$$\begin{pmatrix} z & 0 \\ y_1 s_1 + y_2 s_2 & z^{-1} \end{pmatrix} \mapsto \begin{pmatrix} z & 0 \\ y_i & z^{-1} \end{pmatrix}.$$

Let us try to determine the right-hand side of (14) explicitly. Fix a nonzero  $t \in \widehat{\mathcal{L}}^k_a$ ; then any element of  $\widehat{\mathcal{L}}^k_a \otimes V_{m_1} \otimes V_{m_2}$  is of course of the form  $t \otimes v$  for  $v \in V_{m_1} \otimes V_{m_2}$ . The one-parameter subgroups

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & 0 \\ y_i s_i & 1 \end{pmatrix} : y_i \in \mathbb{C} \right\} \quad \text{for } i = 1, 2$$

generate  $\widehat{\mathcal{G}}_a$ , so for a section  $\phi$  be  $\widehat{\mathcal{G}}_a$ -equivariant it is necessary and sufficient to be equivariant with respect to each of these. Now we see that

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot t = z^k t,$$

so for equivariance we must have

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot v = z^{-k} v$$

for that  $v$  such that  $\phi(a) = t \otimes v$ . In infinitesimal terms, this means that  $v \in (V_{m_1} \otimes V_{m_2})(-k)$ , the  $-k$ -weight space of  $V_{m_1} \otimes V_{m_2}$  as a representation of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ . A typical element  $\begin{pmatrix} 1 & 0 \\ y_1 s_1 & 1 \end{pmatrix}$  of the second subgroup acts trivially on  $\widehat{\mathcal{L}}_a^k$  and on  $V_{m_2}$ ; on  $V_{m_1}$  it acts as  $\begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ . Hence for equivariance  $v$  must belong to  $L_{m_1} \otimes V_{m_2}$ , where  $L_{m_1} \subset V_{m_1}$  is the subspace on which  $\begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix}$  acts trivially. Infinitesimalizing again we recognize  $L_{m_1}$  as the lowest weight space  $V_{m_1}(-m_1)$ . In a similar way we deduce  $v \in V_{m_1} \otimes V_{m_2}(-m_2)$ . Now since  $v$  determines  $\phi$ , we can identify

$$\begin{aligned} \Gamma_{\widehat{\mathcal{G}}_a}(\widehat{\mathcal{L}}_a^k \otimes V_{m_1} \otimes V_{m_2}) &= (V_{m_1} \otimes V_{m_2})(-k) \cap V_{m_1}(-m_1) \otimes V_{m_2} \cap V_{m_1} \otimes V_{m_2}(-m_2) \\ &= (V_{m_1} \otimes V_{m_2})(-k) \cap V_{m_1}(-m_1) \otimes V_{m_2}(-m_2) \\ &= (V_{m_1} \otimes V_{m_2})(-k) \cap (V_{m_1} \otimes V_{m_2})(-m_1 - m_2), \end{aligned}$$

which clearly has dimension  $\delta_k^{m_1+m_2}$ .

**(16) Corollary.**  $\hat{z}_k(\Sigma_g; m_1, m_2) = (M^g J)_{m_1, m_2}$ , where  $J$  is the  $(k+1) \times (k+1)$  matrix with  $(J)_{i,j} = \delta_k^{i+j}$ .

*Proof.*  $\hat{z}_k(\Sigma_g; m_1, m_2) = \sum_{m=0}^k z_k(\Sigma_g; m_1, m) \cdot \hat{z}_k(\Sigma_0; m, m_2)$ ; then use (7) and (13). (In particular, this corollary proves (5), because  $J^2 = I$ .)

**(17) Corollary.** For  $g \geq 1$ ,  $\hat{z}_k(\Sigma_g) = \text{tr} M^{g-1}$ , where we define  $\text{tr} A = \sum_{i=0}^k (A)_{i, k-i}$ .

Of course, this “twisted trace” is not preserved by the diagonalization, so to actually evaluate  $\hat{z}_k$  we notice instead that  $J$  is also diagonalized by  $S$ ; in fact,  $SJS = \text{diag}(-1)^n$ . This gives us an analogue of (12).

**(18) Corollary.**

$$\hat{z}_k(\Sigma_g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=1}^{k+1} \frac{(-1)^{m+1}}{(\sin(m\pi/(k+2)))^{2g-2}}.$$

This formula is the only result obtained so far which will be needed in the sequel. Hence to simplify notation we will in future denote  $\hat{z}_k(\Sigma_g)$  simply by  $D(g, k)$ .

Although it is far from apparent from (18), for fixed  $g$   $D(g, k)$  is a polynomial in  $k$  with rational coefficients. We shall conclude the section by proving this. We shall closely imitate similar calculations for the untwisted case which appear in an unpublished letter by Don Zagier.

**(19) Proposition.** *For fixed  $g \geq 2$  and even  $k$ ,  $D(g, k)$  equals the coefficient of  $x^{3g-3}$  in the power series expansion of*

$$\left(-\frac{k+2}{2}x\right)^{g-1} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{(k+2)x}{\sinh(k+2)x}.$$

*Proof.* We first rewrite (18) in terms of roots of unity:

$$D(g, k) = \frac{1}{2}(-2k-4)^{g-1} \sum_{\substack{\zeta^{2k+4}=1 \\ \zeta \neq \pm 1}} \frac{-\zeta^{k+2}}{(\zeta - \zeta^{-1})^{2g-2}}.$$

Substituting  $\zeta^2 = \lambda$ , we get

$$D(g, k) = (-2k-4)^{g-1} \sum_{\substack{\lambda^{k+2}=1 \\ \lambda \neq 1}} \frac{\lambda^{g-1} \lambda^{(k+2)/2}}{(\lambda-1)^{2g-2}}.$$

Hence

$$\begin{aligned} \frac{D(g, k)}{(-2k-4)^{g-1}} &= \sum_{\substack{\lambda^{k+2}=1 \\ \lambda \neq 1}} \operatorname{Res}_{z=\lambda} \left[ \frac{-z^{g-1}}{(z-1)^{2g-2}} \frac{(k+2)z^{(k+2)/2}}{z^{k+2}-1} \frac{dz}{z} \right] \\ &= \operatorname{Res}_{z=1} \left[ \frac{z^{g-1}}{(z-1)^{2g-2}} \frac{(k+2)z^{(k+2)/2}}{z^{k+2}-1} \frac{dz}{z} \right] \end{aligned}$$

for  $g \geq 2$  by the residue theorem, since then the only poles of the expression in square brackets are at the  $(k+2)$ nd roots of unity. Substituting  $z = e^{2x}$  in the final expression gives

$$\begin{aligned} \frac{D(g, k)}{(-2k-4)^{g-1}} &= \operatorname{Res}_{x=0} \left[ \left(\frac{1}{2 \sinh x}\right)^{2g-2} \frac{k+2}{\sinh(k+2)x} dx \right] \\ &= \text{coefficient of } x^{2g-2} \text{ in} \\ &\quad \frac{1}{2^{2g-2}} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{(k+2)x}{\sinh(k+2)x}. \end{aligned}$$

#### 4. The cohomology of $\mathcal{N}_g$

Let  $\Sigma_g$  be a Riemann surface of genus  $g \geq 2$ . Fix a holomorphic line bundle  $\Lambda$  over  $\Sigma_g$  of degree 1. Then the moduli space  $\mathcal{N}_g$  of stable rank 2 holomorphic vector bundles  $V$  over  $\Sigma_g$  with  $\Lambda^2 V = \Lambda$  is a compact complex  $(3g - 3)$ -manifold. We wish to study the structure of the rational cohomology ring  $H^*(\mathcal{N}_g)$ . In §5, this will essentially be determined completely using Proposition (19). First, though, we need to review what is already known about  $\mathcal{N}_g$ , and to prove some necessary lemmas. A broader discussion of some of the topics mentioned can be found in [1].

Roughly speaking, we wish to identify  $\mathcal{N}_g$  as a space of representations of  $\pi_1(\Sigma_g)$ . So let us choose loops  $e_1, e_2, \dots, e_{2g}$  on  $\Sigma_g$  which generate  $\pi_1(\Sigma_g)$  in the usual way, so that  $\prod e_i e_{i+g} e_i^{-1} e_{i+g}^{-1} \sim 1$ . If we cut out a small disc  $D \subset \Sigma_g$ , then  $\prod e_i e_{i+g} e_i^{-1} e_{i+g}^{-1}$  is homotopic to the boundary circle of  $D$ , but is no longer contractible. Hence we lose the relation, and  $\pi_1(\Sigma_g - D)$  is the free group on the generators  $e_i$ . Now consider the map  $\mu: \text{SU}(2)^{2g} \rightarrow \text{SU}(2)$  defined by

$$(A_1, A_2, \dots, A_{2g}) \mapsto \prod_{i=1}^g A_i A_{i+g} A_i^{-1} A_{i+g}^{-1},$$

and in particular the subspace  $S_g = \mu^{-1}(-I) \subset \text{SU}(2)^{2g}$ . It can be shown [10] that  $-I$  is a regular value of  $\mu$ , so  $S_g$  is a smooth  $(6g - 3)$ -submanifold of  $\text{SU}(2)^{2g}$ . We may regard an element of  $S_g$  as a representation of  $\pi_1(\Sigma_g - D)$  sending the boundary circle to  $-I$ . It must be irreducible, since if it were reducible to an abelian subgroup, it would send the boundary circle to  $I$ . Such a representation gives us a flat connection on an  $\text{SU}(2)$  bundle over  $\Sigma_g - D$  having holonomy  $-I$  around  $D$ . By passing to the associated rank 2 vector bundle and gluing in a fixed twisted unitary connection over  $D$ , we can extend our flat connection to a connection on a unitary vector bundle  $V \rightarrow \Sigma_g$  of degree 1. Then the  $(0, 1)$  part of the associated covariant derivative is a Cauchy-Riemann operator which induces a holomorphic structure on  $V$ . Now the diagonal conjugation action of  $\text{SU}(2)/\pm 1 = \text{SO}(3)$  on  $\text{SU}(2)^{2g}$  clearly preserves  $S_g$ , and by Schur's lemma the restriction of the action is free. Hence the quotient  $S_g/\text{SO}(3)$  is a smooth  $(6g - 6)$ -manifold. But if two representations in  $S_g$  are conjugate, then the induced connections on  $V$  are isomorphic, and hence so are the holomorphic structures. Hence to each point in  $S_g/\text{SO}(3)$  we can associate an isomorphism class of holomorphic bundles over  $\Sigma_g$ . In this context, the celebrated theorem of Narasimhan and Seshadri [18] can be stated as follows.

(20) **Theorem.** *All the holomorphic bundles constructed in this manner are stable, and the resulting map  $\phi: S_g/\mathrm{SO}(3) \rightarrow \mathcal{N}_g$  is a diffeomorphism.*

**Remark.** We will in future identify  $\mathcal{N}_g$  with  $S_g/\mathrm{SO}(3)$ . Under this identification, any diffeomorphism  $f: \Sigma_g \rightarrow \Sigma_g$  induces a diffeomorphism  $\hat{f}: \mathcal{N}_g \rightarrow \mathcal{N}_g$ . (To be precise, we should require that  $f$  preserves a small neighborhood of the disc  $D$ ; but this is not a serious restriction, as any diffeomorphism of  $\Sigma_g$  is isotopic to one of this form.)

To obtain distinguished cohomology classes in  $\mathcal{N}_g$ , we construct a vector bundle  $\mathcal{U}$  over  $\mathcal{N}_g \times \Sigma_g$ , as follows. Let  $\tilde{\Sigma}_g$  be the universal cover of  $\Sigma_g$ , and consider the twisted quotient

$$\tilde{\Sigma}_g \times_{\pi_1(\Sigma_g)} (S_g \times \mathfrak{sl}(2, \mathbb{C})).$$

The action on the right-hand factor is given by  $h(\rho, v) = (\rho, \mathrm{ad}(\rho(h)) \cdot v)$ , where we regard  $\rho \in S_g$  as determining an  $\mathrm{SO}(3)$ -representation of  $\pi_1(\Sigma_g)$ . This gives us a vector bundle over  $S_g \times \Sigma_g$ ; to see that it descends to  $\mathcal{N}_g \times \Sigma_g$ , note that the conjugation action of  $\mathrm{SO}(3)$  on  $S_g$  lifts to an action on this twisted quotient, given by

$$T \cdot (s \times (\rho, v)) = (s \times (T\rho T^{-1}, \mathrm{ad} T \cdot v)).$$

The resulting vector bundle  $\mathcal{U}$  is clearly *natural* in the sense that, for any diffeomorphism  $f: \Sigma_g \rightarrow \Sigma_g$ , we have  $(\hat{f} \times f)^*(\mathcal{U}) \cong \mathcal{U}$ . (In fact, it is also *universal*, that is, it has a connection in the fiber direction whose restriction to  $\{\rho\} \times \Sigma_g$  is isomorphic to  $\mathrm{ad} \rho$ . It is even possible to perform a similar construction in the holomorphic setting; for details see [1].) Because  $\mathcal{N}_g$  is simply connected [12], we may write

$$c_2(\mathcal{U}) = -2\alpha\sigma + \beta - 4\psi,$$

where  $\sigma$  denotes the fundamental class in  $H^2(\Sigma_g)$  and

$$\alpha \in H^2(\mathcal{N}_g); \quad \beta \in H^4(\mathcal{N}_g); \quad \psi \in H^3(\mathcal{N}_g) \otimes H^1(\Sigma_g).$$

(The scalar factors are inserted to agree with the conventions of [13].) The Poincaré duals of our loops  $e_i$  form a basis  $e^1, e^2, \dots, e^{2g}$  of  $H^1(\Sigma_g, \mathbb{Z})$ ; using this, we can decompose

$$\psi = \sum_{i=1}^{2g} \psi_i e^i,$$

where  $\psi_i \in H^3(\mathcal{N}_g)$ . We can now state the following crucial result.

(21) **Theorem** (Newstead). *The ring  $H^*(\mathcal{N}_g)$  is generated by  $\alpha$ ,  $\beta$ , and the  $\psi_i$ .*

**Remark.** Atiyah and Bott [1] later showed that the integral cohomology  $H^*(\mathcal{N}_g, \mathbb{Z})$  is torsion-free, so that rational multiples of the same classes generate the integral cohomology. Moreover, the  $\psi_i$  are actually integral generators of  $H^3(\mathcal{N}_g, \mathbb{Z})$ , a fact which we will need later.

Since we have a set of generators, to determine the ring structure completely it suffices to find the relations. The usual commutation relations hold, so this amounts to deciding for which nonnegative integers  $m_j, n_j, p_{i,j}$  we have

$$(22) \quad \sum_j \alpha^{m_j} \beta^{n_j} \left( \prod_i \psi_i^{p_{i,j}} \right) = 0.$$

But according to Poincaré duality,  $\mu \in H^q(\mathcal{N}_g) = 0$  if and only if  $\mu\nu = 0$  for all  $\nu \in H^{6g-6-q}(\mathcal{N}_g)$ . Hence in principle to decide when equality holds in (22) we need only evaluate the integrals

$$(23) \quad \alpha^m \beta^n \left( \prod_i \psi_i^{p_i} \right) [\mathcal{N}_g]$$

for those  $m, n, p_i$  such that

$$2m + 4n + 3 \sum_i p_i = 6g - 6.$$

A more general discussion of these top-dimensional pairings, and of their analogues in four dimensions, can be found in [4]. Our goal will be to find an explicit formula for (23).

The first thing to notice is the following.

(24) **Proposition.** *Let  $1 \leq i_0 \leq g$ . If  $p_{i_0} > 1$ , or if  $p_{i_0} \neq p_{i_0+g}$ , then the pairing in (23) is zero.*

*Proof.* The first part is easy, since the commutation relations imply  $\psi_{i_0}^2 = 0$ . This only leaves the case  $p_{i_0} + p_{i_0+g} = 1$  for the second part. Now if  $f: \Sigma_g \rightarrow \Sigma_g$  is an orientation-preserving diffeomorphism, then by naturality

$$\hat{f}^*(\alpha)^m \hat{f}^*(\beta)^n \left( \prod_i \hat{f}^*(\psi_i)^{p_i} \right) [\mathcal{N}_g] = \alpha^m \beta^n \left( \prod_i \psi_i^{p_i} \right) [\mathcal{N}_g].$$

However, since  $(\hat{f} \times f)^* \mathcal{U} \cong \mathcal{U}$ , we have

$$\hat{f}^*(\alpha) = \alpha, \quad \hat{f}^*(\beta) = \beta, \quad \sum_i \hat{f}^*(\psi_i) f^*(e^i) = \sum_i \psi_i e^i.$$

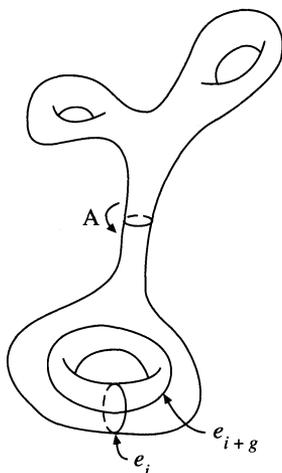


FIGURE 1

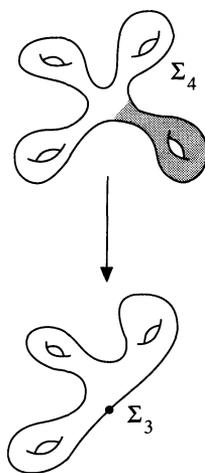


FIGURE 2

Now it is easy to find a diffeomorphism of  $\Sigma_g$  such that  $f^*(e^{i_0}) = -e^{i_0}$ ,  $f^*(e^{i_0+g}) = -e^{i_0+g}$ , but  $f^*(e^i) = e^i$  for all other  $i$ . For example, a half twist of the surface in Figure 1 below the loop labelled has the desired properties. If  $p_{i_0} + p_{i_0+g} = 1$ , we conclude

$$\alpha^m \beta^n \left( \prod_i \psi_i^{p_i} \right) [\mathcal{N}_g] = -\alpha^m \beta^n \left( \prod_i \psi_i^{p_i} \right) [\mathcal{N}_g] = 0. \quad \text{q.e.d.}$$

For  $1 \leq i \leq g$ , define  $\gamma_i = \psi_i \psi_{i+g}$ . Then the proposition above shows that all the pairings (23) are zero except those of the form

$$(25) \quad (\alpha^m \beta^n \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_p}) [\mathcal{N}_g],$$

where  $m + 2n + 3p = 3g - 3$  and  $1 \leq i_1 < i_2 < \cdots < i_p \leq g$ . Actually, the value of (25) is independent of the choice of  $i_j$ . This follows from a diffeomorphism argument similar to that in the proof of (24), because any permutation of the handles in Figure 1 can be realized by a diffeomorphism. Consequently, any pairing of the form (25) is equal to  $(g - p)! / (2^p g!)$  times the even simpler expression

$$(\alpha^m \beta^n \gamma^p) [\mathcal{N}_g],$$

where  $\gamma = 2 \sum_{i=1}^g \gamma_i$ . The class  $\gamma$  has the advantage of being independent of our choice of basis  $\{e^i\}$ , since  $\psi^2 = \gamma\sigma$ .

Still using the identification  $\mathcal{N}_g = S_g / \text{SO}(3)$ , we can perform another construction that would not have made sense in the holomorphic setting.

The map from  $\Sigma_g$  to  $\Sigma_{g-1}$  which collapses the  $i$ th handle to a point (see Figure 2) is not holomorphic, but it induces an embedding  $\eta_i: \mathcal{N}_{g-1} \rightarrow \mathcal{N}_g$ . The image of this embedding consists precisely of those conjugacy classes of representations of  $\pi_1(\widehat{\Sigma}_g)$  which send the generators  $e_i$  and  $e_{i+g}$  to the identity. That is,  $\eta_i(\mathcal{N}_{g-1}) = p(\pi_i^{-1}(I) \cap \pi_{i+g}^{-1}(I))$ , where  $\pi_i$  is the restriction to  $S_g$  of the projection of  $SU(2)^{2g}$  on the  $i$ th factor, and  $p$  is the quotient map  $S_g \rightarrow \mathcal{N}_g$ . Since  $\dim_{\mathbb{R}} \mathcal{N}_g = 6g - 6$ ,  $\mathcal{N}_{g-1}$  has real codimension 6.

**(26) Proposition.**  $\eta_i(\mathcal{N}_{g-1})$  is Poincaré dual to  $\gamma_i$ . Consequently,

$$\alpha^m \beta^n \gamma^p [\mathcal{N}_g] = 2g \alpha^m \beta^n \gamma^{p-1} [\mathcal{N}_{g-1}].$$

*Proof.* If we fix an element  $(A_1, A_2, \dots, A_{2g-2}) \in S_{g-1}$ , then the map  $SU(2) \rightarrow S_g$  given by

$$T \mapsto (A_1, \dots, A_{i-1}, T, A_i, \dots, A_{i+g-1}, T^{-1}, A_{i+g}, \dots, A_{2g-2})$$

is a right inverse for  $\pi_i$ . There is a similar right inverse for  $\pi_{i+g}$ . Hence the pullbacks by  $\pi_i$  and  $\pi_{i+g}$  of the fundamental cohomology class of  $SU(2)$  are indivisible classes  $\chi_i, \chi_{i+g} \in H^3(S_g, \mathbb{Z})$ , Poincaré dual to  $\pi^{-1}(I)$  and  $\pi_{i+g}^{-1}(I)$  respectively. We claim that  $\chi_i = \pm p^*(\psi_i)$  and  $\chi_{i+g} = \pm p^*(\psi_{i+g})$ .

To prove this, note that  $\pi_i^{-1}(I)$  is a union of fibers of the  $SU(2)$ -action on  $S_g$ , so  $\chi_i = p^*(\hat{\psi}_i)$  for some  $\hat{\psi}_i \in H^3(\mathcal{N}_g, \mathbb{Z})$ . This  $\hat{\psi}_i$  is unique, because we can see from the Leray-Serre spectral sequence (or even the Gysin sequence) that the natural map  $H^3(\mathcal{N}_g, \mathbb{Z}) \rightarrow H^3(S_g, \mathbb{Z})$  is injective. Now for any  $e^j$  in our basis for  $H^1(\Sigma_g, \mathbb{Z})$  with  $j \neq i$ , there exist diffeomorphisms  $f: \Sigma_g \rightarrow \Sigma_g$  such that  $f^*(e^i) = e^i$  but  $f^*(e^j) \neq e^j$ : the half twists and the Dehn twist in Figure 3 (next page) do the trick. Thus the only classes in  $H^3(\mathcal{N}_g, \mathbb{Z})$  which are invariant under  $f^*$  for all such  $f$  are the integer multiples of  $\psi_i$ . But  $\chi_i$  is invariant under such  $f^*$  by construction, and hence so is  $\hat{\psi}_i$ . Hence by the second part of (21),  $\hat{\psi}_i$  is an integer multiple of  $\psi_i$ . Since both  $\psi_i$  and  $\hat{\psi}_i$  are indivisible, we must have  $\hat{\psi}_i = \pm \psi_i$ . Likewise,  $\hat{\psi}_{i+g} = \pm \psi_{i+g}$ .

We shall not bother to pin down the sign, because for our purposes it is sufficient to note that the same sign holds for  $\hat{\psi}_i$  and  $\hat{\psi}_{i+g}$ . This can be deduced from a diffeomorphism argument like the one above, using a Dehn twist that induces  $e^i \mapsto e^{i+g}$  and  $e^{i+g} \mapsto e^i$  (see Figure 4). Then  $\chi_i \chi_{i+g} = p^*(\gamma_i)$ , and so descending to  $\mathcal{N}_g$  we obtain the first result.

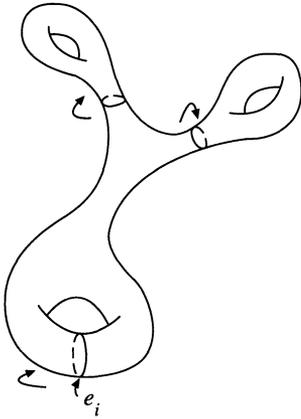


FIGURE 3

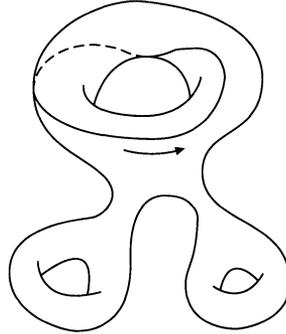


FIGURE 4

As for the second, it is sufficient to show that  $\eta^*(\alpha) = \alpha$ ,  $\eta^*(\beta) = \beta$ , and  $\eta^*(\gamma) = \gamma$ . This is straightforward from the construction of  $\mathcal{U}$ . q.e.d.

We can use this proposition recursively to eliminate  $\gamma$  from the pairings. Hence it now suffices to evaluate  $\alpha^m \beta^n [\mathcal{N}_g]$  when  $m + 2n = 3g - 3$ . This will be carried out in the next section.

### 5. A formula for the pairings

So far, we have practically ignored the complex structure on  $\Sigma_g$  and the complex structure it induces on  $\mathcal{N}_g$ . In this section we shall use that complex structure, together with what we already know, to calculate  $(\alpha^m \beta^n \gamma^p) [\mathcal{N}_g]$ . Recall that in §3 we gave a formula for  $D(g, k) = \dim H^0(\mathcal{N}_g; L^{k/2})$ , where  $L$  was a positive line bundle over  $\mathcal{N}_g$  and  $k$  was an even number. However, instead of the approach we took through mathematical physics, we could have gone through algebraic geometry, using the Hirzebruch-Riemann-Roch theorem. This tells us that

$$(27) \quad \sum_i (-1)^i \dim H^i(\mathcal{N}_g; L^{k/2}) = (\text{ch } L^{k/2} \text{ td } \mathcal{N}_g) [\mathcal{N}_g].$$

Since the canonical bundle of  $\mathcal{N}_g$  is negative [13], the left-hand side equals  $D(g, k)$  by Kodaira vanishing. On the other hand, we can calculate the right-hand side in terms of  $\alpha$  and  $\beta$ . Newstead [13] showed that  $c_1(\mathcal{N}_g) = 2\alpha$  and  $p(\mathcal{N}_g) = (1 + \beta)^{2g-2}$ . By restricting  $L$  to a projective subspace

of  $\mathcal{N}_g$  as in [13], it is not hard to show that  $c_1(L) = \alpha$ . As for  $\text{td } \mathcal{N}_g$ , the identity

$$\frac{x}{1 - e^{-x}} = \exp\left(\frac{1}{2}x\right) \frac{\frac{1}{2}x}{\sinh \frac{1}{2}x}$$

implies (see p. 117 of [15]) that, if  $y_i$  are the Pontrjagin roots,

$$\text{td} = \exp\left(\frac{1}{2}c_1\right) \prod_i \frac{\frac{1}{2}\sqrt{y_i}}{\sinh \frac{1}{2}\sqrt{y_i}}.$$

Noting that  $2g - 2$  of the Pontrjagin roots of  $\mathcal{N}_g$  are  $\beta$  and the rest are 0, we obtain

$$\text{td } \mathcal{N}_g = \exp(\alpha) \left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh \frac{1}{2}\sqrt{\beta}}\right)^{2g-2}.$$

Then by (27)

$$(28) \quad D(g, k) = \left(\exp\left(\frac{k+2}{2}\alpha\right) \left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh \frac{1}{2}\sqrt{\beta}}\right)^{2g-2}\right) [\mathcal{N}_g]$$

for even  $k \gg g$ . For fixed  $g$ , this is a polynomial in  $k$  whose coefficients involve the pairings. On the other hand, our original formula (19) is likewise a polynomial in  $k$ . Substituting  $l = k + 2$  in (19) and (28), and equating coefficients of  $l^m$ , we get

$$\begin{aligned} & - \left(-\frac{1}{2}\right)^{g-1} \frac{(2^{m-g+1} - 2)}{(m - g + 1)!} B_{m-g+1} P_{3g-3-m} \\ & = \frac{1}{m! 2^{3g-3}} P_{3g-3-m} (\alpha^m \beta^{(3g-3-m)/2}) [\mathcal{N}_g], \end{aligned}$$

where  $B_i$  is the  $i$ th Bernoulli number, and  $P_i$  is the coefficient of  $x^i$  in the power series expansion of  $(x/\sinh x)^{2g-2}$ . (We use the conventions under which  $B_2 = 1/6$ ,  $B_4 = -1/30$ , etc., and we interpret  $B_i = 0$  if  $i < 0$ , because  $(k+2)x/\sinh(k+2)x$  has no pole at 0.) Cancelling  $P_{3g-3-m}$  and rearranging, we obtain

$$(29) \quad (\alpha^m \beta^n) [\mathcal{N}_g] = (-1)^{g-1} \frac{m!}{(m - g + 1)!} 2^{2g-2} (2^{m-g+1} - 2) B_{m-g+1}.$$

In particular, we get the volume formula:

$$\text{vol } \mathcal{N}_g = \frac{1}{(3g-3)!} \alpha^{3g-3} [\mathcal{N}_g] = \frac{1}{(2g-2)!} 2^{2g-2} (2^{2g-2} - 2) |B_{2g-2}|.$$

Finally, we combine (29) with (26) to obtain, for  $m + 2n + 3p = 3g - 3$ ,

$$(30) \quad (\alpha^m \beta^n \gamma^p) [\mathcal{N}_g] = (-1)^{p-g} \frac{m!}{q!} \frac{g!}{(g-p)!} 2^{2g-2-p} (2^q - 2) B_q,$$

where  $q = m + p - g + 1$ . This agrees with the results published for genus 2 [12] and 3 [14]. For example, the number 224 obtained by Ramanan is just  $\alpha^6[\mathcal{N}_3]$ . It also enables us to prove Newstead's conjecture that  $\beta^g = 0$ . This conjecture first appeared in [13] and is discussed at greater length in [1]; there is no proof in the literature, but Frances Kirwan [11] has recently found a proof using equivariant cohomology. In any case, though, it is manifest from (30), since  $n > g - 1$  implies  $m + p - g + 1 < 0$ . Indeed, we get the more general and unsuspected result that  $\beta^{g-q}\gamma^q = 0$  whenever  $0 \leq q \leq g$ . However, Newstead's other conjecture, proved by Gieseker [8], that  $c_r(\mathcal{N}_g) = 0$  for  $r > 2g - 2$ , is not at all obvious from this equation, since we do not know of a systematic way to compute the Chern classes in terms of  $\alpha, \beta, \gamma$ .

It was promised in the introduction that the determination of the ring structure would be reduced to a problem in linear algebra, and this ought to be explained. To compute the relations in any degree  $q$ , we construct a matrix whose columns are indexed by the monomials in  $\alpha, \beta, \psi_i$  of total degree  $q$  and whose rows are indexed by similar monomials of complementary total degree  $6g - 6 - q$ . The matrix entry at row  $\mu$ , column  $\nu$  is given by  $\mu\nu[\mathcal{N}_g]$ , as determined from the formula above and (24). The rank of this matrix is just the  $q$ th Betti number of  $\mathcal{N}_g$ . (To prove that this rank agrees with the formula for the Betti number given in [1] could be regarded as a hard exercise in number theory.) Finding the relations at degree  $q$  is equivalent to finding a basis for the null space of this matrix. Such a basis could be obtained by row-reduction, but without any short cuts this would be prohibitively cumbersome. Luckily, though, for many applications, such as those employing the Riemann-Roch theorem, the top-dimensional pairings are exactly what we need.

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