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DEFORMATIONS OF CONFORMAL STRUCTURES ON HYPERBOLIC MANIFOLDS

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Abstract

This paper deals with the geometry of group representations, namely with some geometric approach (from the viewpoint of the $(n+1)$-dimensional hyperbolic geometry) to the space of uniformized conformally flat structures on a hyperbolic n-manifold M of finite volume. In fact three kinds of deformations are studied: bendings, stampings, and stampings-with-torsion along totally geodesic submanifolds of M. The constructions of the last two deformations disprove a conjecture of C. Kourouniotis. The third kind of deformations yields at first time the existence of quasi-Fuchsian groups in space with "maximal" round conic domains in the discontinuous set. Also the problems of nonconnectivity and generation of the deformation space are discussed—they are related to results on the geometry of Nielsen hull and on nontrivial hyperbolic homology cobordisms in four dimensions.

1. Introduction

We will describe here some geometric approaches to the theory of deformations of conformal structures on a hyperbolic n-manifold M, $n \geq 3$, of finite volume, i.e., a complete Riemannian manifold M locally modelled on the hyperbolic (Lobachevsky) space $\mathbb{H}^n$ of constant sectional curvature -1.

The hyperbolic metric in $\mathbb{H}^n$ endows the $(n-1)$-dimensional sphere at infinite $S^{n-1} = \partial\mathbb{H}^n$ with a conformal structure, where the group $\operatorname{Isom}\mathbb{H}^n \cong O(n, 1)$ acts as the group of all conformal automorphisms. Taking the Poincaré ball model of the hyperbolic n-space (in the unit ball $B^n(0, 1) \subset \mathbb{R}^n$), we have the isomorphism (cf. [3]):

$$\{\mathbb{H}^n, \partial\mathbb{H}^n, O(n, 1)\} \cong \{B^n(0, 1), S^{n-1}, \text{Möb}(n-1)\},$$

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where $\text{Möb}(n-1)$ is the Möbius group and S^{n-1} has the standard conformal structure induced by the Euclidean metric of $\mathbb{R}^n$. The Möbius group $\text{Möb}(n-1)$ is generated by reflections in 1-codimensional subspheres of S^{n-1}.

Given a hyperbolic n-manifold M, by conformal structure on M (-conformally flat structure) we mean a structure locally modelled on the standard conformal structure of the n-sphere $S^n \cong \mathbb{R}^n \cup \{\infty\}$, i.e., a maximal atlas on M with all changes of charts in the Möbius group. Extending some chart to the universal covering $\widetilde{M}$ of M, we obtain the developing map

$$d\colon \widetilde{M} \to S^n \tag{1.1}$$

inducing the holonomy homomorphism

$$d_*\colon \pi_1(M) \to \text{Möb}(n) \cong O(n+1, 1). \tag{1.2}$$

A conformal structure c on M will be called a uniformized structure (compare [18] and [9]) if its development d is not surjective while the holonomy group

$$G_* = d_*(\pi_1(M)) \subset \text{Möb}(n)$$

acts discontinuously in the domain $\Omega_0 = d(\widetilde{M}) \subset S^n$, i.e., G_* is a Kleinian group with an invariant connected component Ω_0 of the discontinuity set $\Omega(G_*)$ (see [16]). The manifold Ω_0/G_* with the natural conformal structure is conformally equivalent to the conformal manifold (M, c). Using the fundamental group $\pi_1(M)$ for the marking of the structures, we obtain the basic object of our study, the space $\mathscr{C}(M)$ of uniformized marked conformal structures on a hyperbolic manifold $M = \mathbb{H}^n/G$ $(G \subset \text{Isom}\,\mathbb{H}^n \cong O(n, 1))$ of finite volume.

The space $\mathscr{C}(M)$ has a natural topology. Namely, let

$$\mathscr{R}_n(G) = \text{Hom}(G, O(n+1, 1))$$

be the space of all representations of the group G into $O(n+1, 1)$ with the algebraic convergence topology defined as follows. Representations are close if they are close on a finite generating set. Inside $\mathscr{R}_n(G)$ we have the subspace $\mathscr{D}_n(G)$ of faithful representations with discrete images which act discontinuously somewhere on the n-sphere $S^n = \partial H^{n+1}$. The hyperbolic isometry group $O(n+1, 1)$ acts on the space $\mathscr{R}_n(G)$ by conjugation leaving the subspace $\mathscr{D}_n(G)$ invariant. The quotient space

$$\mathscr{T}_n(G) = \mathscr{D}_n(G)/O(n+1, 1) \subset \text{Hom}(G, O(n+1, 1))/O(n+1, 1)$$

is the space of conjugacy classes and is naturally identified with the space $\mathscr{C}(M)$ via the holonomy representation d_* of (1.2) induced by the developing map (1.1) (see [20] and [13]). This yields a topology on the space $\mathscr{C}(M)$.

We note that the space $\mathscr{C}(M)$ of uniformized conformal structures can be considered as a subspace of the compact space of $(n+1)$-dimensional hyperbolic structures on the finitely generated group $G \cong \pi_1(M)$ (see [21] and [7]).

For a finite volume hyperbolic n-manifold M, $n \geq 3$, the Mostow rigidity [22] states that the space of hyperbolic structures on M reduces to a point. The first results showing nontriviality of the space $\mathscr{C}(M)$ of conformal structures on such an n-manifold M, $n \geq 3$, were contained in [1] (for a finitely generated group G; they also answer a question of A. Borel & H. Wallach [12]) and, for infinitely generated G, in [24] (see also [1]). The matter was greatly clarified by Thurston's "Mickey Mouse example" in [25], which demonstrated that the deformation obtained above in the two-dimensional case corresponds to bending the Riemannian surface $\mathbb{H}^2/G$ along a closed geodesic. This idea of Thurston in n dimensions, $n \geq 3$, corresponds to bending of the manifold $M = \mathbb{H}^n/G$ along its totally geodesic hypersurface (see later [19] and [23]).

In §2 we observe how this idea implies a lower bound for the dimension of the space $\mathscr{C}(M)$. This was done independently and by different methods in [2], [14], and [17]. Moreover, as Johnson and Millson [14] discovered, in the general case, one cannot make simultaneous bendings of M along its intersecting totally geodesic hypersurfaces S_1 and S_2. Namely, if $c_1, c_2 \in H^1(G, \mathscr{M}_n)$ are infinitesimal deformations (elements of the Eilenberg-Mac Lane cohomology group with coefficients in the Lie algebra $\mathscr{M}_n$ of Möb(n)) of such bendings, then a composition $c \in H^1(G, \mathscr{M}_n)$ of c_1 and c_2 is a nonintegrable infinitesimal deformation. This fact motivated a conjecture of C. Kourouniotis (Oberwolfach, September 1985) that spatial deformations of $\mathbb{H}^n/G$ are exactly bendings.

§3 disproves this conjecture. The author's study [4] of the geometry of the Nielsen hull H_G in the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ yields a three-dimensional construction of a new "stamping deformation" of a hyperbolic manifold M along an intersection of totally geodesic hypersurfaces in M (see also [5], [8]). The fact that bending and stamping deformations are distinct is due to the existence of conic singularity along a geodesic on the boundary of the Nielsen hull H_{G_*} for every group $G_* \subset$ Möb(n) obtained by stampings from the Fuchsian group

$G \cong \pi_1(M)$. Moreover, §4 contains a construction of another kind of deformation of a closed hyperbolic 3-manifold M ("stamping-with-torsion" along a closed isolated geodesic in M), which are distinct form bending and stamping deformations.

Finally, in §5 we discuss the following questions.

First, is the space $\mathscr{C}(M)$ of uniformized conformal structures on a hyperbolic n-manifold M of finite volume (especially, for $n = 3$) connected?

Second, do the deformations of a hyperbolic manifold M of finite volume by bendings, stampings, and stampings-with-torsion generate the whole space $\mathscr{C}(M)$?

These questions are closely related to the author's results [4], [8] on the geometry of the Nielsen hull for the limit set of a geometrically finite Kleinian group in the n-sphere S^n and on nontrivial four-dimensional homology cobordism with geometrically finite hyperbolic structures (see [10], [6], and [7]).

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2. Bendings along totally geodesic hypersurfaces

Let $H^n \subset H^{n+1}$ be Poincaré models of Lobachevsky (hyperbolic) spaces in the unit balls or in the half-spaces with the distance function $d(*, *)$. Their isometry groups act on the spheres S^{n-1} and S^n at infinity as the Möbius groups $\text{Möb}_{n-1} \subset \text{Möb}_n$.

Consider a discrete cofinite group $G \subset \text{Möb}_{n-1}$ such that H^n/G contains a totally geodesic hypersurface N covered by a hyperplane $S \subset H^n$. Existence of such a N is equivalent to the existence of a fundamental polyhedron $P = P(G) \subset H^n$ having an $(n-1)$-side $s \subset S$ which is orthogonal to other intersecting sides of P and with the property that, for $g \in G$, $g(S) \cap S$ is either equal to S or is empty. The set of such sides s is denoted $b(G)$.

Let us denote

$$a = \inf(d(S, g(S)) : g \in G \backslash G_S).$$

The stabilizer G_S of S is different from G and $a > 0$ since $\text{vol}(H^n/G) < \infty$. Denote by $\mathscr{W}$ an infinite (since $G_S \neq G$) set of planes $g(S)$, $g \in G$,

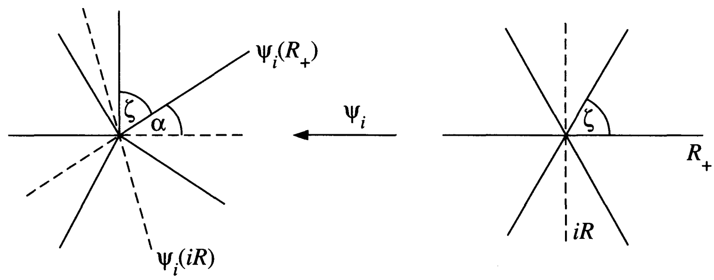

FIGURE 1

and by $S_0 \subset H^{n+1}$ the hyperplane intersecting H^n orthogonally along the hyperplane $S \subset H^n$. We will assume that the origin lies in a component of $H^n \backslash \mathscr{W}$ adjacent to S.

We define a partial order on $\mathscr{W}$. To do this for every plane $R \in \mathscr{W}$ we define a hyperplane $R_0 \subset H^{n+1}$ (as for S) and let R_0^+ be the half-space in H^{n+1} bounded by R_0 and disjoint from 0. Then define, for any $R, Q \in \mathscr{W}$, $R \gg Q \Leftrightarrow R_0^+ \subset Q_0^+$.

For the Poincaré ball model $H^{n+1} = B^{n+1}(0, 1)$, consider a sequence of balls $B_i = B^{n+1}(0, 1 - 1/i)$ and a bijection $q: N \to \mathscr{W}$ such that q^{-1} is order-preserving (subsets $\{R \in \mathscr{W} : R_0 \cap B_i \neq \emptyset\}$, $i \geq 2$, are finite since G is discrete). Consider $\alpha, \zeta \in \mathbb{R}$ such that

$$0 < \zeta < \pi/2, \qquad 0 \leq \alpha < \pi - 2\zeta, \tag{2.1}$$

and define a quasiconformal bending $\psi_j = \varphi_j(\alpha, \zeta)$, $\psi_j: H^{n+1} \to H^{n+1}$, which bends H^n in H^{n+1} with the angle α along the hyperplane $R = q(j) \in \mathscr{W}$, leaving fixed the half-space in $H^n \backslash R$ containing 0 and conformal in the ζ-angle neighborhood (in H^{n+1}) of the complement $H^n \backslash R$ (see Figure 1).

Studying the properties of bending supports and set with the linear dilatation $K(\psi_j, x)$ distinct from 1, C. Kourouniotis [17, Theorem 4.7] proved that in the case when both (2.1) and condition

$$\cosh a/2 > 1/\sin\zeta \tag{2.2}$$

are satisfied, the iterations of these bendings

$$f_j = \psi_1 \circ \psi_2 \circ \cdots \circ \psi_j : H^{n+1} \to H^{n+1}$$

converge to a quasiconformal homeomorphism $f_\alpha: H^{n+1} \to H^{n+1}$ compatible with the group G: $f_\alpha G f_\alpha^{-1} \subset \text{Möb}_n$. Moreover, the bending deformation, i.e., the family of the homomorphisms

$$f_\alpha^*: G \to G(\alpha) = f_\alpha G f_\alpha^{-1} \tag{2.3}$$

does not depend on ζ and defines on our manifold M new conformal uniformized structures obtained from the distinguished structure induced by the hyperbolic metric on H^n/G by bending along the hypersurface $N = S/G_S \subset H^n/G = M$. These new structures are induced by natural projections $\pi: \Omega_0 \to \Omega_0/G(\alpha)$, where $\Omega_0 \subset S^n$ is the component of the discontinuity set of the quasi-Fuchsian group $G(\alpha)$ equal to the quasiconformal ball $\hat{f}_\alpha(B)$, $B \subset S^n = \partial H^{n+1}$, where $\partial B = \partial H^n$ and $\hat{f}_\alpha$ is the quasiconformal extension of f_α to ∂H^{n+1}.

The author [1], [2] independently obtained a different direct approach to the construction of conjugating quasiconformal homeomorphisms f_α (and homomorphism f_α^* from (2.3)). This approach does not use a limiting process for iterations f_j but defines the mapping f_α directly on a fundamental polyhedron $P^{n+1} \subset H^{n+1}$, $P^{n+1} \cap H^n = P(G)$, of the group G, which maps it to another polyhedron $P^{n+1}(\alpha) \subset H^{n+1}$. The polyhedron $P^{n+1}(\alpha)$ has the same combinatorial type and the same angles as P^{n+1}. The distinction between these polyhedra is in the fact that hyperplanes which form the orbit $G(S_0)$, whose intersections s^* with H^n contain sides of the polyhedron P from the set $b(G)$, and turning isometrically in H^{n+1} by the angle α around the planes s^*. The map $f_\alpha^0: P^{n+1} \to P^{n+1}(\alpha)$ is obtained as uniform stretchings along the orbits (= circles) of the rotations mentioned above. The extension process of f_α^0 to the whole space H^{n+1} (compatible with G) defines the sought for quasiconformal bending homeomorphisms f_α.

These constructions by the author [2] and Kourouniotis [17] clearly show that $\{f_\alpha : \alpha \in I\}$ is a smooth (real-analytic) family for the open interval I with conditions (2.1) and (2.2). Moreover, these constructions give a dimension of a ball which embeds into the space $\mathscr{C}(M)$. This dimension is equal to the number of nonintersecting totally geodesic hypersurfaces in the hyperbolic manifold $M = H^n/G$ (see [2], [14], and [17]).

Using A. Weil's approach [27] one can define an infinitesimal deformation of the quasiconformal deformation (a smooth curve in $\mathscr{C}(M)$)

$$\beta: I \to \operatorname{Hom}(G, \text{Möb}_n)/\text{Möb}_n : \alpha \mapsto f_\alpha^* . \tag{2.4}$$

This infinitesimal deformation c is an element of the Eilenberg-Mac Lane cohomology group $H^1(G, \mathscr{M}_n)$, where $\mathscr{M}_n$ is the Lie algebra of Möb_n. Such an element c is determined (for bending see [17], [19]) by an assignment of a vector field $c(g)$ to each element $g \in G$ so that

$$c(g)(x) = \xi - g \cdot \xi(x), \qquad x \in H^{n+1}. \tag{2.5}$$

Here $\xi(x) = \frac{d}{d\alpha} f_\alpha(x)|_{\alpha=0}$ and the action of g on a vector field ξ is defined by the rule

$$g \cdot \xi(x) = Dg(g^{-1}(x)) \cdot \xi(g^{-1}(x)). \tag{2.6}$$

3. "Pea-pod" groups and stamping deformations

To define a new class of deformations of control structure on a manifold H^n/G, $\operatorname{vol} H^n/G < \infty$, distinct from bending deformations we consider special manifolds $M = H^n/G$. They correspond to quasi-Fuchsian groups on the conformal sphere $S^n = \partial H^{n+1}$ called "pea-pod" groups [4]. Their limit set resembles a pod of a certain plant (see Figure 2).

For a definition of these groups consider balls $D_1, \cdots, D_k \subset S^n$ whose intersection

$$D(x, y) = \bigcap_{i=1} D_i$$

is a k-hedral spherical angle with two vertices x and y. Let $h \in \text{Möb}_n$, $\operatorname{fix}(h) = \{x, y\}$, to be a loxodromic transformation, leaving $D(x, y)$ invariant. As a fundamental domain P_h of the cyclic group $\langle h \rangle$ one can take the exterior of two nonintersecting spheres S and S', $P_h \cap \{x, y\} = \varnothing$, which are orthogonal to $\partial D(x, y)$.

Now suppose that there is a finite family $\mathscr{W}$ of $(n-1)$-spheres decomposed into subfamilies $\mathscr{W}_i$ and $\mathscr{W}_{ij}$ $(i, j = 1, \cdots, k)$, where the spheres $S \in \mathscr{W}_{ij}$ are orthogonal to $\partial D_i \cap \partial D_j$ and the spheres $S \in \mathscr{W}_i$ are orthogonal to ∂D_i. In addition, spheres from different subfamilies $\mathscr{W}_{ij}, \mathscr{W}_{ml}$ and $\mathscr{W}_i, \mathscr{W}_j$ do not intersect each other and intersection angles to these spheres along themselves and with S, S' are equal to π/m, $m \in Z$.

If the balls $\operatorname{int} S$, $S \in \mathscr{W}$, cover the surface $P_h \cap \partial D(x, y)$, then the group $G = \langle G_0, h \rangle$, where G_0 is the group generated by reflections in spheres of the family $\mathscr{W}$. The group G is a geometrically finite quasi-

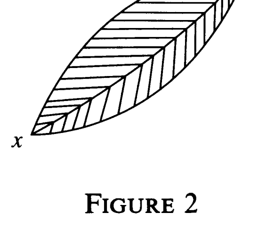

FIGURE 2

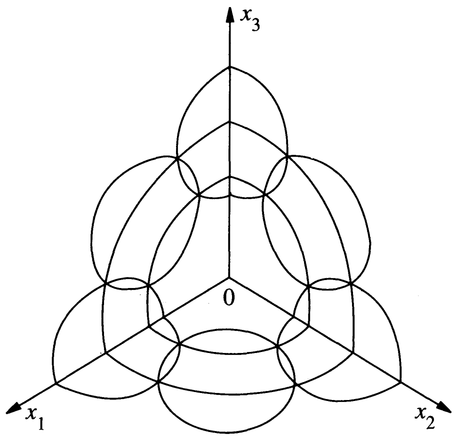

FIGURE 3

Fuchsian subgroup in Möb_n (a "pea-pod" group); its limit set is shaped like a finely breaking k-hedral pod with given vertices x and y.

Let us give a construction of a one-parameter family of such groups in three-dimensions, pointing out the corresponding "stamping" deformation at the same time. Assume that $x = 0$, $y = \infty$ and let $D(t) = D_t(0, \infty)$ be a regular trihedral angle in $\mathbb{R}^3$ with unit edge vectors $v_1(t) = (1, 0, 0)$, $v_2(t)$, $v_3(t)$ such that the scalar product $(v_i(t), v_j(t)) = \cos t$ for $i \neq j$ (t is the value of the side angle at vertex 0). For $t = 2\pi/3$ the boundary of $D(2\pi/3)$ is the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$.

For the definition of the "pea-pod" group $G(t) \subset \text{Möb}_3$ take eight spheres bounding a spherical polyhedron $P(t)$. Their intersection with the angle $D(t)$ is indicated in Figure 3.

Let i, j, k be a permutation of numbers $1, 2, 3$. Define spheres

$$\begin{aligned} S_{ij}(t) &= S^2(b_t v_t^k, r_t), \quad S(t) = S^2(0, R_t), \quad S'(t) = S^2(0, R'_t), \\ S_k(t) &= S^2(b_t(v_t^i + v_t^j)/(2 + 2\cos t)^{1/2}, r_t). \end{aligned} \tag{3.1}$$

Here we denote by $S^2(x, r)$ the two-sphere with center x and radius $r > 0$, and the numbers $R(t)$, $R'(t)$, b_t, and r_t are determined (modulo a positive factor) by the equalities

$$\begin{aligned} &2b_t \sin(t/4) = \sqrt{3}r_t, \qquad R_t = \left(-r_t + \sqrt{4b_t^2 - 3r_t^2}\right)/2, \\ &R'_t = (b_t^2 - r_t^2)/R_t. \end{aligned} \tag{3.2}$$

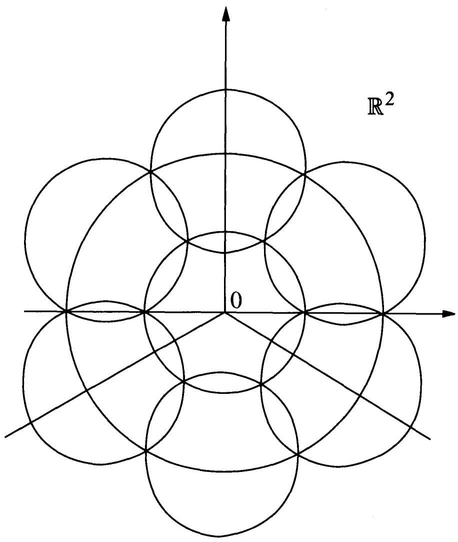

FIGURE 4

Taking a normalization $b_t = \sqrt{3}$ we obtain:

$$(3.3)\qquad \begin{aligned} &r_t = \sin(t/4), \qquad R_t = \sqrt{3}\cos(t/4) - \sin(t/4), \\ &R'_t = \sqrt{3}\cos(t/4) + \sin(t/4). \end{aligned}$$

A direct calculation proves that all dihedral angles of the polyhedron

$$P(t) = \bigcap_{i,j}(\operatorname{ext} S_{ij}(t) \cap \operatorname{ext} S_i(t)) \cap \operatorname{int} S'(t) \cap \operatorname{ext} S(t)$$

do not depend on t, $t_0 < t \leq 2\pi/3$, and are equal to $\pi/3$, where the value $t_0 < \pi/2$ corresponds to the tangency of the spheres $S_{ij}(t_0)$ and $S_{ik}(t_0)$, $j \neq k$.

Thus we can apply to the family (3.1) our "pea-pod" group construction and get as a result the family $G(t)$ of quasi-Fuchsian groups on S^3 which is generated by reflections in spheres in (3.1) and depends smoothly (real-analytically, due to (3.3)) on the parameter t. Moreover, the group G, $G = G(2\pi/3) \subset \text{Möb}_3$, acts isometrically in the Poincaré model of the hyperbolic space in $D(2\pi/3) = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$ as a cofinite group: $\operatorname{vol}\mathbb{R}^3_+/G < \infty$. Figure 4 shows the intersection of $\partial\mathbb{R}^3_+$ with the spheres (= planes in H^3) which bound the fundamental polyhedron $P = P(2\pi/3)$.

Consider polyhedra P^4 and $P^4(t)$ in the hyperbolic space H^4 (Poincaré model in $\mathbb{R}^4_+$) bounded by three-planes spanned on two-spheres (3.1).

They are fundamental polyhedra in H^4 for G and $G(t)$ respectively. Define a $q(t)$-quasiconformal mapping $f_t\colon \operatorname{cl} P^4 \to \operatorname{cl} P^4(t)$, $\lim_{t\to 2\pi/3} q(t) = 1$, which gives a correspondence between their sides and, for some $K_1(t), K_2(t) > 0$, $\lim K_i(t) = 1$ (here t tends to $2\pi/3$), we have

$$f_t\,|_S\colon x \mapsto K_1(t)\cdot x; \qquad f_t\,|_{S'}\colon x \mapsto K_2(t)\cdot x.$$

This mapping f_t extends to a $q(t)$-quasiconformal mapping $F_t\colon H^4 \to H^4$ by the rule

$$F_t(x) = \begin{cases} f_t(x) & \text{if } x \in \operatorname{cl} P^4, \\ i_t(g)\cdot f_t g^{-1}(x) & \text{if } x \in H^4\backslash P^4. \end{cases} \tag{3.4}$$

Here $i_t\colon G \to G(t)$ is the natural isomorphism of the reflection groups, and g is an element of G for which $x \in g(\operatorname{cl} P^4)$. This extension is compatible with the action of G.

The mapping F_t extends to the sphere at infinity $\partial H^4 = S^3$ and gives a $q(t)$-quasiconformal automorphism of it. Therefore we obtain a smooth curve in the space of quasi-Fuchsian presentations of the group G:

$$\begin{gathered} \sigma\colon (t_0, 2\pi/3] \to \operatorname{Hom}(G, \text{Möb}_3)/\,\text{Möb}_3, \\ \sigma(t) = F_t^*\colon G \to G(t) = F_t G F_t^{-1}. \end{gathered} \tag{3.5}$$

If Q_1, Q_2, Q_3 are planes in $H^3 = \mathbb{R}^3_+$, whose lines at infinity ∂Q_i are closures of $\{x : x_0 = 0\}$, $\{x : x_1 = \sqrt{3}x_2\}$, $\{x : x_1 = -\sqrt{3}x_2\}$, respectively, then for them one defines (as in §2) the bending deformations β_i of H^3/G (along Q_i/G_{Q_i}) with infinitesimal elements $c_i \in H^1(G, \mathscr{M}_3)$. It is clear from the stamping construction (3.5) that it defines simultaneous bendings (at equal angles $t_2 = t_2 = t_3 = a(t)$) along totally geodesic surfaces $Q_i/G_{Q_i} \subset H^3/G$. However the stamping deformation does not reduce to bendings β_i due to the presence of stretching along closed geodesics in H^3/G covered by the axis $\{x : x_1 = x_2 = 0\} \subset H^3$ of the hyperbolic element $h \in G$. This shows that the infinitesimal deformation c_σ of stamping σ is such that the crossed-homomorphism $c_\sigma - c_1 - c_2 - c_3$ determines a nonzero element of $H^1(G, \mathscr{M}_3)$.

Theorem 3.1. *For the stamping deformation σ from* (3.5) *and for every t, $t_0 < t < 2\pi/3$, the representation $\sigma(t) = F_t^*$ has an open neighborhood in* $\operatorname{Hom}(G, \textit{Möb}_3)$ *without bending representations of the group G.*

Proof. If the representation $\rho\colon G \to \text{Möb}_n$ is defined by a bending deformation of a certain group $G \subset \text{Möb}_{n-1}$ (along a family of nonintersecting totally geodesic hypersurfaces in H^n/G), then it follows from the bending construction in §2 that the limit set $L(\rho G)$ of the quasi-Fuchsian group $\rho G \subset \text{Möb}_n$ is the image of the sphere $S^{n-1} = L(G)$ by a 1-quasi-conformal mapping, i.e., has the following property (see [4]). Let $H_\rho \subset H^{n+1}$ be the convex Nielsen hull of the limit set $L(\rho G)$ of the group ρG. Then the boundary $\partial H_\rho \subset H^{n+1}$ is isometrically developable in the plane H^n.

At the same time the boundary H_t^* of the Nielsen hull $H_{G(t)}$ for any "pea-pod" group $G(t) = \sigma(t)G = F_tGF_t^{-1}$, $t_0 < t < 2\pi/3$, has a singular geodesic $l(t) \subset H_t^*$ which is the axis of the hyperbolic translation $F_thF_t^{-1}$. Namely, as shown in Lemmas 4.1 and 4.4 in [4], a certain neighborhood of this axis $l(t)$ in H_t^* can be developed in the hyperbolic three-plane $H^3 \subset H^4$ only be means of a quasi-isometry of H^4 stretching its dihedral three-dimensional angles of value t to $2\pi/3$. This shows that the quasi-isometry coefficient of any such mapping $F: H^4 \to H^4$, conjugating the groups $G(t)$ and G, is no less than $2\pi/3t > 1$. This completes the proof.

The proof of the existence of singularity of the Nielsen hull $H_{G(t)}$ mentioned above has the following outline.

Let $p: \overline{H}^4 \to H_{G(t)}$ be the natural retraction mapping a point $x \in H^4 \backslash H_{G(t)}$ to the nearest point $p(x) \in \partial H_{G(t)}$. Denote by B_i, $i = 1, 2, 3$, half-spaces in $\mathbb{R}^3$ complementary to the half-spaces $\operatorname{cl} D_i$ from the definition of $D(t)$. Every $B_i \subset \Omega(G(t))$ is a (strictly) maximal ball of the group $G(t)$ in the sense of [4], i.e., its boundary has limit points which are not contained in any circle or a line. Let $V_i \subset B_i$ be dihedral angles with edges orthogonal to ∂B_i and containing 0. The angles V_i are products of the sides of $D(t)$ and half-lines. The restriction of the retraction p to V_i outside of a certain neighborhood $U(D(t))$ is a conformal homeomorphism and, moreover, the domain $p(\operatorname{cl}(V_1 \cup V_2 \cup V_3) \backslash U(D(t)))$ forms a full neighborhood of the geodesic ray on $\partial H_{G(t)}$ lying on the axis $l(t)$ of hyperbolic translation $F_thF_t^{-1}$ in the space H^4.

In other words, a full neighborhood of geodesic $l(t)$ on the three-dimensional surface $\partial H_{G(t)}$ has a conic singularity and consists of the union of three dihedral angles of magnitude t, $t_0 < t < 2\pi/3$. This completes the proof.

Remark 3.2. The stamping deformation of H^3/G, $\operatorname{vol} H^3/G < \infty$, along a closed geodesic constructed above can also be carried out in the case of a closed manifold as well as for four-dimensional manifolds. The question of the existence of similar stamping deformations along submanifolds of codimension 2 in the manifold H^n/G of dimension $n \geq 4$ remains open.

4. Stamping-with-torsion along an isolated geodesic

In this section we prove the existence of another kind of deformation of conformal structures on a closed hyperbolic three-manifold, i.e., of deformations of H^3/G which are distinct from bendings and stampings described in §§2 and 3. To define this class of deformations we consider manifolds H^3/G of a special kind, associated with certain quasi-Fuchsian groups on S^3 which are similar, in a sense, to the Jørgensen group G_J. The manifold H^3/G_J is the first example of a closed hyperbolic manifold fibered over the circle (see [15]). Namely, the Jørgensen group and our groups have a form of a semidirect product of geometrically infinite normal subgroups and subgroups of similarities with co-presentation $\langle a, b : a^k \cdot b^m = 1\rangle$, where k and m are certain integers. The construction of such groups was given by A. V. Tetenov and the author.

As in §3 we define a group $G \subset \text{Möb}_2$ and its deformation $G(t) \subset \text{Möb}_3$ by means of certain families of spheres in $\mathbb{R}^3$. Namely, for any number $\lambda > 1$ consider three circles $s_i = S^1(z_i, r_i)$ in the complex plane $\mathbb{C} = \mathbb{R}^2$ with centers z_i and radii r_i satisfying the following conditions:

(1) $r_1 = 1$, $r_2 = \lambda$, $r_3 = \lambda^3$;

(2) Arcs of the circles s_i bound a triangle whose angles are equal to $\pi/4$.

For the chosen circles, we denote by z_0 the points of the intersection of two circles:

$$\{z \in \mathbb{C} : |z_2 - z|/|z_1 - z| = 1\}, \qquad \{z \in \mathbb{C} : |z_3 - z|/|z_1 - z| = \lambda^3\},$$

farther from z_1.

Denote by $a > 0$ and by $b > 0$ respectively the angles between vectors $(z_2 - z_0)$ and $(z_3 - z_0)$ with a vector $(z_1 - z_0)$ (see Figure 5).

Applying the translation of the plane $\mathbb{C}$ one may assume that z_0 coincides with the origin, and $(z_i - z_0)$ are the radius-vectors of the points z_i, $i = 1, 2, 3$. Now define loxodromic transformations

$$T_1(z) = \lambda \cdot e^{ia} \cdot z, \qquad T_2(z) = \lambda^3 \cdot e^{-ib} \cdot z. \tag{4.1}$$

For them we have that $T_1(s_1) = s_2$ and $T_2(s_1) = s_3$. In addition, from the definition of the angles $a = a(\lambda)$ and $b = b(\lambda)$ we see that

$$\lim_{\lambda \to 1} a(\lambda) = \lim_{\lambda \to 1} b(\lambda) = 0,$$

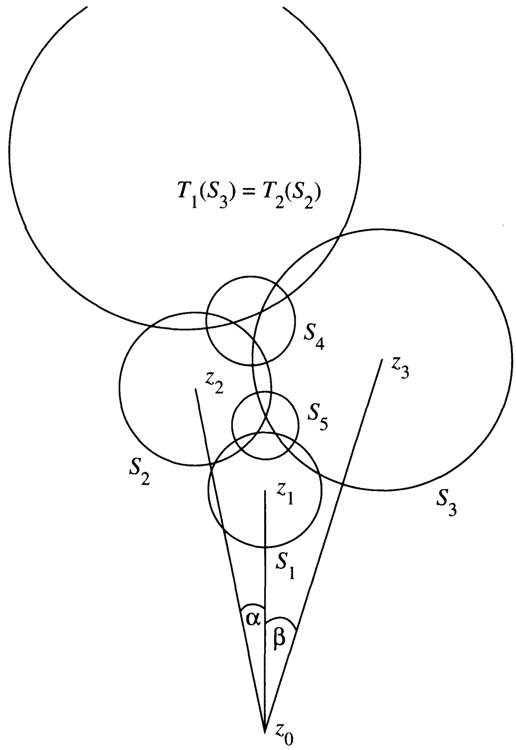

FIGURE 5

and for sufficiently large values of λ their sum $a(\lambda) + b(\lambda)$ becomes larger than $\pi/2$. This shows that there exist an integer m and a corresponding value $\lambda = \lambda(m) > 1$ such that the following conditions are satisfied:

(3) $3a(\lambda) + b(\lambda) = 2\pi/m$;

(4) The transformations T_1 and T_2 generate an elementary group H such that

$$H = \langle T_1, T_2 : T_1^{3m} \cdot T_2^{-m} = 1 \rangle . \tag{4.2}$$

The numbers $m \in \mathbb{N}$, $\lambda = \lambda(m)$, $a = a(\lambda(m))$, and $|z_1|$ (for $z_0 = 0$) obtained as a result of our constructions satisfy the system of equations

$$\begin{aligned} &|z_1|^2(1 - 2\lambda \cos(a) + \lambda^2) = \lambda^2 + \lambda\sqrt{2} + 1 , \\ &|z_1|^2(1 - 2\lambda^3 \cos(2\pi/m - 3a) + \lambda^6) = \lambda^6 + \lambda^3\sqrt{2} + 1 , \\ &|z_1|^2(\lambda^2 - 2\lambda^4 \cos(2\pi/m - 2a) + \lambda^6) = \lambda^6 + \lambda^4\sqrt{2} + \lambda^2 , \end{aligned} \tag{4.3}$$

due to the law of cosines.

Now consider the circle

$$T_1(s_3) = T_2(s_2), \tag{4.4}$$

and denote by s_4 the circle orthogonal to it and to the circles s_2 and s_3. Similarly denote by s_5 the circle disjoint from the circle s_4 and orthogonal to the circles s_1, s_2, s_3.

For the circles s_i, $i = 1, \cdots, 5$, consider the two-spheres $S_i \subset \mathbb{R}^3$ which intersect the plane $\mathbb{C}$ orthogonally along these circles s_i. Let $\mathscr{C}_0$ be a family of spheres in $\mathbb{R}^3$ forming the H-orbit (see (4.2)) of the spheres S_1, S_4, S_5. These spheres define planes in the hyperbolic space H^3 and bound a conical (infinite-sided) spherical polyhedron $P \subset S^3$ whose dihedral angles are equal to either $\pi/2$ or $\pi/4$:

$$P = \bigcap \{\operatorname{ext} S : S = h(S_i),\ h \in H;\ i = 1, 4, 5\}. \tag{4.5}$$

The connected component of P in the half-space $\mathbb{R}^3_+ = H^3$ is a hyperbolically convex polyhedron.

Now let a discrete group $F_0 \subset \text{Möb}_3$ be generated by reflections in the sides of the polyhedron P, i.e., in spheres of the family $\mathscr{C}_0$. P is a fundamental polyhedron for F_0 (see [3, 4.2]). Since elements of the group H transpose the sides of P, they preserve the group F_0, i.e., $hF_0h^{-1} = F_0$ for all $h \in H$. This fact shows that the semidirect product

$$G = H \ltimes F_0 \tag{4.6}$$

is a discrete subgroup of Möb_3 acting discontinuously in $S^3 \backslash \overline{\mathbb{R}}^3$ (isometrically in the space $H^3 = \mathbb{R}^3_+$). Moreover, the group (4.6) is a geometrically finite group with compact quotient $H^3/G = (H^3 \cap \operatorname{cl} P)/G$ and is generated by five elements: T_1, T_2, and three reflections in the spheres S_1, S_4, and S_5.

To construct the required quasiconformal deformation of the group G we define (as in §3) a family of quasi-Fuchsian groups $G(t) \subset \text{Möb}_3$ depending smoothly on parameter t in a certain ε-neighborhood of zero (ε to be defined below), $G(0) = G$.

Namely, just as for G, consider for any $t \in (-\varepsilon, \varepsilon)$ and for any number $\lambda > 1$ three spheres $S_i(t) = S^2(x_t^i, r_t^i)$, $i = 1, 2, 3$, with the centers $x_t^i \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ and radii r_t^i such that:

(1_t) $r_t^1 = 1$, $r_t^2 = \lambda$, $r_t^3 = \lambda^3$; $x_t^1 = (z_t^1, t)$, $x_t^2 = (x_t^2, \lambda t)$, $x_t^3 = (z_t^3, \lambda^3 t)$;

(2_t) Each sphere $S_i(t)$ intersects two other spheres at the angles $\pi/4$.

In the exterior of the spheres $S_i(t)$ consider an analogue of the point z_0, i.e., the point $x_t^0 = (z_t^0, 0)$ farther from the intersection point x_t^1 of the plane $\mathbb{C} \times \{0\}$ with the circle of intersection of two spheres:

$$\{x \in \mathbb{R}^3 : \lambda^3 = |x_t^3 - x|/|x_t^1 - x|\}, \qquad \{x \in \mathbb{R}^3 : \lambda = |x_t^2 - x|/|x_t^1 - x|\}.$$

The point x_t^0 depends smoothly on parameter t and $\lim_{t\to 0} x_t^0 = z_0$.

Denote by $a_t = a(t, \lambda) > 0$ and $b_t = b(t, \lambda) > 0$ angles between the orthogonal projection of the vectors $(x_t^2 - x_t^0), (x_t^1 - x_t^0)$ and $(x_t^3 - x_t^0), (x_t^1 - x_t^0)$, respectively, onto the plane $\mathbb{C} \times \{0\}$. Applying a translation $x \mapsto (x - x_t^0)$ we can assume that the point x_t^0 is at the origin. Analogously to (4.1), define orthogonal matrices

$$A_t = \begin{pmatrix} \cos a_t & \sin a_t & 0 \\ -\sin a_t & \cos a_t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_t = \begin{pmatrix} \cos b_t & -\sin b_t & 0 \\ \sin b_t & \cos b_t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and loxodromic transformations

$$T_{1t}(x) = \lambda \cdot A_t \cdot x, \qquad T_{2t}(x) = \lambda^3 \cdot B_t \cdot x. \tag{4.7}$$

As before, we have $T_{1t}(S_1(t)) = S_2(t)$ and $T_{2t}(S_1(t)) = S_3(t)$. Moreover, for the integer m, determined while defining the group H (condition (3)), one can find a number $\lambda_t = \lambda(m, t) > 1$ such that:

(3_t) $3a(t, \lambda(m, t)) + b(t, \lambda(m, t)) = 2\pi/m$;

(4_t) the transformations T_{1t} and T_{2t} generate an elementary group H^t with co-presentation

$$H_t = \langle T_{1t}, T_{2t} : T_{1t}^{3m} \cdot T_{2t}^{-m} = 1 \rangle. \tag{4.8}$$

The numbers $\lambda_t = \lambda(m, t)$, $a_t = a(t, \lambda(m, t))$, and $|z_t^1|$ (here $x_t^0 = 0$) are solutions of the system of equations obtained (as in (4.3)) from our geometric construction by applying the law of cosines:

(4.9)

$$\lambda_t^2 + \lambda_t\sqrt{2} + 1 = |z_t^1|^2(1 - 2\lambda_t \cos(a_t) + \lambda_t^2) + t^2(\lambda_t - 1)^2,$$

$$\lambda_t^6 + \lambda_t^3\sqrt{2} + 1 = |z_t^1|^2\left(1 - 2\lambda_t^3 \cos\left(\frac{2\pi}{m} - 3a_t\right) + \lambda_t^6\right) + t^2(\lambda_t^3 - 1)^2,$$

$$\lambda_t^6 + \lambda_t^4\sqrt{2} + \lambda_t^2 = |z_t^1|^2\left(\lambda_t^2 - 2\lambda_t^4 \cos\left(\frac{2\pi}{m} - 2a_t\right) + \lambda_t^6\right) + t^2(\lambda_t^3 - \lambda_t)^2.$$

It should be noted that due to the smoothness of this system with respect to t the above parameters of the family $(S_i(t))$ and of the group H_t depend smoothly on t and they tend to the parameters from (4.3) determining the group $H = H_0$ when t approaches zero.

Define two nonintersecting spheres $S_4(t)$ and $S_5(t)$ which are, first, orthogonal to the triples of spheres $S_2(t)$, $S_3(t)$, $T_{1t}(S_3(t)) = T_{2t}(S_2(t))$ and $S_2(t)$, $S_3(t)$, $S_1(t)$, respectively, and, second, disjoint from other spheres of the H_t-orbit of spheres $S_i(t)$, $i = 1, 2, 3$. The second condition determines the number $\varepsilon > 0$ bounding the parameter t.

Consider a family $\mathscr{C}_t = H_t\{S_i(t) : i = 1, 4, 5\}$ whose spheres bound conic infinitely-sided spherical polyhedron $P(t) \subset S^3$ with dihedral angles $\pi/2$ and $\pi/4$ (as for $P = P(0)$). As demonstrated above, a discrete group $F_t \subset \text{Möb}_3$ generated by reflections in sides of $P(t)$ is preserved by elements of the group H_t (i.e., by automorphisms of the boundary of $P(t)$): $hF_th^{-1} = F_t$, $h \in H_t$, and therefore their semidirect product

$$G(t) = H_t \ltimes F_t \tag{4.10}$$

is a discrete (quasi-Fuchsian) group isomorphic to the group G.

Moreover, by extending spheres of the families $\mathscr{C}_0$ and $\mathscr{C}_t$ to three-planes of the hyperbolic space $H^4 = \mathbb{R}^4_+$ which bound polyhedra P^4 and $P^4(t)$ in H^4, we can construct in a similar way a quasiconformal homeomorphism $f_t \colon H^4 \to H^4$ compatible with G and inducing the isomorphism $f_t^* : G \to G(t) = f_tGf_t^{-1}$.

Thus, in the space of quasi-Fuchsian representations of the group G we have a smooth curve defining another (third) kind of (quasi-Fuchsian) deformation of conformal structure on H^3/G. This deformation is said to be stamping-with-torsion along a closed geodesic $l \subset H^3/G$ covered by the axis $\{x \in \mathbb{R}^3_+ : x_1 = x_2 = 0\}$ of the loxodromic elementary subgroup $H \subset G$. The choice of this name is motivated by the following facts. First, the vector field defined by (2.5) and (2.6) determining an infinitesimal deformation $c^* \in H^1(G, \mathscr{M}_3)$ for stamping-with-torsion σ_l along l,

$$\begin{aligned} &\sigma_l \colon (-\varepsilon, \varepsilon) \to \text{Hom}(G, \text{Möb}_3)/\,\text{Möb}_3, \\ &\sigma_l(t) = f_t^* : G \to G(t) = f_tGf_t^{-1}, \end{aligned} \tag{4.11}$$

leaves the geodesic l invariant. Second, for the quasi-Fuchsian group $G(t)$ there exists a maximal circular cone $C(0, \infty)$, ("maximal conic domain" in the sense of [4]) in the discontinuity set $\Omega(G(t))$ on whose boundary limit points $\partial C(0, \infty) \cap L(G(t))$ form an irrationally twisted spiral. This is impossible in the case of the simple stamping from §3. The last fact is due to the presence of an isometrically undevelopable singularity of the circular cone type at the boundary of the Nielsen hull for the quasi-Fuchsian group $G(t)$, $t \neq 0$ (see also [8]).

5. General properties of the space of uniformized conformal structures on hyperbolic manifolds

It is a natural question to ask the following:

Do all deformations of the conformal structure of a hyperbolic manifold $M = H^n/G$, $\operatorname{vol} M < \infty$, (even if $n = 3$) become exhausted by the deformations constructed above, i.e., by bendings, stampings, and stampings-with-torsion?

Note that the space $\mathscr{C}(H^2/G)$ in the case of a closed surface H^2/G of the genus $g > 1$ is well known to be a connected space of dimension $6g-6$. The bending deformations yield a $(3g-3)$-dimensional submanifold. Here $(3g-3)$ is the number of closed nonintersecting geodesics on the surface H^2/G. The remaining deformations are determinated by the Dehn twists along these geodesics. However, in contrast to the closed geodesic on H^2/G, a number of conformal homeomorphisms of a totally geodesic surface in a manifold H^3/G is finite. Therefore the number of conformal structures on H^3/G corresponding to distinct conformal gluings along a bending surface in H^3/G can be at most finite.

On the other hand, the study of the limit set $L(G')$ of a quasi-Fuchsian group $G' \subset \text{Möb}_3$ (compare [4, Theorem 3.3] and [8, Theorem 3.2]) shows that the discontinuity set $\Omega(G') = S^3\backslash L(G')$ is covered by a family consisting of (strictly) maximal balls, and maximal conic domains which is finite modulo G'. In other words, all the possible singularities of the limit set $L(G')$ (or, equivalently, of the boundary of its Nielsen hull) can be obtained by applying the three kinds of deformations of a Fuchsian group $G \subset \text{Möb}_3$ constructed above in §§2–4.

On this basis we conjecture the positive answer to the question above, i.e., bendings, stampings, and stampings-with-torsion exhaust all possibilities.

The second question deals with connectedness of the space $\mathscr{C}(M)$ of uniformized marked conformal structures on a hyperbolic three-manifold $M = H^3/G$ of finite volume. In contrast to the two-dimensional case, the three-dimensional conformal structure arising in the following theorem is likely to give a negative answer to this question.

Theorem 5.1. *On a closed hyperbolic 3-manifold with a sufficiently large number of nonintersecting totally geodesic surfaces there exists a nonstandard conformal structure uniformized by a non-quasi-Fuchsian Kleinian group G without parabolic elements acting on a contractible component of the discontinuity set $\Omega(G)$ as the holonomy group $d_*(\pi_1(M))$.*

Proof. To prove the theorem it is sufficient to use our construction [10] of the hyperbolic four-manifolds with the following homotopy pathology.

These manifolds H^4/G are geometrically finite and after the compactification by two boundary components $(\partial H^4 \backslash L(G))/G = \Omega(G)/G$, $\Omega(G) \subset \partial H^4$, they become compact cobordisms $(M^4; N_0, N_1)$, $\partial M^4 = N_0 \cup N_1$. These cobordisms are homologically trivial, i.e., they have trivial relative homology groups

$$H_*(M^4, N_0) = H_*(M^4, N_1) = 0.$$

Still these cobordisms are not products. Moreover, they are not h-cobordisms. This is because of the fact that although the embedding $N_0 \subset M^4$ of the first component N_0 of the boundary ∂M^4 induces a homotopy equivalence, $\pi_*(M^4, N_0) = 0$, for the second boundary component N_1 we have $\pi_2(M^4, N_1) \neq 0$. Moreover, it is proved in [10] that there exists a cocompact group $\Gamma \subset \operatorname{Isom} H^3$ acting on the ball $B^3 = H^3$ as a hyperbolic isometry group and a quasiconformal embedding $f: B^3 \hookrightarrow \mathbb{R}^3$ compatible with Γ which defines on the hyperbolic manifold B^3/Γ the natural conformal structure of the manifold $N_0 = \Omega_0/G$, i.e., $f_*(\Gamma) = G$. This representation $\Gamma \to G$ is non-quasi-Fuchsian due to homotopical nontriviality of the cobordism.

Remark 5.2. The natural conformal structure on N_0 induced by the natural projection $\Omega_0 \to \Omega_0/G = N_0$ of the component Ω_0 of the discontinuity set $\Omega(G)$ (Ω_0 is the quasiconformal ball) determines a point of the space $\mathscr{C}(H^3/\Gamma)$ which presumably cannot be joined with the distinguished point corresponding natural (hyperbolic) structure on H^3/Γ by a curve in this space. It is likely that this follows from the fact that the quasiconformal ball $\Omega_0 \subset S^3$ which universally covers N_0 has as its boundary a wildly knotted on a dense subset two-sphere in S^3. (Added in proof.) Recently the author constructed (see [5], [6]) nonstandard conformal structures on a closed hyperbolic three-manifold M (topologically similar to the structure of Theorem 5.1) which are uniformized but cannot be approximated by quasi-Fuchsian structures on M obtained from the hyperbolic structure by any of the presently known deformations, i.e., by bending, stampings, or stampings-with-torsion along totally geodesic submanifolds. This construction uses a modification of nontrivial four-dimensional cobordisms in [10] to obtain certain additional geometric properties of the holonomy group G (more precisely, of the convex Nielsen hull H_G in the four-dimensional hyperbolic space H^4).

Remark 5.3. It is seen from the construction [10] that the manifold used above is obtained as a result of the bending deformations of the closed hyperbolic manifold H^3/Γ along a large number β, $\beta \geq 70$, of its disjoint totally geodesic surfaces. The curve $[\Gamma, G]$ corresponding to these deformations is a lift of a curve in representations variety lying in the image of the following embedding

$$S^1 \times \cdots \times S^1 = T^{\mathscr{M}} \hookrightarrow \mathrm{Hom}(\Gamma, \text{Möb}_3)/\text{Möb}_3 .$$

This curve $[\Gamma, G]$ joins the conformal structure of $N_0 = \Omega_0/G$ and $H^3\Gamma$ but it is not wholly contained in the space $\mathscr{C}(H^3/\Gamma)$ of uniformized conformal structures since many of its points have holonomies $d_* : \Gamma \to \text{Möb}_3$ whose images $d_*(\Gamma)$ act nondiscretely on the sphere S^3.

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