SINGULARITIES OF THE CURVE SHRINKING FLOW FOR SPACE CURVES

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Abstract

Singularities for space curves evolving by the curve shrinking flow are studied. Asymptotic descriptions of regions of the curve where the curvature is comparable to the maximum of the curvature are given.

PART I. OVERVIEW

1. Introduction. In this work, we will study singularity formation for space curves evolving by the curve shortening flow

$$(1.1) \qquad\qquad \frac{\partial \gamma}{\partial t} = k \cdot N,$$

where $\gamma : S^1 \times [0, \omega) \to R^3$, $\gamma(\cdot, 0)$ is a smooth curve, and $k \cdot N$ is the curvature times the normal to the curve. N is not always defined, though $k \cdot N$ always makes sense.

Although one has short time existence of solutions on a small open interval in time [7], solutions do not exist for infinite time. In a previous paper [2], it has been shown that solutions to the space curve flow exist until the curvature becomes unbounded. In this work, we study the limiting shape of the curve along forming singularities.

Space curves, in general, behave in a more complicated manner than plane curves. For example, they may not remain embedded (see Figure 1) and inflection points may develop [2]. We prove the rather surprising conjecture, due to Matt Grayson, that singularity formation is a planar phenomenon. We then give asymptotic descriptions of the solution.
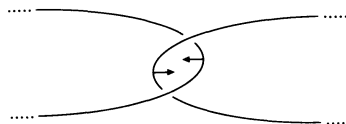

FIGURE 1. CURVES CAN CROSS

Assume that a solution to $\dfrac{\partial \gamma}{\partial t} = kN$ exists on the maximal time interval $[0, \omega)$. Our main results, briefly stated, are the following two theorems.

Theorem (Type-I Singularities). *If* $\lim\limits_{t \to \omega} \|k^2(\cdot, t)\|_\infty (\omega - t)$ *is bounded, then* γ *is asymptotic to a planar solution which is moving by homothety. These planar solutions are given by Abresch and Langer* [1].

Theorem (Type-II Singularities). *If* $\lim\limits_{t \to \omega} \|k^2(\cdot, t)\|_\infty (\omega - t)$ *is unbounded, then there exists a sequence of points and times* $\{p_n, t_n\}$ *on which the curvature blows up such that:*

(1) *a rescaling of the solution along this sequence converges in* C^∞ *to a planar, convex limiting solution* γ_∞;

(2) γ_∞ *is a solution which moves by translation called the Grim Reaper.*

In §7, we will make precise the notion of *rescaling* the solution along a sequence $\{(p_n, t_n)\}$. For now, let it suffice to say that we obtain new solutions γ_n to the curve shortening flow from γ by translating $t_n \mapsto 0$, $\gamma(p_n, t_n) \mapsto 0 \in R^3$, and dilating the solution in space and time so that $k^2(p_n, t_n) \mapsto 1$.

The notion of planarity will be made precise in §2 by considering the quantity τ/k where τ is the torsion of the curve. Another way of stating the planarity result, without recourse to the language of rescalings, is that $\lim\limits_{n \to \infty} (\tau/k)(p_n, t_n) \to 0$.

Planar curves, which evolve under the curve shortening flow by homothetically shrinking, were studied and classified by Abresch and Langer [1] (see also [5]). Huisken [11] and Angenent [3] have shown that if a type-I singularity develops on a planar curve, then the entire curve is asymptotic to an Abresch–Langer solution. The most trivial case of a curve moving by self-similarity is the circle shrinking down to a point. Another example of an Abresch–Langer curve is given in Figure 2.
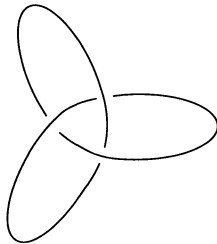

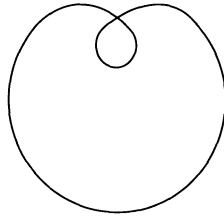FIGURE 2. EXAMPLE OF A DEVELOPING TYPE-I SINGU-LARITY

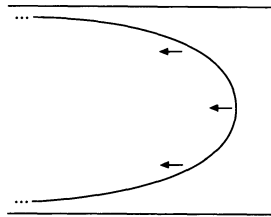FIGURE 3. EXAMPLE OF A DEVELOPING TYPE-II SINGU-
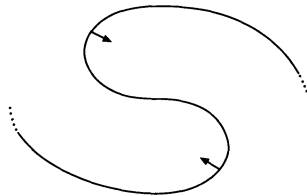LARITY

FIGURE 4. PICTURE OF A GRIM REAPER

FIGURE 5. PICTURE OF THE YIN-YANG CURVE

A standard example for a type-II singularity is given by a loop pinching off to a cusp (Figure 3). Angenent [3], in the case of convex-planar curves, showed that these singularities are asymptotic to Grim Reaper curves. $y = -\log\cos x$ is known as the Grim Reaper (Figure 4) and moves setwise by translation.

We mention one other noncompact curve (besides the straight line) which moves in a self-similar manner. This is the nonconvex yin-yang curve which spirals out to infinity (note that $\int |k|\,ds = \infty$). This curve moves by rotation (Figure 5).

The work is organized as follows:

In Part I we will introduce the indicatrix of a space curve. From some elementary computations, we will derive a precise notion of planarity.

In Part II we will prove some dilation-invariant estimates which will bound derivatives of the tangent vector to the curve in terms of the maximum of the curvature a short time earlier. We will use these estimates to

prove that the torsion and curvature cannot dissipate away from a region too quickly. In addition, we will derive some dilation-invariant integral estimates.

In Part III we will prove that $\lim_{n\to\infty} (\tau/k)\,(p_n, t_n) \to 0$. We will then examine the sequence of rescaled solutions and prove convergence to a limit solution along a subsequence of $\{(p_n, t_n)\}$. We will see that this limit solution is a family of planar, convex curves. For the case of a type-I singularity, we will use Huisken's monotonicity formula to prove that the limit solution is one given by Abresch and Langer. In the case of a type-II singularity, the limit of rescalings of the singularity is a Grim Reaper solution. It is interesting to note that each result on singularities follows from a dilation-invariant integral estimate together with a dilation-invariant pointwise estimate.

It is with pleasure that I thank my thesis advisor, Richard Hamilton, and Matthew Grayson for many enlightening and informative discussions; I owe much of my insight to them. I also wish to express special appreciation to Sigurd Angenent, Mike Gage and Lang-Fang Wu for their many useful comments and to thank B. Chow, D.T.G., and Michael Freedman for their encouragement.

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2. Terminology. For the remainder of the paper, we will assume that our solution exists on the maximal time interval of $[0, \omega)$.

The following definitions will be useful:

Definitions 2.1.

(1) The *maximum of the curvature squared* will be denoted by $M_t = \sup k^2(\cdot, t)$.

(2) $\{(p_n, t_n)\}$ is a *blow–up sequence* if $\lim_{n\to\infty} t_n \to \omega$ and $\lim_{n\to\infty} k^2\,(p_n, t_n) \to \infty$.

(3) $\{(p_n, t_n)\}$ is an *essential blow–up sequence* if (i) $\{(p_n, t_n)\}$ is a blow–up sequence and (ii) $\exists \rho \in R^+$, independent of n, such that $\rho M_t \le k^2\,(p_n, t_n)$ when $t \le t_n$.

We will use the following dilation–invariant categorization of singularity formation:

(4) *type*-I if $\lim_{t\to\omega} M_t \cdot (\omega - t)$ is bounded;

(5) *type*-II if $\lim_{t\to\omega} M_t \cdot (\omega - t)$ is unbounded.

It is not hard to see that an essential blow-up sequence always exists.

In order to fix notation, the Frenet matrix for a space curve γ, with arc

length parameter s, will be written as:

$$(2.2) \qquad \frac{\partial}{\partial s} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where T is the tangent vector, N is the normal vector, and B is the binormal vector. Assume, temporarily, that the curve has no inflection points (i.e., $k \neq 0$ everywhere).

Definition 2.3. The *tangent indicatrix* is the curve described on the unit sphere given by $\Gamma(s) = T(s)$.

Notice that the curve $\Gamma(s) : \mathrm{Image}(\gamma) \to S^2$ is not itself parametrized by arc length since

$$(2.4) \qquad \frac{\partial \Gamma}{\partial s} = \frac{\partial T}{\partial s} = kN.$$

So we define:

$$(2.5) \qquad \frac{\partial}{\partial \sigma} = \frac{1}{k} \frac{\partial}{\partial s}, \qquad d\sigma = k\,ds.$$

Derivatives with respect to this operator give

$$(2.6) \qquad \frac{\partial \Gamma}{\partial \sigma} = N, \qquad \frac{\partial^2 \Gamma}{\partial \sigma^2} = -T + \frac{\tau}{k} B.$$

We see now that the indicatrix curve Γ has T for its position vector, N for its tangent vector, and $\frac{\tau}{k} B$ for its geodesic curvature times the tangential component of its normal vector. The space curve is obviously planar along an arc if the indicatrix of the arc lies on a great circle, that is, if the tangent vectors to the arc on the space curve lie on the same plane.

Definition 2.7. A space curve is said to be *planar* at a point p if $\frac{\tau}{k}(p) = 0$.

Observation 2.8. Note the following:

(1) Dilating the space curve leaves the indicatrix unchanged.

(2) During the evolution, the torsion may go to infinity in a region where the curvature is blowing up, but all we are interested in is the ratio of τ/k.

PART II. MAIN ESTIMATES

3. Dilation invariant estimates. In a previous paper, we have shown:

Theorem (*Long Time Existence* [2]). *Let γ be a solution to* (1.1) *on the time interval* $[0, \alpha)$. *If k is bounded on* $[0, \alpha)$, *then* $\exists \epsilon > 0$ *such that $C(\cdot, t)$ is a smooth solution on the time interval* $[0, \alpha + \epsilon)$.

Throughout this work, we will assume that we have solutions to $\dfrac{\partial \gamma}{\partial t} = k \cdot N$ on the maximal time interval $[0, \omega)$.

We will now derive estimates for derivatives of k and τ on short time periods. These estimates will be dilation–invariant and depend only upon the maximum of the curvature at the starting time. See [12, §7] for analagous estimates on the Ricci-curvature flow.

Theorem 3.1 (*Dilation-invariant estimates*). *Fix* $t_n \in [0, \omega)$. *There exist constants* $\tilde{c}_l < \infty$ *independent of* t_n *such that for* $t \in (t_n, t_n + 1/\left(8M_{t_n}\right)]$ *we have*

$$(3.2) \qquad \left| \frac{\partial^l T}{\partial s^l} \right|^2 \leq \frac{\tilde{c}_l \cdot M_{t_n}}{\left(t - t_n\right)^{l-1}}.$$

Proof. Without loss of generality, we may assume $t_n = 0$, and then translate the estimates.

(1) The operators $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial s}$ do not commute [7]. In fact:

$$(3.3) \qquad \frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + \left| \frac{\partial T}{\partial s} \right|^2 \frac{\partial}{\partial s}.$$

Using the commutator for derivatives given above, we may derive the evolution equation for the tangent vector

$$(3.4) \qquad \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + \left| \frac{\partial T}{\partial s} \right|^2 T,$$

and, from this, the evolution of curvature squared

$$(3.5) \qquad \frac{\partial}{\partial t}\left(\left| \frac{\partial T}{\partial s} \right|^2 \right) = \frac{\partial^2}{\partial s^2}\left(\left| \frac{\partial T}{\partial s} \right|^2 \right) - 2\left| \frac{\partial^2 T}{\partial s^2} \right|^2 + 4\left| \frac{\partial T}{\partial s} \right|^4.$$

It follows from the maximum principle that M_t satisfies

$$(3.6) \qquad -\frac{1}{M_t} + \frac{1}{M_0} \leq 4t.$$

If $t \leq 1/\left(8 \cdot M_0\right)$, then

$$(3.7) \qquad\qquad M_t \leq 2M_0.$$

We may choose $\tilde{c}_0 = 2$.

(2) Our previous computations imply

(3.8)

$$\frac{\partial}{\partial t}\left(t \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^2\right)$$

$$\leq \frac{\partial^2}{\partial s^2}\left(t \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^2\right) + \left|\frac{\partial^2 T}{\partial s^2}\right|^2$$

$$- 2t \cdot \left(\left|\frac{\partial^3 T}{\partial s^3}\right| - \left|\frac{\partial T}{\partial s}\right|\left|\frac{\partial^2 T}{\partial s^2}\right|\right)^2$$

$$+ 16t \cdot \left|\frac{\partial T}{\partial s}\right|^2 \left|\frac{\partial^2 T}{\partial s^2}\right|^2 - 8\left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 16\left|\frac{\partial T}{\partial s}\right|^4$$

$$\leq \frac{\partial^2}{\partial s^2}\left(t \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^2\right) + (32M_0 t - 7) \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 64M_0^2.$$

Since $32M_0 t - 7 < 0$ on this time interval, we have

$$(3.9) \quad \frac{\partial}{\partial t}\left(t \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^2\right) \leq \frac{\partial^2}{\partial s^2}\left(t \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^2\right) + 64M_0^2.$$

Thus it follows that

$$(3.10) \qquad t \cdot \left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^2 \leq 4M_0 + 64M_0^2 t \leq 12M_0,$$

and we may conclude on this time interval that

(3.11)
$$\left| \frac{\partial^2 T}{\partial s^2} \right|^2 \leq \frac{12 M_0}{t}.$$

So we may choose $\tilde{c}_1 = 12$.

(3) The evolution for the higher derivatives satisfy [2]:

(3.12)
$$\begin{aligned}
\frac{\partial}{\partial t} \left(\left| \frac{\partial^n T}{\partial s^n} \right|^2 \right) &\leq \frac{\partial^2}{\partial s^2} \left(\left| \frac{\partial^n T}{\partial s^n} \right|^2 \right) - 2 \left(\left| \frac{\partial^{n+1} T}{\partial s^{n+1}} \right| - \left| \frac{\partial T}{\partial s} \right| \left| \frac{\partial^n T}{\partial s^n} \right| \right)^2 \\
&\quad + 2(n+2) \left| \frac{\partial T}{\partial s} \right|^2 \left| \frac{\partial^n T}{\partial s^n} \right|^2 \\
&\quad + 4n \left\langle \frac{\partial^n T}{\partial s^n}, \frac{\partial^2 T}{\partial s^2} \right\rangle \left\langle T, \frac{\partial^n T}{\partial s^n} \right\rangle \\
&\quad + 4(n+1) \left\langle \frac{\partial^n T}{\partial s^n}, \frac{\partial T}{\partial s} \right\rangle \left\langle \frac{\partial T}{\partial s}, \frac{\partial^n T}{\partial s^n} \right\rangle \\
&\quad + 2 \sum_{i=1}^{n-1} N_i \left\langle \frac{\partial^i T}{\partial s^i}, \frac{\partial^n T}{\partial s^n} \right\rangle,
\end{aligned}$$

where the coefficients N_i represent inner products which have been previously bounded.

The induction hypothesis and repeated usage of the Peter-Paul inequality allow us to find constants a_i and A on our time interval such that:

$$\frac{\partial}{\partial t} \left(\sum_{i=1}^{n} a_i t^{i-1} \left| \frac{\partial^i T}{\partial s^i} \right|^2 \right) \leq A M_0^2.$$

Thus, as before, we obtain $\tilde{c}_l$.   q.e.d.

Note that these estimates also reprove the long time existence result stated in [2]. That is, as long as the curvature remains bounded on an open time interval $[0, \alpha)$, one can define a smooth limit for the tangent vectors T at time α. Thus, by integrating the tangent vectors, one can obtain a smooth limit curve. Short time existence allows us to find solutions for a further time.

Recall that

(3.13)

$$\left|\frac{\partial T}{\partial s}\right|^2 = k^2,$$

$$\left|\frac{\partial^2 T}{\partial s^2}\right|^2 = k^4 + \left(\frac{\partial k}{\partial s}\right)^2 + (k\tau)^2,$$

$$\left|\frac{\partial^3 T}{\partial s^3}\right|^2 = \left(3k\frac{\partial k}{\partial s}\right)^2 + \left(\frac{\partial^2 k}{\partial s^2} - k^3 - k\tau^2\right)^2 + \left(2\frac{\partial k}{\partial s}\tau + k\frac{\partial \tau}{\partial s}\right)^2,$$

$$\left|\frac{\partial^4 T}{\partial s^4}\right|^2 = \left(k\frac{\partial^2 \tau}{\partial s^2} + 3\frac{\partial k}{\partial s}\frac{\partial \tau}{\partial s} + 3\frac{\partial^2 k}{\partial s^2}\tau - k^3\tau - k\tau^3\right)^2$$

$$\cdots + \left(\text{terms involving} \quad \left\{k, \tau, \frac{\partial k}{\partial s}, \frac{\partial \tau}{\partial s}, \frac{\partial^2 k}{\partial s^2}\right\}\right)^2.$$

We will often refer to the dilation-invariant estimates in the following form.

Corollary 3.14. *Let* $t_n = \tilde{t}_n + 1/\left(32M_{\tilde{t}_n}\right)$ *for* $\tilde{t}_n \in [0, \omega)$. *Assume that* $\rho M_t \le M_{t_n}$ *for* $t \le t_n$. *Then there exist constants* $c_1, c_2, c_3, c_4 < \infty$, *depending only on* ρ, *such that for* $t \in \left[t_n, t_n + 3\rho/\left(64M_{t_n}\right)\right]$ *we have*

$$k^2 \le c_1 M_{t_n},$$

$$\left(\frac{\partial k}{\partial s}\right)^2 \le c_2 M_{t_n}^2,$$

(3.15)
$$(k\tau)^2 \le c_2 M_{t_n}^2,$$

$$\left(\frac{\partial^2 k}{\partial s^2} - k^3 - k\tau^2\right)^2 \le c_3 M_{t_n}^3,$$

$$\left(2\frac{\partial k}{\partial s}\tau + k\frac{\partial \tau}{\partial s}\right)^2 \le c_3 M_{t_n}^3,$$

$$\left(k\frac{\partial^2 \tau}{\partial s^2} + 3\frac{\partial k}{\partial s}\frac{\partial \tau}{\partial s} + 3\frac{\partial^2 k}{\partial s^2}\tau - k^3\tau - k\tau^3\right)^2 \le c_4 M_{t_n}^4.$$

Proof. It is not hard to see that $\exists \{\tilde{t}_n\}$ such that $t_n = \tilde{t}_n + 1/(32M_{\tilde{t}_n})$. Define the time interval

(3.16) $I_n = [t_n, t_n + 3\rho/(64M_{t_n})]$ where $t_n = \tilde{t}_n + 1/\left(32M_{\tilde{t}_n}\right)$.

Since $M_{t_n} \geq \rho M_{\tilde{t}_n}$, I_n will be a subset of the region in time $[\tilde{t}_n, \tilde{t}_n + 1/(8M_{\tilde{t}_n})]$ on which our dilation-invariant estimates are useful and the result follows easily.   q.e.d.

The importance of this corollary is that the bounds on the right-hand side are given in terms of M at the time t_n instead of at the earlier time $\tilde{t}_n$.

4. Controlling the dissipation of curvature and torsion. Before we study singularities of the flow, we will examine points on the curve where the curvature is comparable to the maximum of the curvature. Estimates can be made on the amount of curvature which can escape from a small region in a short amount of time.

A "forward" space-time cylinder around a point on the curve will be defined as follows:

Definition 4.1. For $d \in R^+$ and $(p_n, t_n) \in S^1 \times [0, \omega)$ define:

(4.2)
$$\mathbf{N}(p_n, t_n, d) = \left\{ (p, t) \in S^1 \times [t_n, \omega) \mid \operatorname*{dist}_{t_n} \{p_n, p\} \right.$$
$$\left. \leq \sqrt{\frac{d}{M_{t_n}}}, |t_n - t| \leq \frac{d}{M_{t_n}} \right\}$$

Note that $k \sim \text{distance}^{-1}$ and $t \sim \text{distance}^2$ so the neighborhood makes dimensional sense. Also note that we are defining the neighborhood in terms of a time-independent metric $\operatorname*{dist}_{t_n}\{\cdot, \cdot\}$ where $\operatorname*{dist}_{t_n}\{u_1, u_2\} = \left| \int_{u_1}^{u_2} ds_{(t_n)} \right|$.

Theorem 4.3. *Let* $\{(p_n, t_n)\}$ *be an essential blow-up sequence. Then there exist constants* $d_0, d_1, d_2 \in R^+$ *depending only on* ρ *so that the following hold*:

(1) *The temporal loss of* $k(p_n, \cdot)$ *is bounded from below*:

(4.4) $|k(p_n, t)| \geq \dfrac{1}{\sqrt{2}} |k(p_n, t_n)|$ *for* $t \in [t_n, t_n + \dfrac{d_1}{M_{t_n}}]$.

(2) *The spatial loss of* $k(\cdot, t)$ *is bounded from below*:

$$|k(p, t)| \geq \frac{1}{\sqrt{2}} |k(p_n, t)| \quad \text{for } \operatorname*{dist}_{t}\{p, p_n\} \leq \sqrt{\frac{d_2}{M_{t_n}}};$$

(4.5) $t \in [t_n, t_n + \dfrac{d_1}{M_{t_n}}].$

(3) *Hence,*

(4.6) $|k(p, t)| \geq \dfrac{1}{2} |k(p_n, t_n)|$ *for* $(p, t) \in \mathbf{N}(p_n, t_n, d_0)$.

Proof.   As in Corollary (3.14), we will consider the time interval $[t_n, t_n + 3\rho/(64M_{t_n})]$.

First we bound the temporal loss of k.

From Corollary (3.14) there exists a constant a_1, depending only on ρ, such that

(4.7) $\left| \dfrac{\partial k}{\partial t} \right| \leq a_1 M_{t_n}^{3/2}$

away from where the curvature vanishes. This, in addition to the fact that $k^2(p_n, t_n) \geq \rho \cdot M_{t_n}$ imply

(4.8) $|k(p_n, t)| \geq \dfrac{|k(p_n, t_n)|}{\sqrt{2}}$ when $|t - t_n| \leq \dfrac{a_2}{M_{t_n}}$,

where a_2 is a constant depending only on ρ.

Hence, we define

(4.9) $d_1 = \min \left\{ \dfrac{3\rho}{64}, a_2 \right\}$.

Next, we bound the spatial loss of k.

Again, from Corollary (3.14) there exists a constant a_3, depending only on ρ such that

(4.10) $\left(\dfrac{\partial k}{\partial s} \right)^2 \leq a_3 M_{t_n}^2$.

As above, we may conclude that there exists a constant a_4, depending only on ρ, such that

(4.11) $|k(p, t)| \geq \dfrac{|k(p_n, t)|}{\sqrt{2}}$ when $\operatorname*{dist}_{t} \{p, p_n\} \leq \dfrac{a_4}{\sqrt{M_{t_n}}}$.

Define

(4.12) $d_2 = \min \{d_1, a_4\}$.

Notice that the second assertion of the theorem is a statement concerning a changing metric $\operatorname*{dist}_{t} \{\cdot, \cdot\}$. Our definition of neighborhood, however, utilizes a metric $\operatorname*{dist}_{t_n} \{\cdot, \cdot\}$ which has been frozen at time t_n. Since

distances decrease under the curve shortening flow, this claim follows from the previous two.   q.e.d.

In [2] we found that inflection points ($k = 0$) could develop for an evolving space curve. Where the curvature becomes zero, the torsion becomes unbounded. Of course, the Frenet frame is not even defined at such a point. One result of [2] (or alternately, the dilation–invariant estimates of this work) is that the flow ignores these types of singularities in the torsion. Let us now discuss the behavior of torsion near a point where the curvature is nonzero.

We showed above that the curvature could not dissipate away too quickly from a region of nonzero curvature. Here we will state, without proof, the analogous result for torsion. One uses the evolution equation for τ ([2]) and constants c_3, c_4 from Corollary 3.16. Otherwise, the proof is essentially the same as the one for curvature.

Theorem 4.13. *Let* $\{(p_n, t_n)\}$ *be an essential blow–up sequence. Assume that* $\exists \mu > 0$ *a constant such that* $\forall n$ *we have* $\tau^2(p_n, t_n) \geq \mu k^2(p_n, t_n)$. *Then there exist constants* $d_3, d_4, d_5 \in R^+$ *depending only on* μ *and* ρ *such that the following hold:*

(1) *The temporal loss of* $\tau(p_n, \cdot)$ *is bounded from below:*

$$(4.14) \qquad |\tau(p_n, t)| \geq \frac{1}{\sqrt{2}} |\tau(p_n, t_n)| \quad \text{for } t \in [t_n, t_n + \frac{d_4}{M_{t_n}}].$$

(2) *The spatial change of* $\tau(\cdot, t)$ *is bounded from below:*

$$(4.15)$$
$$|\tau(p, t)| \geq \frac{1}{\sqrt{2}} |\tau(p_n, t)| \quad \text{for } \operatorname*{dist}_t \{p, p_n\} \leq \sqrt{\frac{d_5}{M_{t_n}}}; \qquad t \in [t_n, t_n + \frac{d_4}{M_{t_n}}].$$

(3) *Hence*

$$(4.16) \qquad |\tau(p, t)| \geq \frac{1}{2} |\tau(p_n, t_n)| \quad \text{for } (p, t) \in \mathbf{N}(p_n, t_n, d_3).$$

5. The integral $\int |k| ds$. The dilation-invariant integral $\int |k| ds$ turns out to be a very useful quantity to study. In the case of a general space curve, we will show that $\frac{d}{dt} \int |k| ds \leq -\int \tau^2 |k| ds$. This estimate will be used to prove that singularities are asymptotically planar.

Theorem 5.1. *For a solution* γ *to the curve shortening flow we have*

$$(5.2) \qquad \frac{d}{dt} \int_{\gamma} |k| \, ds \leq -\int_{\gamma} \tau^2 |k| \, ds.$$

Proof. From

(5.3)
$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 - \tau^2 k,$$

we may derive

(5.4)
$$\frac{\partial}{\partial t}\left(k^2\right) = \frac{\partial^2}{\partial s^2}\left(k^2\right) - 2\left(\frac{\partial k}{\partial s}\right)^2 + 2k^4 - 2\tau^2 k^2.$$

A technical difficulty of this theorem is that for space curves $k^2 = \left|\frac{\partial T}{\partial s}\right|^2$ is a more natural function to study than k. In the special case of a planar curve, by choosing a consistent normal field one may define an "inside" and an "outside". Then $k > 0$ or $k < 0$ makes sense.

Hence, following a suggestion of R. Hamilton, we will make use of the function $\sqrt{k^2 + \epsilon}$. For simplicity, denote $K_\epsilon = \sqrt{k^2 + \epsilon}$ where $\epsilon > 0$. The derived equation for this quantity is

(5.5)
$$\frac{\partial K_\epsilon}{\partial s} = \frac{\partial^2 K_\epsilon}{\partial s^2} + \frac{1}{K_\epsilon^3} k^2 \left(\frac{\partial k}{\partial s}\right)^2 + \frac{1}{K_\epsilon}\left(-\left(\frac{\partial k}{\partial s}\right)^2 + k^4 - \tau^2 k^2\right).$$

Since $k < K_\epsilon$ for all $\epsilon > 0$, we have

(5.6)
$$\frac{d}{dt}\int_\gamma K_\epsilon \, ds$$

$$= \int_\gamma \left[\frac{\partial^2 K_\epsilon}{\partial s^2} + \frac{1}{K_\epsilon^3} k^2 \left(\frac{\partial k}{\partial s}\right)^2\right] ds$$

$$+ \int_\gamma \left[\frac{1}{K_\epsilon}\left(-\left(\frac{\partial k}{\partial s}\right)^2 + k^4 - \tau^2 k^2\right) - K_\epsilon k^2\right] ds$$

$$= \int_\gamma \left[\frac{1}{K_\epsilon^3} k^2 \left(\frac{\partial k}{\partial s}\right)^2 + \frac{1}{K_\epsilon}\left(-\left(\frac{\partial k}{\partial s}\right)^2 + k^4 - \tau^2 k^2\right) - K_\epsilon k^2\right] ds$$

$$\leq -\int_\gamma \frac{1}{K_\epsilon} \tau^2 k^2 \, ds.$$

The result follows from letting $\epsilon \to 0$.   q.e.d.

Using this we obtain the following dilation-invariant integral estimate.

Corollary 5.7. $\exists C > 0$ *such that*

(5.8)
$$\int_0^\omega \int_\gamma \tau^2 |k| \, ds \, dt \leq C.$$

Proof. We have already shown that

$$(5.9) \qquad \frac{d}{dt} \int_\gamma |k| \, ds \leq 0.$$

Note that if the flow is defined on the time interval $[0, \omega)$, then the integral $\int_\gamma |k| \, ds$ is defined on $[0, \omega]$; a limit at time ω exists and is unique, since the integral is bounded from below and is monotonically decreasing. Hence, we have

$$(5.10) \qquad \int_\gamma |k| \, ds(\omega) \leq \int_\gamma |k| \, ds(0),$$

and therefore

$$(5.11) \qquad \int_0^\omega \int_\gamma \tau^2 |k| \, ds \, dt \leq \int_\gamma |k| \, ds(0) - \int_\gamma |k| \, ds(\omega) \leq \int_\gamma |k| \, ds(0).$$

q.e.d.

We will use the following formulation of this result.

Corollary 5.12. $\forall \epsilon, \exists \theta$ *such that*

$$(5.13) \qquad \int_{\omega-\theta}^\omega \int_\gamma \tau^2 |k| \, ds \, dt < \epsilon.$$

For a planar curve, the decreasing integral $\int |k| ds$ measures the total change in angle. An estimate on how this integral evolves will be used to show that the limit of the rescaled solutions is convex. In the special case of convex planar curves, $\int |k| ds = \int k ds$ measures the winding number of the curve and is an invariant of the flow (until a singularity develops).

Theorem 5.14. *For a planar solution γ to the curve shortening flow we have*

$$(5.15) \qquad \frac{d}{dt} \int_\gamma |k| \, ds = -2 \sum_{p | k(p, \cdot) = 0} \left| \frac{\partial k}{\partial s} \right|.$$

Proof. Use

$$(5.16) \qquad \int_\gamma |k| ds = \int_{k \geq 0} k ds - \int_{k \leq 0} k ds,$$

differentiate with respect to time, and then integrate by parts. See [8] (§1, §2) for details and justification.

PART III. SINGULARITY FORMATION

6. Forming singularities become planar. In this section, we shall see that $\lim\limits_{n \to \infty} \frac{\tau}{k} (p_n, t_n) \to 0$ along an essential blow-up sequence. We will prove this fact first without using the language of rescalings because this was the formulation of the conjecture first posed to the author. In the next section, we will see that this fact simply implies that the limit of rescalings is a planar solution to the curve shortening flow.

Theorem 6.1. *If* $\{(p_n, t_n)\}$ *is an essential blow-up sequence, then*

$$(6.2) \qquad \lim_{n \to \infty} (\tau/k) (p_n, t_n) \to 0.$$

Proof of Planarity. To obtain a contradiction, we will assume that on a subsequence of points and times $\{p_{n_j}, t_{n_j}\} \subset \{p_n, t_n\}$, $\exists \mu, 0 < \mu < \infty$ such that $\forall n_j$ we have

$$(6.3) \qquad \tau^2 \{p_{n_j}, t_{n_j}\} \geq \mu k^2 \{p_{n_j}, t_{n_j}\}.$$

That is, on this sequence the limit of $(\tau/k) (p_n, t_n)$ is not zero. For ease of notation, we will drop the subscript j and assume that we are working with the subsequence.

As before, it is not hard to see that $\exists \{\tilde{t}_n\}$ where $t_n = \tilde{t}_n + \frac{1}{32 M_{\tilde{t}_n}}$ (for n large). We need this sequence only because of the time-delay necessary for the dilation-invariant estimates.

Our work concerning the dissipation of curvature and torsion in §4 implies that $\exists d$, where $d = d(\mu, \rho)$, such that

$$(6.4) \qquad \left. \begin{array}{l} k(p, t) \geq \dfrac{1}{2} k (p_n, t_n) \\[2mm] \tau(p, t) \geq \dfrac{1}{2} \tau (p_n, t_n) \end{array} \right\} \quad \text{for } (p, t) \in \mathrm{N}(p_n, t_n, d).$$

Next, we will see that distances do not decrease "too quickly" on short time intervals. That is, assume $t_n \leq t \leq t_n + 1/\left(8 M_{t_n}\right)$ and $u_1, u_2 \in S^1$ are two points given in a time-independent parametrization u. Recall that we may write $ds = v du$, from which it follows that $\dfrac{\partial v}{\partial t} = -k^2 du$. Using

$$\mathrm{dist}_t \{u_1, u_2\} = \left| \int_{u_1}^{u_2} ds \right|$$

we obtain

(6.5)
$$\frac{d}{dt}\left(\operatorname*{dist}_t\{u_1, u_2\}\right) = -\int_{u_1}^{u_2} k^2(\cdot, t)ds \geq -M_t \cdot \int_{u_1}^{u_2} ds$$
$$= -2M_{t_n} \cdot \operatorname*{dist}_t\{u_1, u_2\}.$$

Here, we have used the fact that $k^2(\cdot, t) \leq M_t \leq 2M_{t_n}$. Integrating this inequality yields

(6.6)
$$\operatorname*{dist}_t\{u_1, u_2\} \geq \operatorname*{dist}_{t_n}\{u_1, u_2\} \cdot e^{-2M_{t_n}(t-t_n)}.$$

Of course, $t - t_n < 1/(8M_{t_n})$ so

(6.7)
$$\operatorname*{dist}_t\{u_1, u_2\} \geq \operatorname*{dist}_{t_n}\{u_1, u_2\} \cdot e^{-1/4}.$$

Thus we see that distance does not change drastically on time intervals of the order M^{-1}. Therefore, there exists a constant $C = C(\mu, \rho, d)$ independent of n, such that

(6.8)
$$\int_{\tilde{t}_n}^{\tilde{t}_n + \frac{1}{8M_{\tilde{t}_n}}} \int_\gamma \tau^2 |k| \, ds dt \geq \iint_{N(p_n, t_n, d)} \tau^2 |k| \, ds dt > C.$$

In the preceding section we saw that $\forall \epsilon$, $\exists \theta$ such that

$$\int_{\omega-\theta}^{\omega} \int_\gamma \tau^2 |k| \, ds dt < \epsilon.$$

Choose $\epsilon \ll C$ and an n large enough that $\tilde{t}_n > \omega - \theta$. Thus

(6.9)
$$C < \int_{\tilde{t}_n}^{\tilde{t}_n + \frac{1}{8M_{\tilde{t}_n}}} \int_\gamma \tau^2 |k| \, ds dt \leq \int_{\omega-\theta}^{\omega} \int_\gamma \tau^2 |k| \, ds dt < \epsilon$$

yields a contradiction.

7. **The limit of the rescaled solutions.** In this section, we will prove some basic facts about limits of rescaled solutions along an essential blow-up sequence. We will show first that a limit γ_∞ exists, and then that γ_∞ is a family of planar, convex curves.

First, we will give a precise definition of what we mean by rescaling a solution.

Definition 7.1. We define the *rescaled solutions* γ_n of γ along a blow-up sequence $\{(p_n, t_n)\}$, $(p_n, t_n) \in S^1 \times [0, \omega)$, to be as follows: let

$\gamma_n : S^1 \times [-\lambda_n^2 t_n, \lambda_n^2(\omega - t_n)) \to R^3$ be a solution to the curve shortening flow defined by

$$(7.2) \qquad \gamma_n(\cdot, \bar{t}) = \lambda_n \left(A_n \gamma(\cdot, t) + B_n \right) ; \qquad \bar{t} = \lambda_n^2 \left(t - t_n \right) ,$$

where $\lambda_n \in R^+$, $A_n \in SO(3)$, $B_n \in R^3$ are chosen such that (i) $\gamma_n(p_n, 0) = 0 \in R^3$, (ii) the unit-length tangent vector $T_n(p_n, 0) = (1, 0, 0)$ and (iii) $k_n \cdot N_n(p_n, 0) = (0, 1, 0)$. The subscript n will be used to denote quantities on γ_n.

It is clear that rotating and translating a solution give another solution. If one dilates space and time, scaling time as space squared, then one also obtains another solution to the curve shortening flow. We must now prove that a limit of rescalings along an essential blow-up sequence exists.

Theorem 7.3. *Assume* $\{(p_n, t_n)\}$ *is an essential blow-up sequence. Then, there exists a subsequence of* $\{(p_n, t_n)\}$ *along which the rescaled solutions converge to a smooth, nontrivial limit* γ_∞. *The solution* γ_∞ *exists at least on the time interval* $[-\infty, 0]$.

Proof. If we denote $\alpha_n = -\lambda_n^2 t_n$ and $\omega_n = \lambda_n^2(\omega - t_n)$, it follows from the fact that $M_t \to \infty$ that $\lim_{n \to \infty} \alpha_n \to -\infty$. Note that $\lim_{n \to \infty} \omega_n$ is finite if a type-I singularity occurs. In the case of a type-II singularity, an essential blow-up sequence will be chosen such that $\omega_n \to \infty$.

A limit solution, if it exists, may be a family of noncompact curves in space. It will be more convenient, when considering questions of convergence, to think of our solutions as a family of periodic curves in space $\tilde{\gamma}_n : R^1 \times [-\alpha_n, \omega_n) \to R^3$ such that $\tilde{\gamma}(0, \cdot) = \gamma_n(p_n, \cdot)$. We will then parametrize the curves by arclength from the origin $0 \in R^1$.

Now define the operator

$$(7.4) \qquad \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \phi_n(s) \frac{\partial}{\partial s} ,$$

where $\dfrac{\partial \phi_n}{\partial s} = k_n^2$. Then

$$
\begin{aligned}
[\frac{\delta}{\delta t}, \frac{\partial}{\partial s}] &= \frac{\partial}{\partial t}\frac{\partial}{\partial s} + \phi_n \frac{\partial}{\partial s}\frac{\partial}{\partial s} - \frac{\partial}{\partial s}\frac{\partial}{\partial t} - \frac{\partial \phi_n}{\partial s}\frac{\partial}{\partial s} - \phi_n \frac{\partial}{\partial s}\frac{\partial}{\partial s} \\
(7.5) \qquad &= k_n^2 \frac{\partial}{\partial s} - k_n^2 \frac{\partial}{\partial s} \\
&= 0.
\end{aligned}
$$

ϕ_n is chosen in such a way as to make $\dfrac{\delta s}{\delta t} = 0$. We are essentially finding a good coordinate system on the space-time solution.

Since $\{(p_n, t_n)\}$ is an essential blow-up sequence, $\forall \tilde{\gamma}_n$ and $t \leq 0$ we have the property that $\rho \cdot \sup k_n^2(\cdot, t) \leq 1$. The invariance of our estimates under the rescaling procedure implies that there exist constants $\{c_l\}$ independent of n such that

$$(7.6) \qquad \left| \frac{\partial^l T_n}{\partial s^l} \right|^2 \leq c_l \quad \text{for } t \leq 0.$$

A simple computation shows that this gives bounds on all of the higher time derivatives $\left| \dfrac{\partial^l T_n}{\partial t^l} \right|^2$ for $t \leq 0$. The fact that $\phi(s) = \int_0^s k_n^2 ds + C \leq \rho^{-1}s + C$, for some constant C, yields bounds on $\left| \dfrac{\delta^l T_n}{\delta t^l} \right|^2$. Therefore, for a given compact set in $R^1 \times [\alpha_n, \omega_n)$, we have bounds, independent of n, on all mixed derivatives $\left| \dfrac{\delta^j \partial^k T_n}{\delta^j t \partial^k s} \right|^2$.

The Ascoli–Arzela theorem then implies that there exists a subseqence of $\{(p_n, t_n)\}$ on which the tangent vectors $T_n(s, t)$ converge uniformly on compact sets of $R^1 \times [-\infty, \omega_\infty)$ to a smooth limit $T_\infty(s, t)$. We may thus recover a smooth solution $\tilde{\gamma}_\infty$ by integrating the tangent vectors. We will denote by γ_∞ one period, possibly infinite, of $\tilde{\gamma}_\infty$.

The process of rescaling does not allow the limit solution to be trivial, i.e., a straight line. This is of course because the construction of γ_∞ enforces the condition $k_\infty^2(0,0) = 1$.

Theorem 7.7. γ_∞ *is a family of convex planar curves.*

Proof. The integral estimate of the previous section implies

$$(7.8) \qquad \int_{-\infty}^{0} \int_{\gamma_\infty} |k_\infty| \tau_\infty^2 \, ds \, dt = 0,$$

which shows that $\tau_\infty = 0$ wherever $k_\infty \neq 0$. There are examples of C^∞ curves which do not lie on one plane yet still satisfy this condition. We appeal, however, to the fact that our solutions are analytic [8] and may thus conclude that our limit solution is globally planar.

Since we know that our sequence of rescaled solutions is converging in C^∞ to a planar, limit solution γ_∞, it follows that

$$(7.9) \qquad \int_{-\infty}^{0} \sum_{\{p \mid k_\infty(p, t) = 0\}} \left| \frac{\partial k_\infty}{\partial s} \right| dt = 0.$$

Therefore, any inflection point for the limit curve must be degenerate (i.e., $k_\infty = \dfrac{\partial k_\infty}{\partial s} = 0$). Results of Angenent [4] imply that if a solution has degenerate inflection points for any interval in time, the solution must be a line. Since γ_∞ is not trivial, the family of curves must have no inflection points, and therefore must all be convex.

8. Type-I and Type-II blow-up sequences. In this section we will attempt to understand the limiting shapes of rescaled solutions along essential blow-up sequences.

Type-I Singularities. In this section, we will not need the assumption that a blow-up sequence is essential. We will, in fact, prove convergence to Abresch–Langer solutions along a subsequence of any blow-up sequence.

The proof is an argument of Huisken [11] in which he uses a backwards heat kernel in R^{n+1} to prove this result for hypersurfaces moving by the mean curvature flow. For the sake of completeness, we will outline the proof, but we refer the reader to his paper for details. In the case of convex planar curves, Angenent [3] also has a proof of this result.

It is convenient, in the study of type-I singularities, to consider a modification of the original solution γ defined by:

$$(8.1) \qquad \bar{\gamma}(s, \bar{t}) = (2(\omega - t))^{-\frac{1}{2}} \gamma(s, t) \quad \text{where } \bar{t} = -\frac{1}{2}\log(\omega - t).$$

This leads us to define the following scale-invariant operators

$$(8.2) \qquad \frac{\partial}{\partial \bar{t}} = (2(\omega - t))\frac{\partial}{\partial t}; \qquad \frac{\partial}{\partial \bar{s}} = (2(\omega - t))^{\frac{1}{2}}\frac{\partial}{\partial s}.$$

The modified flow satisfies the evolution equation

$$(8.3) \qquad \frac{\partial}{\partial \bar{t}}\bar{\gamma} = \frac{\partial^2}{\partial \bar{s}^2}\bar{\gamma} + \bar{\gamma}.$$

The curvature for the modified solution is

$$(8.4) \qquad \bar{k}(s, \bar{t}) = (2(\omega - t))^{\frac{1}{2}} k(s, t).$$

Since we are assuming that the forming singularity is of type-I, we have $\bar{k}^2 < C$ for some constant $C < \infty$ on the entire modified curve for $\forall \bar{t} \in [\bar{t}_0, \infty)$. Here, $\bar{t}_0 = -\frac{1}{2}\log(\omega - t_0)$.

Following Huisken, we define the backwards heat kernel.

Definition 8.5. Define $\rho(x, t)$ to be the *backwards heat kernel* flowing out of the point in space-time: $(\vec{0}, \omega) \in R^3 \times [0, \omega]$. More precisely, let

$$(8.6) \qquad \rho(x, t) = \frac{1}{\sqrt{4\pi(\omega - t)}} \cdot e^{-|x|^2/4(\omega - t)}, \qquad t < \omega.$$

The modified kernel will be then be as follows.

Definition 8.7. Define $\bar{\rho}(x, \bar{t})$ to be the *modified backwards heat kernel* given by

$$(8.8) \qquad \bar{\rho}(x, \bar{t}) = e^{-|x|^2} \qquad x \in R^3.$$

(Note that this kernel is independent of time.)

We may now state the monotonicity formula.

Theorem 8.9 (Huisken [11]).

(1) *For γ, when $t \in [0, \omega)$, we have:*

$$(8.10) \qquad \frac{d}{dt} \int_\gamma \rho\, ds = -\int_\gamma \rho \left| \frac{\partial^2 \gamma}{\partial s^2} + \frac{1}{2(\omega - t)} \gamma^\perp \right|^2 ds.$$

(2) *For γ_∞, when $t \in [t_0, \infty)$, we have:*

$$(8.11) \qquad \frac{d}{d\bar{t}} \int_{\bar{\gamma}} \bar{\rho}\, d\bar{s} = -\int_{\bar{\gamma}} \bar{\rho} \left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} + \bar{\gamma}^\perp \right|^2 d\bar{s}.$$

Here, $\gamma^\perp = \gamma - \gamma^\top$, and $\gamma^\top$ is the tangential component of the position vector.

Proof of Monotonicity Formula. We will only show the formula in the case of the modified flow.

$$
\begin{aligned}
(8.12) \qquad \frac{d}{d\bar{t}} \int_{\bar{\gamma}} \bar{\rho}\, d\bar{s} &= \int_{\bar{\gamma}} \bar{\rho} \left(-\left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} \right|^2 + 1 \right) - \bar{\rho} \left\langle \bar{\gamma}, \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} + \bar{\gamma} \right\rangle d\bar{s} \\
&= \int_{\bar{\gamma}} -\bar{\rho} \left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} + \bar{\gamma} \right|^2 + \bar{\rho} + \bar{\rho} \left\langle \bar{\gamma}, \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} \right\rangle d\bar{s} \\
&= \int_{\bar{\gamma}} -\bar{\rho} \left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} + \bar{\gamma} \right|^2 + \bar{\rho} \left\langle \bar{\gamma}, \frac{\partial \bar{\gamma}}{\partial \bar{s}} \right\rangle^2 d\bar{s} \\
&= -\int_{\bar{\gamma}} \bar{\rho} \left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} + \bar{\gamma}^\perp \right|^2 d\bar{s}.
\end{aligned}
$$

We have made use of the computation $\frac{\partial}{\partial \bar{s}} d\bar{s} = \left(-\left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} \right|^2 + 1 \right) d\bar{s}$, integration by parts, and the fact that $\left\langle \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2}, \bar{\gamma}^\top \right\rangle = 0$.   q.e.d.

Therefore, integrating the monotonicity formula in time gives the following corollary:

Corollary 8.13. $\forall \epsilon > 0$, $\exists \alpha < \infty$ such that

(8.14)
$$\int_{\alpha}^{\infty} \int_{\gamma} \bar{\rho} \left| \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} + \bar{\gamma}^{\perp} \right|^2 d\bar{s} d\bar{t} < \epsilon.$$

Therefore, for the unmodified solution we have

Theorem 8.15.   *Let* $\{ p_n, t_n \}$ *be any blow-up sequence. Then a rescaling of* γ *converges along a subsequence of* $\{ p_n, t_n \}$ *to a solution* γ_{∞} *moving by homothety. Furthermore, each* γ_{∞} *is planar and has the same winding number.*

Proof. Note that for a type-I singularity, all blow-up sequences are essential. The proof that the rescaled modified solutions converge along some subsequence of $\{ p_n, t_n \}$ is essentially the same as for the unmodified solutions. This is, of course, due to the fact that our estimates are dilation invariant and the fact that the modified curvature is bounded for all time. As in [11], the monotonicity formula implies that γ_{∞} must be moving by homothety.

Our comments in §2 imply that the entire solution is planar. Furthermore, since the limit of the scale-invariant integral $\int_{\gamma} |k| ds = \int |\bar{k}| d\bar{s}$ is unique and represents the total change in angle of a planar curve, we know that the curve cannot converge to an Abresch–Langer solution with infinitely many loops. In fact, any two limiting solutions obtained by rescaling along different subsequences of $\{ (p_n, t_n) \}$ differ only by dilation and a rigid motion.

Type–II Singularities. We will now assume a type-II singularity is forming at time ω. Recall that our model for this type of behavior is a cusp forming. The author wishes to acknowledge many useful comments and suggestions from S. Angenent in this final section. The arguments are generalizations of the arguments given in [10].

Theorem 8.16.   *There exists an essential blow-up sequence* $\{ (p_n, t_n) \}$ *such that a limit of rescalings along* $\{ (p_n, t_n) \}$ *converges to the Grim Reaper.*

Proof. Our theorem in the previous section implies that any limit curve of an essential blow-up sequence γ_{∞} exists since $t = -\infty$. We must also insure the fact that the curve lasts until time $+\infty$. R. Hamilton brought the following argument to our attention.

Lemma.   *For a type-II singularity, there exists an essential blow-up sequence along which the rescaled solutions converge to a solution whose curvature is bounded on* $[-\infty, +\infty]$.

Proof of Lemma. Let $\delta > 0$, and set

(8.17) $\sigma_\delta(t) = \min\{\sigma \in [t, \omega) : M_\sigma = (1 + \delta)M_t\}.$

Define the set of " ρ-essential times" to be

(8.18) $E = \{t \leq \omega : \rho M_{\bar{t}} \leq M_t \text{ for } \forall \bar{t} \in [0, t]\}.$

It is clear that $t \in E$ implies $\sigma_\delta(t) \in E$. So pick $t = t_0 \in E$ and define
$t_n = \sigma_\delta(t_{n-1}) \in E$.
    The lemma follows from the assertion:

(8.19) $N = \sup_{n \to \infty} M_{t_n}(\sigma_\delta(t_n) - t_n) = +\infty.$

Now $M_{t_n} = (1 + \delta)^n M_{t_0}$. Clearly $t_n \to \omega$ (or else a singularity would
have to be occurring before time ω). Therefore $t_n - t_{n-1} \leq M_{t_{n-1}}^{-1} N =
(1 + \delta)^{1-n} M_{t_0}^{-1} N$. Hence

$$(\omega - t_0)M_{t_0} = \sum_{n=0}^{\infty}(t_{n+1} - t_n)M_{t_0} \leq N \sum_{n=0}^{\infty}(1 + \delta)^{-n} = N(1 + \frac{1}{\delta}),$$

and the assertion is proved. Thus, when the solution is rescaled to keep
$M_{t_n} = 1$, it follows $\sigma_\delta(t_n) \to \infty$. This *a priori* control on the curvature is
enough to conclude that the γ_∞ exists on $[-\infty, +\infty]$ and has the max-
imum of the curvature bounded by $1 + \delta$ for all time. (In fact, it is not
difficult to show that δ can be made to go to zero.)   q.e.d.
    This limiting curve is planar and convex. The solution does not cross
itself or else a loop would pinch and the curvature would not be bounded
for all time in the future. By [6], the curve must turn at least π or else
the curve would not be ancient (that is, it could not exist since $t = -\infty$).
Thus the curve must turn exactly π and is embedded.
    In order to show that the curve is actually the Grim Reaper, we must
modify the arguments of R. Hamilton [10] used on planar, compact, con-
vex curves. The dilation-invariant estimates, in addition to the fact that
$\int k\,ds = \pi$, imply that the curvature goes to zero at the ends of the curve.
It is not hard to show that all of the derivatives of k must also decay to
zero near the ends.
    It will be convenient to compute using (θ, t) coordinates (see [7]).
Consider the quantity $g = k_\tau$ where $\dfrac{\partial}{\partial \tau}$ is the time derivative that fixes
angles. Note that $[\dfrac{\partial}{\partial \tau}, \dfrac{\partial}{\partial \theta}] = 0$ and $k_\tau = k^2 k_{\theta\theta} + k^3$. Then

(8.20) $g_\tau = \left(k^2 k_{\theta\theta} + k^3\right)_\tau = k^2\left(g_{\theta\theta} + g\right) + \dfrac{2g^2}{k}.$

Letting $h = 2tg + k$, we obtain

(8.21)
$$h_\tau = 2tk^2 g_{\theta\theta} + 2tk^2 g + \frac{4tg^2}{k} + 2g + k^2 k_{\theta\theta} + k^3$$
$$= k^2 h_{\theta\theta} + \left(k^2 + \frac{2g}{k}\right) h.$$

For any compact convex curve it is easily seen that the maximum principle implies $k_\tau \geq -\frac{k}{2t}$. This follows from the fact that $h \geq 0$ at $t = 0$. One may apply this estimate to the noncompact curve by closing up γ_∞ a large distance away from the point of maximum curvature in such a way as to give an embedded convex curve. As the curves better approximate γ_∞, they enclose more area and exist for longer and longer times. These new solutions converge in C^∞ to γ_∞, and by sending their starting times back to $-\infty$, we obtain $k_\tau \geq 0$ for γ_∞.

Let Γ be a planar, compact, convex curve with $\int_\Gamma d\theta = 2\pi$. For $Z = \int_\Gamma (\log k)_\tau d\theta$, we have

(8.22)
$$\frac{d}{dt}Z = \int_\Gamma k \left(k_{\tau\theta\theta} + k_\tau + \frac{2k_\tau^2}{k}\right) - \frac{k_\tau}{k^2}\left(k^2 k_{\theta\theta} + k^3\right) d\theta$$
$$= 2\int_\Gamma \frac{k_\tau^2}{k} d\theta \geq 2Z^2 / \int_\Gamma d\theta \geq Z^2/\pi.$$

Thus, on the interval $[0, T)$, we have $Z(t) \leq \pi/(T-t)$. So, again we may approximate γ_∞ by curves which last for longer and longer times into the future. We may conclude that $Z = 0$ for γ_∞.

It is now clear that $k_\tau = 0$ on γ_∞. Such a curve is the Grim Reaper.

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