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CONVEX HYPERSURFACES WITH PRESCRIBED GAUSS-KRONECKER CURVATURE

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In [14, Problem 59] Yau raised the general question when a function F defined in a Euclidean 3-space $\mathbb{R}^3$ is the mean curvature or Gaussian curvature of a closed surface with prescribed genus. For mean curvature it was proposed to minimize the functional

$$\text{Area}(X) - \int_{\widetilde{X}} F$$

among all surfaces X of the same genus ($\widetilde{X}$ is the set bounded by X). However, it is not clear how the minimum, if it ever exists, should have the same genus. In this paper we study the latter problem for the Gauss-Kronecker curvature of a closed hypersurface in a Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ $(n \geq 1)$. If the given function is positive, the solution hypersurface is a priori convex. The difficulty of determining its topology is subdued and we can concentrate on the analysis.

Let X be a smooth hypersurface embedded in $\mathbb{R}^{n+1}$ and oriented with respect to its inner normal. Denote σ_k $(k = 0, 1, \cdots, n+1)$ the normalized kth elementary symmetric function of its principal curvatures. (Set $\sigma_0 = 1$ and $\sigma_{n+1} = 0$.) The following first variation formulas for the kth curvature integral, $k = 0, \cdots, n$, $I_k(X) = (n-k)^{-1} \int_X \sigma_k$, are valid [10]:

$$\delta I_k(X)\xi = \int_X \sigma_{k+1} \langle \xi, \nu \rangle. \tag{1}$$

Here ξ is a smooth variation vector field on X, ν is the unit outer normal, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^{n+1}$. If we let $J_k(X) = I_k(X) - \int_{\widetilde{X}} F$, where F is a function defined in $\mathbb{R}^{n+1}$ and $\widetilde{X}$ is the compact subset bounded by X, we have

$$\delta J_k(X)\xi = \int_X (\sigma_{k+1} - F)\langle \xi, \nu \rangle.$$

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Consequently, at least in a formal sense, any critical point of the functional J_k is a hypersurface whose $(k+1)$th mean curvature is equal to F; in other words, it solves the equation

$$\sigma_{k+1}(X) = F. \tag{2}$$

When $k = 0$, this variational formulation of the problem coincides with the one outlined in [14].

In this paper we study the case $k = n - 1$. From now on we write I_{n-1} for I, J for J_{n-1}, J and K for σ_n. Our first result is:

Theorem A. *Let F be a smooth, positive, and integrable function in $\mathbb{R}^{n+1}$. Then there exists an absolute minimum of J which is a smooth solution of* (2) *if and only if there exists a smooth uniformly convex hypersurface X satisfying $J(X) \leq 0$.*

Recall that a hypersurface is uniformly convex if all its principal curvatures are bounded between two positive constants. Here the minimization is taken over all smooth uniformly convex hypersurfaces. Theorem A will be proved by deforming a minimizing sequence of hypersurfaces along the logarithmic gradient flow

$$X_t = -(\log K - \log F)\nu$$

to an absolute minimum. Such a method was developed in our study of Monge-Ampère equations [13]. Although a direct method may be applicable and, in fact, has been successful in the two-dimensional case [12], the method of gradient flow is more flexible in dealing with this problem. It enables us to find critical points other than minima. We shall establish

Theorem B. *Let F be a smooth function which satisfies* (a) *$\Omega = \{x\colon F(x) > 0\}$ is bounded and* (b) *$F^{1/n}$ is concave in Ω and the ratio between the minimal and maximal eigenvalues of the Hessian of $F^{1/n}$ is uniformly bounded above. Then* (2) *admits two solutions if there exists a convex hypersurface X lying inside Ω and satisfying $J(X) \leq 0$.*

Further existence results can be found in §4, after we prove Theorems A and B in §§2 and 3.

The problem of finding closed convex hypersurfaces with a given Gauss-Kronecker curvature function was studied in [7] (see also [2]). For the prescribed function F it is assumed that (a) there exist R_1 and R_2, $0 < R_1 < R_2$, such that $F(x) > R_1^{-n}$ for $|x| = R_1$ and $F(x) < R_2^{-n}$ for $|x| = R_2$, and (b) $\frac{\partial}{\partial \rho}\rho^n F(\rho x) \leq 0$, $x \in S^n$, $\rho > 0$. Then it was shown that there exists a unique convex hypersurface whose Gauss-Kronecker curvature is equal to F. Subsequently Delanoë [5] showed that (a) alone is sufficient for existence. As the reader will see, our proof of Theorem

A also establishes Oliker and Delanoë's result after some obvious modification. In fact one can show that the solution hypersurface minimizes all convex hypersurfaces bounded by the sphere $S(R_2)$ and containing the ball $B(R_1)$. More recently, Oliker [8] found a variational approach to (2) and established the existence of a certain generalized solution under conditions similar to (a). Both his formulation and proof are very different from ours.

In concluding the introduction we give a geometric interpretation of the functional I and the associated variational formula. For a uniformly convex X, $I(X)$ is equal to $\int_{S^n} \widehat{\sigma}_1$, where $\widehat{\sigma}_1$ is the (normalized) sum of all principal radii. By Minkowski's formula we may again write I as $\int_{S^n} H$, where H is the support function of X, or $2^{-1}\int_{S^n} B$, where $B(\nu) = H(\nu)+H(-\nu)$ is the distance between two supporting hyperplanes of X with normals ν and $-\nu$. Thus I is nothing but a constant multiple of the average of width of X. Now the variational formula (1) is evident: Let ξ be a normal vector field and $\varphi = \langle \xi , \nu \rangle$. Then the family of convex hypersurfaces which fits in ξ has support function given by $H + \varepsilon\varphi$. So obviously the first variation of I is equal to $\int_{S^n} \varphi = \int_X K\varphi$.

1. A logarithmic gradient flow

Denote the class of all smooth uniformly convex hypersurfaces by $\mathscr{K}$. We consider the logarithmic gradient flow of J:

$$X_t = -(\log K - \log F(X))\nu , \tag{1.1}$$

where $X(0)$ belongs to $\mathscr{K}$. Obviously $X(t)$ stays in $\mathscr{K}$ as long as it exists. Along this flow J is decreasing:

$$\frac{d}{dt} J(X(t)) = -\int_{X(t)} (K - F)\log\frac{K}{F} \leq 0 \tag{1.2}$$

and equality holds if and only if $X(t)$ solves (2). To study (1.1) we shall first reduce (1.1) to an initial value problem for a parabolic Monge-Ampère equation for the support function of X. Next we shall derive a priori estimates for the solution of this equation. They will be used in the proof of Theorem A in the next section.

Recall that for a convex hypersurface X (or, more generally, the boundary of a convex body $\widetilde{X}$) its support function H is defined by

$$H(x) = \sup\{\langle x , p\rangle \colon p \in X\}, \qquad x \in \mathbb{R}^{n+1}\backslash\{0\}.$$

It is convex and is of homogeneous degree 1. If X is uniformly convex, H is differentiable and X can be recovered from H by

$$p^i(x) = \frac{\partial H}{\partial x_i}(x), \qquad x \in S^n, \ i = 1, \cdots, n+1,$$

where $p = (p^1, \cdots, p^{n+1})$ is the point on X with unit outer normal x. Thus all geometric quantities can be described in terms of H. For instance, the principal curvatures of X are the n positive eigenvalues of the Hessian $(H_{\alpha\beta} + H\delta_{\alpha\beta})$ where subscripts α and β denote covariant differentiations with respect to an orthonormal frame on S^n. Henceforth there is a one-to-one correspondence between the class of smooth uniformly convex hypersurfaces and the class of smooth convex functions of homogeneous degree one in $\mathbb{R}^{n+1}\backslash\{0\}$ whose Hessians possess exactly n many positive eigenvalues (the remaining eigenvalue is zero due to homogeneity).

Now, suppose (1.1) has a solution $X(\cdot, t)$ which is uniformly convex for each t. We can parametrize it by the inverse of its Gauss map N_t. Let $\widehat{x} = N_t(x)$, $X(\widehat{x}, t) = X(N_t^{-1}(\widehat{x}), t)$,and $K(\widehat{x}, t) = K(N_t^{-1}(\widehat{x}), t)$. Then (1.1) becomes

$$\frac{\partial X}{\partial t}(\widehat{x}, t) = \left\langle \frac{\partial X}{\partial x'}, \frac{\partial N_t^{-1}}{\partial t} \right\rangle - (\log K - \log F)\widehat{x}.$$

Taking the inner product with $\widehat{x}$, we obtain, after replacing $\widehat{x}$ by x,

$$\frac{\partial H}{\partial t} = -\log K + \log F(\nabla H), \qquad (x, t) \in S^n \times (0, \infty), \tag{1.3}$$

where $H(x, 0) = \Phi(x)$ is the support function of $X(0)$. Conversely, it is not hard to show that (1.3) implies (1.1). (We have established this fact for a similar equation in [11].) Therefore, (1.1) and (1.3) are equivalent. In view of the formula

$$K = (\det(H_{\alpha\beta} + H\delta_{\alpha\beta}))^{-1}$$

we see that (1.3) is a parabolic equation for $H(t)$. Although (1.3) is defined on S^n, in practice it is more convenient to act as follows: Extend K as a function of degree 0 over $\mathbb{R}^{n+1}\backslash\{0\}$ and consider the equation satisfied by the restriction of H on a tangent space of S^n. As a typical case we consider the hyperplane passing the south pole. Let $h(x) = H(x_1, \cdots, x_n, -1)$. Then h satisfies

$$\begin{aligned} h_t(x, t) &= (1 + |x|^2)^{1/2} \log\det \nabla^2 h + g(x, h, \nabla h), \\ h(x, 0) &= (1 + |x|^2)^{1/2}\Phi(x, -1) \end{aligned} \tag{1.4}$$

in $\mathbb{R}^n \times (0, \infty)$. Here [3]

$$g = \left(\frac{n+2}{2}\right)(1+|x|^2)^{1/2}\log(1+|x|^2) + (1+|x|^2)^{1/2}\log F(\nabla h, h^*),$$

$$h^* = \sum_1^n x_\alpha \frac{\partial h}{\partial x_\alpha} - h.$$

(1.5) Proposition. *Suppose that H is a solution of (1.3) in $Q = S^n \times [0, T]$ which satisfies $r < H < R$ in Q for some positive r and R. For t_0, $0 < t_0 < T$, there exist constants C and C' such that*

$$\|H\|_{\widetilde{C}^2(\overline{Q}')} \leq C, \qquad Q' = S^n \times [t_0, T],$$

and

$$(H_{\alpha\beta} + H\delta_{\alpha\beta})\xi^\alpha\xi^\beta \geq C'|\xi|^2 \qquad (\xi \in \mathbb{R}^n) \text{ in } Q'.$$

Here C and C' depend on n, t_0, r, R, the C^2-norm of F in the ball $B(2R)$, and a positive lower bound of F in $B(2R)$. Furthermore, for each $k \geq 2$, there exists C_k which also depends on higher derivatives of F such that $\|H\|_{\widetilde{C}^k(\overline{Q}')} \leq C_k$.

Recall that $\|H\|_{\widetilde{C}^k(\overline{Q}')} = \sup_{Q'} \sum_{2l+m\leq k} |D_t^l D_x^m H|$.

Proof. In the following proof positive constants C_1, C_2, etc. have the same dependence as C in the statement of the theorem. We also fix a smooth function g in $[0, T]$ such that it is positive in $(0, T]$, $g(0) = 0$, $0 < g < 1$ in $[0, t_0)$, $g(t) = 1$ for $t \geq t_0$, and its derivative g' is bounded between 0 and $2t_0^{-1}$.

First from the assumption it is clear that X is contained in $B(2R)$. Therefore we have $|\nabla H| \leq 2R$.

Step 1: $H_t \geq -C_1$. Consider the function $H_t(H-\delta)^{-1}g$, where $\delta = r/2$. Suppose it has a negative minimum which attains at (x_1, t_1) with $t_1 > 0$. Without loss of generality we may assume x_1 is the south pole and the matrix $(H_{\alpha\beta})$, $\alpha, \beta = 1, \cdots, n$, is diagonal at this point. Let

$$\eta(x, t) = \frac{H_t(x, 1-|x|^2)^{1/2}, t)}{H(x, -(1-|x|^2)^{1/2}, t) - \delta} g(t).$$

At (x_1, t_1) we have

$$0 = \frac{\partial \eta}{\partial x_\alpha} = \left(\frac{H_{t\alpha}}{H-\delta} - \frac{H_t H_\alpha}{(H-\delta)^2}\right) g,$$

$$0 \geq \frac{\partial \eta}{\partial t} = \left(\frac{H_{tt}}{H-\delta} - \frac{H_t^2}{(H-\delta)^2}\right) g + \frac{H_t}{H-\delta} g',$$

$$0 \leq \frac{\partial^2 \eta}{\partial x_\alpha^2} = \frac{1}{H-\delta}\left[H_{t\alpha\alpha} - \frac{2H_{t\alpha}H_\alpha}{H-\delta} - \frac{H_t H_{\alpha\alpha}}{H-\delta} + \frac{2H_t H_\alpha^2}{(H-\delta)^2} + \frac{\delta H_t}{H-\delta}\right] g$$

for $\alpha = 1, \cdots, n$. On the other hand, differentiating (1.4) gives

$$H_{tt} = \sum \frac{H_{t\alpha\alpha}}{H_{\alpha\alpha}} + \frac{\partial g}{\partial z} H_t + \sum \frac{\partial g}{\partial p_\alpha} H_{\alpha t} \tag{1.6}$$

at (x_1, t_1). As a result

$$\begin{aligned}
\frac{H_t^2}{H-\delta} &\geq H_{tt} + \frac{g'}{g} H_t \\
&\geq \frac{nH_t}{H-\delta} - \frac{\delta H}{H-\delta}\sum \frac{1}{H_{\alpha\alpha}} + \frac{H_t}{H-\delta}\sum \frac{\partial g}{\partial p_\alpha} H_\alpha + \frac{\partial g}{\partial z} H_t + \frac{g'}{g} H_t \\
&\geq \left(\frac{n}{H-\delta} + \frac{\partial g}{\partial z} + \frac{1}{H-\delta}\sum \frac{\partial g}{\partial p_\alpha} H_\alpha + \frac{g'}{g}\right) H_t \\
&\quad - n\frac{\delta H_t}{H-\delta} F^{1/n} \exp\left(-\frac{H_t}{n}\right),
\end{aligned}$$

where in the last step we have used the arithmetic-geometric inequality. This implies

$$\eta^2 \geq \left[\left(\frac{n}{H-\delta} + \frac{\partial g}{\partial z} + \frac{1}{H-\delta}\sum \frac{\partial g}{\partial p_\alpha} H_\alpha\right) g + \frac{g'}{H-\delta}\right]\eta - \frac{\delta^2}{2n} F^{1/n} \eta^3,$$

which clearly gives a lower bound for η and hence for H_t.

Step 2: $H_t \leq C_2$. Consider the function $H_t H^{-1} g$. Assume it has a positive maximum at (x_2, t_2), where t_2 is positive and x_2 is the south pole. Rotate the coordinate so that $(H_{\alpha\beta})$ is diagonal at this point. The function

$$\zeta(x, t) = \frac{H_t(x, -(1-|x|^2)^{1/2}, t)}{H(x, -(1-|x|^2)^{1/2}, t)} g(t)$$

satisfies

$$0 = \frac{\partial \zeta}{\partial x_\alpha} = \left(\frac{H_{t\alpha}}{H} - \frac{H_t H_\alpha}{H^2}\right) g,$$
$$0 \le \frac{\partial \zeta}{\partial t} = \left(\frac{H_{tt}}{H} - \frac{H_t^2}{H^2}\right) g + \frac{H_t}{H} g',$$
$$0 \ge \frac{\partial^2 \zeta}{\partial x^\alpha} = \frac{1}{H}\left(H_{t\alpha\alpha} - \frac{2H_{t\alpha}H_\alpha}{H} - \frac{H_t H_{\alpha\alpha}}{H} + \frac{2H_t H_\alpha^2}{H^2}\right) g$$

at (x_2, t_2). Putting these conditions into (1.5), we have

$$\begin{aligned} \frac{H_t^2}{H} &\le H_{tt} + \frac{g'}{g} H_t \\ &\le \left(\frac{n}{H} + \frac{\partial g}{\partial z} + \frac{1}{H} \sum \frac{\partial g}{\partial p_\alpha}\right) H_t + \frac{g'}{g} H_t, \end{aligned}$$

that is,

$$\zeta^2 \le \left[\left(\frac{n}{H} + \frac{\partial g}{\partial z} + \frac{1}{H} \sum \frac{\partial g}{\partial p_\alpha} H_\alpha\right) g + \frac{g'}{H}\right] \zeta,$$

which yields an upper bound for H_t.

Step 3: $|\nabla^2 H| \le C_3$.

Lemma. *Let D be a bounded domain in $\mathbb{R}^n \times [0, T]$ such that $\Omega_t = \{x : (x, t) \in D\}$ is a nonempty convex set for each t in $[0, T]$. Suppose that u is a smooth solution of*

$$\begin{aligned} &p(x) \log \det \nabla^2 u - u_t = q(x, u, \nabla u) \quad \text{in } D, \\ &u(x, 0) = \varphi(x) \quad \text{on } \Omega_0, \\ &u(x, t) = 0 \quad \text{on } \partial\Omega_t \times [0, T], \end{aligned} \tag{1.7}$$

whose Hessian is positive definite for each t. Here p and q are smooth and p is positive. Then for any subdomain D' of D such that $\overline{\Omega'_t} \subseteq \Omega_t$ ($\Omega'_t = \Omega_t \cap D'$) for each t and t_0, $t_0 < T$, there exists a constant K_0 such that

$$\sup_{t_0 \le t \le T} \|\nabla^2 u(\cdot, t)\|_{C(\overline{\Omega'_t})} \le K_0,$$

where K_0 depends on n, t_0, $\inf_{t_0 \le t \le T}\{\operatorname{dist}(\Omega'_t, \partial\Omega_t)\}$, $\sup_{t_0 \le t \le T}\{\operatorname{diam} \Omega_t\}$, $\|u\|_{\tilde{C}^1(\overline{D})}$, $\|u_t\|_{C(\overline{D})}$, a positive lower bound of p in $\overline{D}$, and C^2-norms of p and q.

Taking this lemma for granted for this moment, we finish the proof of this step as follows:

Consider the restriction of $H(\cdot, t)$ to the tangent space of S^n at x. As usual we may take x to be the south pole. From the assumption we have

$$r(1+|x|^2)^{1/2} \leq h(x, t) \leq R(1+|x|^2)^{1/2},$$

where h satisfies (1.4). The set $D(h_0) = \{(x, t): h(x, t) \leq h_0, 0 \leq t \leq T\}$ has Ω_t satisfying $B(((h_0/R)^2 - 1)^{1/2}) \subseteq \Omega_t \subseteq B(((h_0/r)^2 - 1)^{1/2})$. As a result, if we choose $h_0 = 2R$ and let $D = D(2h_0)$ and $D' = B(3^{1/2}) \times [0, T]$, we can apply the lemma to $u = h - h_0$ to obtain a second-order estimate for h and hence for H. Since x is an arbitrary point on S^n, we finish the proof of Step 3.

It remains to prove the lemma. We adapt the method of Pogorelov [9] (see also [7]). Let $\Phi_\xi = -u\exp(2^{-1}\lambda|\nabla u|^2)u_{\xi\xi}g$, where ξ, $|\xi| = 1$ and λ is a large number to be chosen. We assume the maximum of Φ_ξ over ξ and $\overline{D}$ attains at (x_0, t_0), $t_0 > 0$, for some Φ_ξ. By a rotation we may assume $\xi = (1, 0, \cdots, 0)$ and $(\nabla^2 u)$ is diagonal at (x_0, t_0). Writing $\Phi = \Phi_\xi$, we have, for $\alpha = 1, \cdots, n$,

$$(1.8) \qquad 0 = \frac{\partial \log \Phi}{\partial x_\alpha} = \frac{u_{11\alpha}}{u_{11}} + \lambda u_\alpha u_{\alpha\alpha} + \frac{u_\alpha}{u},$$

$$0 \leq \frac{\partial \log \Phi}{\partial t} = \frac{u_{11t}}{u_{11}} + \sum \lambda u_\beta u_{\beta t} + \frac{u_t}{u} + \frac{g'}{g},$$

$$(1.9) \qquad 0 \geq \Phi^{-1}\frac{\partial^2 \Phi}{\partial x_\alpha^2}$$

$$= \frac{u_{11\alpha\alpha}}{u_{11}} - \frac{u_{11\alpha}^2}{u_{11}^2} + \lambda \sum u_\beta u_{\beta\alpha\alpha} + \lambda u_{\alpha\alpha}^2 + \frac{u_{\alpha\alpha}}{u} - \frac{u_\alpha^2}{u^2}$$

at (x_0, t_0). Multiplying the last inequality by $u_{11}u_{\alpha\alpha}^{-1}$ and then summing over α, we obtain

$$\sum \frac{u_{11\alpha\alpha}}{u_{\alpha\alpha}} \leq \sum \frac{u_{11\alpha}^2}{u_{11}u_{\alpha\alpha}} = \lambda u_{11}^2 = \lambda u_{11} \sum \frac{u_\beta u_{\beta\alpha\alpha}}{u_{\alpha\alpha}} - \frac{nu_{11}}{u} + \frac{u_{11}}{u^2}\sum \frac{u_\alpha^2}{u_{\alpha\alpha}}.$$

On the other hand, differentiating (1.7) gives

$$p\sum \frac{u_{\alpha\beta\beta}}{u_{\beta\beta}} + p_\alpha \log\det \nabla^2 u - u_{\alpha t} = \frac{\partial q}{\partial x_\alpha}, \qquad \alpha = 1, \cdots, n,$$

and

$$p\sum \frac{u_{1\alpha\alpha 1}}{u_{\alpha\alpha}} - p\sum \frac{u_{\alpha\beta 1}^2}{u_{\alpha\alpha}u_{\beta\beta}} + 2p_1 \sum \frac{u_{1\alpha\alpha}}{u_{\alpha\alpha}} + p_{11}\log\det \nabla^2 u - u_{11t} = \frac{\partial^2 q}{\partial x_1^2}.$$

Combining the above equations we have

$$p\sum_{\alpha,\beta}\frac{u_{1\alpha\beta}^2}{u_{\alpha\alpha}u_{\beta\beta}} - 2p_1\sum_{\alpha}\frac{u_{1\alpha\alpha}}{u_{\alpha\alpha}} - p_{11}\log\det\nabla^2 u + u_{11t} + \frac{\partial^2 q}{\partial x_1^2}$$

$$\leq p\sum_{\alpha}\frac{u_{11\alpha}^2}{u_{11}u_{\alpha\alpha}} - \lambda p u_{11}^2 - \lambda u_{11}\sum_{\alpha}\left(-p_\alpha\log\det\nabla^2 u + u_{\alpha t} + \frac{\partial q}{\partial x_\alpha}\right)$$

$$-\frac{np}{u}u_{11} + p\frac{u_{11}}{u^2}\sum_{\alpha}\frac{u_\alpha^2}{u_{\alpha\alpha}}.$$

Using (1.8), (1.9), and then the inequality

$$\left|2p_1\sum_{\alpha>1}\frac{u_{1\alpha\alpha}}{u_{\alpha\alpha}}\right| \leq p\sum_{\alpha>1}\left[\frac{u_{1\alpha\alpha}}{u_{\alpha\alpha}}\right]^2 + \frac{(n-1)p_1^2}{p},$$

we arrive at

$$\begin{aligned}&(1-n)\frac{p_1^2}{p} + 2p_1\left(\lambda u_1 u_{11} + \frac{u_1}{u}\right)\\ &-p_{11}\left(\frac{u_t+q}{p}\right) + \frac{\partial^2 q}{\partial x_1^2} - \frac{u_t u_{11}}{u} - \frac{g'}{g}u_{11}\\ &\leq -\lambda p u_{11}^2 + \lambda u_{11}\sum u_\alpha\left[\frac{p_\alpha(u_t+q)}{p} - \frac{\partial q}{\partial x_1}\right] - \frac{npu_{11}}{u} + \frac{p}{u^2}u_1^2.\end{aligned}\tag{1.10}$$

On the other hand,

$$\frac{\partial q}{\partial x_1} = \frac{\partial q}{\partial p_1}u_{11} + q_z u_1 + q_1,$$

and

$$\frac{\partial^2 q}{\partial x_1^2} = \sum\frac{\partial q}{\partial p_\alpha}u_{11\alpha} + \frac{\partial^2 q}{\partial p_1^2}u_{11}^2 + T,$$

where T is at most linear in $\nabla^2 u$. Substituting these into (1.10) and using (1.8) we obtain

$$(\lambda p - K_1)\Phi^2 \leq K_2 + K_3\Phi$$

at (x_0, t_0) for some constants K_1, K_2, and K_3. If we choose λ such that $\lambda p - K_1 \geq 1$, we have a bound of Φ_ξ in D. For a subdomain D' of D with $\rho = \inf_{t_0\leq t\leq T}\{\operatorname{dist}(\Omega_t', \partial\Omega_t)\} > 0$,

$$-u \geq \rho\|u\|_{C(\overline{D})}\left(\sup_{t_0\leq t\leq T}\operatorname{diam}\Omega_t\right)^{-1}$$

by convexity. Consequently, we deduce a bound on $\nabla^2 u$ in $\overline{\Omega}'_t$, $t_0 \le t \le T$, and the proof of the lemma is complete.

Combining (1.3), Step 2, and Step 3 implies immediately that

$$(H_{\alpha\beta} + H\delta_{\alpha\beta})\xi^\alpha \xi^\beta \ge C'|\xi|^2.$$

As a result, the restriction of H to a tangent space satisfies the uniformly parabolic equation (1.4) in $B(3) \times [t_0, T]$. Applying Krylov's Hölder estimate [6] for uniformly parabolic equations we obtain a Hölder estimate on H_t and $\nabla^2 H$ in $B(2) \times [t_1, T]$ for $t_1 > t_0$. Thus parabolic regularity yields estimates of all orders in $B(1) \times [t_2, T]$ for $t_2 > t_1$. Since a finite number of these domains covers $S^n \times [t_2, T]$, we have established the last assertion of this proposition.

2. Absolute minima

In this section we prove Theorem A. Let c be the infinum of J over all uniformly convex hypersurfaces. It is evident that c is nonpositive, for the value of J at a sphere tends to zero as the sphere shrinks to a point. On the other hand, if c is equal to zero and there exists X such that $J(X) \le 0$, then X is an absolute minimum and a solution of (2) in view of (1.2) and the short time existence of (1.3). So we may assume c is negative and show that it is attainable.

First we observe that there is a uniform upper bound for the diameter of X with $J(X) \le 0$. For, we have

$$\int_{S^n} H \le \int_{\widetilde{X}} F \le \|F\|_{L^1(\mathbb{R}^{n+1})}.$$

Let L be the line segment in $\widetilde{X}$ such that its length is equal to $D(X)$. Denote its support function by H_L. By a direct computation the integral $\int_{S^n} H_L$, which is apparently greater than $\int_{S^n} H$, is equal to $(2n+2)^{-1}\tau_{n-1}D(X)$, where τ_{n-1} is the volume of S^{n-1}. Therefore $D(X)$ is bounded. Fix $\varepsilon_0 > 0$ such that $c + \varepsilon_0 < 0$. Next we claim that for all X with $J(X) \le c + \varepsilon_0$, there exist r and R which depend on ε_0 and a uniform upper bound on $D(X)$ such that the inradius of X, $r_{\text{in}}(X)$, is greater than r and all X are contained in the ball $B(R)$. To see this suppose that there exists a sequence of X^k with $J(X^k) \le c + \varepsilon_0$ but $r_{\text{in}}(X^k)$ tending to zero. But as there is a uniform upper bound for the diameter of X^k, $\overline{\lim}_{k\to\infty} J(X^k) \ge 0$, which gives a contradiction. Hence there must be a positive lower bound for the inradius. Similarly one can prove that X is contained in $B(R)$ for some large R.

For a convex body $\widetilde{X}$ (or its boundary X) with support function H we define its *support center* to be

$$z = z(X) = \int_{S^n} Hx.$$

If $\widetilde{X}$ is a ball, z is its center. In general, z is contained in the interior of $\widetilde{X}$ for a nondegenerate $\widetilde{X}$. To see this let B be a ball inside $\widetilde{X}$ and connect it to $\widetilde{X}$ by $t\widetilde{X} + (1-t)B$, $0 \le t \le 1$. In case the support center of $\widetilde{X}$ lies outside $\widetilde{X}$, by continuity there is some t such that the support center of $t\widetilde{X} + (1-t)B$ lies on its boundary. So it suffices to show that it is impossible for $z(X)$ to lie on X. For simplicity we may assume z has outer normal $(1, 0, \cdots, 0)$. Then, taking z as the origin, it is not hard to see that $H(x_1, \cdots, x_n) \le H(-x_1, \cdots, x_n)$ for $x_1 > 0$, and $H(1, 0, \cdots, 0) < H(-1, 0, \cdots, 0)$. Therefore, $\int_{S^n} Hx_1$ is not equal to zero, a contradiction.

Denote by $\mathscr{C}(r, D)$ the class of convex bodies whose inradii are not less than r and diameters are not greater than D. We claim that there exists $\delta > 0$ such that $B(z(X), \delta)$ is contained inside $\widetilde{X}$ for $\widetilde{X}$ in $\mathscr{C}(r, D)$. For, if not, we can find a sequence $\{\widetilde{X}^k\}$ in $\mathscr{C}(\delta, D)$ whose centers z^k satisfy that $\operatorname{dist}(z^k, X^k)$ tends to zero as k tends to infinity. By a suitable translation we may assume all X^k's are bounded in a ball. By Blaschke's selection theorem it contains a subsequence (still denoted by $\{X^k\}$) converging to a nondegenerate convex body X in the Hausdorff metric, that is, the corresponding support functions H^k converge uniformly to H on S^n. But this immediately implies that $z(X)$ lies on the boundary of $\widetilde{X}$. Hence there is a contradiction and our claim is established.

Now, we are ready to establish the long time existence of (1.3). Let H be a solution of (1.3) with $J(X(0)) \le c + \varepsilon$, $\varepsilon \le \varepsilon_0$. We derive a priori estimates of all orders of H on any finite interval as follows. Denote by $[0, T^*)$ the maximal interval of existence. By the local existence of parabolic equations we know that T^* is positive. Denote by $z(t)$ the support center of $X(t)$. For t and t' in $[0, T^*)$ with $|t - t'| \le 1$,

$$|z(t) - z(t')| \le \int_{t'}^{t} \int_{S^n} |H_t| \le \int_{t'}^{t} \int_{\{|H_t| \le \delta'\}} |H_t| + \int_0^{T^*} \int_{\{|H_t| > \delta'\}} |H_t|,$$

where $\delta' = (4\tau_n)^{-1}\delta$. Integrating (1.2) from 0 to T^*, we obtain

$$\varepsilon \ge \int_0^{T^*} \int_{\{|H_t| \le \delta'\}} H_t(1 - e^{-H_t})F \ge m(e^{\delta'} - 1) \int_0^{T^*} \int_{\{|H_t| \le \delta'\}} |H_t|,$$

where $m = \inf\{F(x) \colon x \in B(R)\}$. Therefore, for

$$\varepsilon < \min\{\varepsilon_0, 4^{-1} m\delta(e^{\delta'} - 1)\},$$

we have

$$|z(t) - z(t')| < \delta/2.$$

In other words, the ball $B(z(t), \delta/2)$ is contained in $\widetilde{X}(t')$ for all t', $|t - t'| \leq 1$. We may write $[0, T^*)$ as a suitable union of intervals of length less than or equal to one and apply Proposition (1.5) to each such interval taking its left endpoint as the origin. In this way we obtain a uniform estimate of H of all orders in the interval $[2^{-1}T^*, T^*)$ in case T^* is finite. However, by a standard argument one can extend H beyond T^*. Since this contradicts the maximality of T^*, T^* must be infinity. Proposition (1.5) thus yields a uniform estimate of all orders of H as well as a positive lower bound for the curvature K of the convex hypersurfaces determined by H in, say, $[1, \infty)$.

In view of the inequality

$$\varepsilon > \int_0^\infty \int_{S^n} H_t(F - K) = \int_0^\infty \int_{S^n} (F - K) \log \frac{F}{K}$$

we can extract a sequence $\{X^j\}$, $X^j = X(t_j)$, such that in self-explanatory notation,

$$\lim_{t_j \to \infty} \int_{S^n} (F(\nabla H^j) - K^j) \log \frac{F(\nabla H^j)}{K^j} = 0.$$

By compactness we may assume $\{H^j\}$ converges smoothly to a convex function H on S^n. Clearly H solves (2).

We have shown that for each sufficiently small ε, there corresponds a smooth solution X_ε of (2) with $J(X_\varepsilon) \leq c + \varepsilon$. Since we have a uniform estimate of all orders on X_ε, $\varepsilon < \varepsilon_0$, by letting ε tend to zero, by compactness again there is a subsequence of $\{X_\varepsilon\}$ which converges smoothly to a solution X. Clearly $J(X) = c$. Hence the proof of Theorem A is complete.

3. Another gradient flow

In this section we shall first study the gradient flow

$$X_t = -(K^{1/n} - F^{1/n})\nu \tag{3.1}$$

and then use it to prove Theorem B. We shall derive the equations satisfied by the Gauss-Kronecker curvature K and the mean curvature M of X.

By applying the maximum principle it will be shown that K has a positive lower bound (hence convexity is preserved) and all principal curvatures are comparable as long as X stays in the region where F is positive. The choice of this flow is inspired from Chow [4]. The computation below has been carried out in [4] in case F is identically equal to zero.

Throughout this section the function $F = F^{1/n}$ *is always assumed to satisfy* (i) *the set* $\Omega = \{x\colon G > 0\}$ *is bounded and* (ii) *there exist positive constants a and* b *such that*

$$a|\xi|^2 \le \frac{\partial^2 G}{\partial x_\alpha \partial x_\beta}\xi^\alpha \xi^\beta \le b|\xi|^2, \qquad \alpha, \beta = 1, \cdots, n+1,$$

in Ω.

We regard X as an embedding of S^n into $\mathbb{R}^{n+1}$. The induced metric and the second fundamental form, written in terms of a local coordinate, are given respectively by

$$g_{ij} = \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle \quad \text{and} \quad b_{ij} = \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial \nu}{\partial x_j} \right\rangle,$$

where $\langle\cdot, \cdot\rangle$ is the Euclidean metric in $\mathbb{R}^{n+1}$. Denoting by ∇_i the covariant differentiation in x_i and the moving indices up or down by contraction with the metric as usual, we compute

$$\text{(3.2)} \qquad \begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2(K^{1/n} - G)b_{ij}, \\ \frac{\partial b_{ij}}{\partial t} &= \nabla_i \nabla_j (K^{1/n} - G) - (K^{1/n} - G) b_{ik} b_j^k. \end{aligned}$$

As

$$\begin{aligned} \nabla_i \nabla_j K^{1/n} &= \frac{1}{n}\nabla_i (K^{1/n} \nabla_j K) \\ &= \frac{1}{n} K^{1/n-1} \nabla_i \nabla_j K + \frac{1}{n}\left(\frac{1}{n} - 1\right) K^{1/n-2} \nabla_i K \nabla_j K \end{aligned}$$

and

$$\nabla_i \nabla_j K = K h^{kl} \nabla_i \nabla_j b_{kl} + K^{-1} \nabla_i K \nabla_j K + K \nabla_i h^{kl} \nabla_j b_{kl},$$

where h^{kl} satisfies $h^{kl} b_{lm} = \delta_m^k$, we have

$$\begin{aligned} \nabla_i \nabla_j K^{1/n} = \frac{1}{n} K^{1/n} h^{kl} \nabla_i \nabla_j b_{kl} + \frac{1}{n^2} K^{1/n-2} \nabla_i K \nabla_j K \\ + \frac{1}{n} K^{1/n} \nabla_i h^{kl} \nabla_j b_{kl}. \end{aligned}$$

Using Gauss' equation $R_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk}$ leads to

$$h^{kl}\nabla_i\nabla_j b_{kl} = h^{kl}(\nabla_k\nabla_l b_{ij} + R_{ijkm}b_{ml} + R_{ijlm}b_{mk}) = \Delta_h b_{ij} + Mb_{ij} - nA_{ij},$$

where $\Delta_h = \frac{1}{n}K^{1/n}h^{kl}\nabla_k\nabla_l$ and $A_{ij} = b_{ik}b_j^k$. Therefore,

$$\frac{\partial b_{ij}}{\partial t} = \Delta_h b_{ij} + \frac{1}{n}K^{1/n}\left(Mb_{ij} - nA_{ij} + \frac{1}{n}K^{-2}\nabla_i K\nabla_j K + \nabla_i h^{kl}\nabla_j b_{kl}\right) - (K^{1/n} - G)A_{ij} - \nabla_i\nabla_j G.$$

By means of (3.2) and (3.3), we obtain

$$\frac{\partial K}{\partial t} = \Delta_h K + \frac{1}{n}\left(\frac{1}{n} - 1\right)K^{1/n-1}|\nabla K|_h^2 + (K^{1/n} - G)KM - K\Delta_h G. \tag{3.4}$$

Using the identity

$$\frac{1}{nK^2}\nabla_i K\nabla_j K + \nabla_i h^{kl}\nabla_j b_{kl} = \frac{M^{2n}}{nk^2}\nabla_i\frac{K}{M^n}\nabla_j\frac{K}{M^n} - M^{-2}h^{km}h^{ln}(M\nabla_i b_{mn} - \nabla_i M b_{mn})(M\nabla_j b_{kl} - \nabla_j M b_{kl}),$$

we also have

$$\frac{\partial M}{\partial t} = \Delta_h M + \frac{1}{n^2}K^{1/n-1}M^{2n}\left|\nabla\frac{K}{M^n}\right|^2 + \frac{1}{n}K^{1/n}(M^2 - n|A|^2) - (K^{1/n} - G)|A|^2 - \Delta G - \frac{K^{1/n}}{nM^2}g^{ij}h^{km}h^{ln}(M\nabla_i b_{mn} - \nabla_i M b_{mn})(M\nabla_j b_{kl} - \nabla_j M b_{kl}).$$

Combining this equation with (3.4) we finally arrive at

$$\frac{\partial}{\partial t}\frac{K}{M^n} = \Delta\frac{K}{M^n} + \frac{K}{M^{n-1}}\left(\frac{n|A|^2}{M^2} - 1\right) - \frac{K}{M^n}\Delta_h G + \frac{nK}{M^{n+1}}\Delta G + \frac{K^{1/n+2}}{M^{n+1}}g^{ij}h^{km}h^{ln}(M\nabla_i b_{mn} - \nabla_i M b_{mn})(M\nabla_j b_{kl} - \nabla_j M b_{kl}). \tag{3.5}$$

(3.6) **Proposition.** *Let X be a solution of (3.1), which lies in Ω. Then there exist positive constants l and K_0 depending on a, b, and initial data such that* (a) $l\kappa_{\min} \geq \kappa_{\max}$ *and* (b) $K \geq K_0$, *where $\kappa_{\max}$ and $\kappa_{\min}$ are respectively the maximal and minimal principal curvatures of X.*

Proof. Taking special coordinates so that $g_{ij} = \delta_{ij}$, h_{ij} and $G^{\alpha\beta}$ are diagonal at a point, we obtain

$$\begin{aligned} &- nK^{-1/n}\Delta_h G + \frac{n}{M}\Delta G \\ &= \sum_{\alpha, 1}\left(- \frac{1}{\kappa_i}G_{\alpha\alpha}\nabla_i X^\alpha \nabla_i X^\alpha + \frac{n}{M}G_{\alpha\alpha}\nabla_i X^\alpha\nabla_i X^\alpha\right) \\ &= \sum_i \frac{1}{\kappa_i}a_i - \frac{n}{M}\left(\sum_i a_i\right), \end{aligned}$$

where $a_i = -\sum G_{\alpha\alpha}\nabla_i X^\alpha \nabla_i X^\alpha$ bounded between a and b. Thus we can find λ such that

$$-nK^{-1/n}\Delta_h G + \frac{1}{M}\Delta G > 0$$

if $\kappa_{\max}/\kappa_{\min} \geq \lambda$. Applying the maximum principle to (3.5) we immediately conclude that either $\frac{K}{M^n} \geq \min \frac{K}{M^n}(0)$ or $1 \leq \kappa_{\max}/\kappa_{\min} \leq \lambda$. In both cases (a) holds.

To prove (b) we look at the coefficient of K in (3.4). By (a) we have

$$\begin{aligned} &(K^{1/n} - F^{1/n})M + nG_\alpha \nu^\alpha - h^{ij}G_{\alpha\beta}\nabla_i X^\alpha \nabla_j X^\beta \\ &\quad > -nC \max_{\{F>0\}} G\kappa_{\min} - n\max_{\{F>0\}} |G_\alpha \nu^\alpha| + \kappa_{\min}^{-1}a > 0 \end{aligned}$$

if $\kappa_{\min}$ is less than a constant κ_0 depending on l and G. Applying the maximum principle to (3.4) we conclude that either $K \geq \min K(0)$ or $K \geq \kappa_0^n$. Hence the proof of the proposition is finished.

As in the case for the logarithmic gradient flow, it can be shown that (3.1) is equivalent to the following equation for the support function H of X:

$$H_t = -K^{1/n} + G. \tag{3.7}$$

We shall derive a priori estimates for solutions of (3.7). Since G is smooth and uniformly concave, we can find a small ε_0 such that the principal curvatures of the boundary of the set $\Omega' = \{x\colon G(x) > \varepsilon_0\}$ are greater than ε_0.

(3.8) Proposition. *Suppose that X is a solution of space (3.1) with $X(0)$ lying in Ω'. Then X remains in Ω'.*

Proof. If X touches $\partial\Omega'$ for the first time at a point, then at the normal of this point H_t is nonnegative. On the other hand, since $X(t)$ is convex and touches $\partial\Omega'$ from inside, we have

$$H_t = -K^{1/n} + G < -\varepsilon_0 + \varepsilon_0 < 0,$$

a contradiction.

(3.9) Proposition. *Suppose that H is a solution of* (3.7) *in $Q = S^n \times [0, T]$ with $\widetilde{X}(0)$ contained inside Ω', and further that $H > r$ for some positive r. Then there exist constants C and C' such that*

$$\|H\|_{\widetilde{C}^2(\overline{Q})} \le C$$

and

$$(H_{\alpha\beta} + H\delta_{\alpha\beta})\xi^\alpha\xi^\beta \ge C'|\xi|^2.$$

Here C and C' depend on n, r, $\operatorname{diam}(\Omega')$, *the C^2-norm of F in Ω', a positive lower bound of F in Ω', and initial data. Consequently for each k, $k \ge 2$, and t_0, $0 < t_0 < T$, there exists C_k which also depends on higher derivatives of F such that*

$$\|H\|_{\widetilde{C}^k(\overline{Q}')} \le C_k, \qquad Q' = S^n \times [t_0, T].$$

Proof. By (3.7) and (3.8), $X(\cdot, T)$ is uniformly convex and stays in Ω'. We have $|\nabla H| \le \operatorname{Diam}(\Omega')$, and estimate other derivatives of H as follows. The argument is similar to that of Proposition (1.5) except now we take g to be the constant function 1.

Step 1: $H_t \ge -C_1$. This is similar to the proof of Proposition (1.5). By differentiating (3.7) we obtain

$$H_{tt} = \frac{1}{n} K^{1/n} \sum \frac{H_{t\alpha\alpha}}{H_{\alpha\alpha}} + G_z H_t + \sum \frac{G}{p_\alpha} H_{\alpha t}.$$

Consider the same auxiliary function η as before. At a negative minimum we have

$$\begin{aligned}
\frac{H_t^2}{H-\delta} &\ge H_{tt} \\
&\ge \frac{1}{n} K^{1/n} \left(\frac{nH_t}{H-\delta} - \frac{\delta H_t}{H-\delta} \sum \frac{1}{H_{\alpha\alpha}} \right) + \frac{H_t}{H-\delta} \sum \frac{\partial G}{\partial p_\alpha} H_\alpha + G_z H_t \\
&\ge \frac{1}{n}(-H_t + G)\left[\frac{nH_t}{H-\delta} - \frac{n\delta H_t}{H-\delta}(-H_t + G) \right] \\
&\quad + \frac{H_t}{H-\delta} \sum \frac{\partial G}{\partial p_\alpha} H_\alpha + G_z H_t \\
&\ge C_2(-H_t)^3 - C_3.
\end{aligned}$$

This implies a lower bound for η and hence for H_t.

Step 2: $H_t \le C_4$. This follows immediately from the equation itself.

Step 3: $|\nabla^2 H| \le C_5$.

Step 4: $(H_{\alpha\beta} + H\delta_{\alpha\beta})\xi^\alpha\xi^\beta \geq C'|\xi|^2$. Both steps are consequences of Proposition (3.6).

Now we can argue as in the proof of Proposition (1.5) to conclude the rest of the proof.

We have come to the main body of the proof of Theorem B. Let Y be an element in $\mathscr{K}$ which is contained in Ω' and satisfies $J(Y) \leq 0$. By assumption, Y exists. For X with $D(X) = \rho$, we have

$$J(X) \geq \int_{S^n} \widehat{H} - \left(\max_{\overline{\Omega}} F\right)\omega_n\rho^{n+1} = \delta_n\rho - \left(\max_{\overline{\Omega}} F\right)\omega_n\rho^{n+1},$$
$$\delta_n = (2n+2)^{-1}\tau_{n-1},$$

where $\widehat{H}$ is the support function for a line segment of length ρ contained in $\widetilde{X}$, and ω_n is the volume of the unit ball in $\mathbb{R}^{n+1}$. As a result, we can fix a ρ, $\rho < D(Y)$, such that

$$J(X) \geq 2^{-1}\delta_n\rho,$$

whenever $D(X) = \rho$. We can also fix a small sphere Z in Ω' such that $D(Z)$ is less than ρ and $J(Z)$ is less than $2^{-1}\delta_n\rho$. Suppose that $\Gamma = \{\gamma\colon \gamma\colon [0, 1] \to \mathscr{K}$ is continuous, $\gamma(s) \subseteq \Omega'$ and satisfies $D(\gamma(0)) < \rho$, $D(\gamma(1)) > \rho$, and $J(\gamma(0))$, $J(\gamma(1)) < 2^{-1}\delta_n\rho\}$, and set

$$c = \inf_{\gamma\in\Gamma}\max_s J(\gamma(s)).$$

Clearly Γ is nonempty and $c \geq 2^{-1}\delta_n\rho$. We shall show that c is a critical value of J.

For $\varepsilon > 0$, $c - \varepsilon > 0$, we can find γ in Γ such that

$$J(\gamma(s)) \leq c + \varepsilon.$$

Using this γ as initial data we can solve the gradient flow (3.1) and obtain a family of solutions $\gamma(t, s)$ in a set of the form $\{(t, s)\colon 0 \leq t < T(s),\ 0 \leq s \leq 1\}$. For each (t, s) in this set, $\gamma(t, s)$ is uniformly convex and stays in Ω'. For the life-span of γ we claim that with a further restriction of ε, a flow of (3.1), which satisfies initially $J(X) \leq c + \varepsilon$, exists as long as $J(X(t)) \geq c - \varepsilon$. To prove this we first observe that $D(X(t))$ and $r_{\text{in}}(X(t))$ tend to zero simultaneously if this ever happens. For, if $\widetilde{X}$ collapses into a degenerate convex body but not a pont as time evolves, there would be some point on X with arbitrary small principal curvature at some direction. However, by Proposition (3.6)(a) this implies that the Gauss-Kronecker curvature at this point is also small—a contradiction with (b) of the same proposition. As a result, the inradius of X has a uniform positive lower bound as long as $J(X) \geq c - \varepsilon$. As we have shown in the proof of Theorem A we can find $\delta > 0$ such that $B(z(t), \delta)$ is contained in $\widetilde{X}(t)$, where $z(t)$ is the convex center of $\widetilde{X}(t)$. Now, for t and t', $|t - t'| \leq 1$, we have

$$|z(t) - z(t')| \leq \int_{t'}^{t} \left(\int_{\{|H_t| \leq \delta'\}} + \int_{\{|H_t| > \delta'\}} \right) |H_t|,$$

where $\delta' = (4\tau_n)^{-1}\delta$. Along the flow,

$$\frac{dJ}{dt} = - \int_{S^n} (K - F)(K^{1/n} - F^{1/n}) \leq 0.$$

Integrating it from t' to t gives

$$2\varepsilon \geq \int_{t'}^{t} \int_{\{|H_t| \leq \delta'\}} (F - K) H_t \geq \delta'^n \int_{t'}^{t} \int_{\{|H_t| \leq \delta'\}} |H_t|.$$

Thus $|z(t) - z(t')| < \delta/2$ if we restrict ε to be less than $2^{-2n-3}\tau_n^n \delta^{n+1}$. Now using Proposition (3.9) rather than Proposition (1.5) we can argue as in the proof of Theorem A that $X(t)$ exists as long as $J(X(t)) \geq c - \varepsilon$.

It makes sense to define $t^*(s) = \sup\{t\colon J(\gamma(t, s)) \geq c - \varepsilon\}$ (set $t^*(s)$ to be zero if $J(\gamma(0, s)) < c - \varepsilon$). Notice that t^* cannot be continuous, otherwise $\gamma(t^*(s), s)$ defines a curve in Γ and yet $J(\gamma(t^*(s), s)) \leq c - \varepsilon$. Let s_0 be a point of discontinuity. Then we claim that $t^*(s_0) = \infty$. Suppose on the contrary there exists $\{s_j\}$ such that s_j tends to s_0 and $t^*(s_j)$ tends to t_1, which is not equal to $t^*(s_0)$. For any $t < t^*(s_0)$, $J(\gamma(t, s_0)) > c - \varepsilon$. Hence, for large j, $J(\gamma(t, s_j)) > c - \varepsilon$. This implies that $t_1 > t^*(s_0)$. However, fix t' with $t_1 > t' > t^*(s_0)$. Then $J(\gamma(t', s_0)) < c - \varepsilon$. ($\gamma(t^*(s_0), s_0)$ cannot be a critical point otherwise $t^*(s_0)$ would be infinite.) But this yields that $t_1 \leq t'$, a contradiction. Similarly one can draw a contradiction in the case $t^*(s_0) = 0$.

Let $X(t) = \gamma(t, s_0)$. Then X satisfies (3.1) for all time. Using the fact

$$2\varepsilon \geq \int_0^{\infty} \int_{S^n} (K^{1/n} - G)(K - F),$$

we can follow the proof of Theorem A (a positive lower bound of K is given by Proposition (3.6)) to conclude that there exists a smooth solution X_ε of (2) with $J(X_\varepsilon)$ lying between c and $c + \varepsilon$. Since all X_ε's are contained in Ω' and their Gauss-Kronecker curvatures are pinched between two positive constants, it is not hard to see that the inradii of X_ε are bounded below and above by two positive constants. Thus a sequence $\{X^j = X_{\varepsilon_j}\}$ converges to a nondegenerate convex hypersurface X

in Hausdorff metric as ε_j tends to zero. We claim X is smooth and solves our problem. Since there is $\delta > 0$ such that $B(z(X^j), \delta)$ is contained in X^j, $B(z(X), \delta/2)$ is contained inside X^j for sufficiently large j. Using $z(X)$ as the origin, we have

$$\frac{\delta}{2}(1+|x|^2)^{1/2} \le h^j \le 2R(1+|x|^2)^{1/n}$$

for sufficiently large j. Here h^j is the restriction of H^j to the tangent space at the south pole and satisfies the equation

$$\det \nabla^2 h^j = F(\nabla h^j, h^{j^*})$$

in $\mathbb{R}^n$. Since the domain $\{x\colon h^j < 4R\}$ contains the ball $B(3^{1/2})$ and the oscillatin of h^j over this domain is at least $2R$, we may apply Pogorelov's C^2-interior estimate [9] to obtain a uniform estimate of $\{\nabla^2 h^j\}$ in the unit ball. Appealing to further interior regularity for solutions of uniformly elliptic equations in nondivergence form we obtain uniform estimates of all orders of $\{\nabla^2 h^j\}$ in $B(2^{1/2})$. Therefore h is smooth and satisfies

$$\det \nabla^2 h = F(\nabla h, h^*)$$

in $B(2^{1/2})$. Similarly one can show that the restriction of H on any tangent space is smooth and satisfies a corresponding equation in a ball of radius $1/2$ centered at the base point. Hence we conclude that H is a smooth solution of (2). Clearly $J(X) = c$. Thus the proof of Theorem B is finished.

4. Functionals with symmetry

Perhaps the simplest candidate for a curvature function that one can imagine is the constant function 1. By a characterization of spheres the solution of (2) is the unit sphere. Next one may ask whether (2) is solvable if the prescribed function is bounded between two positive constants. Although it is easy to verify that the setting for a mountain pass argument is valid, we have not been able to establish this result. Similarly a mountain pass argument should be able to apply under the assumption in Theorem A, that is, F is integrable in the whole space, to produce a solution other than an absolute minimum. A main technical obstacle is to ensure the logarithmic gradient flow, though apparently preserves convexity, behaves in a nice way. Specifically one would like to see that the inradius and the diameter tend to zero simultaneously provided that it really happens.

Nevertheless, this desirable property can be established if we impose certain symmetry assumptions on the prescribed function and seek solutions among hypersurfaces invariant under this symmetry. To formulate the result let G be a subgroup of the orthogonal $\mathscr{O}(n+1)$ which acts on $\mathbb{R}^{n+1}$ from the right. F is called G-invariant if $F(xg) = F(x)$ for all x in $\mathbb{R}^{n+1}$ and g in G. A hypersurface X is called G-symmetric if $Xg = X$ for all g in G. Denote the class of all G-symmetric hypersurfaces in $\mathscr{K}$ by $\mathscr{K}_G$. We assume G satisfies the condition that $\{x\colon xg$ spans $\mathbb{R}^{n+1}$ for some $x\}$. It is easy to see this condition is equivalent to

$$\beta = \sup\left\{\frac{D(X)}{r_{\text{in}}(X)}\colon X \in \mathscr{K}_G\right\} < \infty. \tag{4.1}$$

(4.2) **Proposition.** *Let G be a subgroup of $\mathscr{O}(n+1)$ such that* (4.1) *holds and let F be a positive G-invariant function. Then* (i) *there exists a solution for* (2) *if*

$$\varliminf_{R\to\infty}\left(\frac{\beta\tau_n}{2} - \frac{1}{R}\int_{B_R} F\right)R \le 0,$$

and (ii) *there exist two solutions for* (2) *if*

$$\varliminf_{R\to\infty}\left(\frac{2\tau_n}{\beta} - \frac{1}{R}\int_{B_R} F\right)R = \infty$$

and $\inf\{X\colon X \in \mathscr{K}_G\}$ *is nonpositive.*

In particular, (2) is solvable if F is bounded between two positive constants or F is integrable and there exists some X in $\mathscr{K}_G$ with $J(X) \le 0$.

Proof. In both cases it is easy to see that there exist $\rho > 0$ and Z, Y in $\mathscr{K}_G$ such that $J(X) \ge 2^{-1}\delta_n\rho$ on $D(X) = \rho$, $J(z) < 2^{-1}\delta_n\rho$, $D(Z) < \rho$ and $J(Y) < 2^{-1}\delta_n\rho$, $D(Y) > \rho$. We may define Γ (replace Ω' by $\mathbb{R}^{n+1}$) and c as in the proof of Theorem B and show that c is a critical value of J in $\mathscr{K}_G$.

As usual we decrease J along a negative gradient flow. For the present case we choose the logarithmic gradient flow (1.1) as it preserves convexity. Also by the uniqueness of solution to a parabolic equation $X(t)$ still belongs to $\mathscr{K}_G$ for each t. We claim that the solution exists as long as $J(X(t))$ is bounded between $c-\varepsilon > 0$ and $c+\varepsilon$. For, in case (i) we have

$$J(X) \le \frac{\tau_n D}{2} - \int_{B(r_{\text{in}})} F \le r_{\text{in}}\left(\frac{\beta\tau_n}{2} - \frac{1}{r_{\text{in}}}\int_{B(r_{\text{in}})} F\right).$$

Hence $J(X)$ becomes nonpositive as $r_{\text{in}}(X)$ and $D(X)$ go to infinity. In case (ii) we first observe, by the Brunn-Minkowski inequality, that the following isoperimetric inequality [1] holds:

$$\left(\frac{1}{n+1}\int_{S^n} H\right)^{n+1} \geq \omega_n^n |\widetilde{X}|$$

and equality holds if and only if X is a unit sphere. Using this we have

$$\begin{aligned} J(X) &\geq \omega_n^{-(1/(n+1))} |\widetilde{X}|^{(n+1)^{-1}} \left(\tau_n - \frac{\beta}{D}\int_{B(D/2)} F\right) \\ &\geq \frac{D}{2}\left(\frac{2\tau_n}{\beta} - \frac{2}{D}\int_{B(D/2)} F\right). \end{aligned}$$

Hence as in case (i) we conclude that the diameter of X is bounded as long as $|J(X)-c| < \varepsilon$. Now by symmetry the support center of $X(t)$ is always the origin. Arguing as in the proof of Theorem B and using Proposition (1.5) instead of Proposition (3.9), we find that there exists $X(t)$ with $c \leq J(X) \leq c+\varepsilon$ which contains a subsequence $\{X(t_j)\}$ converging smoothly to a solution X_ε of (2). Letting ε go to zero, we obtain a solution X of (2) with $J(X) = c$.

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