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HYPERBOLIZATION OF POLYHEDRA

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Introduction

Hyperbolization is a process for converting a simplicial complex into a metric space with "nonpositive curvature" in the sense of Gromov. Several such processes are described in [19, §3.4]. One of the purposes of this paper is to elaborate this idea of Gromov. Another purpose is to use it to construct the three examples described below.

Our approach to hyperbolization is based on the following construction of Williams [32]. Suppose that X is a space and that $f\colon X \to \sigma^n$ is a map onto the standard n-simplex. Suppose, also, that K is an n-dimensional simplicial complex. To these data Williams associates a space $X\Delta K$, constructed by replacing each n-simplex in the barycentric subdivision of K by a copy of X. The pair (X, f) is a "hyperbolized n-simplex" if X^n is a nonpositively curved manifold with boundary and f has appropriate properties. (It is proved in §4 that hyperbolized simplices exist.) If (X, f) is a hyperbolized n-simplex, then $X\Delta K$ is nonpositively curved; it is called a "hyperbolization of K."

In all three examples we begin with a polyhedral homology manifold having a desired feature; a hyperbolization then has the added feature of nonpositive curvature. The first example is a closed aspherical four-manifold which cannot be triangulated. Taking the product of this example with a n-torus, we obtain an aspherical manifold of any dimension ≥ 4 which is not homotopy equivalent to a PL manifold. The second example is a closed smooth manifold of dimension $n \geq 5$ which carries a topological metric of nonpositive curvature, while its universal cover, though contractible, is not homeomorphic to a Euclidean n-space $\mathbb{R}^n$. As we shall see, such a manifold cannot carry a PL or smooth metric of nonpositive curvature. The third example is a further refinement: M^n carries a topological metric of strict negative curvature and its universal

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cover $\widetilde{M}^n$ is homeomorphic to $\mathbb{R}^n$, yet M^n cannot carry a PL or smooth metric of strict negative curvature. The invariant in this case is the "ideal boundary" of the universal cover.

The construction in the first example goes as follows. Let K^4 be a triangulation of the "E_8 homology four-manifold." (By this we mean the polyhedral homology four-manifold formed by taking the smooth, simply connected, four-manifold with boundary with the E_8 form as its intersection form and then attaching the cone on the boundary.) A neighborhood of the cone point in K^4 is isomorphic to the cone on Poincaré's homology three-sphere Σ^3. For suitably chosen hyperbolized four-simplex X, we will have that (i) $X\Delta K$ is an orientable polyhedral homology four-manifold with vanishing second Stiefel-Whitney class, (ii) $X\Delta K$ has one singular point, a neighborhood of which is isomorphic to the cone on Σ^3, and (iii) the signature of $X\Delta K$ is 8. Since $X\Delta K$ is nonpositively curved, it is aspherical, i.e., $X\Delta K$ is a $K(\pi, 1)$. Properties (i) and (iii) imply that $X\Delta K$ is not homotopy equivalent to a closed PL manifold (by Rohlin's Theorem). On the other hand, $X\Delta K$ is homotopy equivalent to a closed topological manifold, namely, the manifold N^4 formed by replacing a neighborhood of the singular point by a contractible manifold bounded by Σ^3. (This uses [16].) It follows from recent work of Casson that N^4 is not homeomorphic to a simplicial complex, i.e., it cannot be triangulated.

Before describing the remaining two examples we need to discuss some properties of universal covers of nonpositively curved polyhedral homology manifolds which are piecewise flat or piecewise hyperbolic (our examples are of this type). In the PL setting we reprove the following version of the Cartan-Hadamard Theorem (Theorem 3b.2), the first part of which is a result of David Stone [31].

Theorem. *Let $\widetilde{M}^n$ be a simply connected, nonpositively curved, piecewise flat (or piecewise hyperbolic), PL manifold.*

(i) (*Stone*) *$\widetilde{M}^n$ is homeomorphic to $\mathbb{R}^n$.*
(ii) *The ideal boundary of $\widetilde{M}^n$ is homeomorphic to the $(n-1)$-sphere S^{n-1}.*

This result is false for polyhedral homology manifolds which are not PL manifolds; for example, the universal cover of the hyperbolization of the E_8 homology manifold is not simply connected at infinity and its ideal boundary is not S^3 (it is not even an ANR). (It follows that the universal cover of our example of a nontriangulable aspherical four-manifold N^4 is not homeomorphic to $\mathbb{R}^4$.)

The fact that polyhedral homology manifolds which are not PL manifolds have something to do with exotic universal covers was first recognized in [11], through the use of reflection groups. In the recent Ph.D. thesis of G. Moussong [24], it is shown that some of the results of [11] on reflection groups can be recovered using nonpositive curvature. In particular, Moussong proves that the natural contractible simplicial complex on which a Coxeter group W acts properly with compact quotient can be given a piecewise flat structure with *nonpositive* curvature. Sometimes this simplicial complex is a polyhedral homology manifold and one can use the results of §3 to see that its "fundamental group at infinity" can be nontrivial. We should mention, in this regard, that Ancel and Siebenmann have announced some related results concerning the ideal boundary of these reflection group examples; in particular, they have pointed out that the ideal boundary need not be a sphere (or even an ANR).

It has been known since 1975 that polyhedral homology manifolds which are not PL manifolds can unexpectedly and miraculously be topological manifolds; for example, the double suspension of any homology n-sphere is homeomorphic to S^{n+2}. The definitive result is Edwards' Characterization Theorem (cf. [13]): *A polyhedral homology manifold of dimension ≥ 5 is a topological manifold if and only if the link of each vertex is simply connected.*

In our second example we hyperbolize a certain non-PL triangulation of S^n, $n \geq 5$. By Edwards' Theorem the resulting nonpositively curved space Q^n is a topological manifold. We show that the universal cover $\widetilde{Q}^n$ is not simply connected at infinity.

In our third example we are concerned with the ideal boundary. When the curvature is strictly negative the ideal boundary is a quasi-isometry invariant. Hence, there is an obstruction for a manifold which admits a topological metric of strict negative curvature to have a PL metric of strict negative curvature: the ideal boundary of its universal cover must be homeomorphic to a sphere (by the previously stated version of the Cartan-Hadamard Theorem). As we shall see, the ideal boundary of the universal cover is a finer invariant than its fundamental group at infinity. We apply a "strict" hyperbolization procedure to the double suspension of Σ^3. ("Strict" means that the curvature is strictly negative.) The result is a negatively curved topological five-manifold N^5. The universal cover $\widetilde{N}^5$ is simply connected at infinity, hence, by a theorem of Stallings [30], it is homeomorphic to $\mathbb{R}^5$. However, its ideal boundary is not homeomorphic to S^4.

The hyperbolization process is conceptually as simple as the reflection group techniques of [11]; however, it is a more potent source of examples. One reason is that hyperbolization provides more flexibility with characteristic classes and characteristic numbers, while reflection group constructions generally yield stably parallelizable manifolds. Thus, one cannot use reflection groups to produce a four-manifold of nonzero signature as in the first example. Another drawback of the reflection group techniques is that it is impossible to use them to produce examples of manifolds of strict negative curvature in dimensions ≥ 30. (This follows from results of Vinberg [33], as has been pointed out to us by Gabor Moussong.) Thus, it would seem to be very difficult to find something like our third example by means of reflection groups. On the other hand, an analog of the second example can be produced using reflection groups.

A hyperbolization procedure is interesting as a purely topological process. In this context it makes more sense to call it "asphericalization." The first such asphericalization procedure is due to Kan and Thurston [20]. They associate to each simplicial complex K a space $a(K)$ and a map $f_K\colon a(K) \to K$ with the following properties.

(1) $a(K)$ is aspherical.

(2) f_K induces an isomorphism on homology (with local coefficients).

As pointed out in [21], if K is n-dimensional, then one can find such an asphericalization of the form $a(K) = X\Delta K$ for an n-complex X which is suitably acyclic and aspherical. A basic problem with this type of asphericalization is that property (2) prevents such a procedure from taking manifolds to manifolds; for example, no two-manifold asphericalization of the two-sphere can satisfy (2). Suppose, however, that we weaken (2) as follows.

(2′) $f_{K*}\colon H_*(a(K)) \to H_*(K)$ is into.

Then one can produce hyperbolizations satisfying (1), (2′) and the following additional properties.

(3) If K is an n-manifold, then so is $a(K)$.

(4) If K is a manifold, then its stable tangent bundle pulls back (via f_K) to the stable tangent bundle of $a(K)$.

In the terminology of surgery theory (cf. [7]), properties (2′), (3), and (4) mean that f_K is a "degree one normal map." These hyperbolization procedures also have the following property.

(5) If K is a manifold, then the normal map $f_K\colon a(K) \to K$ is normally bordant to the identity.

Property (5) has the following interesting consequence.

Theorem A. *Let Ω_* be any bordism theory of smooth or PL manifolds. Then each element of Ω_n can be represented by an aspherical manifold.*

For example, Ω_* could be PL or smooth, unoriented, oriented, or framed bordism, etc. Properties $(2')$ and (4) have the following consequence.

Theorem B. *If K is a closed n-manifold, then $f_K: a(K) \to K$ induces a surjection on any generalized homology theory.*

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1. The Williams functor

The standard n-simplex is denoted by σ^n. A *space over* σ^n is a pair (X, f), where X is a topological space and $f: X \to \sigma^n$ is a continuous map. Suppose that K is an n-dimensional simplicial complex and that (X, f) is a space over σ^n. From these data Williams [32] constructs a space $X\Delta K$ together with a map $X\Delta K \to K$.

Part (a) of this section consists of some preliminary material concerning simplicial complexes. In part (b) we explain Williams' construction and its naturality properties. In part (c) we list various conditions on (X, f). In parts (d), (e), and (f) and we impose these conditions on (X, f) and consider the effect on $X\Delta K$. We are primarily interested in the case where X is an oriented n-manifold with $\partial X = f^{-1}(\partial\sigma^n)$ and with $f: (X, \partial X) \to (\sigma^n, \partial\sigma^n)$ a map of degree one (and with a similar condition for each face of σ^n). In part (g) we discuss a relative version of Williams' construction. Using this, we find that (when K is a manifold) $X\Delta K$ and K are bordant. Finally, in part (h) we show that $X\Delta K$ is aspherical provided that X satisfies appropriate conditions of asphericity.

(1a) Simplicial complexes over σ^n. A simplicial map is *nondegenerate* if its restriction to each simplex is injective (i.e., if no edge is collapsed to a vertex).

(1a.1) **Definitions.** A *simplicial complex over* σ^n is a pair (L, π), where L is an abstract simplicial complex and $\pi: L \to \sigma^n$ is a nondegenerate simplicial map. (This implies that $\dim L \leq n$.) If (L_i, π_1) and (L_2, π_2) are simplicial complexes over σ^n, then a nondegenerate simplicial map $g: L_1 \to L_2$ is a *map over* σ^n if the following diagram commutes:

$$\begin{tikzcd} L_1 \arrow[rr, "g"] \arrow[dr, "\pi_1"'] & & L_2 \arrow[dl, "\pi_2"] \\ & \sigma^n & \end{tikzcd}$$

Let $\mathscr{K}(\sigma^n)$ denote the category with simplicial complexes over σ^n as objects and with the maps over σ^n as morphisms.

(1a.2) **Example.** Let K be an abstract simplicial complex of dimension $\leq n$ and let K' be its derived complex. Let $d: K \to \{0, 1, \cdots, n\}$ be the function which assigns to each simplex its dimension. As an abstract simplicial complex, σ^n can be identified with the poset of nonempty subsets of $\{0, 1, \cdots, n\}$. The function d defines a simplicial map $K' \to \sigma^n$. Hence, the derived complex of any n-dimensional simplicial complex is naturally a complex over σ^n. Moreover, if $g: K \to L$ is any nondegenerate simplicial map of n-dimensional complexes, then $g': K' \to L'$ is a map over σ^n.

(1b) The construction. Suppose that (X, f) is a space over σ^n and that (L, π) is a simplicial complex over σ^n. The fiber product of X and $|L|$ over σ^n will be denoted by $X\tilde{\Delta}L$. ($|L|$ denotes the geometric realization of L.) In other words, $X\tilde{\Delta}L$ is the subspace of $X \times |L|$ consisting of all pairs (x, y) with $f(x) = \pi(y)$. The natural projections are denoted by $f_L: X\tilde{\Delta}L \to |L|$ and $p: X\tilde{\Delta}L \to X$:

$$\begin{array}{ccc} X\tilde{\Delta}L & \xrightarrow{f_L} & |L| \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & \sigma^n \end{array}$$

(1b.1) **Examples.** Suppose that L is a boundary complex of an octahedron (so that L is a triangulation of the two-sphere). There is a natural simplicial projection $\pi: L \to \sigma^2$. We will consider three different examples where X is an orientable two-manifold with boundary, $f^{-1}(\partial\sigma^2) = \partial X$, and $f|\partial X$ is transverse to $\partial\sigma^2$; it will then follow from Corollary 1f.2, below, that $X\tilde{\Delta}L$ is a closed orientable two-manifold.

(i) X is a surface of genus g with one hole and the map $f|\partial X \to \partial\sigma^2$ is a homeomorphism. Obviously, $X\tilde{\Delta}L$ is then an octahedron with each two-simplex replaced by a genus g surface, i.e., $X\tilde{\Delta}L$ is a surface of genus $8g$.

(ii) X is a hexagon and the map $f: X \to \sigma^2$ is a two-fold branched cover (branched at the center of σ^2). A pair of adjacent two-simplices in L corresponds to a pair of hexagons in $X\tilde{\Delta}L$ which intersect in two edges (opposite edges on each hexagon). The Euler characteristic χ of $X\tilde{\Delta}L$ is given by $\chi = 12 - 24 + 8 = -4$; so $X\tilde{\Delta}L$ is a closed surface of genus 3.

(iii) X is $\partial\sigma^2 \times I$ and the map f restricts to the identity on each component of ∂X. The Euler characteristic of $X\tilde{\Delta}L$ is easily computed to be -12; hence, $X\tilde{\Delta}L$ is a closed surface of genus 7.

If K is any n-dimensional simplicial complex, then put

$$X\Delta K = X\tilde{\Delta}K',$$

where K' is a complex over σ^n as in Example 1a.2.

Notation. If J is any subset of the standard simplex σ^n, then put $X_J = f^{-1}(J)$.

In particular, if α is a closed face of σ^n, then X_α is called a *face* of X.

(1b.2) **Definition.** Suppose that (X, f) and (Y, g) are spaces over σ^n. A map $\varphi: X \to Y$ is *face-preserving* if $\varphi(X_\alpha) \subset Y_\alpha$ for all faces α of σ^n.

If (L_1, π_1) and (L_2, π_2) are simplicial complexes over σ^n and $h: L_1 \to L_2$ is a map over σ^n, then the restriction of $\mathrm{id}_X \times |h|: X \times |L_1| \to X \times |L_2|$ to $X\tilde{\Delta}L_1$ is denoted by $1\tilde{\Delta}h: X\tilde{\Delta}L_1 \to X\tilde{\Delta}L_2$. Similarly, if $k: K_1 \to K_2$ is any nondegenerate map, then put $1\Delta k = 1\tilde{\Delta}k'$. If $\varphi: X \to Y$ is a map such that $g \circ \varphi = f: X \to \sigma^n$, then $\varphi \times \mathrm{id}_{|L|}: X \times |L| \to Y \times |L|$ restricts to a map $\varphi\tilde{\Delta}1: X\tilde{\Delta}L \to Y\tilde{\Delta}L$. Even if φ is only required to be face-preserving, then one can still define, for any simplicial complex K, a map $\varphi\Delta 1: X\Delta K \to Y\Delta K$ as in [32, p. 320]. It is then easy to see that Williams' fiber product construction is functorial in both X and L. We state this as the following lemma.

(1b.3) **Lemma.** (i) *There is a functor $(L, \pi) \rightsquigarrow X\tilde{\Delta}L$ from $\mathcal{K}(\sigma^n)$ (the category of simplicial complexes over σ^n) to the category of topological spaces.*

(ii) *The construction is also functorial in the first variable. Thus, $(X, f) \rightsquigarrow X\tilde{\Delta}L$ defines a functor from the category of spaces over σ^n and face-preserving maps to the category of topological spaces.*

The following result is also obvious.

(1b.4) **Lemma.** *Let (X, f) be a space over σ^n. For any simplicial complex (L, π) over σ^n let $f_L: X\tilde{\Delta}L \to |L|$ be the natural projection. Thus, to each object (L, π) in $\mathcal{K}(\sigma^n)$ we have associated a continuous map ($=$ morphism of spaces) $f_L: X\tilde{\Delta}L \to |L|$. This is a natural transformation from the functor $X\tilde{\Delta}(\ )$ to the geometric realization functor $|\ |$.*

(1c) Conditions on (X, f). NP(C0) X is path connected and for each codimension-one face α of σ^n, the face X_α is nonempty.

(C1) X is a compact n-dimensional PL manifold with boundary. Moreover, for each k-dimensional face α of σ^n, X_α is a k-dimensional PL submanifold of ∂X and $\partial(X_\alpha) = X_{\partial\alpha}$. The map $f: X \to \sigma^n$ is also required to be piecewise linear.

A smooth n-dimensional *manifold with corners* X is a manifold with boundary which is locally differentiability modelled on $\mathbb{R}_+^n$ $(= [0, \infty)^n)$. If $\varphi: U \to \mathbb{R}_+^n$, $U \subset X$, is some coordinate chart, and $x \in U$, then the number of zeros in the vector $(\varphi_1(x), \cdots, \varphi_n(x))$ is denoted by $d(x)$; it is independent of φ. A *k-dimensional stratum* of X is the closure of a component of $\{x \in X | d(x) = k\}$. Let $f: X \to Y$ be a smooth map between n-dimensional manifolds with corners such that the inverse image of each k-dimensional stratum F of Y is a union of k-dimensional strata of X. The map f is *transverse* to F if for each $x \in f^{-1}(F)$ the differential Df_x induces a linear isomorphism $T_xX/T_xf^{-1}(F) \to T_{f(x)}Y/T_{f(x)}F$.

The smooth version of (C1) is the following.

(C1′) X is a compact smooth n-dimensional manifold with corners. Moreover, for each k-dimensional face α of σ^n, X_α is a union of k-dimensional strata. The map $f: X \to \sigma^n$ is required to be smooth and transverse to each proper face of σ^n.

(C2) X satisfies (C1) (or (C1′)) and, in addition, the map $f: (X, \partial X) \to (\sigma^n \partial\sigma)$ is degree one mod 2.

(C2′) X satisfies (C1) and, in addition, X is oriented and the map $f: (X, \partial X) \to (\sigma^n, \partial\sigma^n)$ is degree one.

Notation. If X is a smooth or PL manifold, then let τ_X denotes its stable tangent bundle. (In the smooth case, τ_X is the Whitney sum of the tangent vector bundle with a trivial vector bundle. In the PL case, τ_X is the "stable PL tangent block bundle" (cf. [27]). It can be regarded as a stabilized regular neighborhood of the diagonal in $X \times X$.)

(C3) X satisfies (C1) or (C1′) and τ_X is trivial. (Note that (C3) implies that X is orientable.)

We suppose that (L, π) is a connected complex over σ^n and consider the effect of imposing our conditions on $X\tilde{\Delta}L$. The proof of the next lemma is left to the reader.

(1c.1) **Lemma.** *Suppose that* (X, f) *satisfies* (C0). *Then*

(i) $X\tilde{\Delta}L$ *is path connected.*

(ii) *The homomorphism* $(f_L)_*: \pi_i(X\tilde{\Delta}L) \to \pi_1(L)$ *is surjective.*

(1d) Homological surjectivity. Suppose that (X, f) satisfies (C2′). Let α be an oriented k-face of σ^n. Since f is transverse to α, the orientation on α induces one on X_α and the map $f|_{X_\alpha}: (X_\alpha, \partial X_\alpha) \to (\alpha, \partial\alpha)$ is of degree one. Let $\langle X_\alpha \rangle$ denote the orientation cycle in $C_k(X_\alpha, \partial X_\alpha)$ ($\langle X_\alpha\rangle$ is the sum of oriented k-simplices in X_α in some triangulation of X in which X_α is a subcomplex.) Let γ be an oriented k-simplex in L projecting to α in σ^n. Let $\langle X\widetilde{\Delta}\gamma\rangle$ be the corresponding orientation chain ($X\widetilde{\Delta}\gamma$ can be identified with X_α). Define a chain map $j\colon C_*(L) \to C_*(X\widetilde{\Delta}L)$ by sending an oriented k-simplex γ to the k-chain $\langle X\widetilde{\Delta}\gamma\rangle \in C_k(X\widetilde{\Delta}\gamma) \subset C_k(X\widetilde{\Delta}L)$. Obviously, the map j splits the chain map $(f_L)_\#\colon C_k(X\widetilde{\Delta}L) \to C_k(L)$. Under the weaker assumption that (X, f) satisfies (C2), similar remarks hold with $\mathbb{Z}/2$ coefficients. We have therefore, proved the following result.

(1d.1) **Lemma [W, p. 323].** (i) *If* (X, f) *satisfies* (C2), *then the map* $f_{L*}\colon H_*(X\widetilde{\Delta}L; \mathbb{Z}/2) \to H_*(L; \mathbb{Z}/2)$ *is onto.*

(ii) *If* (X, f) *satisfies* (C2$'$), *then* $f_{L*}\colon H_*(X\widetilde{\Delta}L; F_L^*A) \to H_*(L; A)$ *is onto, where* A *is any local coefficient system on* $|L|$.

(1e) Local structure of $X\Delta K$. In this subsection we suppose that (X, f) is a space over σ^n satisfying condition (C1) or (C1$'$). Let α be a k-face of σ^n and let $\mathring{\alpha}$ denote its relative interior. Then $\mathring{\alpha}$ has an open neighborhood in σ^n which is homeomorphic to a product bundle of the form $\mathring{\alpha} \times \mathbb{R}_+^{n-k}$. Identifying the $(n-k)$-simplex σ^{n-k} with $\{(x_1, \cdots, x_{n-k}) \in \mathbb{R}_+^{n-k} \mid \sum x_i \leq \varepsilon\}$ we have a smaller product bundle neighborhood of the form $\mathring{\alpha} \times \sigma^{n-k} \subset \mathring{\alpha} \times \mathbb{R}_+^{n-k}$ (analogous to a closed disk bundle neighborhood of a submanifold). Since X is an n-manifold with corners, it has a similar local structure to σ^n, i.e.,

(1) $X_{\mathring{\alpha}}$ has a product bundle open neighborhood in X of the form $X_{\mathring{\alpha}} \times \mathbb{R}_+^{n-k}$.

The transversality condition in (C1$'$) implies that the differential of $f\colon X \to \sigma^n$ induces a bundle map $X_{\mathring{\alpha}} \times \mathbb{R}_+^{n-k} \to \mathring{\alpha} \times \mathbb{R}_+^{n-k}$ which covers $f|X_{\mathring{\alpha}}\colon X_{\mathring{\alpha}} \to \mathring{\alpha}$ and is a linear isomorphism on each fiber. Similarly, if (C1) holds, then the fact that f is a PL map implies that it induces a bundle map as above. Thus, if either (C1) or (C1$'$) holds we may assume, after possibly altering f by an isotopy, that the following holds:

(2) The map f takes $X_{\mathring{\alpha}} \times \mathbb{R}_+^{n-k}$ to $\mathring{\alpha} \times \mathbb{R}_+^{n-k}$ by a bundle map of the form $g \times \mathrm{id}$, where $g = f|X_{\mathring{\alpha}}$.

Our goal in this subsection is to show (cf. Lemma 1e.1, below) that statements (1) and (2) imply that for any simplicial complex K, the spaces $X\Delta K$ and K have similar local structures, i.e., "they have isomorphic links."

Before stating this lemma we recall some basic notions from PL topology. Suppose that α is a k-simplex in a simplicial complex K. The *link of*

α *in* K, denoted by $\mathrm{Link}(\alpha, K)$ is the abstract simplicial complex consisting of all simplices β in K such that $\alpha < \beta$. (If β in an n-simplex in K and $\alpha < \beta$, then $\mathrm{Link}(\alpha, \beta)$ is isomorphic to an $n-k-1$ simplex; thus, the n-simplex β becomes an $(n-k-1)$-simplex in $\mathrm{Link}(\alpha, K)$. The *dual cone* of α *in* K, denoted by $\mathrm{Dual}(\alpha, K)$, is the cone on $\mathrm{Link}(\alpha, K)$. The *open dual cone,* denoted by $\mathrm{Dual}^\circ(\alpha, K)$ is the complement of $\mathrm{Link}(\alpha, K)$ in $\mathrm{Dual}(\alpha, K)$. For example, if α is a k-face of an n-simplex σ^n, then $\mathrm{Dual}(\alpha, \sigma^n) \cong \sigma^{n-k}$ while $\mathrm{Dual}^\circ(\alpha, \sigma^n) \cong \mathbb{R}^{n-k}_+$. From this it follows that if α is any k-simplex in a simplicial complex K, then $\mathring{\alpha}$ has an open product bundle neighborhood of the form $\mathring{\alpha} \times \mathrm{Dual}^\circ(\alpha, K)$.

(1e.1) **Lemma.** *Suppose that* (X, f) *is a space over* σ^n *satisfying* (C1) *or* (C1′), *that* L *is a simplicial complex over* σ^n, *and that* α *is a* k*-simplex in* L.

(1) *The manifold* $X_{\mathring{\alpha}}$ *has a product bundle open neighborhood in* $X\widetilde{\Delta}L$ *of the form* $X_{\mathring{\alpha}} \times \mathrm{Dual}^\circ(\alpha, L)$.

(2) *The map* $f_L \colon X\widetilde{\Delta}L \to L$ *induces a bundle map*

$$X_{\mathring{\alpha}} \times \mathrm{Dual}^\circ(\alpha, L) \to \mathring{\alpha} \times \mathrm{Dual}(\mathring{\alpha}, L)$$

from a neighborhood of $X_{\mathring{\alpha}}$ *in* $X\widetilde{\Delta}L$ *to a neighborhood of* $\mathring{\alpha}$ *in* L. *Moreover, this map has the form* $g \times \mathrm{id}$, *where* $g \colon X_{\mathring{\alpha}} \to \mathring{\alpha}$ *is* $f_L | X_{\mathring{\alpha}}$.

In other words the "link of a face in $X\widetilde{\Delta}L$" is equal to the link of the corresponding face of L.

Proof. Statements (1) and (2) in the lemma follow immediately from the corresponding statements preceding the lemma.

(1f) Tangential properties of $X\Delta K$.

(1f.1) **Definition.** Let K be an n-dimensional simplicial complex. Then K is a *PL* n*-manifold* if the dual cone of each k simplex is an $(n-k)$-cell, i.e., for each k-simplex $\beta \in K$, $\mathrm{Link}(\beta, K)$ is PL homeomorphic to the standard $(n-k-1)$-sphere. The complex K is a *homology* n*-manifold* if for each k simplex $\beta \in K$, the homology of $\mathrm{Link}(\beta, K)$ is isomorphic to that of S^{n-k-1}.

By an abuse of language we will say that K is a *smooth* n*-manifold* if there is a smooth n-manifold M^n and a smooth triangulation $\varphi \colon |K| \to M^n$.

From Lemma 1e.1 we deduce the following.

(1f.2) **Corollary.** *Suppose* X *satisfies* (C1) *or* (C1′).

(i) *If* L *is a homology* n*-manifold, then so is* $X\widetilde{\Delta}L$.
(ii) *If* L *is a PL* n*-manifold, then so is* $X\widetilde{\Delta}L$.

Part (ii) of this corollary can also be proved by a transversality argument. This argument also yields a smooth version of (1f.2). The argument runs as follows. Let P^n denote the image of $\sigma^n \times \sigma^n$ under the map $\sigma^n \times \sigma^n \to \mathbb{R}^n$ given by $(u, v) \to u - v$. The set P^n, being the projection of a product of two simplices, is a convex polyhedron. It obviously contains the origin in its interior. Recall that $X\widetilde{\Delta}L$ is the subset of $X \times |L|$ consisting of all (x, y) such that $f(x) = \pi(y)$. Let $\varphi \colon X \times |L| \to P^n$ be defined by $\varphi(x, y) = f(x) - \pi(y)$ so that

$$X\widetilde{\Delta}L = \varphi^{-1}(0).$$

As $\varphi \colon X \times |L| \to P^n$ is piecewise linear and transverse to 0, part (ii) of the corollary follows. This argument also gives the following result.

(1f.3) **Proposition.** *Suppose that X satisfies* (C1) *and that L is a PL n-manifold. Then $X\widetilde{\Delta}L$ is an n-dimensional PL submanifold of $X \times |L|$ and the normal bundle of $X\widetilde{\Delta}L$ in $X \times L$ is trivial.*

(1f.4) **Corollary.** *Suppose that X satisfies* (C3) (*stable tangential triviality*) *and that L is a PL n-manifold. Then the stable tangent PL block bundle of $X\widetilde{\Delta}L$ is the pullback of the stable tangent PL block bundle of L, i.e.,*

$$\tau_{X\widetilde{\Delta}L} = (f_L)^* \tau_L .$$

Proof. The restriction $\tau_{X \times |L|}$ to $X\widetilde{\Delta}L$ is $\tau_{X\widetilde{\Delta}L}$ plus the normal block bundle of $X\widetilde{\Delta}L$. This normal block bundle is trivial. Hence, $\tau_{X\widetilde{\Delta}L}$ is stably equivalent to the restriction of $\tau_{X \times |L|}$. Since, by (C3), τ_X is trivial and since the following diagram commutes,

$$\begin{tikzcd} X\widetilde{\Delta}L \arrow[rr, hook] \arrow[dr] & & X \times |L| \arrow[dl] \\ & |L| & \end{tikzcd}$$

we have that $\tau_{X\widetilde{\Delta}L}$ is stably equivalent to the pullback of τ_L. q.e.d.

There are also following smooth versions of these results.

(1f.5) **Proposition.** *Suppose that X satisfies* (C1$'$) *and that K is a smooth n-manifold. Then $X\Delta K$ is a smooth n-dimensional submanifold of $X \times |K'|$ with trivial normal bundle.*

Proof. The map $d \colon |K'| \to \sigma^n$ takes a top dimensional simplex γ^n in K, folds it onto a simplex in its barycentric subdivision, and then identifies that simplex with σ^n. Thus $d|\gamma^n$ can be identified with the orbit map of the symmetric group Σ_{n+1} on γ^n. Since this orbit map can be regarded as a smooth map, so can d. The map $f \colon X \to \sigma^n$ is smooth by hypothesis (C1$'$). Hence, $\varphi = f - d \colon X \times |K'| \to \mathbb{R}^n$ is smooth. If $y \in |K'|$ is an interior point of a k-simplex β of K', then the differential of $d\colon |K'| \to \sigma^n$ at y takes $T_y\beta$ onto $T_{d(y)}d(\beta)$. Condition (C1′) implies that if $x \in X\widetilde{\Delta}\overset{\circ}{\beta}$, then the differential of f at x is onto the normal space to $d(\beta)$ in σ^n. Thus, φ is transverse to $0 \in \mathbb{R}^n$.

(1f.6) **Corollary.** *Suppose that X satisfies* (C1′) *and* (C3) *and that K is a smooth n-manifold. Then the stable tangent vector bundle of $X\Delta K$ is the pullback of the stable tangent bundle of K, i.e., $\tau_{X\Delta K} = (f_K)^*\tau_K$.*

Remark. If the stable tangent bundle of X is trivial, then X is orientable. Hence, if K is a orientable n-manifold, then so is $X\Delta K$. Moreover, if (C2′) holds, then the map $f_K\colon X\Delta K \to K$ is of degree one. Corollaries 1f.4 and 1f.6 assert that f_K is covered by a map of stable tangent bundles; hence, in the language of surgery theory, f_K is a "normal map."

Remark. There is no hope that $f_K\colon X\Delta K \to K$ can be covered by a map of unstable tangent bundles which is a fiberwise isomorphism. The reason is that $X\Delta K$ and K may have different Euler characteristics and hence, the Euler classes of their tangent bundles may differ.

(1g) A relative construction. Next we discuss a relative version of the Williams functor $K \rightsquigarrow X\Delta K$. Suppose, from now on, that X satisfies (C1) or (C1′).

Let J be a subcomplex of K. Let $R(J, K)$ denote the standard derived neighborhood of J in K', $R^\circ(J, K)$ its relative interior, and $\partial R(J, K) = R(J, K) - R^\circ(J, K)$.

Let $\widehat{K}$ denote the simplicial complex formed by deleting the interior of $R(J, K)$ from K' and attaching the cone on $\partial R(J, K)$, i.e.,

$$\widehat{K} = (K' - R^\circ(J, K)) \cup \mathrm{Cone}(\partial R(J, K)).$$

Let c_0 denote the cone point. The complex $K' - R^\circ(J, K)$ is a simplicial complex over σ^n (cf. (1a.2) moreover, under the map $d\colon K \to \{0, 1, \cdots, n\}$ of (1a.2), no vertex of $\partial R(J, K)$ is mapped to the vertex 0 in σ^n. Hence, the structure on $K' - R^\circ(J, K)$ as a complex over σ^n extends to a structure on $\widehat{K}$ by sending c_0 to 0. Consider a point v_0 in $X\widetilde{\Delta}\widehat{K}$ which maps to c_0 in $\widehat{K}$. By Lemma 1e.1, v_0 has a neighborhood in $X\widetilde{\Delta}\widehat{K}$ of the form $\mathrm{Cone}(\partial R(J, K))$. Remove the interior of this neighborhood and paste in $R(J, K)$. The result is denoted by $X\Delta(K, J)$ and called *the relative Williams construction on* (K, J), i.e.,

$$X\Delta(K, U) = (X\widetilde{\Delta}\widehat{K} - \mathrm{Dual}^\circ(c_0, \widehat{K})) \cup R(J, K).$$

Remark. Suppose that K is a PL manifold. Then $X\tilde{\Delta}\hat{K}$ is also a PL manifold except at c_0. Since $R^\circ(J, K)$ is a PL manifold, we conclude that $X\Delta(K, J)$ is always a PL n-manifold (whether or not the subcomplex J is a submanifold).

(1g.1) **Example.** Suppose that K is a closed PL n-manifold. Extend the triangulation to $K \times I$ $(I = [0, 1])$. Then $X\Delta(K \times I, K \times 1)$ is a PL $(n+1)$-manifold with boundary. The boundary has two components $X\Delta(K \times 0)$ and $K \times 1$. Thus, $X\Delta(K \times I, K \times 1)$ is a bordism between $X\Delta K$ and K.

(1h) Asphericalization. Another condition we can impose on (X, f) is the following.

(C4) X is an aspherical CW-complex and if P is any subcomplex of σ^n, then each component of X_P is aspherical and the inclusion $i: X_P \to X$ induces a monomorphism

$$i_*: \pi_1(X_P, x_0) \to \pi_1(X, x_0)$$

(where the base point x_0 can be chosen in any component of X_P).

(1h.1) **Proposition.** *Suppose that (L, π) is a finite complex over σ^n and that X satisfies* (C4). *Then $X\tilde{\Delta}L$ is aspherical. Moreover, if J is any subcomplex of L over σ^n, then the inclusion induces a monomorphism $\pi_1(X\tilde{\Delta}J) \to \pi_1(X\tilde{\Delta}L)$ (again for any choice of base point).*

Before discussing the proof, we recall some well-known results. A *graph of groups* consists of a finite graph Γ with vertex set V and edge set E together with groups G_v and H_e for each $v \in V$ and $e \in E$. Moreover, whenever v is an endpoint of e, we should be given a monomorphism $\varphi_{v,e}: H_e \to G_v$. Suppose that $(\Gamma, \{G_v, H_e, \varphi_{v,e}: H_e \to G_v\})$ is a graph of groups. We construct a space Z as follows. Start with a disjoint union of $K(G_v, 1)$, $v \in V$, and $K(H_e, 1)\times[0, 1]$, $e \in E$. The homomorphism $\varphi_{v,e}$ has a geometric realization $K(H_e, 1) \to K(G_v, 1)$. Use these maps to paste $K(H_e, 1) \times \{0\}$ and $K(H_e, 1) \times 1$ to the appropriate $K(G_v, 1)$. The resulting space is Z. For a proof of the following well-known lemma, see [28, pp. 156–157].

(1h.2) **Lemma.** *Let $(\Gamma, \{G_v, H_e, \varphi_{v,e}: H_e \to G_v\})$ be a graph of groups and let Z be the space constructed above. Then Z is aspherical.*

The proof of Proposition 1h.1 follows from Lemma 1h.2 by induction on the number of simplices in L. If $\dim L = n$, then in the inductive step we are gluing a copy of X to $(X\tilde{\Delta}L) - X$ along a subspace of the form $X\tilde{\Delta}J$. If the subcomplex J is connected, then we can apply (1h.2) in the case where the graph is an interval, the vertex groups are $\pi_1(X)$ and $\pi_1(X\tilde{\Delta}L - X)$, and the edge group is $\pi_1(X\tilde{\Delta}J)$. If J is not connected, then the graph has two vertices (with vertex groups as before) and one edge for each component of J. If $\dim L < n$, then the graph has a vertex for each component of $X\widetilde{\Delta}L - X_\alpha$ and a vertex for each component of $X\widetilde{\Delta}\alpha$, where α is a top dimensional simplex of L, and an edge for each component of $X\widetilde{\Delta}J$.

In summary, if (X, f) is a space over σ^n satisfying conditions (C0), (C1), (C2$'$), (C3), and (C4), then the Williams functor $K \rightsquigarrow X\Delta K$ is an asphericalization procedure with properties (1), (2$'$), (3), (4), and (5) listed in the Introduction. In §4 we construct such a space (X, f).

2. Spaces of nonpositive curvature

The concept of nonpositive curvature can be extended to metric spaces more general than Riemannian manifolds (for example, see [2], [6], [24], [31] and, in particular, [17], [19], and [5]). Much of the recent interest in this area has been sparked by the spectacular collection of ideas in [19]. Since the original version of this paper was written (in the summer of 1988) several excellent expositions of parts of [19] have appeared in preprint form, most notably [5]. In particular, the article by Ballman (Chapter 10 in [5]) gives simple and clear explanations for the facts we summarize in subsections (2a) and (2c), below.

In part (2a) we define the notion of "nonpositive curvature" via the so-called "CAT-inequalities." In part (2b) we discuss the ideal boundary (also called the "sphere at infinity" or the "visual sphere") of the universal cover of a nonpositively curved space. Interesting examples of nonpositively curved spaces are provided by polyhedra which are "piecewise flat" or "piecewise hyperbolic." For the polyhedra, nonpositive curvature is equivalent to the condition that all "links are large." These piecewise constant curvature metrics on polyhedra are discussed in part (2c). In part (2d) we define the important concept of the "infinitesimal shadow" of an endpoint of a geodesic segment.

(2a) Basic definitions: the CAT-inequalities. A *geodesic segment* in a metric space X is an isometric map from an interval into X. A *triangle* in X consists of three points (the *vertices*) together with three geodesic segments (the *edges*) connecting them. A metric space X is *geodesic* if it is complete and if any two points in it can be connected by a geodesic segment. A subset Y of a geodesic space X is *totally geodesic* if, locally, every geodesic segment in X with endpoints in Y is actually contained in Y.

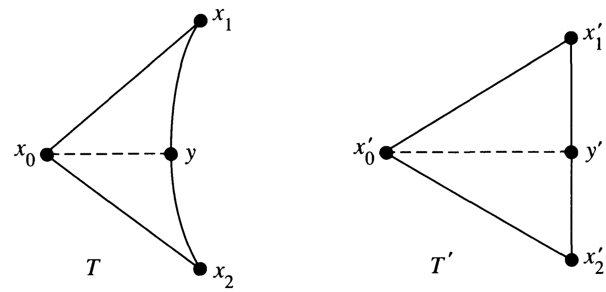

FIGURE 1

For each real number ε, let $M^2(\varepsilon)$ be the complete, simply connected, Riemannian two-manifold of curvature ε. If T is a triangle in X, then a *comparison triangle* in $M^2(\varepsilon)$ is a triangle T' with the same edge lengths as T. Of course, a comparison triangle is unique up to an isometry of $M^2(\varepsilon)$. Comparison triangles always exist for any fixed $\varepsilon \leq 0$. If $\varepsilon > 0$, then T has a comparison triangle provided T has perimeter $\leq 2\pi/\sqrt{\varepsilon}$.

Suppose that T is a triangle in X with vertices x_0, x_1, x_2 and that y is a point on the geodesic segment $[x_1, x_2]$ (see Figure 1). Let T' be a comparison triangle in $M^2(\varepsilon)$ with corresponding vertices x_0', x_1', x_2' and let y' be the point in $[x_1', x_2']$ corresponding to y (i.e., $d(y, x_i) = d'(y', x_i')$, $i = 1, 2$, where d and d' denote the distance functions on X and $M^2(\varepsilon)$, respectively). The pair (T, y) "satisfies CAT(ε)" if $d(x_0, y) \leq d(x_0', y')$. The space X "satisfies CAT(ε)" if (T, y) satisfies CAT(ε) for every triangle T in X and point $y \in T$. (If $\varepsilon > 0$, then we only consider triangles of perimeter $\leq 2\pi/\sqrt{\varepsilon}$).

A smooth Riemannian manifold with sectional curvature $\leq \varepsilon$ satisfies CAT(ε) locally; if it is simply connected and complete then it satisfies CAT(ε) globally (cf. [5, Chapter 3, §2]). This motivates the following definition.

(2a.1) **Definition** ([19, p. 107]). A geodesic space X has "curvature $\leq \varepsilon$" if it satisfies CAT(ε) locally.

If X has curvature $\leq \varepsilon$, then, obviously, any totally geodesic subspace of X also has curvature $\leq \varepsilon$.

A function $\lambda: X \to \mathbb{R}$ on a geodesic space X is *convex* if its restriction to each geodesic segment is a convex function on the interval.

(2a.2) **Remark.** Suppose X is a simply connected geodesic space of curvature ≤ 0. Then according to [19], X satisfies CAT(0) globally (see also [5, Theorem 7, Chapter 10, §1]). This implies that the distance function $d: X \times X \to \mathbb{R}$ is convex. Thus, if $f_i: [a_i, b_i] \to X$, $i = 1, 2$, is a geodesic segment, then the function $\varphi\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ defined by $\varphi(s, t) = d(f_1(s), f_2(t))$ is a convex function in the usual sense (cf. [19, p. 119]).

The convexity of the distance function implies that X is contractible. (For X a smooth Riemannian manifold this is the well-known Cartan-Hadamard Theorem. In our context, Gromov [19, p. 119] attributes this fact to Cartan, Hadamard, and Alexandrov.) Applying this version of the Cartan-Hadamard Theorem to the universal cover of a nonpositively curved space, we have the following result.

(2a.3) **Theorem** ([19, p. 119] or [5, Theorem 14, Chapter 10, §2]). *A nonpositively curved geodesic space is aspherical.*

(2a.4) **Gluing Lemma** ([19, p. 124] or [5, Corollary 5, Chapter 10, §1]). *Suppose that either*

(a) *X is the disjoint union of two geodesic spaces X_1 and X_2 and that $Y_i \subset X_i$, for $i = 1, 2$, is a totally geodesic closed subspace, or*

(b) *X is a geodesic space and Y_1 and Y_2 are two disjoint totally geodesic closed subspaces.*

Let $f\colon Y_1 \to Y_2$ be an isometry and let $\widehat{X}$ be the space formed from X by identifying Y_1 with Y_2 via f. Then $\widehat{X}$, with the obvious metric, is a geodesic space. If the curvature of each component of X is $\leq \varepsilon$, with $\varepsilon \leq 0$, then the same is true for $\widehat{X}$.

Proof. Only the last sentence of the lemma needs to be proved. Suppose that T is a small triangle in $\widehat{X}$ with vertices x_0, x_1, x_2 and that y is a point in the segment $[x_1, x_2]$. We must show that (T, y) satisfies $\mathrm{CAT}(\varepsilon)$. Let $\widehat{Y}$ denote the image of Y_1 $(= \text{image } Y_2)$ in $\widehat{X}$. The crucial case is when $x_0 \in \widehat{Y}$. If T is sufficiently small, then the segments $[x_0, x_1]$ and $[x_0, x_2]$ can be identified with geodesic segments in X. The segment $[x_1, x_2]$ might intersect $\widehat{Y}$ in some segment $[y_1, y_2]$, where $y_i \in \widehat{Y}$. Construct geodesics from y_i to x_0 and consider the triangles T_0, T_1, T_2 with vertex sets $\{x_0, y_1, y_2\}$, $\{x_0, x_1, y_1\}$, $\{x_0, x_2, y_2\}$, respectively. Let T' and T'_i, $i = 0, 1, 2$, be comparison triangles in $M^2(\varepsilon)$ for T and T_i (see Figure 2).

The three triangles T'_0, T'_1, and T'_2 fit together to give a pentagon S in $M^2(\varepsilon)$, as indicated in Figure 2. It follows from arguments in [2, p. 19] that the angles at y'_1 and y'_2 are not convex. (An English translation of [2] by J. Stallings exists in preprint form.) Hence the distance from a point on a side of S opposite to x'_0 is smaller than the distance between the corresponding points of T'. Since, by hypothesis, $\mathrm{CAT}(\varepsilon)$ holds for each T_i, $i = 0, 1, 2$, it follows that it holds for T. This proves the lemma.

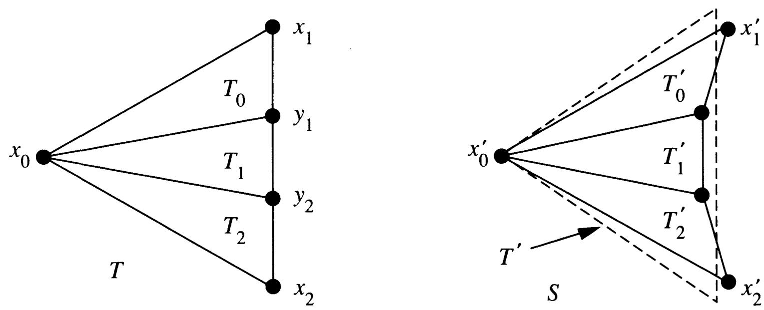

FIGURE 2

(2b) The ideal boundary. Let us introduce the following notation. For a point x in a metric space P, denote by $B_x(r)$ (respectively, $\overline{B}_x(r)$ or $S_x(r)$) the open metric ball (respectively, closed metric ball or sphere) of radius r about x.

Suppose P is a CAT(0) geodesic space. Since the distance function is convex, any two points can be joined by a unique geodesic. Define a map $c_r \colon P - B_x(r) \to S_x(r)$, called *geodesic contraction*, by sending a point y to the point on the geodesic joining x and y of distance r from x. It is easy to see that for each $r \in (0, \infty)$ the map c_r is continuous deformation retraction.

(2b.1) **Definition.** The *visual sphere of P at x* is the set of geodesic rays emanating from x. We denote it by $S_x(\infty)$.

Since every geodesic ray, beginning at x, intersects $S_x(r)$ in a unique point, we have the following tautological identification:

$$S_x(\infty) = \varprojlim S_x(r),$$

where the maps defining the inverse limit are given by geodesic contraction; to be precise, if $r_1 > r_2$, then we have a natural map $c_{r_2} \mid S_x(r_1) \colon S_x(r_1) \to S_x(r_2)$. This gives a topology on the visual sphere, namely, the natural topology on the inverse limit.

A notion which is closely related to the visual sphere is that of the "ideal boundary," which we shall describe below.

Suppose that P is a CAT(0) geodesic space. Embed P into the space $C(P)$ of continuous functions on P (with the topology of uniform convergence on compact sets) by sending x to the function d_x, where $d_x(y) = d(x, y)$. The divide $C(P)$ by the linear subspace L of constant functions. Let $\overline{P}$ denote the closure of the image of P in $C(P)/L$. The *ideal boundary* of P is $\overline{P} - P$. It is easy to show that $\overline{P}$ is compact (cf. [6, §3]).

Functions projecting to the points in $\overline{P} - P$ are called *horofunctions.*

To every geodesic ray $g(t)$ emanating from a point $x \in P$, we can assign a horofunction, h_g, defined by

$$h_g(y) = \lim_{t\to\infty}[d(y, g(t)) - t]$$

and called the *ray function* of $g(t)$. In this way, by sending a geodesic ray to its ray function, we get an injection $\psi: S_x(\infty) \to \overline{P} - P$. It is proved in [6, pp. 21–32] that ψ is a homeomorphism whenever P is a simply connected, nonpositively curved, Riemannian manifold. It seems likely that ψ is always a homeomorphism for any CAT(0) geodesic space; however, we shall only need the following weaker result, the proof of which is a modification of the argument in [6].

(2b.2) **Theorem.** *Suppose that P is a* CAT(0) *geodesic space and that P is a Riemannian manifold on the complement of a set of codimension* 2. *Then for any $x \in P$, the natural map $\psi: S_x(\infty) \to \overline{P} - P$ is a homeomorphism.*

(2b.3) **Remarks.** (1) The polyhedral homology manifolds of piecewise constant curvature, discussed in §3, are Riemannian manifolds on the complement of a set of codimension 2.

(2) The theorem implies that the visual sphere $S_x(\infty)$ is independent of the choice of basepoint x.

The proof is based on Lemmas and Corollaries 2b.4–2b.11, below. In all of these we only assume that P is a CAT(0) geodesic space.

(2b.4) **Lemma.** *Suppose that X is a closed convex subset of P. Then*

(i) *For any point $p \in P$, there is a unique point $x \in X$ which is closest to p.*

(ii) *Let $\pi: P \to X$ be the map which sends p to the closest point in X. Then π is distance decreasing.*

Proof. (i) Suppose the $x_1, x_2 \in X$ are of minimal distance from p. Let y be the midpoint of the segment from x_1 to x_2. If $x_1 \neq x_2$, then it follows from the CAT(0) inequality that $d(y, p) < d(x_i; p)$; hence, $x_1 = x_2$. (Compare [6, p. 8].)

(ii) To prove the second statement one needs the concept of angle between two geodesic segments. Suppose that $f_1, f_2: [0, 1] \to P$ are two geodesic segments with $f_1(0) = f_2(0) = p$. Let $\theta(s, t)$ be the angle at the point corresponding to p in a comparison triangle for $pf_1(s)f_2(t)$. Since P is nonpositively curved, it follows from [2] that $\lim_{t\to 0}\theta(s, t)$ exists; this limit is denoted by θ and called the *angle* between f_1 and f_2 at p. Now suppose that $p, q \in P$ and let $f: [0, a] \to P$ and $g: [0, b] \to P$ be geodesics from $\pi(p)$ to p and $\pi(q)$ to q, respectively. Put $\varphi(s, t) = d(f(s), g(t))$. We must show $\varphi(a, b) \geq \varphi(0, 0)$.

This follows from convexity of φ once it is known that $\varphi(t, t) \geq \varphi(0, 0)$ for small values of t. Consider the quadrilateral $f(t)g(t)\pi(q)\pi(p)$. Since $\pi(p)$ is the closest point to p, the angle at $\pi(p)$ is $\geq \pi/2$. Similarly, the angle at $\pi(q)$ is $\geq \pi/2$. By [2] the angles of any triangle in P are $\leq$ the corresponding angles in a comparison triangle. It follows that the angles at $f(t)$ and $g(t)$ are $\leq \pi/2$. Let $x = d(\pi(p), g(t))$. Then, for small t, we have that $\varphi(t, t)^2 + t^2 \geq x^2 \geq \varphi(0, 0)^2 + t^2$; hence, $\varphi(t, t) \geq \varphi(0, 0)$ and the lemma follows.

(2b.5) **Lemma.** *Any horofunction $h: P \to \mathbb{R}$ has the following properties.*

(1) *h is convex (i.e., the restriction of h to any geodesic segment is a convex function).*

(2) $h(x) - h(y) \leq d(x, y)$.

(3) *For any $x \in P$ and positive real number r, there exist points y_1 and $y_2 \in S_x(r)$ such that $h(y_1) - h(y_2) = 2r$.*

Proof. These properties are clearly closed conditions and they hold for distance functions d_x.

(2b.6) **Lemma.** *Let h be a horofunction, $x \in P$, and $r \in (0, \infty)$. There is a unique minimum of h on $S_x(r)$. If we normalize h so that $h(x) = 0$, then the minimum value of h on $S_x(r)$ is $-r$.*

Proof. By (2) and (3) of Lemma 2b.5 the minimum value has to be $-r$. A point where such a minimum value is obtained is a point on the convex set $h^{-1}((-\infty, -r])$ which is closest to x, such a point is unique by Lemma 2b.4(i).

(2b.7) **Lemma.** *Let h be a horofunction, $x \in P$, and r and R positive real numbers with $r \leq R$. The geodesic segment joining the minimum of h on $S_R(x)$ to x intersects $S_r(x)$ at the minimum of h on $S_r(x)$.*

Proof. This follows from the triangle inequality and the uniqueness of the minimum.

(2b.8) **Corollary.** *Let h be a horofunction. Then*

(i) *For any $x \in P$ there is a geodesic ray $c_{h,x}: [0, \infty) \to P$ joining x to the minima of h on the concentric spheres around x.*

(ii) *If y is a point on $c_{h,x}$, then the ray $c_{h,y}$ coincides with a forward section of $c_{h,x}$.*

(iii) *The restriction of h to the image of $c_{h,x}$ is a linear function (essentially the arc-length parametrization).*

Two geodesic rays are *asymptotic* if they stay a bounded distance apart. It is obvious that this is an equivalence relation and that two asymptotic rays define the same ray function (up to a constant).

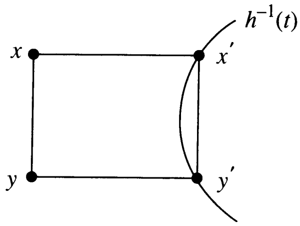

FIGURE 3

(2b.9) **Lemma.** *Let h be a horofunction and $x, y \in P$. Then the rays $c_{h,x}$ and $c_{h,y}$ are asymptotic.*

Proof. By sliding y along its geodesic ray $c_{h,y}$, we can assume $h(x) = h(y)$. We then have the quadrilateral pictured in Figure 3, where x' and y' are projections onto the convex sets $h^{-1}((-\infty, -t])$. By Lemma 2b.4(ii), $d(x', y') \leq d(x, y)$ and therefore, the distance stays bounded. q.e.d.

The set $\{c_{h,x}\}_{x \in P}$ is called the *family of asymptotic rays associated to the horofunction h*.

(2b.10) **Lemma.** *Let g_x be a geodesic ray beginning at x. For any $y \in P$, there is a unique geodesic ray g_y beginning at y and asymptotic to g_x.*

Proof. Uniqueness follows from CAT(0). If h is the ray function of g_x, then obviously $c_{h,x} = g_x$ and, hence, $c_{h,y}$ will serve for g_y.

(2b.11) **Corollary.** *Let h be a horofunction.*

(i) *The family of asymptotic geodesic rays associated to h is a single equivalence class of asymptotic rays.*

(ii) *Let $x \in P$ and let $g = c_{h,x}$. Then the horofunctions h and h_g define the same family of asymptotic rays.*

Proof of Theorem 2b.2. We wish to show that the injection $\psi: S_x(\infty) \to \overline{P} - P$ is a homeomorphism. We show ψ is onto. Let h be a horofunction corresponding to a point in the ideal boundary. Let $g = c_{h,x}$. It follows from the above lemmas and their corollaries that the restrictions of h and h_g to any ray asymptotic to g differ by a constant (a priori, the constant depends on the ray). On the smooth part of P the level surfaces of h and h_g are both orthogonal to the family of asymptotic rays. It follows that h and h_g are equal up to a constant on the smooth part of P which is open dense in every path component. Hence the map ψ takes the point in $S_x(\infty)$ corresponding to g to the point in $\overline{P} - P$ corresponding to h. Thus, ψ is onto. It is straightforward to see that it is continuous and hence, a homeomorphism. q.e.d.

For space satisfying $\mathrm{CAT}(\varepsilon)$, with $\varepsilon < 0$, the ideal boundary has a strong invariance property.

(2b.12) **Theorem.** *Any coarse quasi-isometry between* $\mathrm{CAT}(\varepsilon)$ *spaces,* $\varepsilon < 0$, *extends to a homeomorphism of the ideal boundaries.*

According to [17] this is due to Efremovich and Tichomiriva [14]. We will only use the following significantly weaker fact.

(2b.13) **Corollary.** *If* X *and* Y *are compact geodesic spaces of curvature* $\leq \varepsilon$, *with* $\varepsilon < 0$, *then the lift of any homeomorphism* $f: X \to Y$ *to the universal coverings extends to a homeomorphism of the ideal boundaries.*

(2c) Polyhedra of piecewise constant curvature. Let $M^n(\varepsilon)$ denote the complete, simply connected, Riemannian manifold of constant sectional curvature ε. Suppose that K is an n-dimensional, locally finite, abstract simplicial complex with vertex set V. Suppose further that we have a function $\psi: V \to M^n(\varepsilon)$ such that if α is any k-simplex in K, then ψ maps the vertex set of α to $k+1$ points in $M^n(\varepsilon)$ spanning a geometric k-simplex (denoted by $\psi(\alpha)$) in $M^n(\varepsilon)$. The function ψ gives us a way of identifying each simplex in K with a geometric simplex in $M^n(\varepsilon)$. This allows us to define arc-length: the length of a curve is the sum of lengths of its intersection with each simplex. Let P denote the geometric realization of K. Using arc-length one defines a metric d on the polyhedron P: the distance from x to y is the infimum of the lengths of all curves connected x to y. (This is called the *intrinsic metric* on P.) The polyhedron P together with the metric d is called a polyhedron of *piecewise constant curvature* ε. We say that P is *piecewise spherical, piecewise flat*, or *piecewise hyperbolic* as ε is $+1$, 0, or -1, respectively. The simplicial complex K together with the function ψ is called a *geometric triangulation of* P.

Basic facts about polyhedra of piecewise constant curvature can be found in [10], [24], [31], and [5, Chapter 10, §3].

Next we want to establish that links in such polyhedra have a natural piecewise spherical structure.

Suppose that α is a geometric n-simplex in $M^n(\varepsilon)$ and that v is a vertex of α. The set of unit tangent vectors to geodesic rays, which emanate from v and enter α, is naturally parametrized by a spherical $(n-1)$-simplex, denoted by $\mathrm{Link}(v, \alpha)$. More generally, if β is a k-face of α and $x \in \beta$, then the intersection of α with the normal space to β at x is a geometric $(n-k)$-simplex, denoted by $\beta^{\perp}$. The spherical $(n-k-1)$-simplex $\mathrm{Link}(x, \beta^{\perp})$ will be denoted by $\mathrm{Link}(\beta, \alpha)$.

Let P be a polyhedron of piecewise constant curvature with geometric triangulation K and let β be a simplex in K. An $(n-k-1)$-simplex in the abstract simplicial complex $\mathrm{Link}(\beta, K)$ is an n-simplex $\alpha \in K$ with $\beta < \alpha$. In the preceding paragraph we saw how to identify this $(n-k-1)$-simplex with a spherical simplex. Hence, $\mathrm{Link}(\beta, K)$ is naturally a piecewise spherical simplicial complex.

Suppose that $\beta \subset S^k$ is a spherical k-simplex and $\gamma \subset S^l$ is a spherical l-simplex. Regard S^k and S^l as the unit spheres in the orthogonal subspaces $\mathbb{R}^{k+1}$ and $\mathbb{R}^{l+1}$ of $\mathbb{R}^{k+l+2}$. The *orthogonal join* of β and γ is the spherical $(k+l+1)$-simplex in S^{k+l+1} spanned by the vertices of β and γ. The *l-fold suspension* of β is the orthogonal join of β and S^l, i.e., it is the union of all geodesic segments in S^{k+l+1} from a point in S^l to one in β.

If α is an n-simplex in $M^n(\varepsilon)$ and $x \in \alpha$, then $\mathrm{Link}(x, \alpha)$ is the convex subset of S^{n-1} defined as the set of unit tangent vectors to geodesic rays emanating from x and going into α. If x belongs to the relative interior of a k-face β, then $\mathrm{Link}(x, \alpha)$ is the $(k-1)$-fold suspension of the $(n-k-1)$-simplex $\mathrm{Link}(\beta, \alpha)$.

Using the above notions it makes sense to define $\mathrm{Link}(x, P)$ for any point x in a polyhedron P of piecewise constant curvature. (One might call $\mathrm{Link}(x, P)$ the "sphere of radius 0" about x and denote it by $S_x(0)$.)

(2c.1) **Definition.** A piecewise spherical polyhedron L is *large* if any two points x and y in L with $d(x, y) < \pi$ can be joined by a unique geodesic segment in L.

For example, if L is homeomorphic to a circle, then L is large if and only if its circumference is $\geq 2\pi$.

(2c.2) **Remark.** Suppose that L is a piecewise spherical polyhedron. For a point v in L let $B_v(\pi)$ denote of open ball (in L) of radius π about v. If L is large, then each point w in $B_v(\pi)$ is connected to v by a unique geodesic in L. It follows that $B_v(\pi)$ is contractible.

A piecewise spherical polyhedron L is large if and only if its l-fold suspension is large. From this, one can deduce the following.

(2c.3) **Lemma.** *Let P be a piecewise constant curvature polyhedron and let K be a geometric triangulation of P. Then the following statements are equivalent:*

(i) $\mathrm{Link}(x, P)$ *is large for all* $x \in P$.
(ii) $\mathrm{Link}(\beta, K)$ *is large for each simplex* β *of* K.

If either condition of the lemma holds then we say that "P has large links."

(2c.4) **Lemma** (Gromov [19, p. 120] and Ballman [5, Chapter 10, §3]). *Suppose that P is a polyhedron of piecewise constant curvature $\varepsilon \leq 0$. Then the curvature of P is $\leq \varepsilon$ if and only if P has large links.*

(2d) Infinitesimal shadows. Suppose that P is a piecewise constant curvature polyhedron and that we have a geodesic segment with endpoint $x \in P$. If $\mathrm{Link}(x, P)$ is small, then it may be impossible to extend the geodesic past x. If $\mathrm{Link}(x, P)$ is large then, unlike the case of a smooth Riemannian manifold, the local extension of the geodesic may not be unique. In other words, a point can cast a "shadow." The nonuniqueness of geodesic extension is measured by a certain subset of $\mathrm{Link}(x, P)$, which we shall define below and call it the "infinitesimal shadow" of x with respect to the geodesic segment.

Suppose that $g: (-\delta, \delta) \to P$ is a geodesic with $g(0) = x$. The geodesic defines two points in $\mathrm{Link}(x, P)$, an incoming direction g'_- and an outgoing direction g'_+ (called the "incoming and outgoing unit tangent vectors"). Let v be a point in $\mathrm{Link}(x, P)$. The *infinitesimal shadow of x with respect to v*, denoted by $\mathrm{Shad}(x, v)$, is the subset of all $w \in \mathrm{Link}(x, P)$ such that there is some geodesic g with $g'_- = v$ and $g'_+ = w$. For example, suppose x is a nonsingular point, i.e., suppose that $\mathrm{Link}(x, P)$ is isometric to the standard sphere S^{n-1}. If v is any point in S^{n-1}, then $\mathrm{Shad}(x, v)$ is the antipodal point. For another example, suppose that $\mathrm{Link}(x, P)$ is a circle of length $2\pi + \theta$. Then $\mathrm{Shad}(x, v)$ is the circular arc of length θ centered at $-v$, i.e., $\mathrm{Shad}(x, v)$ is the complement of the open ball of radius π centered at v. From this example, one can deduce the following lemma.

(2d.1) **Lemma.** *Suppose that P is a piecewise constant curvature polyhedron, that $x \in P$, and that $v \in \mathrm{Link}(x, P)$. Then $\mathrm{Shad}(x, v)$ is the complement of $B_v(\pi)$ in $\mathrm{Link}(x, P)$ where $B_v(\pi)$ denotes the open ball of radius π centered at $v \in \mathrm{Link}(x, P)$.*

Proof. To each spherical k-cell $\sigma^k \subset S^k$ and real number ε, one can associate a convex polyhedral cone, well defined up to isometry, in $M^{k+1}(\varepsilon)$ and denoted by $\mathrm{Cone}_\varepsilon \sigma$. (Recall that $M^n(\varepsilon)$ denotes the simply connected Riemannian manifold of constant curvature ε.) $\mathrm{Cone}_\varepsilon \sigma$ consists of all geodesic rays emanating from a base point and with initial tangent vector lying in σ. (If $\varepsilon > 0$, only consider the geodesic segments of length $< \pi/\sqrt{\varepsilon}$.) If L is a piecewise spherical polyhedron, then let $\mathrm{Cone}_\varepsilon L$ be the space of piecewise constant curvature ε formed by pasting together the $\mathrm{Cone}_\varepsilon \sigma$, $\sigma \in L$. Suppose P has piecewise constant curvature ε. The point $x \in P$ has a neighborhood N which is isometric to a neighborhood of the cone point in $\text{Cone}_\varepsilon \text{Link}(x, P)$. Suppose Γ is a geodesic segment in $\text{Link}(x, P)$ with endpoints v and w. Then $\text{Cone}_\varepsilon \Gamma$ is a totally geodesic subspace of N. It follows that if $l(\Gamma) < \pi$, then there is no geodesic through x, with incoming direction v and outgoing direction w, that is to say, $\text{Shad}(x, v) \subset \text{Link}(x, P) - B_v(\pi)$. If $l(\Gamma) \geq \pi$, then there is a geodesic through x in $\text{Cone}_\varepsilon \Gamma$ with incoming direction v and outgoing direction w. Hence, $\text{Link}(x, P) - B_v(\pi) \subset \text{Shad}(x, v)$.

3. Homology manifolds of piecewise constant curvature

The purpose of this section is to discuss the properties of universal covers of nonpositively curved, piecewise constant curvature, polyhedra which are PL manifolds or (more generally) homology manifolds. The basic definitions are given in part (3a); the PL case is discussed in part (3b), and in part (3c) we review some material on cell-like maps and prove Theorem 3b.2. This result states that the universal cover of a nonpositively curved piecewise constant curvature PL manifold is PL-homeomorphic to Euclidean space. In part (3d) we discuss the non-PL case.

(3a) Basic definitions. An $(n-1)$-dimensional piecewise spherical polyhedron L is a PL-*sphere* if it is PL homeomorphic to S^{n-1}.

Suppose that P^n is a piecewise constant curvature polyhedron and that it is an n-dimensional homology manifold. A point $x \in P$ is *metrically nonsingular* if $\text{Link}(x, P)$ is isometric to the standard $(n-1)$-sphere; it is a *PL nonsingular point* if $\text{Link}(x, P)$ is a PL $(n-1)$-sphere. P is a *PL n-manifold* if its PL singular set is empty. For the remainder of this section Q^n will denote a simply connected, homology n-manifold of piecewise constant curvature $\varepsilon \leq 0$, with large links. In other words, Q^n satisfies $\text{CAT}(\varepsilon)$, with $\varepsilon \leq 0$. We shall be interested in investigating the topology of metric spheres and balls in Q, as well as the topology of the visual sphere.

(3b) The PL case. First, we consider the case where Q is a PL n-manifold.

(3b.1) **Lemma.** *Suppose that L is a large piecewise spherical polyhedron which is a PL n-manifold. Then for any $v \in L$ and $r \in (0, \pi)$, $\overline{B}_v(r)$ is homeomorphic to the standard closed n-ball (in Euclidean space). Consequently, $B_v(\pi)$ is homeomorphic to an open n-ball.*

(3b.2) **Theorem.** *Suppose that Q is a PL n-manifold (and that Q is a simply connected, piecewise flat polyhedron with large links).*

(i) (*Stone* [31]) *For each $x \in Q$ and $r \in (0, \infty)$, $\overline{B}_x(r)$ is homeomorphic to the standard n-ball.*

(ii) (*Stone* [31]) *Q is homeomorphic to* $\mathbb{R}^n$.
(iii) *The visual sphere* $S_x(\infty)$ *is homeomorphic to* S^{n-1}.

(3c) Cell-like maps. Our proof of the above results makes use of a well-known theorem concerning the approximation of cell-like maps by homeomorphisms. Before stating the Approximation Theorem, we need to recall some terminology. (A nice exposition of this material is given in [13].)

A compact metric space C is *cell-like* if there is an embedding of C into the Hilbert cube I^∞ so that for any neighborhood $\mathscr{U}$ of C in I^∞, the space C is null-homotopic in $\mathscr{U}$. A cell-like subspace of a manifold is *cellular* if it has arbitrarily small neighborhoods homeomorphic to a cell. A compact subset of S^m is *pointlike* if its complement is homeomorphic to $\mathbb{R}^m$. A pointlike subset of S^m is cellular in S^m. A continuous surjection $\varphi: X \to Y$ is *cell-like* if each point inverse image is cell-like. The following theorem is due to Siebenmann [29] for $n \geq 5$, Quinn [25] for $n = 4$, Armentrout [4] for $n = 3$, and R. L. Moore [23] for $n = 2$ (see [13] for further discussion).

(3c.1) **Approximation Theorem.** *Suppose that $\varphi: M^n \to N^n$ is a cell-like map of topological n-manifolds, if $n = 3$, further assume that φ is cellular (i.e, each point inverse image is cellular). Then φ can be approximated by a homeomorphism.*

Proof of Lemma 3b.1 *and Theorem* 3b.2. Let (L_n) denote the statement of Lemma 3b.1 in dimension n, and (T_n) the statement of part (i) of Theorem 3b.2 in dimension n. By a theorem of M. Brown [9], part (i) implies part (ii) of Theorem 3b.2. The proof of part (i) will also show that geodesic contraction $S_x(r) \to S_x(s)$, $r > s$, is a cell-like map. By Theorem 3c.1 such a map is approximable by a homeomorphism, i.e., it is a *near homeomorphism.* According to another theorem of M. Brown [8], an inverse limit of near homeomorphisms is a near homeomorphism (see [3] for a short proof). Hence, the visual sphere $S_x(\infty)$ is homeomorphic to the standard sphere, i.e., part (iii) of Theorem 3b.2 will be true.

We shall prove these results according to the inductive scheme $(L_{n-1}) \Rightarrow (L_n)$ and (T_n). To simplify terminology we shall only show that $(L_{n-1}) \Rightarrow (T_n)$, the proof of $(L_{n-1}) \Rightarrow (L_n)$ being entirely similar. Obviously, (L_1) and (T_1) are true.

Suppose, by inductive hypothesis, that (L_{n-1}) is true and that $\dim Q = n$. Let $x \in Q$. We consider the metric ball $\overline{B}_x(r)$. We first claim that this is an n-manifold with boundary (the boundary being $S_x(r)$). This is obvious except near a point $y \in S_x(r)$. Since the distance function is convex, $\overline{B}_x(r)$ is a totally geodesic subspace of Q. Let $g(t)$ be a geodesic from y to a point in $\overline{B}_x(r)$. Let $v \in \operatorname{Link}(y, Q)$ be the outgoing direction of the geodesic from y to x, let $w \in \operatorname{Link}(y, Q)$ be the outgoing direction of $g(t)$, and let α be the distance from v to w in $\operatorname{Link}(y, Q)$. It follows from the CAT(0) inequality and the law of cosines that $r > d(g(t), x) \geq (r^2 + t^2 - 2rt\cos\alpha)^{1/2}$. Therefore, $2r\cos\alpha > t$ and, consequently, $\alpha < \pi/2$. Conversely, if $w \in \operatorname{Link}(y, Q)$ is of distance $< \pi/2$ from v, then there is a geodesic with outgoing direction w which remains in $\overline{B}_x(r)$ for some positive time $s(w)$, where $s(w) \to 0$ as the distance from w to v goes to $\pi/2$. Let $\overline{B}_v(\pi/2)$ denote the closed ball of radius $\pi/2$ about v in $\operatorname{Link}(y, Q)$. Let X be the subset of $\overline{B}_v(\pi/2) \times [0, \varepsilon)$ consisting of all (w, t) such that $t \leq \min(\varepsilon, s(w))$. Then X is an interval bundle over $\overline{B}_v(\pi/2)$ with the interval collapsed to 0 over $S_v(\pi/2)$. Let $\widehat{X}$ be the result of collapsing $\overline{B}_v(\pi/2) \times 0$ to a point. It follows from the above discussion that there is a small neighborhood of y in $\overline{B}_x(r)$ homeomorphic to $\widehat{X}$. By (L_{n-1}), $\overline{B}_v(\pi/2)$ is homeomorphic to a standard $(n-1)$-ball; hence, $\widehat{X}$ is a standard n-ball and, therefore, the metric ball $\overline{B}_x(r)$ is an n-manifold with boundary near y.

The easiest way to understand the remainder of the proof is to consider the "annular region" between $(S_x(r)$ and $S_x(s)$, where $s > r > 0$, defined by $A_{rs} = \overline{B}_x(s) - B_x(r)$. Define a map $\varphi\colon A_{rs} \to S_x(r) \times [r, s]$ by $\varphi(y) = (c_r(y), d(y, x))$, where $c_r\colon Q - B_x(r) \to S_x(r)$ denotes the geodesic contraction. The point inverse images under φ are closely connected to the nonuniqueness of geodesic continuations. If a geodesic segment is contained in the geometric nonsingular set, then the infinitesimal shadow at the endpoint is a point. Similarly, if the segment is contained in a stratum of the geometric singular set, then the infinitesimal shadow is again a singleton (since the link is a spherical suspension). It follows that on any geodesic ray the set of points with nontrivial infinitesimal shadows is discrete. Assume that we have chosen s close enough to r so that any geodesic ray emanating from x has at most one point in A_{rs} with nontrivial infinitesimal shadow. If $z \in S_x(r)$ and z' is a point on a geodesic ray from x through z with nontrivial shadow, then $\varphi^{-1}(z, t)$ is a point for $t \geq d(x, z')$, while for $t > d(x, z')$, $\varphi^{-1}(z, t) \cong \operatorname{Shad}(z', v)$. By (L_{n-1}) this shadow is cellular; hence, φ is cell-like. From this, it follows easily that $\overline{B}_x(r)$ is homeomorphic to the n-disk D^n. (For example, by using a composition of such maps we can find a cell-like map $\overline{B}_x(r) \to D^n$.)

(3c.2) **Corollary.** *Suppose that M is a piecewise flat PL n-manifold with large links. Then the universal cover of M is homeomorphic to $\mathbb{R}^n$.*

(3d) The non-PL case. We return to the general situation where Q is a polyhedral homology manifold.

A polyhedral n-manifold L is a *generalized homology n-sphere* if it has the same homology as does S^n.

An inverse sequence of groups $\{f_i : G_i \to G_{i-1}\}$ is said to satisfy the *Mittag-Leffler condition* if, for each i, the descending chain $G_i \supset f_{i+1}(G_{i+1}) \supset f_{i+2}(G_{i+2}) \supset \cdots$ satisfies the descending chain condition (i.e., it eventually stabilizes). Suppose that a space X can be written as a countable increasing union of compact subspace, $X = \bigcup C_i$, where $C_1 \subset C_2 \subset \cdots$. Then X is *one-ended* if the inverse limit of the inverse sequence $\{\pi_0(X - C_i)\}$ consists of one element. Suppose that X is one-ended. Then one can choose the C_i so that each $X - C_i$ is path connected. The space X is *semistable at infinity* if the inverse sequence $\{\pi_1(X - C_i)\}$ satisfies the Mittag-Leffler condition.

(3d.1) **Theorem.** *Suppose that Q is a polyhedral homology n-manifold and that Q is a simply connected, piecewise flat polyhedron with large links. Then for each $x \in Q$ and $r \in (0, \infty)$, $\overline{B}_x(r)$ is a contractible homology n-manifold with boundary; its boundary being $S_x(r)$. From this we deduce the following*:

(i) *$S_x(r)$ is a generalized homology $(n-1)$-sphere.*

(ii) *If $s > r$, then geodesic contraction $c_r : S_x(s) \to S_x(r)$ is a map of degree one. Hence, the induced map on fundamental groups is surjective.*

(iii) *Q is semistable at infinity.*

(iv) *The "fundamental group at infinity" of Q is the inverse limit*

$$\pi_1^\infty = \varprojlim \pi_1(S_x(r)).$$

Proof. As in the proof of Theorem 3b.2, we prove by induction that, for each large, piecewise spherical, homology manifold, the ball of radius r, $r < \pi$, is a contractible homology manifold (and therefore, that each infinitesimal shadow is acyclic). It follows, as before, that $\overline{B}_x(r)$ is a contractible homology manifold with boundary. By Poincaré-Lefschetz duality, $S_x(r)$ has the homology of S^{n-1}. Consider the geodesic contraction $c_r : A_{rs} \to S_x(r)$. This is a deformation retraction. Also, the map $\varphi : A_{rs} \to S_x(r) \times [r, s]$ has acyclic point inverse images, hence, its restriction to $S_x(s)$ (which is $c_r : S_x(s) \to S_x(r)$) induces an isomorphism in homology. Thus, c_r is degree one and, therefore, surjective on fundamental groups. This proves (ii). The statement that Q is semistable at ∞ means that the inverse system $\{\pi_1(Q - B_x(r))\}$ satisfies the Mittag-Leffler condition. This follows since $Q - B_x(r)$ and $S_x(r)$ are homotopy equivalent and, therefore, $\pi_1(Q - B_x(s)) \to \pi_1(Q - B_x(r))$ is onto by (ii). Statement (iv) follows, since by definition $\pi_1^\infty = \varprojlim \pi_1(Q - B_x(r))$.

(3d.2) **Corollary.** *Suppose Q is as above. If there exists a point $x \in Q$ and a real number r such that $S_x(r)$ is not simply connected, then Q is not simply connected at ∞.*

In many cases we can say more about metric spheres in Q.

(3d.3) **Proposition.** *Let Q be as above (Q is a simply connected, piecewise flat, homology n-manifold with large links). Suppose that the PL singular set of Q is discrete. Let $s_1, \cdots, s_k$ denote the PL singular points in $B_x(r)$ and suppose that r is such that $S_x(r)$ does not contain any PL singular points. Then $S_x(r)$ is homeomorphic to the connected sum*

$$S_x(r) = \mathrm{Link}(s_1, Q) \# \cdots \# \mathrm{Link}(s_k, Q).$$

(Note that each $\mathrm{Link}(s_i, Q)$ is a PL $(n-1)$-manifold with the same homology as S^{n-1}.)

Proof. Consider the annular region between $S_x(r_2)$ and $S_x(r_1)$, $r_1 > r_2$. Suppose that we have chosen r_1 and r_2 so that the only PL singular points in the annular region lie on the inner sphere $S_x(r_1)$. Let $x_1, \cdots x_l$ be these singular points. The intersection of a small neighborhood of x_i with $\overline{B}_x(r_1)$ is homeomorphic to the cone on $\overline{B}_v(\pi/2)$, where $v \in \mathrm{Link}(x_i, Q)$ is the outgoing direction of a geodesic segment from x_i to x and $\overline{B}_v(\pi/2)$ is the ball of radius $\pi/2$ in $\mathrm{Link}(x_i, Q)$. By Lemma 3b.1, $\overline{B}_v(\pi/2)$ is a standard $(n-1)$-ball. The point $x_i \in S_x(r_1)$ has a neighborhood homeomorphic to the cone on $S_v(\pi/2)$. (This cone is an $(n-1)$ disk.) Replace this neighborhood by the thickened shadow,

$$C(x_i) = \mathrm{Link}(x_i, Q) - B_v(\pi/2).$$

The effect on $S_x(r_1)$ is to take the connected sum with $\mathrm{Link}(x_i, Q)$, $i = 1, \cdots, l$. Call the resulting homology sphere $\Sigma_x(r_1)$. We can factor geodesic contraction $c_r \colon S_x(r_2) \to S_x(r_1)$ as $c_{r_1} = \theta \circ \lambda$, where $\lambda \colon S_x(r_2) \to \Sigma_x(r_1)$ is given by

$$\lambda(y) = \begin{cases} c_{r_1}(y) & \text{if } c_{r_1}(y) \neq x_i, \\ v_i & \text{if } c_{r_1}(y) = x_i, \end{cases}$$

where v_i is the direction of an incoming geodesic segment y to x_i. The map $\theta \colon \Sigma_x(r_1) \to S_x(r_1)$ is the natural collapse. As in Theorem 3b.2, the map λ is cell-like. Hence, $S_x(r_2)$ is homeomorphic to $\Sigma_x(r_1)$.

(3d.4) **Corollary.** *Suppose that Q is as above; only now assume that each component of the PL singular set is compact and convex. Let $S_1, \cdots, S_k$ be the components of the singular set in $B_x(r)$ and suppose that the* *sphere $S_x(r)$ does not intersect the singular set. Let R_i be a standard derived neighborhood of S_i contained in $B_x(r)$. Then $S_x(r)$ is homeomorphic to the following connected sum of homology spheres:*

$$S_x(r) = \partial R_1 \# \cdots \# \partial R_k .$$

Proof. Collapse each component of the PL singular set to a single point. The resulting piecewise flat polyhedron Q' satisfies the hypothesis of the previous proposition. If s_i is the collapse of S_i, then $\mathrm{Link}(S_i, Q') = \partial R_i$.

(3d.5) **Remark.** The first examples of aspherical manifolds with exotic universal covers were constructed in [11] using reflection groups. These examples can also be understood via Proposition 3d.3. We recall the construction of [11]. Let (W, S) be a Coxeter system and let $\mathrm{Nerve}(W, S)$ be the abstract simplicial complex with simplices the nonempty subsets T of S such that the subgroup generated by T is finite. Let $K(W, S)$ denote the cone on the derived complex of $\mathrm{Nerve}(W, S)$. As in [11] one can paste together copies of $K(W, S)$ (one for each element of W) to get a contractible polyhedron $\widetilde{K}(W, S)$ with W-action. If W is a right-angled Coxeter group, then Gromov [19, §4.6] has shown how to give $\widetilde{K}(W, S)$ the structure of a piecewise flat polyhedron with large links. This result has been extended to arbitrary Coxeter groups in the Ph.D. thesis of G. Moussong [24]. Given an arbitrary simplicial complex J, one can find a Coxeter system (W, S) with $\mathrm{Nerve}(W, S) = J'$ (cf. [11, Lemma 11.3]). Suppose that $\mathrm{Nerve}(W, S)$ is a nonsingular, homology n-sphere, $n \geq 3$, and that it is not simply connected. ("Nonsingular" means that $(\mathrm{Nerve}(W, S))$ is a PL manifold.) Then $\widetilde{K}(W, S)$ is a homology $(n+1)$-manifold with isolated PL singularities at the cone points. It follows from Proposition 3d.3 that $\widetilde{K}(W, S)$ is not simply connected at infinity. Let C be a compact contractible manifold with $\partial C = |\mathrm{Nerve}(W, S)|$ and let M be the $(n+1)$-manifold formed by pasting together copies of C. The identity map on $|\mathrm{Nerve}(W, S)|$ extends to a homotopy equivalence $C \to K(W, S)$. This induces a proper homotopy equivalence $M \to \widetilde{K}(W, S)$. It follows that M is also not simply connected at infinity; hence, M is not homeomorphic to $\mathbb{R}^{n+1}$.

4. Hyperbolized simplices

(4a) Basic results and definitions. Suppose that (X, f) is a space over σ^n. We can impose the following condition.

(C5) X is a geodesic space of curvature ≤ 0. Moreover, for each connected subcomplex J of σ^n, the subspace X_J is totally geodesic.

(Notice that condition (C5) implies condition (C4) of §(1h).)

(4a.1) **Lemma.** *Suppose that (L, π) is a finite simplicial complex over σ^n, that (X, f) is a space over σ^n, and that (X, f) satisfies* (C5). *Then $X\tilde{\Delta}L$ (with the intrinsic metric) is a geodesic space of curvature ≤ 0. Moreover, if P is any connected subcomplex of L, then $X\tilde{\Delta}P$ is a totally geodesic subspace of $X\tilde{\Delta}L$.*

Proof. First observe that if L is the union of two subcomplexes L_1 and L_2 over σ^n, then $X\tilde{\Delta}L = (X\tilde{\Delta}L_1)\cup(X\tilde{\Delta}L_2)$ and $(X\tilde{\Delta}L_1)\cap(X\tilde{\Delta}L_2) = X\tilde{\Delta}(L_1 \cap L_2)$. Suppose that each piece $X\tilde{\Delta}L_i$, $i = 1, 2$, is a geodesic space of curvature ≤ 0 and that each component of $X\tilde{\Delta}(L_1 \cap L_2)$ is totally geodesic. Then it follows from the Gluing Lemma 2a.4, that the union $X\tilde{\Delta}L$ also has curvature ≤ 0. Using this observation one proves the lemma by induction on the number of simplices in L. The argument is similar to the proof of Proposition 1h.1. At each stage we glue a copy of $X\tilde{\Delta}\omega$ to $X\tilde{\Delta}L_1$ along a space of the form $X\tilde{\Delta}J$, where ω is a simplex in L, L_1 is a subcomplex of L, and $J = \partial\omega \cap L_1$. (If $\dim \omega = n$, then $X\tilde{\Delta}\omega \cong X$.) By induction, $X\tilde{\Delta}L_1$ is nonpositively curved. By (C5) each component of $X\tilde{\Delta}J$ is totally geodesic; hence, $X\tilde{\Delta}(L_1\cup\omega)$ is nonpositively curved and, therefore, so is $X\tilde{\Delta}L$. The argument also shows that the last sentence of the lemma is true.

(4a.2) **Definition.** Suppose that (X, f) is a space over σ^n. Then (X, f) is called a *hyperbolized n-simplex* if it satisfies conditions (C1) (from §(1c)) and (C5). It is *strictly hyperbolized* if its curvature is strictly negative.

We shall often want to impose other conditions on a hyperbolized simplex, for example, we shall say that (X, f) is degree one if it satisfies (C2$'$), that it is *tangentially trivial* if (C3) holds, and that it is *piecewise flat* if it is a piecewise flat polyhedron.

(4a.3) **Theorem.** *Suppose that (X^n, f) is a hyperbolized n-simplex which is degree one and tangentially trivial. For any n-dimensinal simplicial complex K, put $a(K) = X\Delta K$. Then $a(K)$ is an asphericalization procedure satisfying* (1), (2$'$), (3), (4), *and* (5) *of the Introduction.*

Proof. Properties (1) (2$'$), (3), (4), and (5) follow from Proposition 1h.1, Lemma 1c.1(ii), Corollary 1f.2, Corollary 1f.4, and Example 1g.1, respectively.

Remark. The condition of being a hyperbolized simplex is very strong. Under weaker conditions on (X, f), $X\Delta K$ might still be a hyperbolization procedure. For example, if X is a hexagon as in Example 1b.1(ii), then for any two-complex K, $X\Delta K$ is a piecewise flat, nonpositively curved polyhedron. The hexagon, however, does not satisfy (C5). This example works because a regular hexagon has angles $2\pi/3$ and thus all links are large. It would be interesting to find similar "link conditions" in higher dimensions.

The purpose of this section is to show that hyperbolized simplices, as in Theorem 4a.3, exist in every dimension. We give two constructions. Both constructions yield a piecewise flat, nonpositively curved, hyperbolized n-simplex. Both constructions are by induction on n. The first construction, the simplest, is not degree one. It is described in part (4b). The second construction, which is due to Gromov [19, §3.4], is degree one. Neither of the constructions presented is strict.

(4b) Cartesian product with an interval. Suppose, by induction on n, that (X^n, f) is a tangentially trivial, hyperbolized n-simplex. (We can take X^1 to be the one-simplex.) Apply the Williams functor to get a "hyperbolized n-sphere" $Y^n = X^n\Delta(\partial\sigma^{n+1})$. Put $X^{n+1} = Y^n \times [0, 1]$.

We need to define the map $f\colon X^{n+1} \to \sigma^{n+1}$. The restriction of f to the boundary ($= Y^n \times \{0, 1\}$) is defined to be two copies of the natural projection $Y^n \to \partial\sigma^{n+1}$. This is then extended arbitrarily to $X^{n+1} \to \sigma^{n+1}$ (σ^{n+1} is contractible). Then (X^{n+1}, f) clearly does the job. (In dimension 2 this construction was described in Example 1b.1(iii).)

(4c) Gromov's construction. We modify the previous example to be of degree one. To do this we need the idea of a "reflection" on a space and using this a "cylinder construction."

Reflections. Suppose that A is a topological space and that B is a subspace. The *double of A along B*, denoted by $D(A, B)$, is defined as

$$D(A, B) = (A \times \{-1, +1\})/\sim,$$

where the equivalence relation $\sim$ is defined by $(a, \varepsilon) \sim (a', \varepsilon')$ iff $a \in B$ and $a = a'$, or $(a, \varepsilon) = (a', \varepsilon')$. Denote the equivalence class of (a, ε) by $[a, \varepsilon]$. Identify A with the image of $A \times 1$ in $D(A, B)$. There is a natural involution on $D(A, B)$ which sends $[a, \varepsilon]$ to $[a, -\varepsilon]$. The subspace A is a fundamental domain for the involution (in the strong sense that it intersects each orbit in exactly one point). The fixed point set of the involution is B. If A is a geodesic space and B is a closed totally geodesic subspace, then the metric on A induces one on $D(A, B)$ (the distance between two points is the infimum of the lengths of curves connecting them); by the Gluing Lemma (2a.4), $D(A, B)$ is a geodesic space and the canonical involution is an isometry.

Suppose that Y is a topological space and that $r: Y \to Y$ is an involution. The involution r is a *reflection* if there are subspaces $A \subset Y$ and $B \subset A$ such that B is in the fixed point set of r and so that the natural map $g: D(A, B) \to Y$ given by

$$g([a, \varepsilon]) = \begin{cases} a & \text{if } \varepsilon = +1, \\ r(a) & \text{if } \varepsilon = -1, \end{cases}$$

is a homeomorphism. We shall say that A is a *half-space* for r on Y. If Y is a geodesic space, if A and B are totally geodesic subspaces, and if $g: D(A, B) \to Y$ is an isometry, then r is an *isometric reflection.* If Y is a manifold and A is a manifold with boundary B, then r is called a *locally linear* reflection.

(4c.1) **Lemma.** *Suppose that A is a manifold with boundary B, that $Y = D(A, B)$, and that r is the canonical locally linear reflection on Y. Then the following statements are equivalent*:

(i) *The stable tangent bundle of Y is trivial.*
(ii) *The stable tangent bundle of A is trivial.*
(iii) *The stable tangent bundle of Y is $\mathbb{Z}/2$-equivariantly trivial.*

Moreover, the set of trivializations of τ_A is naturally bijective with the set of equivariant trivializations of τ_Y.

Proof. Since A is a submanifold of codimension 0 in Y, (i) $\Rightarrow$ (ii). One way to see that (ii) $\Rightarrow$ (iii) is to notice that Y is a fiber product according to the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "\varphi"] \\ A \arrow[r, "g"'] & {[0, \infty)} \end{tikzcd}$$

where $\varphi(t) = t^2$ and $g: A \to [0, \infty)$ is any map which is transverse to 0 and has $g^{-1}(0) = B$. We then have that Y is embedded as a two-sided submanifold of $A \times \mathbb{R}$; hence, (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is obvious as is the last statement of the lemma.

The cylinder construction. Suppose that r is a reflection on a space Y, that A is a half-space, and that $B = A \cap r(A)$ is the fixed point set. Following [19, p. 116], we let $\Omega(Y, A, r)$ be the space formed from $Y \times [-1, 1]$ by gluing $r(A) \times (-1)$ to $r(A) \times (+1)$ (via the identity map of $r(A)$). The image of $A \times \{\pm 1\}$ in $\Omega(Y, A, r)$ is denoted by $\partial\Omega$; it is naturally identified with $D(A, B)$ and hence, with Y. We list some properties of this construction in the following proposition.

(4c.2) **Proposition.** *Suppose that* r *is a reflection on a space* Y *and that* A *is a half-space for* r. *Put* $\Omega = \Omega(Y, A, r)$.

(1) *Suppose that* Y *is an* n*-dimensional manifold and that the reflection* r *is locally linear. Then* Ω *is an* $(n+1)$*-dimensional manifold with boundary; its boundary is* $\partial\Omega$, *which is naturally identified with* Y. *Suppose further that the stable tangent bundle of* A *is trivial and that* $\psi: \tau_A \to A \times \mathbb{R}^N$ *is a trivialization. Then* ψ *extends to a trivialization* $\tau_\Omega \to \Omega \times \mathbb{R}^{N+1}$ (*in particular, the stable tangent bundle of* Ω *is trivial*).

(2) *Suppose* Y *is aspherical. Then* Ω *is aspherical. Moreover,* $\pi_1(Y) \to \pi_1(\Omega)$ *is an injection.*

(3) *Suppose that* Y *is a geodesic space and that* r *is an isometric reflection. Then the induced metric on* Ω (*that is, the metric induced from the product metric on* $Y \times [-1, 1]$) *makes* Ω *into a geodesic space and* $\partial\Omega$ *is a totally geodesic subspace. Suppose further that the curvature of* Y *is* ≤ 0. *Then the curvature of* Ω *is nonpositive.*

Proof. (1) The only part of (1) which requires proof is the statement about stable tangent bundles. Another way to describe the construction of Ω is as follows. Take $Y \times S^1$ and "slit it open" along $A \times$ point. Since τ_A is trivial, so is τ_Y (by Lemma 4c.1). Since $\tau_{Y \times S^1} = \tau_Y \times \tau_{S^1}$, the stable tangent bundle of $Y \times S^1$ is trivial, so the stable tangent bundle remains trivial when we slit it open. If ψ is a trivialization of τ_A, then it extends to an equivariant trivialization of τ_Y. We can then take the Cartesian product with some standard trivialization of τ_{S^1}. Restricting back to $A \times$ point, we obviously recover ψ.

(2) This statement follows from Lemma 1h.2.

(3) This statement follows from the Gluing Lemma 2a.4.

Remark. If A is a piecewise flat polyhedron, then so are Y and Ω (since S^1 is flat).

We now follow the same inductive procedure as in part (b). Assume, by induction, that (X^n, f) is a piecewise flat, tangentially trivial, degree one, hyperbolized n-simplex. We construct X^{n+1}. Put $Y^n = X^n \Delta(\partial\sigma^{n+1})$. The automorphism group of σ^{n+1} is Σ_{n+2}, the symmetric group of degree $n+2$. It acts on the derived complex $(\partial\sigma^{n+1})'$ through automorphisms over σ^n and the natural map $\pi: (\partial\sigma^{n+1})' \to \sigma^n$ can be identified with the orbit map $S^n \to S^n/\Sigma_{n+2} \cong \sigma^n$. Also, Σ_{n+2} acts on $\partial\sigma^{n+1}$ as a group generated by isometric reflections. By functoriality of the Williams construction, Σ_{n+2} acts on Y^n and the natural map $Y^n \to (\partial\sigma^{n+1})$ is Σ_{n+2}-equivariant. Suppose r is a transposition in Σ_{n+2}. The complement of the fixed set of r on $(\partial\sigma^{n+1})$ has two components. It follows that the same is true for the fixed set of r on Y^n, and hence, that r acts on Y^n (a smooth manifold by Proposition 1f.5) as a smooth reflection. Let A^n be a half space for r on Y^n. Finally using the cylinder construction defined above, we put

$$X^{n+1} = \Omega(Y^n, A^n, r).$$

Extend the natural map $\partial X^{n+1} = Y^n \to \partial\sigma^{n+1}$ to $f: X^{n+1} \to \sigma^{n+1}$ by choosing a collared neighborhood of ∂X^{n+1} in X^{n+1} and then using the fact that σ^{n+1} is the cone on $\partial\sigma^{n+1}$. It follows from Proposition 4c.2 that this gives a hyperbolization procedure with all the right properties. The hyperbolized simplices constructed above are piecewise flat and have curvature ≤ 0. For constructions of hyperbolized simplices with curvature strictly less than zero, see [19, §§3.4B and 4.3A].

5. Applications

In this section we give the details for the three examples discussed in the Introduction.

(5a) A nontriangulable aspherical manifold. Our goal is to prove the following result.

(5a.1) **Theorem.** *There is a closed aspherical topological four-manifold N^4 with the following properties.*

(i) *N^4 is not homotopy equivalent to a PL four-manifold.*

(ii) *N^4 is not homeomorphic to a simplicial complex.*

(iii) *The universal cover $\tilde{N}^4$ is not simply connected at infinity (and hence, $\tilde{N}^4$ is not homeomorphic to $\mathbb{R}^4$).*

We recall some basic facts and examples from four-dimensional topology.

Let $M^4(E_8)$ denote the smooth, simply connected, four-manifold with boundary, constructed by plumbing eight copies of the tangent disk bundle of S^2 according to the Dynkin diagram E_8 (see [7, pp. 116–126]). The boundary of $M^4(E_8)$, denote by Σ^3, is Poincaré's homology three-sphere (i.e., dodecahedral space). The intersection form on $H_2(M^4(E_8); \mathbb{Z})$ is then nonsingular over $\mathbb{Z}$, even, and of signature 8. Choose a smooth triangulation of $M^4(E_8)$ and let $K(E_8)$ denote the four-dimensional simplicial complex formed by attaching the cone on Σ^3 to $M^4(E_8)$. The complex $K(E_8)$ is a polyhedral homology four-manifold; it has one singular point (the cone point). Let $w_i(K(E_8))$ denote the ith Stiefel-Whitney class of $K(E_8)$ (defined via Wu classes as in [22, p. 132]). The homology four-manifold $K(E_8)$ has the following properties:

(a) $K(E_8)$ is orientable.

(b) $w_2(K(E_8)) = 0$ (i.e., $K(E_8)$ is "spin").

(c) The signature $\sigma(K(E_8))$ of the intersection form is 8.

A famous theorem of Rohlin (cf. [26]) asserts that these three properties imply that $K(E_8)$ is not homotopy equivalent to a PL a four-manifold. On the other hand, Freedman proved in [16] that Σ^3 does bound a contractible four-manifold topologically; it follows that $K(E_8)$ is homotopy equivalent to the closed topological four-manifold formed by gluing this contractible manifold onto $M^4(E_8)$. This topological four-manifold will be denoted by $\widehat{M}^4(E_8)$. Recent results of Casson (on his new invariant for homology three-spheres) imply that any four-manifold satisfying properties (a), (b), and (c) above cannot be triangulated (see [1, p. (xvi)]). Thus, $\widehat{M}^4(E_8)$ is not homeomorphic to a simplicial complex.

Next we apply our hyperbolization procedure to the simplicial complex $K(E_8)$. Let (X^4, f) be the hyperbolization of the four-simplex constructed in §(4b). Put

$$P^4 = X^4 \Delta K(E_8). \tag{5a.2}$$

We list some of the properties of P^4 in the following lemma.

(5a.3) **Lemma.** *Let P^4 be the polyhedron defined by* (5a.2).

(i) *P^4 is a piecewise flat, nonpositively curved, polyhedron (and hence, P^4 is aspherical).*

(ii) *P^4 is a polyhedral homology four-manifold. Moreover, it is a PL four-manifold except at one singular point x_0.*

(iii) *P^4 is orientable.*

(iv) $w_2(P^4) = 0$.

(v) $\sigma(P^4) = 8$.

(vi) *The universal cover of P^4 is not simply connected at infinity.*

Proof. Property (i) follows from the fact that X^4 is a hyperbolized four-simplex and the Gluing Lemma 2a.4. Property (ii) is the fact that P^4 and $K(E_8)$ have the same links (cf. Lemma 1e.1). The point x_0 is the "cone point," i.e, it is the unique point in P^4 with link Σ^3. The stable tangent bundle of $K(E_8) - x_0$ is trivial; it follows from Corollary 1f.4 that the same is true for $P^4 - x_0$. Properties (iii) and (iv) follow. Using the relative version of the construction on $K(E_8) \times I$, we get an oriented bordism of homology manifolds from P^4 to $K(E_8)$. Hence, $\sigma(P^4) = \sigma(K(E_8)) = 8$. Property (vi) follows from Proposition 3d.3. q.e.d.

We are now in a position to construct the topological four-manifold N^4 needed to prove Theorem 5a.1. Let N^4 be the four-manifold formed from P^4 by removing a neighborhood of the cone point (i.e., a cone on Σ^3) and replacing it with a contractible four-manifold. Obviously, N^4 is a closed topological four-manifold homotopy equivalent to P^4. Thus, N^4 is aspherical (since P^4 is). Moreover, from Lemma 5a.3, we have that

(a′) N^4 is orientable.
(b′) $w_2(N^4) = 0$.
(c′) $\sigma(N^4) = 8$.

It follows, as before, from Rohlin's Theorem that N^4 is not homotopy equivalent to a PL four-manifold and from Casson's results that N^4 is not homeomorphic to a simplicial complex. A homotopy equivalence $N^4 \to P^4$ induces a proper homotopy equivalence $\widetilde{N}^4 \to \widetilde{P}^4$ of universal covers. Since $\widetilde{P}^4$ is not simply connected at infinity, neither is $\widetilde{N}^4$. In particular, $\widetilde{N}^4$ is not homeomorphic to $\mathbb{R}^4$. This concludes the proof of Theorem 5a.1.

The following corollary improves results of [12].

(5a.4) **Corollary.** *In every dimension $n \geq 4$, there is a closed aspherical n-manifold which is not homotopy equivalent to a PL n-manifold.*

Proof. We claim that $N^4 \times T^k$ is not homotopy equivalent to a PL manifold. Suppose, to the contrary, that $f: M^n \to N^4 \times T^k$ is a homotopy equivalence, where M^n is a PL n-manifold, $k = n-4$. First suppose that $k = 1$. Let $g: M^5 \to S^1$ denote a PL approximation of the composition of f with projection to the second factor, let x_0 be a regular value for g, and let $M_1^4 = g^{-1}(x_0)$. Then $w_1(M_1^4) = w_2(M_1^4) = 0$ and a simple algebraic argument shows $\sigma(M_1^4) = \sigma(N^4) = 8$, contradicting Rohlin's Theorem. If $k > 1$, then we can use Farrell's Fibering Theorem [15] to decompose M^n as $M^5 \times T^{k-1}$, where M^5 is homotopy equivalent to $N^4 \times S^1$ and, hence, again contradict Rohlin's Theorem.

(5b) Nonpositively curved topological manifolds which are not covered by Euclidean space. In [17, p. 187], Gromov asks the following question.

Question. *Are there convex geodesic spaces which are topological manifolds different from $\mathbb{R}^n$?*

Here "convex" means that the distance function is convex (cf. Remark 2a.2). A generalized Cartan-Hadamard Theorem [19, p. 119] asserts that a simply connected, nonpositively curved, geodesic space is convex. Hence, the following result supplies an affirmative answer to Gromov's question.

(5b.1) **Theorem.** *For each $n \geq 5$, there is a piecewise flat, nonpositively curved, polyhedron Q^n with the following properties*:

(1) *Q^n is a closed topological n-manifold.*

(2) *The universal cover $\tilde{Q}^n$ is not simply connected at ∞ and, hence, not homeomorphic to $\mathbb{R}^n$.*

The manifold Q^n will be constructed by applying hyperbolization to certain non-PL triangulations of S^n.

Let A^{n-1} be a compact acyclic smooth manifold with the following two properties:

(1) $\pi_1(\partial A^{n-1}) \to \pi_1(A^{n-1})$ is onto.

(2) The double of A^{n-1} (a homology $(n-1)$-sphere) is not simply connected.

It is fairly easy to construct such A^{n-1} for $n-1 \geq 4$. For example, suppose that H^{n-2} is any nonsimply connected homology $(n-2)$-sphere, that C^{n-2} is the complement of an open $(n-2)$-ball in H^{n-2}, and that $A^{n-1} = C^{n-2} \times I$. Since ∂A^{n-1} is then the double of C^{n-2}, it is obvious that (1) holds. The double of A^{n-1} is homeomorphic to the boundary of $A^{n-1} \times I$. Since $A^{n-1} \times I = C^{n-2} \times I \times I = C^{n-2} \times D^2$, we have that π_1 (double of A^{n-1}) $= \pi_1(\partial(C^{n-2} \times D^2)) = \pi_1(C^{n-2}) \neq 0$; hence, (2) holds.

Suppose that A^{n-1} satisfies (1) and (2). Let $D(A, \partial A)$ denote the double of A^{n-1} (∂A and $D(A, \partial A)$ are both homology spheres). Choose a PL triangulation of A^{n-1} and let Z^{n-1} be the simplicial complex formed by attaching the cone on ∂A to A^{n-1}. (Thus, Z^{n-1} is a homology manifold with the homology of S^{n-1}.) Let ΣZ denote the suspension of Z^{n-1}. The geometric realization of ΣZ is homotopy equivalent to S^n. The PL singular set of ΣZ is an interval (the suspension of the cone point in Z^{n-1}). The link of any vertex in ΣZ is either a PL $(n-1)$-sphere, the suspension of ∂A, or Z^{n-1} (at the suspension points). Thus, every vertex has simply connected link. It follows from Edwards Polyhedral-Topological Manifold Characterization Theorem in [13, p. 119] that $|\Sigma Z|$ is a manifold, hence, an n-sphere ($n \geq 5$). Let R^n be a standard derived neighborhood of the singular interval in ΣZ. Clearly, $\partial R^n = (A^{n-1} \times \partial I) \cup (\partial A \times I)$, i.e., $\partial R^n = D(A, \partial A)$.

We continue as in part (5a). Let X^n be a hyperbolized n-simplex. Put $Q^n = X^n \Delta \Sigma Z$. Then Q^n is a piecewise flat, nonpositively curved, polyhedron. Since Q^n and ΣZ have the same links (Lemma 1e.1), the PL singular set of Q^n is also an interval; its derived neighborhood can be identified with R^n; and Q^n is a topological n-manifold.

Let $\widetilde{Q}^n$ denote the universal cover of Q^n. We note that each component of the PL singular set is a convex interval. (It is convex since the one-skeleton of K^n is a totally geodesic subspace of its hyperbolization, Q^n.) It follows from Theorem 3d.1 and Corollary 3d.4 that $\widetilde{Q}^n$ is semistable at infinity and that its fundamental group at infinity is the "projective free product" of an infinite number of copies $\pi_1(D(A, \partial A))$. This proves Theorem 5b.1.

(5b.2) **Remark.** Theorem 5b.1 can also be proved using reflection groups. With notation as in Remark 3d.5, let (W, S) be a Coxeter system with Nerve$(W, S) = Z'$ (where Z is the generalized homology $(n-1)$-sphere constructed above). Then Z is simply connected. As above, $\widetilde{K}(W, S)$ is a topological manifold and is not simply connected at infinity. The quotient of $\widetilde{K}(W, S)$ by a torsion-free subgroup of W is another example which satisfies properties (1) and (2) in Theorem 5b.1.

We note that the above example provides a counterexample to Theorem 16.1 in [11]. There it is claimed that if Nerve(W, S) is a generalized homology $(n-1)$-sphere, then $\widetilde{K}(W, S)$ is simply connected at infinity if and only if Nerve(W, S) is simply connected. However, all that is proved is that if Nerve(W, S) is not simply connected, then $\widetilde{K}(W, S)$ is not simply connected at infinity. The above construction provides a counterexample to the stronger claim.

(5c) A negatively curved topological manifold with no negatively curved PL metric. The example of part (5b), with its universal cover being nonsimply connected at infinity, does not carry a nonpositively curved PL metric (although it is homeomorphic to a PL manifold). We shall now construct another example of this type: a PL manifold manifold carrying a strictly negatively curved $(\kappa < 0)$ continuous metric, but no PL, $\kappa < 0$, metric.

Let Σ^3 be a nonsimply connected homology three-sphere, with finite fundamental group, and with triangulation K^3. Its double suspension (or equivalently, its join with a circle) is homeomorphic to S^5; however, the double suspension of K^3 is not a PL triangulation: links of one simplices in S^1 (the suspension circle) are not simply connected.

Hyperbolize $S^1 * K^3$ with a strictly hyperbolized five-simplex and call the result N^5.

(5c.1) **Theorem.** *Let N^5 be the strictly negatively curved polyhedron constructed above and $\widetilde{N}^5$ its universal cover. Then*

(i) *N^5 is a topological manifold.*

(ii) N^5 *is homeomorphic to a smooth five-manifold.*

(iii) $\widetilde{N}^5$ *is homeomorphic to* $\mathbb{R}^5$.

(iv) *The ideal boundary* $S(\infty)$ *of* $\widetilde{N}^5$ *is not a manifold.*

(v) N^5 *does not carry a strictly negatively curved PL metric (i.e., there is no strictly negatively curved polyhedral metric on* N^5 *so that the underlying polyhedron is a PL manifold).*

Proof. Statement (i) follows from Edwards' Theorem. Since $N^5 \to S^5$ is covered by a map of stable tangent bundles, the stable tangent bundle of N^5 is trivial. Hence, (ii) follows from smoothing theory. The metric on N^5 has strict negative curvature. The subset $S^1 \subset N^5$ (the hyperbolization of the suspension circle) along which we have PL singularities is totally geodesic. Thus, in the universal covering $\widetilde{N}^5$, we see unbounded geodesics of PL singular points and no other singularities. An argument similar to those in §3 shows that the sphere of radius r, $S_y(r)$, about a nonsingular point y is homeomorphic to a (nonsingular) connected sum of copies of the suspension of Σ^3. Hence $S_y(r)$ is simply connected; it follows that $\widetilde{N}^5$ is simply connected at infinity and, therefore, PL homeomorphic to $\mathbb{R}^5$ (cf. [30]). This proves statement (iii).

We note that (iv) implies (v). Indeed, suppose that N^5 carried a strictly negatively curved PL metric. By Theorem 3b.2(iii), $S(\infty)$ would be homeomorphic to S^4 and we would get a contradiction to the Efremovich-Tichomirova Theorem (Corollary 2b.13).

It remains to prove (iv). By Theorem 3d.1, $S(\infty)$ is a simply connected homology four-manifold with the homology of S^4. Hence, if $S(\infty)$ were a manifold, then, by the four-dimensional Poincaré Conjecture (cf. [16]), it would be homeomorphic to S^4. We will show that $S(\infty)$ is not homeomorphic to S^4 by finding two points γ_+ and γ_- in $S(\infty)$ so that $S(\infty) - \{\gamma_+, \gamma_-\}$ is not simply connected. The points γ_+ and γ_- are endpoints in $S(\infty)$ of a singular geodesic γ. Since $S(\infty)$ is independent of basepoint, choose the basepoint x to lie on a singular geodesic γ. Then $S_x(r) - \gamma$ is not simply connected (its fundamental group is $\pi_1(\Sigma^3)$). Let η_r be a noncontractible loop in $S_x(r) - \gamma$. As preimages of points under geodesic contraction are connected, we can construct a curve η_∞ in $S(\infty)$ which projects to η_r under geodesic contraction. We claim that η_∞ is nontrivial in $\pi_1(S(\infty) - \{\gamma_+, \gamma_-\})$. Suppose not. Then η_∞ can be homotoped to zero in the complement of some open neighborhood of $\{\gamma_+, \gamma_-\}$. Such an open set is a full preimage of some open set U in

$S_x(R) - \gamma$, for some large radius $R > r$. Therefore, we can push the homotopy forward and kill the image of η_∞ in $\pi_1(S_x(R) - \gamma)$. We try to push the homotopy forward to $S_x(r)$. Unfortunately the geodesic contraction $c_r\colon S_x(R) \to S_x(r)$ does not map $S_x(R) - \gamma$ to $S_x(r) - \gamma$. However, we have the following.

(5c.2) **Lemma.** *Let $\overline{\gamma}_+$ and $\overline{\gamma}_-$ be the points where γ intersects $S_x(r)$. Both the geodesic contraction $c_r\colon S_x(R) - c_r^{-1}(\{\overline{\gamma}_-, \overline{\gamma}_-\}) \to S_x(r) - \{\overline{\gamma}_-, \overline{\gamma}\}$ and the inclusion $i\colon S_x(R) - c_r^{-1}(\{\overline{\gamma}_-, \overline{\gamma}_-\}) \to S_x(R) - \gamma$ induce isomorphisms on fundamental groups.*

Proof. Both $S_x(R)$ and $S_x(r)$ are nonsingular connected sums of several copies of the suspension of Σ^3. After collapsing extraneous copies of this suspension to points we see that the restriction of c_r to $S_x(R) - c_r^{-1}(\{\overline{\gamma}_+, \overline{\gamma}_-\})$ is cell-like, hence, a homotopy equivalence. This proves the assertion concerning c_r. Denote by D_+ and D_- the sets $c_r^{-1}(\overline{\gamma}_+)$ and $c_r^{-1}(\overline{\gamma}_-)$. Then D_+ and D_- are acyclic subcomplexes. By excision, $(S_x(R) - \gamma, S_x(R) - (D_+ \cup D_-))$ is acyclic; hence, the inclusion $i\colon S_x(R) - (D_+ \cup D_-) \to S_x(R) - \gamma$ induces an isomorphism on homology. The range of i, $S_x(R) - \gamma$, is homotopy equivalent to $\Sigma^3 \times (0, 1)$. If i_* is not surjective on π_1, then it factors through some covering of $S_x(R) - \gamma$, contradicting the fact that it induces an isomorphism on H_3. The map i_*, being a surjection between two isomorphic finite groups, is therefore an isomorphism. q.e.d.

From the lemma, we see that the image of η_∞ in $\pi_1(S_x(R) - (D_+ \cup D_-))$ is nontrivial; hence, η_∞ is not contractible, completing the proof of (iv) and, thereby, Theorem 5c.1.

Remark. Similar considerations apply to a strict hyperbolization of the double suspension of any smooth homology sphere with nontrivial finite fundamental group. Thus, Theorem 5c.1 holds in any dimension ≥ 5.

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