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# ON THE GLOBAL EXISTENCE AND THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE EINSTEIN-MAXWELL-YANG-MILLS EQUATIONS

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**Abstract**

The Einstein equations coupled to massless (conformally invariant) source field equations are represented by conformal field equations which are regular for any sign of the conformal factor. For the case of four-dimensional Lorentz spaces we introduce an improved technique to reduce Cauchy problems for the conformal Einstein-Maxwell-Yang-Mills equations to Cauchy problems for symmetric hyperbolic systems. In the case where the sign of the cosmological constant is such that conformal infinity is space-like a general stability result for asymptotically simple solutions, a semiglobal existence result for asymptotically simple solutions obtained from arbitrary asymptotic initial data, and a global existence result for asymptotically simple solutions to the Einstein-Maxwell-Yang-Mills equations for data close to De-Sitter data is derived. In the case of vanishing cosmological constant it is shown that hyperboloidal initial sufficiently close to Minkowskian hyperboloidal initial data evolve into a solution of the Einstein-Maxwell-Yang-Mills equations which has a smooth asymptotic structure in future null directions and which is regular at future time-like infinity.

## 1. Introduction

In this article we generalize results obtained in [13], [16], [17] on the global, respectively semiglobal, existence and the asymptotic behavior of solutions to Einstein's field equations in the case of dimension four and Lorentz signature. We consider more general situations than before, present an improved technique to derive the results which exhibits also more clearly those properties of Einstein's field equations which determine the asymptotic behavior of the solutions, and include "matter fields" into the discussion. The possibility of some of these generalizations has been pointed out already in [17].

Our results are based on an analysis of the "conformal structure" of Einstein's field equations, i.e., on a study of the "conformal Einstein

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equations" which must be satisfied by the "nonphysical" metric which is obtained from the "physical" metric by a conformal rescaling. Since the conformal equations provide the starting point for our discussions and since we need to introduce the language necessary to state our results—they will be formulated mostly in terms of the rescaled metric and fields derived from it—we first investigate the conformal properties of the equations. In §2 a certain representation of the conformal field equations is derived under the assumption that the energy momentum tensor is trace-free and shows a particular rescaling behavior under conformal rescalings of the metric field, conditions which are satisfied by various conformally invariant (massless) fields. For our purpose one of the most important properties of the equations which we obtain is their "regularity", i.e., the fact that their principal part remains unchanged if the conformal factor vanishes or becomes negative. This allows us to show the existence of solutions up to and through "conformal infinity". In §§3 and 4 the equations are reformulated to allow an easy discussion of their splitting into constraint and propagation equations and of the hyperbolicity of the propagation equations. In §§5 and 6 the resulting propagation equations are analyzed as far as possible without further assumptions on the matter fields. In §7 a method to deal with the coupled system of Einstein's equations and matter field equations is illustrated by discussing in detail the case of gauge field equations like Maxwell's or the Yang-Mills equations. The analysis of the conformal Einstein-Maxwell-Yang-Mills equations leads to the *reduction theorem* of §8, which shows how Cauchy problems for these equations can be reduced to Cauchy problems for symmetric hyperbolic evolution equations.

Some consequences of our reduction theorem are illustrated in the last two sections, where it is shown how the properties of the conformal field equations combine with the characterization of the asymptotic behavior of gravitational fields given by Penrose ([28], [29], [30]). The case of Einstein-Maxwell-Yang-Mills equations with cosmological constant $\lambda$ such that $\lambda g(\tau, \tau) < 0$ for some time-like vector $\tau$ (which in our signature is the case where conformal infinity, if it is smooth, is space-like) is discussed in §9. It is shown in Theorem (9.5) that "asymptotic initial data", which represent the geometry and the embedding of the hypersurface at "past conformal infinity", determine a unique past asymptotically simple solution of the Einstein-Maxwell-Yang-Mills equations, which is in particular "semiglobal" in the sense that all its "physical" null geodesics are past complete. The freedom to prescribe asymptotic initial data is essentially the same as that to describe data in a standard Cauchy problem for the same equations.

The stability of the property of solutions of the Einstein-Maxwell-Yang-Mills equation to be "asymptotically simple" in the sense that it allows a (complete) smooth conformal infinity in the past as well as in the future is shown in Theorem (9.8). Suppose there is given an asymptotically simple solution, which is such that it has an orientable compact Cauchy hypersurface $S$. Then it follows that Cauchy data on $S$, which in a certain sense are close enough to the Cauchy data implied on $S$ by the given solution, develop into a solution of the Einstein-Maxwell-Yang-Mills equations, which is again asymptotically simple.

Since the De-Sitter space-time represents such a solution, we thus obtain the existence of asymptotically simple solutions of the Einstein-Maxwell-Yang-Mills equations for data close to De-Sitter data. These solutions are "global" in the sense of being null geodesically complete.

The case of the Einstein-Maxwell-Yang-Mills equations with cosmological constant $\lambda = 0$ (in which case conformal infinity, if smooth, represents a null hypersurface) is considered in §10. For reasons discussed there the results are not as complete yet as one would wish. However, for the "hyperboloidal initial value problem", where on an initial surface diffeomorphic to a closed unit ball in $\mathbb{R}^3$ data are given which describe the geometry and the embedding of a space-like hypersurface, which "touches" future (say) conformal infinity at its boundary, the result presented in Theorem (10.2) is fairly satisfactory. It is shown that arbitrary smooth hyperboloidal data develop into a solution for which a smooth piece of conformal infinity" exists in the future on the data surface. Moreover, if the data are in a certain sense sufficiently close to "Minkowskian hyperboloidal data", they develop into a solution of the Einstein-Maxwell-Yang-Mills equations such that the resulting future conformal infinity is future complete and allows a regular point $i^+$ which may be considered to represent future time-like infinity. Thus the solutions are seen to develop in the future of the hyperboloidal surface a similar asymptotic behavior as Minkowski space.

The existence of solutions with a smooth piece of conformal infinity "near" (as measured in conformal space-time) the initial surface can also be shown for hyperboloidal data given on the initial surface which shows a more complicated topology than the unit ball. It is, however, quite a remarkable consequence of the "geometric content" of the conformal field equations that for such more general data the maximal development will necessarily show some kind of singularity, of the conformal structure (cf. the remarks following Theorem (10.2)).

The results on the hyperboloidal initial value problem appear to reduce the problem of the global existence and the asymptotic structure of solutions of Einstein's equations with cosmological constant $\lambda = 0$ to the investigation of the evolution near space-like infinity. In the conformal picture space-like infinity is thought of as being represented by one point $i^0$. The critical open problem ("$i^0$-problem") asks whether the solution allows the construction of smooth pieces of past and future conformal infinity "near" $i^0$. In the cases where this is possible the space-time contains hyperboloidal surfaces and one may expect to be able to use the results on the hyperboloidal initial value problem to obtain global existence results.

As discussed in §10 and in somewhat more detail in [18], the $i^0$-problem is extremely difficult from the point of view of PDE theory as well as from the point of differential geometry. With this in mind, the method used previously to analyze the field equations, which have been developed in the investigation of characteristic initial value problems [12], has been replaced in this article by an extension of a technique developed by Sen and Sommers ([34], [35], [36]). The reason for this is the observation, which should prove useful in many other space-time problems, that the splitting of equations considered by those authors lead to propagation equations which are algebraically equivalent to symmetric hyperbolic systems. The latter are much better adapted to the geometry of the Cauchy problem than the symmetric hyperbolic systems considered in [12]. Furthermore, the fact that one is dealing with essentially covariant spinor equations should be of considerable advantage in analyzing in detail the geometrical implications of the field equation which we shall need to understand before the $i^0$-problem will find its resolution.

The idea that the characterization of the asymptotic behavior of solutions in terms of their conformal structure is correct for a large class of data has found some amplification recently by a result of Cutler and Wald [10]. They construct a class of asymptotically flat standard Cauchy data on $\mathbb{R}^3$ for the Einstein-Maxwell equations, which outside a compact set are Schwarzschild vacuum data with positive mass. As a consequence the asymptotic behavior of the time development can be described explicitly near space-like infinity and the solutions contain hyperboloidal hypersurfaces. It can be shown that some of the resulting hyperboloidal data are as close as one wishes to Minkowskian hyperboloidal data. Invoking Theorem (10.2) the authors are thus able to show for the first time, 25 years after Penrose introduced the idea of "conformal infinity" [29], the existence of nontrivial solutions of Einstein's field equations (case where $\lambda = 0$) which have a smooth and complete structure at past and future conformal infinity and which are regular at past and future time-like infinity.

Since in this article the conformal Einstein equations are derived and analyzed in detail only in the case of a four-dimensional Lorentz space, which is the case of most physical interest, it may be pointed out here that a derivation of regular conformal field equations is possible and may be used with profit also under more general assumptions. In the source-free case regular conformal equations may be obtained for any dimension $n \geq 4$ and any signature of the metric. The definition of the unknowns which have to be introduced and the explicit form of the conformal equations will then depend on the dimension. In the case of Riemannian metrics it may be expected that in suitable dimensions "asymptotically simple" solutions (generalizing Penrose's notion of a conformal boundary) to the Einstein equations with negative "cosmological constant" can be obtained as solutions of suitably defined finite boundary value problems for the conformal Einstein equations. Einstein spaces of this kind have been considered in [11], [24]. Since in three dimensions solutions to the (source-free) Einstein equations are spaces of constant curvature, it may appear that nothing is gained by using the conformal equations. That this is not true is illustrated by the work of Beig and Simon [5] (generalized by Beig and Simon [6] and similarly by Kundu [27]) who analyzed the asymptotic behavior of solutions to the "static Einstein equations", which may be written as Einstein equations for a three-dimensional Riemannian metric coupled to a source field equation, by studying conformal field equations (cf. [18]).

## 2. The conformal Einstein equations with matter fields

In the following we shall deal with Lorentz spaces with four-dimensional manifold $M$ and metric of signature $(+,-,-,-)$, and various fields on $M$ and associated substructures, all of which will be assumed to be of class $C^\infty$ unless specified to the contrary. Furthermore the Lorentz spaces are assumed to be orientable and time-orientable, and a time direction is chosen. Occasionally we state conformal transformation laws for the case of $n$-dimensional Lorentz spaces. In this section Einstein's field equations for the "physical metric" $\tilde{g}_{\mu\nu}$

$$\widetilde{R}_{\mu\nu} - \tfrac{1}{2}\widetilde{R}\tilde{g}_{\mu\nu} + \lambda\tilde{g}_{\mu\nu} = \kappa\widetilde{T}_{\mu\nu}, \tag{2.1}$$

where $\lambda$ is the cosmological constant, $\kappa = 8\pi G$, $G$ is the gravitational constant, $\widetilde{R}_{\mu\nu}$, resp. $\widetilde{R}$, is the Ricci tensor, resp. Ricci scalar, of $\tilde{g}_{\mu\nu}$, and $\widetilde{T}_{\mu\nu}$ is the energy momentum tensor, will be expressed in terms of the "nonphysical metric" $g_{\mu\nu}$, which is obtained from $\tilde{g}_{\mu\nu}$ by a rescaling with conformal factor $\Omega$,

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \tag{2.2}$$

and tensor fields derived from $g_{\mu\nu}$, $\Omega$, and from the "nonphysical energy momentum tensor"

$$T_{\mu\nu} = \Omega^{-2} \widetilde{T}_{\mu\nu}. \tag{2.3}$$

Indices of fields which carry a tilde will be lowered and raised in the following with $\tilde{g}_{\mu\nu}$ and its contravariant form, while all other indices are moved with $g_{\mu\nu}$.

We need the transformation laws of various tensor fields under the conformal rescaling (2.2). If the dimension of our manifold $M$ is $n \geq 3$ and if we decompose the curvature tensor of $g_{\mu\nu}$ according to

$$R_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} + 2\{g_{\mu[\lambda} L_{\rho]\nu} + L_{\mu[\lambda} g_{\rho]\nu}\}, \tag{2.4}$$

where the square brackets denote antisymmetrization, $C^{\mu}{}_{\nu\lambda\rho}$ is the conformal Weyl tensor, and

$$L_{\mu\nu} = \frac{1}{n-2}\left\{R_{\mu\nu} - \frac{1}{2(n-1)} R g_{\mu\nu}\right\}, \tag{2.5}$$

with $R_{\mu\nu}$, resp. $R$, the Ricci tensor, resp. Ricci scalar, of $g_{\mu\nu}$, and if we use the analogous decomposition for the physical curvature tensor $\widetilde{R}_{\mu\nu\lambda\rho}$, then the transformation law of the curvature tensor under (2.2) follows from the transformation laws

$$\widetilde{C}^{\mu}{}_{\nu\lambda\rho} = C^{\mu}{}_{\nu\lambda\rho}, \tag{2.6}$$

$$\begin{aligned} \widetilde{R}_{\mu\nu} &= R_{\mu\nu} + (n-2)\Omega^{-1}\nabla_\mu\nabla_\nu\Omega \\ &\quad + g_{\mu\nu}\{\Omega^{-1}\nabla_\lambda\nabla_\rho\Omega - (n-1)\Omega^{-2}\nabla_\lambda\Omega\nabla_\rho\Omega\}g^{\lambda\rho}, \end{aligned} \tag{2.7}$$

which entail

$$\widetilde{R} = \Omega^2 R + (n-1)\{2\Omega\nabla_\lambda\nabla_\rho\Omega - n\nabla_\lambda\Omega\nabla_\rho\Omega\}g^{\lambda\rho}, \tag{2.8}$$

$$\begin{aligned} \{\widetilde{R}_{\mu\nu} = \tfrac{1}{n}\tilde{g}_{\mu\nu}\widetilde{R}\} &= \{R_{\mu\nu} - \tfrac{1}{n}g_{\mu\nu}R\} \\ &\quad + (n-2)\Omega^{-1}\{\nabla_\mu\nabla_\nu\Omega - \tfrac{1}{n}g_{\mu\nu}g^{\lambda\rho}\nabla_\lambda\nabla_\rho\Omega\}. \end{aligned} \tag{2.9}$$

Here $\nabla_\lambda$ denotes the covariant Levi-Civita operator defined by $g_{\mu\nu}$, and $g^{\mu\nu}$ satisfies $g^{\mu\nu}g_{\mu\lambda} = \delta^{\nu}{}_{\lambda}$. In all these equations the summation convention is assumed, as will be done in the following.

Apart from the fact that we seem to live in a space-time of dimension $n = 4$, this dimension is in the context of our investigation special for various reasons. The field equations for the gauge fields, e.g. Maxwell or Yang-Mills fields, which we want to couple to Einstein's equations are conformally invariant only in dimension 4. If the transformation behavior (2.3) of the energy momentum tensor is assumed, which in fact follows naturally from (2.2) for the type of matter fields we want to consider, we have the transformation behavior

$$(2.10)\quad g^{\lambda\rho}\nabla_\lambda T_{\rho\nu} = \Omega^{-4}\tilde{g}^{\lambda\rho}\{\tilde{\nabla}_\lambda \tilde{T}_{\rho\nu} + \Omega^{-1}((n-4)\tilde{\nabla}_\lambda\Omega\tilde{T}_{\rho\nu} - \tilde{\nabla}_\nu\Omega\tilde{T}_{\lambda\rho})\}.$$

Therefore the "conservation law"

$$(2.11)\qquad \tilde{\nabla}^\lambda \tilde{T}_{\lambda\mu} = 0,$$

which forms part of Einstein's field equations, will entail the analogous equation for the nonphysical energy momentum tensor only if the energy momentum tensor is trace-free and $n = 4$.

If the matter fields and consequently the energy momentum tensor vanish identically, i.e., in the "source-free case", we could derive for suitably defined unknowns (cf. (2.22)) regular conformal field equations also in dimension higher than four, which would be slightly more complicated than the equations for $n = 4$ due to the dependence of equation (2.7) on $n$. However, part of our system of conformal Einstein equations will be constituted by the contracted Bianchi identities and the proof of the reduction theorem (8.1) will exploit the fact that in four dimensions the contracted Bianchi identities carry exactly the same information as the (uncontracted) Bianchi identities. Furthermore the spinor calculus which we shall employ to discuss the hyperbolicity of the conformal field equations is very special in four dimensions. It is therefore not obvious whether an analogue of Reduction Theorem (8.1) can be obtained in dimension higher than four.

Assuming now that $n = 4$ and that the trace-free condition

$$(2.12)\qquad \tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} = 0, \quad \text{whence } g^{\mu\nu}T_{\mu\nu} = 0,$$

is satisfied by the energy momentum tensor, we can derive from Einstein's equations (2.1), (2.11) for the unknown tensor fields

$$(2.13)\qquad \begin{gathered} g_{\mu\nu}, \quad \Omega, \quad s = \tfrac{1}{4}\nabla^\mu\Omega, \quad s_{\mu\nu} = \tfrac{1}{2}(R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}), \\ d^\mu{}_{\nu\lambda\rho} = \Omega^{-1}C^\mu{}_{\nu\lambda\rho}, \quad T_{\mu\nu} \end{gathered}$$

the following "conformal Einstein equations", which are in a domain where $\Omega$ is positive equivalent to Einstein's equations (2.1):

$$(2.14)\qquad R_{\mu\nu\lambda\rho} = \Omega\, d_{\mu\nu\lambda\rho} + 2\{g_{\mu[\lambda}L_{\rho]\nu} + L_{\mu[\lambda}g_{\rho]\nu}\}$$

with $L_{\mu\nu} = s_{\mu\nu} + \Lambda g_{\mu\nu}$, $\Lambda = \frac{1}{24}R$,

$$\nabla_\mu \nabla_\nu \Omega = -\Omega s_{\mu\nu} + g_{\mu\nu} + \tfrac{1}{2}\kappa \Omega^3 T_{\mu\nu}, \tag{2.15}$$

$$\nabla_\mu s = -s_{\mu\nu} \nabla^\nu \Omega - 2\Lambda \nabla_\mu \Omega - \Omega \nabla_\mu \Lambda + \tfrac{1}{2}\kappa \Omega^2 T_{\mu\nu} \nabla^\nu \Omega, \tag{2.16}$$

$$2\nabla_{[\lambda} L_{\rho]\nu} = \nabla_\mu \Omega\, d^\mu{}_{\nu\lambda\rho} + \kappa \Omega t_{\lambda\rho\nu}, \tag{2.17}$$

$$\nabla_\mu d^\mu{}_{\nu\lambda\rho} = \kappa t_{\lambda\rho\nu}, \tag{2.18}$$

$$\nabla^\lambda T_{\lambda\mu} = 0, \tag{2.19}$$

$$6\Omega s - 3\nabla_\mu \Omega \nabla^\mu \Omega + 6\Omega^2 \Lambda = \lambda, \tag{2.20}$$

where we have used the notation

$$t_{\lambda\rho\nu} = \Omega \nabla_{[\lambda} T_{\rho]\nu} + 3\nabla_{[\lambda} \Omega T_{\rho]\nu} - g_{\nu[\lambda} T_{\rho]\mu} \nabla^\mu \Omega. \tag{2.21}$$

These equations are obtained as follows. Equations (2.14), (2.15), (2.20) are just (2.4), (2.8), (2.9), respectively, rewritten in the new unknowns taking into account (2.1), (2.12). Equation (2.16) may be obtained either by taking a derivative of (2.20) and using (2.15) to simplify, or as an integrability condition for (2.15), by taking a covariant derivative of that equation, commuting derivatives, contracting, and using (2.17). After equation (2.16) has been introduced, equation (2.20) acquires the status of a constraint equation, or $\lambda$ may be understood as a constant of integration. To derive (2.18), we rewrite the contracted Bianchi identities

$$\tilde{\nabla}_\mu \tilde{C}^\mu{}_{\nu\lambda\rho} = \tilde{\nabla}_{[\lambda} \tilde{L}_{\rho]\nu}$$

for the physical fields by using (2.3), (2.12), whence

$$\tilde{L}_{\mu\nu} = \tfrac{1}{2}\kappa \tilde{T}_{\mu\nu} + \tfrac{1}{6}\lambda \tilde{g}_{\mu\nu},$$

and using for $n = 4$ the decisive transformation law (given here for arbitrary dimension $n \geq 4$)

$$\nabla_\mu (\Omega^{3-n} C^\mu{}_{\nu\lambda\rho}) = \Omega^{3-n} \tilde{\nabla}_\mu \tilde{C}^\mu{}_{\nu\lambda\rho} \tag{2.22}$$

to obtain an equation for the nonphysical fields. Equation (2.17) is then a consequence of the contracted Bianchi identities

$$\nabla_\mu C^\mu{}_{\nu\lambda\rho} = 2\nabla_{[\lambda} L_{\rho]\nu} \tag{2.23}$$

for the nonphysical fields and equation (2.18).

Finally equation (2.19) follows from (2.10), (2.12).

Besides the unknowns (2.13), the function $\Lambda$ appears in equations (2.14)–(3.20), which represents the unphysical Ricci scalar. This function is the "gauge source function" for the conformal factor and may locally be given as an arbitrary (smooth) function of four space-time coordinates, as has been discussed in [14]. Its choice will depend on the particular problem which one wants to solve. An important property of equations (2.14)–(2.20) is their invariance under transitions

$$(\Omega, g_{\mu\nu}, T_{\mu\nu}) \mapsto (\Theta\Omega, \Theta^2 g_{\mu\nu}, \Theta^{-2} T_{\mu\nu}) \tag{2.24}$$

with positive functions $\Theta$.

Equations (2.14)–(2.20) have to be supplemented by equations for the matter fields and the energy momentum tensor has to be given in terms of these fields. The most conspicuous feature of the conformal Einstein equations is then the occurrence of derivatives of the energy momentum tensor on the right of equations (2.17), (2.18), which means that there will appear first- or even higher-order derivatives of the matter fields in those equations, which usually cannot be eliminated by using the matter field equations. How the resulting difficulties may be resolved depends on the particular form of the energy momentum tensor for a given choice of matter fields. In §§3–6 we shall analyze the field equations (2.14)–(2.20) as far as possible without any assumptions on the matter fields. In §7 it will be shown how the terms on the right of equations (2.17), (2.18) may be handled for gauge field equations like Maxwell's and the Yang-Mills equations.

## 3. The field equations in the spin frame formalism

The sense in which the conformal Einstein equations (2.14)–(2.20) split into constraint equations and hyperbolic propagation equations reveals itself in a most natural way if the equations are written in the two-component spin frame formalism. Since there is available a detailed account of all the associated spinor techniques and results [31], we shall confine ourselves to a few remarks to fix the notation used here, which will be the same as in [12]. In §§3–8 all considerations are of a local nature while global solutions constructed later will carry a global smooth orthonormal frame. Therefore the assumption of a spin structure poses no problems.

Let $\{\delta_a\}_{a=0,1}$ be a spin frame field on our manifold $M$, which is normalized such that

$$\varepsilon(\delta_a, \delta_b) = \varepsilon_{ab} \quad \text{with } \varepsilon_{01} = 1, \tag{3.1}$$

where $\varepsilon$ denotes the antisymmetric form associated with the spin structures. In the following we assume all spinor fields (with the exception of $\delta_a$

and $e_{aa'}$) to be given by their components with respect to the spin frame $\{\delta_a\}$. We define $\varepsilon^{ab}$ by $\varepsilon_{ab}\varepsilon^{cb} = \varepsilon_a{}^c$, where the right-hand side vanishes if $a \neq c$ and gives 1 if $a = c$, and the summation convention is assumed as in the following. Spinor indices are raised or lowered according to

$$\mu^a = \varepsilon^{ab}\mu_b, \qquad \mu_b = \mu^a\varepsilon_{ab}$$

and similar relations hold for the complex conjugate fields.

The bundle morphism, which maps the bundle of normalized spin frames onto the bundle of pseudo-orthonormal frames (see (3.3)), associates with $\{\delta_a\}$ a frame field $\{e_{aa'}\}_{a,a'=0,1}$ on $M$, for which we write $e_{aa'} = \delta_a\bar{\delta}_{a'}$. here the overbar denotes complex conjugation, under which primed indices change into unprimed indices and vice versa (cf. [30]). The frame vector fields $e_{aa'}$ satisfy the reality condition (using the convention of [30])

$$e_{aa'} = \bar{e}_{aa'} \tag{3.2}$$

and the normalization condition

$$g(e_{aa'}, e_{bb'}) = \varepsilon_{ab}\varepsilon_{a'b'}. \tag{3.3}$$

To a covariant differentiation operator $\nabla$, which is metric in the sense that $\nabla g \equiv 0$, is associated a covariant operator $\nabla$ acting on spinor fields. If the connection coefficients $\Gamma_{aa'bc} = \Gamma_{aa'(bc)}$ (the brackets denoting symmetrization) with respect to $\delta_a$, $e_{aa'}$ are given by

$$\nabla_{e_{aa'}}\delta_b \equiv \Gamma_{aa'}{}^c{}_b\delta_c,$$

then on a spinor field $\mu$ which has coefficients $\mu_{bb'}{}^d$ (say) in the spin frame $\{\delta_a\}$, the directional derivative operator $\nabla_{aa'} \equiv \nabla_{e_{aa'}}$ acts according to

$$\nabla_{aa'},\mu_{bb'}{}^d = \mu_{bb'}{}^d{}_{,\lambda}e^\lambda{}_{aa'} - \Gamma_{aa'}{}^c{}_b\mu_{cb'}{}^d - \overline{\Gamma}_{aa'}{}^{c'}{}_{b'}\mu_{bc'}{}^d + \Gamma_{aa'}{}^d{}_c\mu_{bb'}{}^c,$$

where the comma denotes partial differentiation with respect to some coordinate system $x^\mu$, and $e^\mu{}_{aa'}$ are the components of the vector field $e_{aa'}$ with respect to the corresponding coordinate frame.

We can express any tensor equation and any tensor equation by a spinor and a spinor equation, where to each tensor index corresponds a pair consisting of an unprimed and a primed spinor index. We will use this transition to write all the equations needed in the later analysis in terms of spinor fields. We will go back and forth freely between the tensor and the spinor notation to keep the equations short.

The torsion tensor $S_{bb'}{}^{aa'}{}_{cc'}$ of the metric covariant differentiation operator $\nabla$ is defined by

$$
\begin{aligned}
S_{bb'}{}^{aa'}{}_{cc'}e_{aa'} &\equiv \nabla_{bb'}e_{cc'} - \nabla_{cc'}e_{bb'} - [e_{bb'}, e_{cc'}] \\
&= \Gamma_{bb'}{}^{d}{}_{c}e_{dc'} + \overline{\Gamma}_{bb'}{}^{d'}{}_{c'}e_{cd'} - \Gamma_{cc'}{}^{d}{}_{b}e_{db'} \\
&\quad - \overline{\Gamma}_{cc'}{}^{d'}{}_{b'}e_{db'} - [e_{bb'}, e_{cc'}],
\end{aligned}
\tag{3.4}
$$

where the square brackets denote the commutator of vector fields.

The curvature tensor $r^{aa'}{}_{bb'cc'dd'}$ is defined by the Ricci identity

$$(\nabla_{cc'}\nabla_{dd'} - \nabla_{dd'}\nabla_{cc'})\mu^{aa'} = r^{aa'}{}_{bb'cc'dd'}\mu^{bb'} - S_{cc'}{}^{ee'}{}_{dd'}\nabla_{ee'}\mu^{aa'}, \tag{3.5}$$

which holds for any vector field $\mu^{aa'}$. It can be decomposed in the form

$$r_{aa'bb'cc'dd'} = \varepsilon_{a'b'}r_{abcc'dd'} + \varepsilon_{ab}r_{abcc'dd'}, \tag{3.6}$$

where the curvature spinor $r_{abcc'dd'} = r_{(ab)cc'dd'}$ is given in terms of the connection coefficients by

$$
\begin{aligned}
r_{abcc'dd'} = {} & e_{dd'}(\Gamma_{cc'ab}) - e_{cc'}(\Gamma_{dd'ab}) \\
& + \Gamma_{dd'as}\Gamma_{cc'}{}^{s}{}_{b} + \Gamma_{sd'ab}\Gamma_{cc'}{}^{s}{}_{d} - \Gamma_{cc'as}\Gamma_{dd'}{}^{s}{}_{b} \\
& - \Gamma_{sc'ab}\Gamma_{dd'}{}^{s}{}_{c} + \Gamma_{ds'ab}\overline{\Gamma}_{cc'}{}^{s'}{}_{d'} \\
& - \Gamma_{cs'ab}\overline{\Gamma}_{dd'}{}^{s'}{}_{c'} - S_{cc'}{}^{ee'}{}_{dd'}\Gamma_{ee'ab}
\end{aligned}
\tag{3.7}
$$

and satisfies

$$(\nabla_{cc'}\nabla_{dd'} - \nabla_{dd'}\nabla_{cc'})\omega^{a} = -r^{a}{}_{bcc'dd'}\omega^{b} - S_{cc'}{}^{ee'}{}_{dd'}\nabla_{ee'}\omega^{a} \tag{3.8}$$

for any spinor $\omega^a$. In equation (3.7) we understand $e_{aa'}$ to be the directional derivative operator acting on functions.

The conformal Einstein equations (2.14)–(2.21) will now be written as spinor equations. Here the role of the metric $g$ will be played by the frame $\{e_{aa'}\}$, which allows to determine the metric by (3.3). The unknowns in the equations are given by

$$e_{aa'}, \quad \Gamma_{aa'bc}, \quad \Omega, \quad s, \quad \Sigma_{aa'}, \quad \Phi_{aa'bb'}, \quad \varphi_{abcd}, \quad T_{aa'bb'}, \tag{3.9}$$

where $\Sigma_{aa'}$ represents the differential of $\Omega$ which has been introduced to obtain a first-order system, while the spinor fields $\Phi_{aa'bb'}$ and $\varphi_{abcd}$ have the symmetry and reality properties

$$\varphi_{abcd} = \varphi_{(abcd)}, \qquad \Phi_{aa'bb'} = \Phi_{ba'ab'} = \Phi_{ab'ba'} = \overline{\Phi}_{aa'bb'}, \tag{3.10}$$

and define together with $\Lambda$ the spinor field

$$R_{abcc'dd'} = -\Omega\varphi_{abcd}\varepsilon_{c'd'} - \Phi_{ac'bd'}\varepsilon_{cd} + \Lambda\varepsilon_{c'd'}(\varepsilon_{bd}\varepsilon_{ac} + \varepsilon_{ad}\varepsilon_{bc}), \tag{3.11}$$

which will by equation (3.13) represent the decomposition of the curvature spinor into irreducible parts. The field $\varphi_{abcd}$ will represent the rescaled Weyl tensor $d^{\mu}{}_{\nu\lambda\rho}$ while $\Phi_{aa'bb'}$ is the traceless part $s_{\mu\nu}$ of the Ricci tensor.

Equations (2.14)–(2.18) and the condition that the connection be torsion free are represented equivalently by

$$0 = S_{aa'}{}^{bb'}{}_{cc'}e_{bb'}, \tag{3.12}$$

$$0 = K_{abcc'dd'} \equiv r_{abcc''dd'} - R_{abc'dd'}, \tag{3.13}$$

$$0 = Q_{aa'} \equiv \nabla_{aa'}\Omega - \Sigma_{aa'}, \tag{3.14}$$

$$0 = Q_{aa'bb'} \equiv \nabla_{aa'}\Sigma_{bb'} + \Omega\Phi_{aa'bb'} - s\varepsilon_{ab}\varepsilon_{a'b'} - \tfrac{1}{2}\kappa\Omega^3 T_{aa'bb'}, \tag{3.15}$$

$$\begin{aligned} 0 &= P_{aa'} \\ &\equiv \nabla_{aa'}s + \Phi_{aa'bb'}\Sigma^{bb'} + 2\Lambda\Sigma_{aa'} + \Omega\nabla_{aa'}\Lambda - \tfrac{1}{2}\kappa\Omega^2 T_{aa'bb'}\Sigma^{bb'}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} 0 &= U_{aa'bb'cc'} \\ &\equiv \nabla_{aa'}(\Phi_{bb'cc'} + \Lambda\varepsilon_{bc}\varepsilon_{b'c'}) - \nabla_{bb'}(\Phi_{aa'cc'} + \Lambda\varepsilon_{ac}\varepsilon_{a'c'}) \\ &\quad + \Sigma^{d}{}_{c'}\varphi_{dabc}\varepsilon_{a'b'} + \Sigma_{c}{}^{d'}\bar{\varphi}_{d'a'b'c'}\varepsilon_{ab} - \kappa\Omega t_{aa'bb'cc'}, \end{aligned} \tag{3.17}$$

$$0 = H_{aa'bb'cc'} \equiv \nabla^{d}{}_{c'}\varphi_{dcab}\varepsilon_{a'b'} + \nabla_{c}{}^{d'}\bar{\varphi}_{d'c'a'b'}\varepsilon_{ab} + \kappa t_{aa'bb'cc'}. \tag{3.18}$$

Here the torsion tensor and the curvature spinor $r_{abcc'cc'}$ are thought of as being given by the right-hand sides of (3.4) and (3.7), and (3.12), (3.13) are considered as differential equations for $e_{aa'}$, $\Gamma_{aa'bc}$.

For later use we have introduced the "zero-quantities"

$$S_{bb'}{}^{aa'}{}_{cc'},\ K_{abcc'dd'},\ Q_{aa'},\ Q_{aa'bb'},\ P_{aa'},\ U_{aa'bb'cc'},\ H_{aa'bb'cc'}, \tag{3.19}$$

which are defined by (3.4) respectively by the right members of equations (3.13)–(3.18) as functionals of the unknowns (3.9) and the function $\Lambda$. The field equations are then expressed as the requirement that the zero-quantities vanish.

Because of the antisymmetry in its first two indices, we may represent the tensor $t_{\mu\nu\lambda}$ in the form

$$t_{aa'bb'cc'} = t_{abcc'}\varepsilon_{a'b'} + \bar{t}_{a'b'c'c}\varepsilon_{ab}. \tag{3.20}$$

Using the fact that the nonphysical energy momentum tensor is symmetric, trace-free, and, by equation (2.19), divergence-free, we find

$$2t_{abcc'} = T_{ah'b}{}^{h'}{}_{cc'} = \Omega\nabla_{h'(a}T_{b}{}^{h'}{}_{c)c'} + 3\Sigma_{h'(a}T_{b}{}^{h'}{}_{c)c'}. \tag{3.21}$$

The decomposition (3.20) allows us to represent equations (3.17), (3.18) by the equivalent equations

$$(3.22)\quad 0 = V_{abcc'} \equiv \nabla_a{}^{h'} \Phi_{bh'cc'} + 2\varepsilon_{a(b} \nabla_{c)c'} \Lambda - \Sigma^d{}_{c'} \varphi_{abcd} + \kappa \Omega t_{abcc'},$$

$$(3.23)\quad 0 = H_{abcc'} \equiv \nabla^h{}_c \varphi_{abch} + \kappa t_{abcc'}.$$

The lengthy expressions (3.17), (3.18) have been given here for later use.

## 4. General remarks about our analysis

To discuss the hyperbolicity of the field equations we introduce an arbitrary time-like vector field $\tau = \tau^{aa'} e_{aa'}$. The choice of this vector field allows a reduction of the spin structure to the group $SU(2)$ and consequently a treatment of the equations in terms of unprimed spinors. Furthermore, due to the splitting of the tangent spaces into the subspaces generated by $\tau$ and the hyperplanes $S_\tau$ orthogonal to $\tau$, we obtain a splitting of the spinor algebra, which can be dealt with in an algebraically simple manner in terms of unprimed spinors and their symmetries. The resulting "space-spinor" formalism and some corresponding splittings of field equations have been discussed in [34], [35], [36], to which the reader is referred for further details. In the following we present those facts which are relevant for our later considerations.

It is convenient to assume that the vector field $\tau$ satisfies

$$(4.1)\quad \tau^{aa'} \text{ is real and future directed, } \tau_{aa'}\tau^{aa'} = 2.$$

It implies a complex linear map

$$(4.2)\quad \mu_{a'} \mapsto \mu_a = \tau_a{}^{a'} \mu_{a'}$$

of primed onto unprimed spinors, which extends to spinors of higher valence and different index position in such a way as to commute with contractions, and has an inverse which maps $\mu_a$ onto $-\tau_{a'}{}^a \mu_a$. Any tensor field may be expanded in terms of $\tau$ and the (complex) basis of the hyperplane $S_\tau$ orthogonal to $\tau$, which is provided by

$$(4.3)\quad e_{ab} = \tau_{(b}{}^{a'} e_{a)a'}.$$

If the tensor field $\mu$ has components $\mu_{a_1 a'_1 \cdots a_k a'_k}$ in the frame $e_{aa'}$, it can be expanded in terms of the vector field $e_{ab}$ if and only if $\mu$ is "spatial", i.e., "lives on $S_\tau$", in the sense that $\tau^{a_j a'_j} \mu_{a_1 a'_1 \cdots a_j a'_j \cdots a_k a'_k} = 0$, $j = 1, \cdots, k$, and then its components in the basis $e_{ab}$ are given by

$$(4.4) \qquad \mu_{a_1 b_1 \cdots a_k b_k} = \tau_{b1}{}^{a'_1} \cdots \tau_{b_k}{}^{a'_k} \mu_{a_1 a'_1 \cdots a_k a'_k} = \mu_{(a_1 b_1) \cdots (a_k b_k)} .$$

Conversely, a spinor field with $2k$ unprimed indices, which has the symmetries in the index pairs indicated above, corresponds to a spatial spinor.

We have the projector $h^{aa'}{}_{bb'} = \varepsilon^a{}_b \varepsilon^{a'}{}_b - \frac{1}{2}\tau^{aa'}\tau_{bb'}$, which projects any tensor $t_{aa'_1 \cdots a_k a'_k}$ onto the spatial tensor $t_{a_1 a'_1 \cdots a_k a'_k} h^{a_1 a'_1}{}_{b_1 b'_1} \cdots h^{a_k a'_k}{}_{b_k b'_k}$, and the inner metric $h_{aa'bb'} = \varepsilon_{ab}\varepsilon_{a'b'} - \frac{1}{2}\tau_{aa'}\tau_{bb'}$ implied by $g$ on $S_\tau$, which is a spatial tensor, is mapped under (4.2) onto

$$(4.5) \qquad h_{abcd} = g(e_{ab}, e_{cd}) = -\varepsilon_{a(c} e_{d)b} .$$

It should be noted that the positions of primed and unprimed indices of a spinor are of importance and must be carefully respected if the map (4.2) and its extensions are applied.

Reality properties of unprimed space-spinors may be dealt with in terms of the hermitian conjugation map

$$(4.6) \qquad \mu_a \mapsto \mu^+{}_a = \tau_a{}^{a'} \bar{\mu}_{a'} ,$$

which extends to a complex semilinear map on spinors of higher valence and various index positions, commutes with contractions, satisfies

$$(4.7) \qquad \mu^{++}{}_{a_1 \cdots a_k} = (-1)^k \mu_{a_1 \cdots a_k} ,$$

and serves to define a positive definite hermitian product by

$$(4.8) \qquad (\mu_{a_1 \cdots a_k}, \nu_{b_1 \cdots b_k}) \mapsto \mu_{a_1 \cdots a_k} \nu^{+a_1 \cdots a_k} .$$

If the spinor $\mu$ allows the expansion (4.4), then $\mu$ represents a real tensor in the tensor algebra of $S_\tau$ if and only if

$$(4.9) \qquad \mu^+{}_{a_1 b_1 \cdots a_k b_k} = \tau_{a_1}{}^{a'_1} \tau_{b_1}{}^{b'_1} \cdots \tau_{a_k}{}^{b'_k} \bar{\mu}_{a'_1 b'_1 \cdots b'_k} = (-1)^k \mu_{a_1 b_1 \cdots a_k b_k} .$$

The inner metric on $S_\tau$ given by (4.5) satisfies this condition. The rescaled Weyl spinor $\varphi_{abcd}$, which corresponds to a spatial spinor, does not in general satisfy this condition but can be represented in the form

$$(4.10) \qquad \varphi_{abcd} = \eta_{abcd} + i\mu_{abcd}$$

with symmetric spinors $\eta_{abcd}$ and $\mu_{abcd}$, which satisfy the reality condition (4.9) and represent the electric, respectively the magnetic, parts of the rescaled Weyl tensor $d_{\mu\nu\lambda\rho}$ with respect to $\tau$.

The trace-free Ricci spinor is translated by (4.2) into the spinor

$$\Phi_{abcd} = \tau_b{}^{a'}\tau_d{}^{c'}\Phi_{aa'cc'} \tag{4.11}$$

with symmetries

$$\Phi_{abcd} = \Phi_{cbad} = \Phi_{adcb}. \tag{4.12}$$

We shall often use the fact that any spinor with these symmetries can be expanded in the form

$$\Phi_{abcd} = \Phi^*_{abcd} + \tfrac{1}{3}\Phi^* h_{acbd} + \tfrac{1}{2}\{\varepsilon_{a(b}\Phi^*_{d)c} + \varepsilon_{c(b}\Phi^*_{d)a}\} \tag{4.13}$$

with symmetric spinors $\Phi^*_{abcd}$, $\Phi^*_{ab}$ and a function $\Phi^*$, which satisfy the reality condition if $\Phi_{abcd}$ arises as in (4.11) from a real spinor. Here

$$\Phi^* = \Phi_{abcd}h^{acbd} = -\tfrac{1}{2}\tau^{aa'}\tau^{bb'}\Phi_{aa'bb'}, \qquad \Phi^*_{abcd} = \Phi_{(abcd)} \tag{4.14}$$

represent the trace (with respect to $h$), respectively the trace-free, part of the projection of (4.11) onto $S_\tau$, while

$$\Phi^*_{ab} = \tau_{(a}{}^{a'}\Phi_{b)a'cc'}\tau^{cc'} \tag{4.15}$$

gives the projection of $\Phi_{aa'bb'}\tau^{bb'}$ onto $S_\tau$. Equation (4.15) illustrates the fact that projections by the projector $h^{aa'}{}_{bb'}$ can be translated into the action of the map (4.2) combined with symmetrization in pairs of indices.

The basic differential operator $\nabla_{aa'}$ may be decomposed according to

$$\nabla_{ab} \equiv \tau_b{}^{a'}\nabla_{aa'} = \tfrac{1}{2}\varepsilon_{ab}P + D_{ab} \tag{4.16}$$

with differential operators $P = \tau^{aa'}\nabla_{aa'}$, which acts in the direction of $\tau$, and $D_{ab} = \tau_{(b}{}^{a'}\nabla_{a)a'}$, which act in directions orthogonal to $\tau$, and which satisfy the reality conditions

$$\overline{P} = P, \qquad D^+{}_{ab} = -D_{ab}. \tag{4.17}$$

In translations of the spinor equations (3.12)–(3.16), (3.21), (3.22) into equations for unprimed spinors derivatives of the field $\tau$ will appear, which will be represented by

$$\chi_{abcd} \equiv \tau_b{}^{a'}(\nabla_{aa'}\tau_{cc'})\tau_d{}^{c'} = \chi^*_{abcd} + \varepsilon_{ab}\chi^*_{cd} \tag{4.18}$$

with spinors

$$\chi^*_{abcd} = \chi^*_{(ab)(cd)} = (D_{ab}\tau_{cc'})\tau_d{}^{c'}, \qquad \chi^*_{ab} = \chi_{(ab)} = (P\tau_{ac'})\tau_b{}^{c'}, \tag{4.19}$$

which satisfy the reality condition (4.9). In the particular case where

$$\tau^\mu = \sqrt{2} a \nabla^\mu t \tag{4.20}$$

with some time function $t$ and normalizing factor $a$, we have $\chi^*_{ab} = D_{ab}(\ln a)$ and $\chi^*_{abcd} = \sqrt{2}\tau_b{}^{a'}\tau_d{}^{c'}\pi_{aa'cc'}$, where $\pi_{aa'bb'}$ denotes the second fundamental form on the level surfaces of $t$.

The desired splitting of the field equations into constraints and hyperbolic propagation equations will be obtained by the following procedure:

(i) Using (4.2) all unknowns and all zero-quantities are expressed in terms of unprimed spinors.

(ii) These spinors are suitably decomposed into irreducible ones, i.e., into sums of products of symmetric spinors with $\varepsilon_{ab}$'s.

(iii) To fix the coordinates, the frame, the conformal factor, and possibly some gauge for the gauge fields among the matter fields, suitable gauge conditions have to be imposed. There are a variety of possibilities. The final choice will depend on the problem under consideration, as will be seen later.

(iv) Equating to zero those zero-quantities which do not involve the "time derivative" operator $P$ yields equations which will be called "constraint equations" in the following. It may be noted, however, that these equations cannot be regarded as equations intrinsic to some submanifold, unless the field $\tau$ is related to some time function as in (4.20).

(v) Equating to zero certain combinations of zero-quantities which do involve the operator $P$, we obtain the propagation equations. There is in general some freedom here to pick appropriate combinations.

(vi) Assuming now that $\tau$ and the frame $e_{aa'}$ are related such that

$$\tau^{aa'} = \varepsilon_0{}^a\varepsilon_{0'}{}^{a'} + \varepsilon_1{}^a\varepsilon_{1'}{}^{a'}, \tag{4.21}$$

whence the spatially acting operators $D_{ab}$ are given by

$$D_{00} = -\nabla_{01'}, \quad D_{01} = \tfrac{1}{2}(\nabla_{00'} - \nabla_{11'}), \quad D_{11} = \nabla_{10'}, \tag{4.22}$$

we find that the propagation equations are (essentially) symmetric hyperbolic (see equations (4.25), (4.27)) and thus allow well-posed Cauchy problems.

(vii) We may therefore assume that for suitably Cauchy data the fields (3.9), thus by (3.3) the metric $g$ and possibly some matter fields, are obtained on some manifold $M$ as a solution of the propagation equations, such that on some Cauchy surface $S$ of $(M, g)$ the fields imply the given Cauchy data. Forming from this solution the zero-quantities we find that some of them vanish since we solved the propagation equations. Though we will of course give the data in such a way that the constraints are satisfied on $S$, it is a priori not clear whether they will be satisfied everywhere on $M$. It may even be possible that the connection defined by the $\Gamma_{aa'bc}$, which is metric because of the requirement $\Gamma_{aa'bc} = \Gamma_{aa'(bc)}$, is not torsion free. It turns out, however, that by using the Ricci and the Bianchi identities and the symmetries of the fields involved, we can derive a symmetric hyperbolic system of "subsidiary equations" for those zero-quantities which define constraints. Since this system may be read as a linear (homogeneous) system, the uniqueness property of solutions of such systems and the fact that the domain of dependence for a subsidiary system coincides with that defined by $g$, allow us to conclude that the constraints and consequently the complete set of field equations will be solved everywhere in $M$. At the same time it is shown that the gauge conditions imposed in the beginning are satisfied.

In carrying out this procedure some lengthy calculations have to be performed and it is easy to loose sight of the essential points. Therefore it may be helpful to illustrate some of the steps by studying the simple example of the massless spin-$\frac{n}{2}$ equation

$$\nabla^{f}{}_{a'}\varphi_{ab\cdots ef} = 0 \tag{4.23}$$

for a symmetric spinor field $\varphi_{a\cdots f}$ with $n \geq 1$ indices.

Since the unknown is symmetric, we use the decomposition (4.16) to write (4.23) equivalently as

$$P\varphi_{ab\cdots f} - 2D^{h}{}_{f}\varphi_{a\cdots eh} = 0. \tag{4.24}$$

Expanding the left-hand side into symmetric spinors we find that (4.23) is equivalent to the propagation equation

$$P\varphi_{ab\cdots f} - 2D^{h}{}_{(f}\varphi_{a\cdots e)h} = 0 \tag{4.25}$$

and the constraint equation

$$D^{hf}\varphi_{a\cdots hf} = 0. \tag{4.26}$$

Assuming now that (4.21) holds, we find that the system of equations

$$\binom{n}{a+b+\cdots+f}\{P\varphi_{ab\cdots f} - 2D^{h}{}_{(f}\varphi_{a\cdots e)h}\} = 0, \qquad a+b+\cdots+f = 0, 1, \cdots, n, \tag{4.27}$$

which is algebraically equivalent to (4.25) since it is obtained from that system by multiplication of each single equation by a suitable binomial coefficient, is symmetric hyperbolic. It has the form of an equation

$$A^{\mu}u_{,\mu} + B(u) = 0 \tag{4.28}$$

for the "vector" $u = (\varphi_0, \varphi_1, \cdots, \varphi_n)$, where $\varphi_k = \varphi_{ab\cdots f}$ if $a + b + \cdots + f = k$, such that the $(n+1) \times (n+1)$ matrices $A^{\mu}$ are hermitian, ${}^t\overline{A}^{\mu} = A^{\mu}$, and $A^{\mu}\xi_{\mu}$ is positive definite if $\xi^{\mu} = \tau^{\mu}$. In fact $A^{\mu}\xi_{\mu}$ is positive definite for any future directed real covector which is time-like with respect to $g$, since we have

$$\det(A^{\mu}\xi_{\mu}) = cQ^{j}(\xi)Q_0{}^{j_0}(\xi)Q_1{}^{j_1}(\xi)\cdots Q_k{}^{j_k}(\xi) \tag{4.29}$$

with $k = (n-2)/2$ and $j = j_0 = \cdots = j_k = 1$ if $n$ is even, $k = (n-1)/2$ and $j = 0$, $j_0 = \cdots = j_k = 1$ if $n$ is odd, and

$$Q(\xi) = \tau^{\mu}\xi_{\mu}, \qquad Q_0(\xi) = g^{\mu\nu}\xi_{\mu}\xi_{\nu}, \tag{4.30}$$

$$Q_j(\xi) = (g^{\mu\nu} + c_j\tau^{\mu}\tau^{\nu})\xi_{\mu}\xi_{\nu}, \qquad 1 \le j \le k \quad \text{if } k > 0, \tag{4.31}$$

where $c$ and $c_j$ are positive real numbers.

In the case of equation (4.25), (resp. (4.27)), we have $0 < c_1 < \cdots < c_k$ and the polynomial (4.29) is seen to be strictly hyperbolic. This property will get lost for the more complicated systems we want to consider later. It follows from (4.30), (4.31) that any characteristic of the system (4.29), i.e., any hypersurface $C = \{x \in M | \Phi(x) = 0,\ d\Phi(x) \neq 0\}$ defined by a smooth function $\Phi$ such that $\det(A^{\mu}\Phi_{,\mu}(x)) = 0$ on $C$, is time-like or null with respect to $g$ and the time-like characteristics depend on the choice of the vector field $\tau$ which has been introduced to perform the splitting.

It may be remarked here that in the case where $n$ is even we could write equation (4.24) also as a system of equations for the real space-spinors which appear in a decomposition of $\varphi_{a\cdots f}$ of the type (4.10) and obtain a symmetric hyperbolic system for these fields in the same way as indicated above.

The integrand appearing in the $L^2$-type "energy estimates" for a system of the form (4.28) is given by $n_{\mu}({}^t\bar{u}A^{\mu}u)$ if $n^{\mu}$ denotes the future directed unit normal of the space-like hypersurface $S$ over which the integration is performed. In the case of equation (4.27) it can be expressed in terms of the tensor

$$T_{aa'\cdots ee'ff'} = \varphi_{a\cdots ef}\bar{\varphi}_{a'\cdots e'f'}. \tag{4.32}$$

In fact, if we have $t^{\mu} = \sqrt{2}n^{\mu}$, the identity

$$n_{\mu}e^{\mu h}{}_{h'}\tau_{(f}{}^{h'}\varphi_{a\cdots e)h}\varphi^{+a\cdots ef} = n^{ff'}\tau^{aa'}\cdots\tau^{ee'}T_{aa'\cdots ee'ff'}$$

with (4.20) holding, gives the relation

$$\sqrt{2}n_\mu({}^t u A^\mu u) = \sum_{k=0}^{n} \binom{n}{k} |\varphi_k|^2 = \varphi_{a\cdots ef}\varphi^{+a\cdots ef} = \tau^{aa'} \cdots \tau^{ff'} T_{aa'\cdots ff'} \,.$$

In the case of Maxwell's equations or the Yang-Mills equations, which have a principal part of the type appearing in (4.23), the tensor (4.32) represents (with a suitable interpretation of the product, see (7.9)) just the energy momentum tensor (which is of course the origin of the name "energy estimate") while in the case of the Bianchi identities for the conformal Weyl spinor (equation (3.23) with $\Omega \equiv 1$), the tensor (4.32) in the source-free situation the Bel-Robinson tensor (cf. [31]).

The analogue of step (vii), namely the derivation of a subsidiary system for the zero-quantity represented by the left-hand side of equation (4.26), which would allow us to show that the constraint equations (4.26) are satisfied by a solution of (4.25), which satisfies the constraints on some Cauchy surface, cannot in general be performed if $n \geq 3$ and the background space-time is not conformally flat. In fact, it is well known that in this case equation (4.23) may admit only a few solutions [7]. In the case of equation (3.23) this problem does not arise, however, since here $\varphi_{abcd}$ is proportional to the Weyl spinor of the background metric.

## 5. The splitting of the field equations and the subsidiary system

In this section we will discuss the splitting of the conformal field equations and the derivation of the subsidiary equations as far as possible without touching the question of geometrical gauge conditions or using properties of specific matter fields.

If we use the functions

$$\begin{aligned} &\Omega,\ s,\ \Sigma_{ab} = \tau_b{}^{a'}\Sigma_{aa'},\ \Phi_{abcd} \text{ as in (4.11)},\ \varphi_{abcd}, \\ &T_{abcd} = \tau_b{}^{b'}\tau_d{}^{d'}T_{ab'cd'},\ t_{abcd} = \tau_d{}^{d'}t_{abcd'} \end{aligned} \tag{5.1}$$

to write equations (3.14)–(3.16), (3.22), (3.23) as equations for unprimed spinors, we are led to consider the zero-quantities

$$Q_{ab} = \nabla_{ab}\Omega - \Sigma_{ab}, \tag{5.2}$$

$$Q_{abcd} = \nabla_{ab}\Sigma_{cd} - \chi_{abd}{}^{h}\Sigma_{ch} + \Omega\Phi_{abcd} - s\varepsilon_{ac}\varepsilon_{bd} - \tfrac{1}{2}\kappa\Omega^3 T_{abcd}, \tag{5.3}$$

$$P_{ab} = \nabla_{ab}s + \Phi_{abcd}\Sigma^{cd} + 2\Lambda\Sigma_{ab} + \Omega\nabla_{ab}\Lambda - \tfrac{1}{2}\kappa\Omega_{abcd}\Sigma^{cd}, \tag{5.4}$$

$$\begin{aligned} V_{abcd} = {} & \nabla_a{}^{f}\Phi_{bfcd} - \chi_a{}^{f}{}_{d}{}^{e}\Phi_{bfce} - \chi_a{}^{e}{}_{e}{}^{f}\Phi_{bfcd} + 2\varepsilon_{a(b}\nabla_{c)d}\Lambda \\ & - \Sigma^{f}{}_{d}\varphi_{abcf} + \kappa\Omega t_{abcd}, \end{aligned} \tag{5.5a}$$

$$\begin{aligned} V^{+}{}_{abcd} = {} & -\nabla^{h}{}_{a}\Phi_{hbdc} + \chi^{h}{}_{ab}{}^{f}\Phi_{hfdc} + \chi^{h}{}_{ac}{}^{f}\Phi_{hfbd} \\ & + 2\nabla_{d(c}\Lambda^{\varepsilon}{}_{b)a} + \Sigma_{d}{}^{e}\varphi^{+}{}_{abce} + \kappa\Omega t^{+}{}_{abcd}, \end{aligned} \tag{5.5b}$$

$$H_{abcd} = \nabla^{h}{}_{d}\varphi_{abch} + \kappa t_{abcd}, \tag{5.6}$$

which have symmetries

$$V_{abcd} = V_{a(bc)d}, \quad V^{+}{}_{abcd} = V^{+}{}_{a(bc)d}, \quad H_{abcd} = H_{(abc)d}. \tag{5.7}$$

In terms of these the constraint equations are given by

$$0 = Q_{(ab)} = D_{ab}\Omega - \Sigma_{(ab)}, \tag{5.8}$$

$$\begin{aligned} 0 = Q_{(ab)cd} = {} & D_{ab}\Sigma_{cd} - \chi_{(ab)d}{}^{h}\Sigma_{ch} + \Omega\Phi_{(ab)cd} + s\varepsilon_{c(a}\varepsilon_{b)d} \\ & - \tfrac{1}{2}\kappa\Omega^3 T_{(ab)cd}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} 0 = P_{(ab)} = {} & D_{ab}s + \Phi_{(ab)cd}\Sigma^{cd} + 2\Lambda\Sigma_{(ab)} + \Omega D_{ab}\Lambda \\ & - \tfrac{1}{2}\kappa\Omega^2 T_{(ab)cd}\Sigma^{cd}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} 0 = V_{abcd} - V^{+}{}_{badc} = {} & D_a{}^{f}\Phi_{bfcd} + D_b{}^{f}\Phi_{facd} \\ = {} & \chi_a{}^{f}{}_{d}{}^{e}\Phi_{bfce} - \chi_a{}^{e}{}_{e}{}^{f}\Phi_{bfcd} \\ & - \chi^{h}{}_{ba}{}^{f}\Phi_{hfcd} - \chi^{h}{}_{hd}{}^{f}\Phi_{hfca} \\ & - \Sigma^{f}{}_{d}\varphi_{abcf} - \Sigma_{c}{}^{e}\varphi^{+}{}_{abde} \\ & + \kappa\Omega(t^{*}{}_{abcd} - t^{*+}{}_{abcd}) + \varepsilon_{ac}D_{bd}\Lambda - \varepsilon_{db}D_{ca}\Lambda, \end{aligned} \tag{5.11}$$

$$0 = H_{abf}{}^{f} = D^{hf}\varphi_{abhf} + \kappa t^{*}{}_{abf}{}^{f}, \tag{5.12}$$

while the propagation equations are given by

$$0 = Q_f{}^f = P\Omega - \Sigma_f{}^f, \tag{5.13}$$

$$\begin{aligned} 0 &= Q_f{}^f{}_{cd} \\ &= P\Sigma_{cd} - \chi_f{}^f{}_d{}^h\Sigma_{ch} + \Omega\Phi_f{}^f{}_{cd} - s\varepsilon_{cd} - \tfrac{1}{2}\kappa\Omega^3 T_f{}^f{}_{cd}, \end{aligned} \tag{5.14}$$

$$\begin{aligned} 0 &= P_f{}^f \\ &= Ps + \Phi_f{}^f{}_{cd}\Sigma^{cd} + 2\Lambda\Sigma_f{}^f + \Omega P\Lambda - \tfrac{1}{2}\kappa\Omega^2 T_f{}^f{}_{cd}\Sigma^{cd}, \end{aligned} \tag{5.15}$$

$$0 = P_{bacd} \equiv V_{abcd} - \tfrac{1}{2}\varepsilon_{ad}V_{fbc}{}^f + V^+{}_{badc} - \tfrac{1}{2}\varepsilon_{bc}V^+{}_{fad}{}^f, \tag{5.16}$$

$$0 = -2H_{(abcd)} = P\varphi_{abcd} - D_{(d}{}^h\varphi_{abc)h} + \kappa t^*{}_{(abcd)}. \tag{5.17}$$

The particular combination of zero-quantities on the right of (5.16) has been chosen such that $P_{abcd}$ contains the term $P\Phi_{abcd}$ and has symmetries $P_{abcd} = P_{cbad} = P_{adcb}$. We may decompose $P_{abcd}$ therefore according to (4.13) to obtain the following system of equations which is equivalent to (5.16):

$$\begin{aligned} 0 &= P_{(abcd)} \\ &= P\Phi^*{}_{abcd} - D_{(ab}\Phi^*{}_{cd)} \\ &\quad - \chi_{(a}{}^f{}_d{}^e\Phi_{b|f|c)e} - \chi_{(a}{}^e{}_{|e|}{}^f\Phi_{b|f|cd)} + 2\chi^h{}_{(ab}{}^f\Phi_{|hf|cd)} \\ &\quad - \Sigma^f{}_{(d}\varphi_{abc)f} + \Sigma_{(d}{}^f\varphi^+{}_{abc)f} + \kappa\Omega(t^*{}_{(abcd)} + t^{*+}{}_{(abcd)}), \end{aligned} \tag{5.16a}$$

$$\begin{aligned} 0 &= P_f{}^f{}_{cd} \\ &= P\Phi^*{}_{cd} + D^{hf}\Phi^*{}_{hfcd} - \tfrac{2}{3}D_{cd}\Phi^* \\ &\quad - 2\chi_{hf(d}{}^e\Phi^{hf}{}_{c)e} - 2\chi^{he}{}_e{}^f\Phi_{hf(cd)} - \chi^h{}_{k(d}{}^f\Phi_{|hf|}{}^k{}_{c)} \\ &\quad - \tfrac{1}{2}(\chi_e{}^{feh} + \chi^h{}_e{}^{ef})\Phi_{dfch} - 4D_{cd}\Lambda \\ &\quad - \tfrac{1}{2}\Sigma^{hf}(\varphi_{hfdc} + \varphi^+_{hfdc}) + \tfrac{1}{2}\kappa\Omega(t^*{}_{hcd}{}^h - t^{*+}{}_{hcd}{}^h), \end{aligned} \tag{5.16b}$$

$$\begin{aligned} 0 = \tfrac{1}{3}P_{bacd}h^{bcad} &= \tfrac{1}{3}P\Phi^* + \tfrac{2}{3}D^{hf}\Phi^*{}_{hf} \\ &\quad + \tfrac{2}{3}(\chi^{hkef}\Phi_{hkef} - \chi^h{}_e{}^{ef}\Phi_{hfk}{}^k) + 2P\Lambda, \end{aligned} \tag{5.16c}$$

where vertical strokes indicate that the indices between them are exempt form the symmetrization.

In equations (5.8)–(5.17) the spinor $t_{abcd}$ has been replaced by a spinor $t^*{}_{abcd}$. Its specific form depends on the treatment of the matter fields and will be given explicitly for the case of gauge fields in §7.

Equations (5.13)–(5.17), which will later be supplemented by further equations, will in the following be referred to as the system of propagation equations.

We will now derive a system of subsidiary equations for some of the zero-quantities (5.2)–(5.6), which will be satisfied, if the fields (3.9) are such that they satisfy the propagation equations and the reality conditions stated before. We will go back and forth between spinor and tensor notation to keep the resulting expressions short.

The first three equations are obtained by expressing

$$\nabla_{[\mu}Q_{\nu]}, \quad \nabla_{[\mu}Q_{\nu]\lambda}, \quad \nabla_{[\mu}P_{\nu]}$$

in terms of the zero-quantities and transvecting suitably with $\tau^{\mu}$. The propagation equations are taken into account in the following way. Equation (5.14), which may serve as an example, can be written as $\tau^{\mu}Q_{\mu\nu} \equiv 0$ and thus implies

$$\tau^{\mu}\nabla_{\nu}Q_{\mu\lambda} = -Q_{\mu\lambda}\nabla_{\nu}\tau^{\mu},$$

which allows us to get rid of certain derivatives of the zero-quantities in the subsidiary equations. We obtain:

$$PQ_{\nu} = -Q_{\mu}\nabla_{\nu}\tau^{\mu} - \tau^{\mu}(S_{\mu}{}^{\lambda}{}_{\nu}\nabla_{\lambda}\Omega + Q_{\mu\nu} - Q_{\nu\mu}), \tag{5.18}$$

$$\begin{aligned} PQ_{\nu\lambda} = & -Q_{\mu\lambda}\nabla_{\nu}\tau^{\mu} \\ & +\tau^{\mu}(-S_{\mu}{}^{\delta}{}_{\nu}\nabla_{\delta}\Sigma_{\lambda} - \Sigma_{\delta}K^{\delta}{}_{\lambda\mu\nu} + \Omega U_{\mu\nu\lambda} - P_{[\mu}g_{\nu]\lambda} \\ & \qquad -\kappa\Omega^{2}g_{\lambda[\mu}T_{\nu]\delta}Q^{\delta} + \kappa\Omega^{2}\{t^{*}{}_{\mu\nu\lambda} - t_{\mu\nu\lambda}\}), \end{aligned} \tag{5.19}$$

(5.20)

$$\begin{aligned} PP_{\nu} = & -P_{\mu}\nabla_{\nu}\tau^{\mu} \\ & +\tau^{\mu}(-S_{\mu}{}^{\lambda}{}_{\nu}(\nabla_{\lambda}s + \Omega\nabla_{\lambda}\Lambda) - 2Q_{[\mu}\nabla_{\nu]}\Lambda \\ & \quad + \Sigma^{\lambda}U_{\mu\nu\lambda} - 2Q_{[\mu}{}^{\lambda}S_{\nu]\lambda} + \tfrac{1}{2}\kappa^{2}\Omega^{2}Q_{[\mu}{}^{\lambda}T_{\nu]\lambda} \\ & \quad + \kappa\Omega Q_{[\mu}T_{\nu]\lambda}\Sigma^{\lambda} - \kappa\Omega\Sigma_{[\mu}T_{\nu]\lambda}Q^{\lambda} + \kappa\Omega\Sigma^{\lambda}\{t^{*}_{\mu\nu\lambda} - t_{\mu\nu\lambda}\}), \end{aligned}$$

where $t_{\mu\nu\lambda}$ is given by (2.21).

The only subsidiary equation whose derivation is somewhat complicated is that for $V_{abcd}$. From (3.22) it follows that $V_{f}{}^{f}{}_{cc'}$ is a real covector. This translated into the reality condition

$$V_{f}{}^{f}{}_{cd} = -V^{+}{}_{f}{}^{f}{}_{cd}. \tag{5.21}$$

The propagation equation (5.16)

(5.22) $$V^{+}{}_{badc} = -V_{abcd} + \tfrac{1}{2}\varepsilon_{ad}V_{fbc}{}^{f} + \tfrac{1}{2}\varepsilon_{bc}V^{+}{}_{fad}{}^{f}$$

yields, in view of (5.21) by contracting on $a$ and $b$, an expression for the last term on the right of (5.22). This can be put back into (5.22) to give the propagation equation (5.16) in the form

(5.23) $$V^{+}{}_{badc} = -V_{abcd} + \tfrac{1}{2}\varepsilon_{ad}V_{fbc}{}^{f} + \varepsilon_{bc}(\tfrac{1}{2}V_{fda}{}^{f} - 2V_{f}{}^{f}{}_{ad}).$$

Contraction with respect to $a$ and $d$ then gives

(5.24) $$V_{f}{}^{f}{}_{h}{}^{h} = 0.$$

To derive the desired equations, we study the identity

$$\nabla^{c}{}_{f}V_{cedh} + \nabla_{e}{}^{c}V^{+}{}_{cfhd} = Y_{fedh}$$

(5.25) $$\equiv -\tfrac{i}{2}\tau_{f}{}^{e'}\tau_{h}{}^{d'}\{\nabla^{cc'}U_{aa'bb'ee'}\varepsilon^{aa'bb'}{}_{cc'dd'}\} + \chi^{c}{}_{fh}{}^{k}V_{cedk}$$
$$- \chi_{ef}{}^{gk}V^{+}{}_{gkhd} + \chi_{e}{}^{g}{}_{f}{}^{k}V^{+}{}_{gkhd} + \chi_{e}{}^{g}{}_{h}{}^{k}V^{+}{}_{gfkd},$$

where $\varepsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric Levi-Civita tensor.

Observing (5.24) and the symmetry (5.7) of $V_{abcd}$, we obtain the representation

(5.26) $$V_{abcd} = X_{abcd} + \tfrac{1}{2}\varepsilon_{ad}(X_{bc} - 4Z_{bc}) + \tfrac{1}{2}\{\varepsilon_{a(b}Z_{c)d} + \varepsilon_{d(b}Z_{c)a}\}$$

with symmetric spinor fields

$$X_{abcd} = V_{(abcd)}, \quad X_{bc} = 4V_{f}{}^{f}{}_{bc} - V_{fbc}{}^{f}, \quad Z_{bc} = V_{f}{}^{f}{}_{bc} - \tfrac{1}{2}V_{fbc}{}^{f}.$$

From the left equality of (5.25) and from (5.23) we find the following relations:

(5.27a) $$PX_{ab} + D_{(a}{}^{f}X_{b)f} = -Y_{f}{}^{f}{}_{(ab)} - 2Y_{(a|f|}{}^{f}{}_{b)},$$

(5.27b) $$PX_{abcd} + D_{(ab}Z_{cd)} = -Y_{(abcd)},$$

(5.27c) $$PZ_{ab} - D^{cd}X_{cdab} = \tfrac{1}{2}(Y_{f(ab)}{}^{f} - Y_{(a|f|}{}^{f}{}_{b)}).$$

Using the definition of zero-quantities and in particular the relation

$$\nabla_{\mu}d_{\nu\lambda\rho\delta}\varepsilon^{\mu\rho\delta}{}_{\eta} = \varepsilon^{\rho\delta}{}_{\nu\lambda}\nabla^{\mu}d_{\rho\delta\mu\eta} = \varepsilon^{\rho\delta}{}_{\nu\lambda}(H_{\rho\delta\eta} + \kappa t^{*}_{\rho\delta\eta})$$

we derive

(5.28)
$$\begin{aligned}\nabla^{\mu}U_{\rho\delta\nu}\varepsilon^{\rho\delta}{}_{\mu\eta}&\\ &= (2L_{\pi(\nu}K^{\pi}{}_{\rho)\mu\delta} + S_{\mu}{}^{\pi}{}_{\delta}\nabla_{\pi}L_{\rho\nu})\varepsilon^{\mu\rho\delta}{}_{\eta}\\ &\quad - Q_{\mu}{}^{\pi}d_{\pi\nu\rho\delta}\varepsilon^{\mu\rho\delta}{}_{\eta} - \Sigma^{\pi}\varepsilon^{\rho\delta}{}_{\pi\nu}H_{\rho\delta\eta}\\ &\quad - \kappa\{(\tfrac{1}{2}\Omega^{3}T_{\mu}{}^{\pi}d_{\pi\nu\rho\delta} + \nabla_{\mu}\Omega t^{*}_{\rho\delta\nu} + \Omega\nabla_{\mu}t^{*}_{\rho\delta\nu})\varepsilon^{\mu\rho\delta}{}_{\eta} - \Sigma^{\pi}\varepsilon^{\rho\delta}{}_{\pi\nu}t^{*}_{\rho\delta\eta}\}.\end{aligned}$$

We may now use (5.23), (5.26), (5.28) and the definition of $Y_{fedh}$ by the right-hand side of (5.25), to express in equations (5.27) the right-hand side by the terms in curly brackets appearing on the right of (5.28) and by zero-quantities. Up to the terms in curly brackets, which will be discussed later, the right-hand side of equations (5.27) may then be read as a linear expression of the zero-quantities appearing in (3.12)–(3.23).

Using the decomposition $H_{abce} = H_{(abce)} - \frac{3}{4}\varepsilon_{e(c}H_{ab)f}{}^{f}$ we get the identity

$$PH_{cdf}{}^{f} - D^{h}{}_{(c}H_{d)hf}{}^{f} + 2D^{be}H_{(cdbe)} + 2\chi^{b}{}_{h}{}^{eh}H_{cdbe} = \nabla^{bb'}H_{cdbb'},$$

which simplifies to

$$PH_{cdf}{}^{f} - D^{h}{}_{(c}H_{d)hf}{}^{f} - \tfrac{3}{2}\chi^{b}{}_{h(b}{}^{h}H_{cd)f}{}^{f} = \nabla^{bb'}H_{cdbb'}, \tag{5.29}$$

if the propagation equations (5.17) are taken into account. On the other hand, using the definition (3.18) of $H_{\mu\nu\lambda}$, the symmetries of the tensors involved, and in particular the fact that right and left duals of $d_{\mu\nu\lambda\rho}$ coincide, we obtain

$$\begin{aligned}\nabla^{\lambda}H_{\mu\nu\lambda} = {} & -\tfrac{1}{4}\varepsilon_{\mu\nu}{}^{\alpha\beta}(d_{\alpha\beta\gamma\delta}K^{\gamma}{}_{\varepsilon\eta\pi} + d_{\varepsilon\delta\gamma\beta}K^{\gamma}{}_{\alpha\eta\pi})e^{\varepsilon\delta\eta\pi} \\ & - \tfrac{1}{2}S_{\alpha}{}^{\delta}{}_{\beta}\nabla_{\delta}\, d^{\alpha\beta}{}_{\mu\nu} + \kappa\{\nabla^{\lambda}t^{*}_{\mu\nu\lambda}\}.\end{aligned} \tag{5.30}$$

Since furthermore

$$H_{cdbb'} = \tfrac{1}{2}H_{ch'd}{}^{h'}{}_{bb'}, \tag{5.31}$$

we can express the right-hand side of equation (5.29), again up to the terms arising from the term in curly brackets on the right of (5.30), as a linear function of the zero-quantities.

We will consider the subsidiary equations (5.18)–(5.20), (5.27), (5.29) again in §8, after the other subsystems of the complete subsidiary system have been derived.

## 6. The geometrical gauge freedom

To complete the system of propagation equations for the geometrical quantities (3.9), we have to impose conditions on the coordinates, the frame field, and the Ricci scalar. They have to be given such that we obtain a useful type of propagation equations and that they are suitably adapted to the geometrical situation we want to consider.

For many investigations it is convenient to choose Gauss coordinates, $\tau$ to be tangent to the family of time-like geodesics which are orthogonal to our space-like initial hypersurface $S$, and the frame to be parallely propagated in the direction of $\tau$ and such that condition (4.21) holds. In local considerations (with respect to "conformal time", which may mean "semiglobal" in physical time, see §9) this is always possible, makes the equations simple, and leads to symmetric hyperbolic propagation equations (see, e.g., [13]). The use of Gauss coordinates may sometimes create unnecessary problems, because the geodesic congruence may develop caustics (it must be noted, however, that we are possibly dealing with timelike geodesics with respect to the nonphysical metric, such that our Gauss system may cover regions which are much larger than those covered by "physically" Gauss systems).

A characterization of gauge conditions, which are more flexible and cover essentially all possibilities, has been given in [14] in terms of "gauge source functions". This method will be used here, and the gauge source functions will be related later to some comparison space-time in such a way that we may derive global existence results.

Let $\{c_A\}_{A=0,\cdots,3}$ be an arbitrary frame (which may, but need not, be a coordinate frame) given on some open subset of our manifold $M$, and let $\{\alpha^A\}_{A=0,\cdots,3}$ be the dual forms. The frame $e_{aa'}$ then has an expansion $e_{aa'} = e^A{}_{aa'} c_A$, where the summations convention with respect to indices $A, B, \cdots$ is assumed as in the following. The coefficients $e^A{}_{aa'}$ may be interpreted in two ways. For fixed indices $a, a'$ they may be understood as above as components of $e_{aa'}$ in the basis $c_A$. For fixed $A$ they may be interpreted as the coefficients of the $\alpha^A$ with respect to the dual forms $\sigma^{aa'}$ of the fields $e_{aa'}$, such that $\alpha^A = e^A{}_{aa'}\sigma^{aa'}$. With the latter understanding we have as usual $\nabla_{cc'}\alpha^A = (\nabla_{cc'} e^A{}_{bb'})\sigma^{bb'}$ with

$$\nabla_{cc'} e^A{}_{bb'} = e^B{}_{cc'} c_B(e^A{}_{bb'}) - \Gamma_{cc'}{}^d{}_b e^A_{db'} - \overline{\Gamma}_{cc'}{}^{d'}{}_{b'} e^A{}_{bd'},$$

where $c_B$ is interpreted as the directional derivative operator. Equation (3.4) may then be written as

$$(6.1)\quad S_{aa'}{}^{cc'}{}_{bb'} e^A{}_{cc'} c_A = \{\nabla_{bb'} e^A{}_{aa'} - \nabla_{aa'} e^A{}_{bb'}\} c_A - e^A{}_{aa'} e^B{}_{bb'} [c_A, c_B].$$

Now let $\widehat{\nabla}$ denote the covariant differential operator defined by the Levi-Civita connection arising from an arbitrary chosen, fixed metric $\hat{g}$ on $M$, and denote by $\widehat{\Gamma}_A{}^B{}_C$ its connection coefficients in the basis $c_A$, such that we have

$$(6.2)\qquad \widehat{\nabla}_{c_A} c_B = \widehat{\Gamma}_A{}^D{}_B c_D, \qquad [c_A, c_B] = (\widehat{\Gamma}_A{}^D{}_B - \widehat{\Gamma}_B{}^D{}_A) c_D.$$

Equation (6.1) may then be written as

$$(6.3)\qquad \begin{aligned}\nabla_{bb'}e^A{}_{aa'} - \nabla_{aa'}e^A{}_{bb'} - e^B{}_{aa'}e^D{}_{bb'}(\widehat{\Gamma}_B{}^A{}_D - \widehat{\Gamma}_D{}^A{}_B)\\ = S_{aa'}{}^{cc'}{}_{bb'}e^A{}_{cc'}\\ = S^{cc'}_{ab}e^A{}_{cc'}\varepsilon_{a'b'} + \overline{S}^{cc}_{a'b'}e^A{}_{cc'}\varepsilon_{ab}\\ = S^A{}_{ab}\varepsilon_{a'b'} + \overline{S}^A{}_{a'b'}\varepsilon_{ab}\end{aligned}$$

with $S^A{}_{ab} = S^{cc'}_{ab}e^A{}_{cc'}$ and

$$(6.4)\qquad \begin{aligned}S^A{}_{ab} &= \frac{1}{2}S_{ah'}{}^{cc'}{}_b{}^{h'}e_{cc'}{}^A\\ &= \nabla_{(b}{}^{h'}e_{a)h}{}^A + \tfrac{1}{2}e_{(a}{}^{Bh'}e^C{}_{b)h'}(\widehat{\Gamma}_B{}^A{}_C - \widehat{\Gamma}_C{}^A{}_B).\end{aligned}$$

The part of the derivatives of $e^A{}_{aa'}$ which is not fixed by $S^A{}_{ab}$ is contained in the coordinate gauge source function

$$(6.5)\qquad F^A = \nabla^{aa'}e^A{}_{aa'} = -\delta\alpha^A.$$

Though this has essentially been discussed already in [14], we will indicate again that locally the function $F^A$ can be specified freely, to motivate our later choice of $F^A$. Let $\varphi$ be a diffeomorphism of the manifold $M$ onto the manifold $N$, $\tilde{\alpha}^A$ a basis of forms on $N$ with dual basis $\tilde{c}_A$, $\widetilde{\nabla}$ a covariant Levi-Civita operator arising from some metric $\tilde{h}$ on $N$, $x^\mu$ local coordinates on $M$, and $z^{\mu'}$ local coordinates on $N$, where $\mu, \mu' = 0, \cdots, 3$, with respect to which $\Phi$ has the local expression $z^{\mu'} = z^{\mu'}(x^\nu)$. Assume furthermore that the pullback of the form $\tilde{\alpha}^A$ via $\varphi$ satisfies $\alpha^A = {}^t\varphi\tilde{\alpha}^A$ such that

$$\alpha^A = \tilde{\alpha}^A{}_{\mu'}\frac{\partial z^{\mu'}}{\partial x^\mu}dx^\mu, \quad \text{where } \tilde{\alpha}^A = \tilde{\alpha}^A{}_{\mu'}dz^{\mu'}.$$

Then the condition that $\alpha^A$ satisfies (6.5) takes the form

$$(6.6)\qquad g^{\mu\nu}\nabla_\mu\nabla_\nu z^{\mu'} + c^{\mu'}{}_A\left\{g^{\mu\nu}\frac{\partial\tilde{\alpha}^A{}_{\lambda'}}{\partial z^{\nu'}}\frac{\partial z^{\lambda'}}{\partial x^\mu}\frac{\partial z^{\nu'}}{\partial x^\nu} - F^A\right\} = 0,$$

which for given $F^A = F^A(x^\mu)$ is a semilinear wave equation for $z^{\mu'}(x^\nu)$ and can locally be solved for data suitably specified on some initial surface. Equation (6.6) is the harmonic map equation for $(M, g)$ and $(N, \tilde{h})$ if and only if

$$(6.7)\qquad F^A = g^{\mu\nu}\frac{\partial z^{\mu'}}{\partial x^\mu}\frac{\partial z^{\nu'}}{\partial x^\nu}\widetilde{\nabla}_{\nu'}\tilde{\alpha}_{\mu'}{}^A.$$

In particular, if $M = N$ it follows that the identity map is harmonic for $(M, g)$, $(M, h)$ if and only if

$$F^A = g^{\mu\nu}\widehat{\nabla}_\mu \alpha^A{}_\nu = -\varepsilon^{ab}\varepsilon^{a'b'} e^B{}_{aa'} e^D{}_{bb'} \widehat{\Gamma}_B{}^A{}_D . \tag{6.8}$$

This is the gauge source function which will be considered in §§9 and 10.

Now setting

$$\tau_b{}^{a'} e^A{}_{aa'} = e^A{}_{ab} + \tfrac{1}{2}\varepsilon_{ab}\tau^A \tag{6.9}$$

with

$$e^A{}_{ab} = \tau_{(b}{}^{a'} e_{a)a'}{}^A , \qquad \tau^A = \tau^{aa'} e_{aa'}{}^A ,$$

we get from (6.4)

$$\begin{aligned} S^A{}_{ab} = {} & \tfrac{1}{2}Pe^A{}_{ab} + D_{(b}{}^f e_a{}^A{}_{)f} - \tfrac{1}{2}D_{ab}\tau^A + e^A{}_{f(a}\chi_{b)h}{}^{fh} \\ & + \tfrac{1}{2}\chi_{(a}{}^h{}_{b)h}\tau^A - \tfrac{1}{2}(e^B{}_{af} e^C{}_b{}^f + e^B{}_{ab}\tau^C)(\widehat{\Gamma}_B{}^A{}_C - \widehat{\Gamma}_C{}^A{}_B), \end{aligned} \tag{6.10}$$

$$\begin{aligned} S^A{}_{ab}{}^+ = {} & -\tfrac{1}{2}Pe^A{}_{ab} + D_{(b}{}^f e_a{}^A{}_{)f} + \tfrac{1}{2}D_{ab}\tau^A - e^A{}_{hf}\chi^h{}_{(ba)}{}^f \\ & - \tfrac{1}{2}\chi_{f(ab)}{}^f \tau^A - \tfrac{1}{2}(e^B{}_{af} e^C{}_b{}^f - e^B{}_{ab}\tau^C))(\widehat{\Gamma}_B{}^A{}_C - \widehat{\Gamma}_C{}^A{}_B), \end{aligned} \tag{6.11}$$

which entail the constraint equation

$$\begin{aligned} 0 & = S^A{}_{ab} + S^A{}_{ab}{}^+ \\ & = 2D_{(b}{}^f e_a{}^A{}_{)f} + e_f{}^A{}_{(a}\chi_{b)h}{}^{fh} - e^A{}_{hf}\chi^h{}_{(ab)}{}^f \\ & \quad + \tfrac{1}{2}(\chi_{(a}{}^h{}_{b)h} + \chi^h{}_{(ab)h})\tau^A - e^B{}_{af} e^C{}_b{}^f(\widehat{\Gamma}_B{}^A{}_C - \widehat{\Gamma}_C{}^A{}_B) \end{aligned} \tag{6.12}$$

and the propagation equation

$$\begin{aligned} 0 & = S^A{}_{ab} - S^{A\,+}_{ab} \\ & = Pe^A{}_{ab} - D_{ab}\tau^A + e^A{}_{f(a}\chi_{b)h}{}^{fh} + e^A{}_{hf}\chi^h{}_{(ba)}{}^f \\ & \quad + \tfrac{1}{2}(\chi_{(a}{}^h{}_{b)h} - \chi^h{}_{(ab)h})\tau^A - e^B{}_{ab}\tau^C(\widehat{\Gamma}_B{}^A{}_C - \widehat{\Gamma}_C{}^A{}_B), \end{aligned} \tag{6.13}$$

which is implemented by equation (6.5) and may be written as

$$0 = \tfrac{1}{2}P\tau^A + D^{ab}e^A{}_{ab} + \tfrac{1}{2}\chi_{fh}{}^{fh}\tau^A + \chi^a{}_h{}^{fh} e^A{}_{af} - F^A . \tag{6.14}$$

To discuss the propagation equations arising from (3.13) in a similar fashion, we introduce some notation. Set

$$\begin{aligned} \nabla_{aa'}\Gamma_{bb'cd} \equiv {} & e_{aa'}(\Gamma_{bb'cd}) - \Gamma_{aa'}{}^f{}_b\Gamma_{fb'cd} - \overline{\Gamma}_{aa'}{}^{f'}{}_{b'}\Gamma_{bf'cd} \\ & - \Gamma_{aa'}{}^f{}_c\Gamma_{bb'fd} - \Gamma_{aa'}{}^f{}_d\Gamma_{bb'cf} , \end{aligned}$$

which is the expression obtained for the covariant derivative of a spinor field, which in the given frame happens to have components identical with $\Gamma_{aa'bc}$. Then (3.7) gives

$$
\begin{aligned}
\gamma^{ab}{}_{cd} &= \tfrac{1}{2} r^{ab}{}_{ch'd}{}^{h'} \\
&= \nabla_{(c}{}^{e'}\Gamma_{d)e'}{}^{ab} + \Gamma_{(c}{}^{e'f(a}\Gamma_{d)e'}{}^{b)}{}_{f} - S_{cd}^{ff'}\Gamma_{ff'}{}^{ab}, \\
r^{ab}{}_{c'd'} &= \tfrac{1}{2} r^{ab}{}_{hc'}{}^{h}{}_{d'} \\
&= \nabla^{e}{}_{(c'}\Gamma_{d')e}{}^{ab} + \Gamma^{e}{}_{(c'}{}^{f(a}\Gamma_{d')e}{}^{b)}{}_{f} - \overline{S}_{c'd'}^{ff'}\Gamma_{ff'}{}^{ab}.
\end{aligned}
\tag{6.15}
$$

The part of the derivatives of $\Gamma_{aa'bc}$ which is not fixed by these expressions is contained in the frame gauge source functions

$$
F^{ab} = \nabla^{dd'}\Gamma_{dd'}{}^{ab} = F^{(ab)}. \tag{6.16}
$$

It has been shown in [14] that these functions can locally be chosen arbitrarily and that their choice implies a semilinear wave equation for the frame field $e_{aa'}$. Setting

$$
\begin{aligned}
&\tau_{d}{}^{c'}\Gamma_{cc'ab} = \Gamma_{cdab} = \Gamma_{(cd)ab} + \tfrac{1}{2}\varepsilon_{cd}\Gamma_{ab}, \\
&\overset{*}{r}_{abcd} = \tau_{c}{}^{c'}\tau_{d}{}^{d'} r_{abc'd'}, \qquad \overset{*}{R}_{abcd} = \tau_{c}{}^{c'}\tau_{d}{}^{d'}\tfrac{1}{2}R_{abhc'}{}^{h}{}_{d'} = -\Phi_{acbd}, \\
&R_{abcd} = \tfrac{1}{2}R_{abch'd}{}^{h'} = -\Omega\varphi_{abcd} - \varepsilon_{a(c}\varepsilon_{d)b}\Lambda, \\
&K_{abcedf} = \tau_{e}{}^{c'}\tau_{f}{}^{d'}K_{abcc'dd'}, \qquad S_{ab}^{cd} = -\tau^{d}{}_{d'}S_{ab}^{cd'}
\end{aligned}
$$

we have

$$
K_{abcedf} = K_{abcd}\varepsilon_{ef} + \overset{*}{K}_{abef}\varepsilon_{cd} \tag{6.17}
$$

with

$$
\begin{aligned}
K^{ab}{}_{cd} = {} & \tfrac{1}{2}P\Gamma_{(cd)}{}^{ab} + \tfrac{1}{2}(D_{c}{}^{f}\Gamma_{(df)}{}^{ab} + D_{d}{}^{f}\Gamma_{(cf)}{}^{ab} - D_{cd}\Gamma^{ab}) \\
& + \chi_{(c|h|}{}^{fh}\Gamma_{d)f}{}^{ab} + \Gamma_{(c}{}^{ef(a}\Gamma_{d)e}{}^{b)}{}_{f} - S_{dc}^{fe}\Gamma_{fe}{}^{ab} \\
& + \Omega\varphi^{ab}{}_{cd} - \varepsilon^{a}{}_{(c}\varepsilon^{b}{}_{d)}\Lambda,
\end{aligned}
\tag{6.18}
$$

(6.19)

$$
\begin{aligned}
K^{*ab}{}_{cd} = {} & -\tfrac{1}{2}P\Gamma_{(cd)}{}^{ab} + \tfrac{1}{2}(D_{c}{}^{f}\Gamma_{(df)}{}^{ab} + D_{d}{}^{f}\Gamma_{(cf)}{}^{ab} + D_{cd}\Gamma^{ab}) \\
& - \chi^{e}{}_{(cd)}{}^{f}\Gamma_{ef}{}^{ab} - \Gamma_{e(c}{}^{f(a}\Gamma^{|e|}{}_{d)}{}^{b)}{}_{f} + S^{+}{}_{dc}^{ef}\Gamma_{fe}{}^{ab} + \Phi^{a}{}_{c}{}^{b}{}_{d}.
\end{aligned}
$$

From these zero-quantities we get the constraint equations

$$
0 = K^{ab}{}_{cd} + K^{*ab}{}_{cd}, \tag{6.20}
$$

which in the case that $\tau$ is orthogonal to a hypersurface $S$ are the Gauss-Codazzi equations, and the propagation equations

$$
\begin{aligned}
0 &= K^{ab}{}_{cd} - K^{*ab}{}_{cd} \\
&= P\Gamma_{(cd)}{}^{ab} - D_{cd}\Gamma^{ab} + \chi_{(c|h|}{}^{fh}\Gamma_{d)f}{}^{ab} \\
&\quad + \chi^{e}{}_{(cd)}{}^{f}\Gamma_{ef}{}^{ab} + \Gamma_{(c}{}^{ef(a}\Gamma_{d)e}{}^{b)}{}_{f} \\
&\quad + \Gamma_{e(c}{}^{f(a}\Gamma^{|e|}{}_{d)}{}^{b)}{}_{f} \\
&\quad + \Omega\varphi^{ab}{}_{cd} - \varepsilon^{a}{}_{(c}\varepsilon^{b}{}_{d)}\Lambda - \Phi^{a}{}_{c}{}^{b}{}_{d},
\end{aligned} \tag{6.21}
$$

which are implemented by equation (6.16) and may be written as

$$
0 = \tfrac{1}{2}P\Gamma^{ab} + D^{ch}\Gamma_{(ch)}{}^{ab} + \chi^{d}{}_{h}{}^{fh}\Gamma_{df}{}^{ab} - F^{ab}. \tag{6.22}
$$

The terms arising from the torsion tensor in equations (6.18), (6.19) drop out in (6.21) because we have assumed that the propagation equation (6.13) is satisfied together with (6.21). Equations (6.13), (6.14), (6.21), (6.22) constitute the system of propagation equations for the frame and connection coefficients.

To derive the subsidiary equations for $S^{\lambda}{}_{\mu\nu}$ and $K^{\lambda}{}_{\mu\nu\rho}$, we use the Bianchi identities

$$
\sum_{(\mu\nu\lambda)} \nabla_{\mu}S^{\rho}{}_{\nu\lambda} = \sum_{(\mu\nu\lambda)} (r^{\rho}{}_{\lambda\mu\nu} + S^{\delta}{}_{\mu\lambda}S^{\rho}{}_{\delta\nu}), \tag{6.23}
$$

$$
\sum_{(\mu\nu\lambda)} \nabla_{\mu}r^{\rho}{}_{\eta\nu\lambda} = \sum_{(\mu\nu\lambda)} (S^{\delta}{}_{\nu\mu}r^{\rho}{}_{\eta\delta\lambda}), \tag{6.24}
$$

where the sum is formed after cyclic permutation of the indices $\mu$, $\nu$, $\lambda$. Writing

$$
S_{ac}{}^{A}{}_{bd} = \tau_{c}{}^{a'}\tau_{d}{}^{b'}S_{aa'}{}^{ee'}{}_{bb'}e^{A}{}_{ee'} = S^{A}{}_{ab}\varepsilon_{cd} + S^{A}{}_{cd}{}^{+}\varepsilon_{ab}
$$

and assuming that equation (6.13) holds, we find

$$
PS^{A}{}_{ab} = -\tfrac{i}{2}\nabla_{ef}S_{cd}{}^{A}{}_{gh}\varepsilon^{efcdgh}{}_{(ab)}, \tag{6.25}
$$

where $\varepsilon_{efcdghab} = \tau_{f}{}^{f'}\tau_{d}{}^{d'}\tau_{h}{}^{h'}\tau_{b}{}^{b'}\varepsilon_{ef'cd'gh'a'b'}$.

The right-hand side of equation (6.25) may be evaluated by using (6.23) and the identity $\sum_{(\mu\nu\lambda)} R^{\rho}{}_{\lambda\mu\nu} = 0$ to obtain an expression which may be considered (somewhat arbitrarily) as a linear function of the zero-quantities with coefficients which are space-time functions.

Observing (6.17) and assuming that the propagation equations (6.18) are satisfied, we find a similar way

$$
PK_{abcd} = -\tfrac{i}{2}\nabla_{ef}K_{abghij}\varepsilon^{efghij}{}_{(cd)}. \tag{6.26}
$$

Using the identity

$$\nabla_{ee'} R_{abcc'dd'} \varepsilon^{ee'cc'dd'}{}_{ff'} = 2i\{Q^e{}_{f'} \varphi_{abef} + \Omega H_{abff'} - V_{fabf'}\}$$

and the identity (6.24), the right-hand side of (6.26) may again be written as a linear function of the zero-quantities.

## 7. The matter field equations

In this section a method to include matter fields into the discussion of the conformal field equations will be illustrated by working out the details for the case of Maxwell and Yang-Mills. Thus, we assume that $M$ is the base manifold of a principal bundle $(P, M, \pi)$ with bundle space $P$, projection $\pi$, and structure group $G$, where $G$ is a group $U(k)$ for some $k \geq 1$, any compact semisimple Lie group, or a product of a finite number of such groups. With respect to some local section of $P$ we may pull back the principal connection form and the associated curvature form to the domain of definition $V'$ of the section. Let $A_\mu$ denote the gauge potential and $F_{\mu\nu}$ the gauge field so obtained, both being considered as considered as functions on $V'$ which take values in the Lie algebra $\mathfrak{g}$ of $G$. We think of these fields as being given with respect to a real representation of $\mathfrak{g}$ such that we are dealing with real fields. The basic equations to be solved are given by

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu] - F_{\mu\nu} = 0, \tag{7.1}$$

$$\boldsymbol{\nabla}^\mu F_{\mu\nu} \equiv \nabla^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0, \tag{7.2}$$

where $\boldsymbol{\nabla}$ denotes the gauge and space-time covariant derivative, and the square brackets denote the product in $\mathfrak{g}$. For our purpose an important property of equations (7.1), (7.2) is their conformal invariance. If $\tilde{g}$ is related to $g$ by the rescaling (2.2) and $\tilde{\nabla}$ denotes the Levi-Civita covariant differential operator associated with $\tilde{g}$, then any solution $A_\mu$, $F_{\mu\nu}$ of (7.1) (7.2) is also a solution of the same equations with $g$, $\nabla$ being replaced by $\tilde{g}$, $\tilde{\nabla}$.

The energy momentum tensor for the gauge field $F_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left\{\frac{1}{4} g_{\mu\nu}(F_{\lambda\rho}/F^{\lambda\rho}) - (F_{\mu\lambda}/F_\nu{}^\lambda)\right\}, \tag{7.3}$$

where $(\ /\ )$ denotes a scalar product on $\mathfrak{g}$ which is invariant under the adjoint representation such that, in particular,

$$([u, v]/w) + (v/[u, w]) = 0 \quad \text{for } u, v, w \in \mathfrak{g}, \tag{7.4}$$

and which is the "usual" product in the case of Maxwells equations. The tensor (7.3) is trace-free and rescales according to (2.3) under conformal rescalings (2.2). As a consequence of (7.1) we note the Bianchi identity

$$(7.5) \qquad \sum_{(\mu\nu\lambda)} \bar{\nabla}_\mu F_{\nu\lambda} = 0 .$$

We translate the gauge field equations now into spinor notation. Since $F_{\mu\nu}$ is antisymmetric, we can write

$$(7.6) \qquad F_{aa'bb'} = \varphi_{ab}\varepsilon_{a'b'} + \bar{\varphi}_{a'b'}\varepsilon_{ab}$$

with a spinor field $\varphi_{ab} = \varphi_{(ab)}$, which takes values in the complexification of $\mathfrak{g}$. Using (7.5), (7.6) and introducing zero-quantities $l_{aa'bb'}$, $H_{a'b}$ in the same manner as before, equations (7.1), (7.2) can be written as

$$(7.7) \qquad 0 = l_{aa'bb'} \equiv \nabla_{aa'}A_{bb'} - \nabla_{bb'}A_{aa'} + [A_{aa'}, A_{bb'}] - F_{aa'bb'},$$

$$(7.8) \qquad 0 = H_{a'b} = \bar{\nabla}^a{}_{a'}\varphi_{ab} = \nabla^a{}_{a'}\varphi_{ab} + [A^a{}_{a'}, \varphi_{ab}]$$

and the energy momentum tensor is given by

$$(7.9) \qquad T_{aa'bb'} = \frac{1}{2\pi}(\varphi_{ab}/\bar{\varphi}_{a'b'}) .$$

In the following we will assume that in the conformal field equations $\kappa$ is replaced by $\frac{\kappa}{2\pi}$ and the factor $\frac{1}{2\pi}$ on the right of (7.9) will be dropped.

Proceeding as before we convert primed indices into unprimed indices and write $A_{ab} = \tau_b{}^{a'}A_{aa'}$ and

$$\begin{aligned} l_{acbd} &= \tau_c{}^{a'}\tau_d{}^{b'}l_{aa'bb'} \\ &= \nabla_{ac}A_{bd} - \nabla_{bd}A_{ac} - \chi_{acd}{}^f A_{bf} + \chi_{bdc}{}^f A_{af} \\ &\quad + [A_{ac}, A_{bd}] + \varphi_{ab}\varepsilon_{cd} + \varphi^+{}_{cd}\varepsilon_{ab} \\ &= l_{ab}\varepsilon_{cd} + l^+{}_{cd}\varepsilon_{ab} \end{aligned}$$

with

$$(7.11) \qquad \begin{aligned} l_{ab} \equiv \tfrac{1}{2}l_{afb}{}^f &= -\tfrac{1}{2}PA_{(ab)} - D^f{}_{(a}A_{b)f} + A_{(a}{}^f\chi_{b)h}{}^h{}_t \\ &\quad + \tfrac{1}{2}[A_{af}, A_b{}^f] + \varphi_{ab}, \\ l^+{}_{ab} \equiv \tfrac{1}{2}l_{fa}{}^f{}_b &= \tfrac{1}{2}PA_{(ab)} + D_{f(a}A^f{}_{b)} + \chi_{h(ab)f}A^{hf} \\ &\quad + \tfrac{1}{2}[A_{fa}, A^f{}_b] + \varphi^+{}_{ab}, \end{aligned}$$

and finally

$$(7.12) \qquad \begin{aligned} H_{ab} = \tau_a{}^{a'}H_{a'b} &= \nabla^f{}_a\varphi_{bf} + [A^f{}_a, \varphi_{bf}] \\ &= -\tfrac{1}{2}P\varphi_{ab} + D^f{}_a\varphi_{bf} + [A^f{}_a, \varphi_{bf}] . \end{aligned}$$

Equation (7.11) entails propagation equation only for the symmetric part of $A_{ab}$. The antisymmetric part is fixed by choosing arbitrary the $\mathfrak{g}$-valued gauge source function (cf. [14])

$$B = \nabla^{aa'} A_{aa'} = \nabla^{ac} A_{ac} + \chi^{a}{}_{f}{}^{fc} A_{ac}. \tag{7.13}$$

Setting

$$A_{ab} = \alpha_{ab} + \alpha \varepsilon_{ab} \quad \text{with } \alpha_{ab} = \alpha_{(ab)}, \tag{7.14}$$

we deduce from (7.7), (7.8), resp. (7.11), (7.12), the constraint equations

$$\begin{aligned} 0 = & -(l_{ab} + l^{+}{}_{ab}) \\ & = 2D^{f}{}_{(a}\alpha_{b)f} - \alpha_{(a}{}^{f}\chi_{b)h}{}^{h}{}_{f} - \chi_{h(ab)f}\alpha^{hf} \\ & \quad - [\alpha_{af}, \alpha_{b}{}^{f}] - \varphi_{ab} - \varphi^{+}{}_{ab} - \alpha(\chi_{(a|h|}{}^{h}{}_{b} + \chi_{h(ab)}{}^{h}), \end{aligned} \tag{7.15}$$

$$0 = H^{f}{}_{f} = D^{hf}\varphi_{hf} + [\alpha^{hf}, \varphi_{hf}], \tag{7.16}$$

and with (7.13) the propagation equations

$$0 = 2\{P\alpha + D^{hf}\alpha_{hf} + \chi_{hk}{}^{k}{}_{f}A^{hf} - B\}, \tag{7.17}$$

$$\begin{aligned} 0 = l^{+}{}_{ab} - l_{ab} = P\alpha_{ab} - 2D_{ab}\alpha + \chi_{h(ab)f}\alpha^{hf} + \alpha_{(a}{}^{f}\chi_{b)hf}{}^{h} \\ + \varphi^{+}{}_{ab} - \varphi_{ab} + \alpha(\chi_{f(ab)}{}^{f} - \chi_{a|f|}{}^{f}{}_{b)}) - 2[\alpha_{ab}, \alpha], \end{aligned} \tag{7.18}$$

$$0 = -2H_{(ab)} = P\varphi_{ab} - 2D^{f}{}_{(a}\varphi_{b)f} - 2[A^{f}{}_{a}, \varphi_{bf}]. \tag{7.19}$$

To obtain solutions of the conformal Einstein equations with the energy momentum tensor given by (7.9), we should solve the coupled system consisting of equations (5.13)–(5.17), (6.13), (6.14), (6.21), (6.22), (7.17)–(7.19). There is, however, a slight difficulty.

Let $\psi_{ah'bc}$ be a spinor field which takes values in $\mathfrak{g} \otimes \mathbb{C}$ with

$$\psi_{a}{}^{h'}{}_{bc} = \psi_{(a}{}^{h'}{}_{bc)} \tag{7.20}$$

and define the spinor field $t^{*}_{abcc'}$ by

$$2t^{*}_{abcc'} = \Omega(\psi_{ah'bc}/\bar{\varphi}^{h'}{}_{c'}) - 3\Sigma_{(a}{}^{h'}(\varphi_{bc)}/\bar{\varphi}_{h'c}). \tag{7.21}$$

Since

$$\nabla_{aa'}T_{bb'cc'} = (\nabla_{aa'}\varphi_{bc}/\bar{\varphi}_{b'c'}) + (\varphi_{bc}/\nabla_{aa'}\bar{\varphi}_{b'c'}), \tag{7.22}$$

we find in the present situation that the spinor field $t_{abcc'}$ given by (3.21) satisfies

$$t_{abcc'} = t^*_{abcc'} \tag{7.23}$$

if $H_{a'b} = 0$ and

$$\psi_a{}^{b'}{}_{bc} = \nabla_{(a}{}^{b'}\varphi_{bc)}\,. \tag{7.24}$$

Thus, on the right-hand sides of equations (5.16), (5.17) first-order derivatives of our basic unknown $\varphi_{ab}$ appear, which cannot be eliminated by using (7.12). We shall deal with this difficulty by deriving for the derivatives (7.24) suitable field equations. Though the idea is simple, its realization, which includes the discussion of an extended subsidiary system, entails some lengthy algebra. The derivation of the field equations for $\psi_{ah'bc}$ will be given such that some of the equations encountered on the way may be used later when we discuss the subsidiary equations.

For any symmetric $(\mathfrak{g}\otimes\mathbb{C})$-valued spinor field $\varphi_{ab}$ we have the identity

$$\begin{aligned} 0 = {} & W_{abcdef} \\ \equiv {} & \nabla_{ab}\nabla_{cd}\varphi_{ef} - \nabla_{cd}\nabla_{ab}\varphi_{ef} \\ & - \{\chi_{abd}{}^h\nabla_{ch}\varphi_{ef} - \chi_{cdb}{}^h\nabla_{ah}\varphi_{ef} + 2\varphi_{h(f}r^h{}_{e)abcd} \\ & \quad - S_{ab}{}^{hk}{}_{cd}\nabla_{hk}\varphi_{ef} + [l_{abcd}, \varphi_{ef}] \\ & \quad + [\varphi_{ac}, \varphi_{ef}]\varepsilon_{bd} + [\varphi^+{}_{bd}, \varphi_{ef}]\varepsilon_{ac}\}\,. \end{aligned} \tag{7.25}$$

Assuming that the $\varphi_{ab}$-propagation equations (7.19) are satisfied we have a decomposition

$$\nabla_a{}^b\varphi_{cd} = \nabla_{(a}{}^b\varphi_{cd)} + \tfrac{1}{3}\mu h_a{}^b{}_{cd} \quad \text{with } \mu = H^f{}_f\,. \tag{7.26}$$

Now let $\psi_{ah'bc}$ be as in (7.20). Assuming for a moment that the relation (7.24) is true and that the gauge field equations (7.7), (7.8) as well as the conformal Einstein equations (3.12)–(3.16), (3.21)–(3.23) are satisfied, the identity (7.25) takes the form

$$\begin{aligned} 0 = w_{abcdef} \equiv {} & \nabla_{ab}\psi_{cdef} - \nabla_{cd}\psi_{abef} \\ & - \chi_{abd}{}^h\psi_{chef} + \chi_{cdb}{}^h\psi_{ahef} - 2\varphi_{h(f}R^h{}_{e)abcd} \\ & - [\varphi_{ac}, \varphi_{ef}]\varepsilon_{bd} - [\varphi^+{}_{bd}, \varphi_{ef}]\varepsilon_{ac}\,, \end{aligned} \tag{7.27}$$

where we have introduced a new zero-quantity $w_{abcdef}$.

Equation (7.27) is the desired field equation for the additional unknown $\psi_{cdef}$, besides which we have to satisfy the equation

$$0 = \omega_{abcd} \equiv \nabla_{ab}\varphi_{cd} - \psi_{abcd}\,, \tag{7.28}$$

which may be considered as a constraint.

Because of (7.20) the field $\psi_{abcd}$ may be represented in the form

$$\psi_{abcd} = \nu_{abcd} - \tfrac{3}{2}\varepsilon_{b(a}\nu_{cd)}, \tag{7.29}$$

where

$$\nu_{abcd} = \psi_{(abcd)}, \qquad \nu_{ab} = \tfrac{1}{2}\psi_{f}{}^{f}{}_{ab} = \nu_{(ab)}.$$

Furthermore we may write

$$w_{abcdef} = w_{acef}\,\varepsilon_{bd} + \hat{w}_{bdef}\,\varepsilon_{ac} \tag{7.30}$$

with

$$w_{acef} = w_{(ac)(ef)} = \tfrac{1}{2}w_{ahc}{}^{h}{}_{ef}, \qquad \hat{w}_{bdef} = \hat{w}_{(bd)(ef)} = \tfrac{1}{2}w_{hb}{}^{h}{}_{def'}.$$

As propagation equations for $\psi_{abcd}$ we now take

$$0 = -w_{(abcd)} + \hat{w}_{(abcd)} = \not{P}\nu_{abcd} - 2\not{D}_{(ab}\nu_{cd)} + \cdots, \tag{7.31a}$$

$$\begin{aligned} 0 &= 6w_{h(ab)}{}^{h} - 2\hat{w}_{h(cb)}{}^{h} \\ &= 4\not{P}\nu_{ab} + 2\not{D}^{ef}\nu_{abef} - 6\not{D}_{(a}{}^{f}\nu_{b)f} + \cdots, \end{aligned} \tag{7.31b}$$

where only the principal parts of the equations have been written out, the dots representing zeroth-order quantities.

We will now derive the subsidiary system for the zero-quantities arising from the matter fields, assuming that the unknowns $\alpha$, $\alpha_{ab}$, $\varphi_{ab}$, $\psi_{abcd}$ satisfy the propagation equations (7.17)–(7.19), (7.31). Using the expansion of $l_{abcd}$ on the right of (7.10) and the propagation equation (7.18), we obtain

$$Pl_{ab} - l_{(a}{}^{h}\psi_{b)fh}{}^{f} - l_{hf}\psi_{(a}{}^{h}{}_{b)}{}^{f} = \tfrac{1}{2}\nabla_{hh'}l_{ff'kk'}\varepsilon^{hh'ff'kk'}{}_{(a}{}^{d'}\tau_{b)d'}, \tag{7.32}$$

while on the other hand we find from the definition (7.7) of $l_{\mu\nu}$

$$\nabla_{\mu}l_{\nu\lambda}\varepsilon^{\mu\nu\lambda}{}_{\rho} = -\{\nabla_{\mu}F_{\nu\lambda} + A_{\delta}K^{\delta}{}_{\mu\nu\lambda} + S^{\delta}{}_{\mu\nu}\nabla_{\delta}A_{\lambda}\}\varepsilon^{\mu\nu\lambda}{}_{\rho}, \tag{7.33}$$

where the identity $R^{\delta}{}_{\mu\nu\lambda}\varepsilon^{\mu\nu\lambda}{}_{\eta} = 0$ has been used. By (7.28) the term $\nabla_{\mu}F_{\nu\lambda}$ can be expressed in terms of $\psi_{abcd}$ and the zero-quantity $\omega_{abcd}$. We find that the terms arising from $\psi_{abcd}$ drop out in (7.33) and the right-hand side of this equation can be read as a linear function of the zero-quantities.

Because of (7.19) we have the decomposition

$$\omega_{abcd} = \mu_{abcd} - \tfrac{3}{2}\varepsilon_{b(a}\mu_{cd)} + \tfrac{1}{3}\mu h_{abcd} \tag{7.34}$$

with

$$\mu_{abcd} = \nabla_{(ab}\varphi_{cd)} - \psi_{(abcd)}, \qquad \mu_{ab} = \tfrac{1}{2}(\nabla_f{}^f \varphi_{ab} - \psi_f{}^f{}_{ab}).$$

Setting

$$w_{acef} = \tfrac{1}{2} W_{ahc}{}^h{}_{ef}, \qquad \widehat{W}_{bdef} = \tfrac{1}{2} W_{hb}{}^h{}_{def},$$

where $W_{abcdef}$ is given by the right of (7.25), we deduce from (7.26), (7.27), (7.31) the equations

(7.35a)

$$\begin{aligned} \not{P}\mu_{abcd} - 2\not{D}_{(ab}\mu_{cd)} + \cdots &= -W_{(abcd)} + w_{(abcd)} + \widehat{W}_{(abcd)} - w_{(abcd)} \\ &= 0, \end{aligned}$$

(7.35b)
$$\begin{aligned} &4\not{P}\mu_{ab} + 2\not{D}^{ef}\mu_{abef} - 6\not{D}_{(a}{}^f\mu_{b)f} + \cdots \\ &\quad = 6(W_{h(ab)}{}^h - w_{h(ab)}{}^h) - 2(\widehat{W}_{h(ab)}{}^h - \hat{w}_{h(ab)}{}^h) = 0, \end{aligned}$$

(7.35c)
$$\not{P}\mu + \cdots = -2W_{hf}{}^{hf} = 0,$$

where again only the principal part has been written out. From (7.25), (7.27) it can be seen that the part of equations (7.35) indicated by the dots can be considered again as a linear function of the zero-quantities.

Finally we derive the subsidiary equation for the zero-quantity $w_{abcdef}$. By direct calculation we find $w_{bf}{}^{bf} = 0$. Due to the symmetries (7.30) of $\hat{w}_{bdef}$, $w_{acef}$ and equations (7.31) it is sufficient to derive equations for $w^*{}_{abcd} = \hat{w}_{(abcd)}$, $w^*{}_{ab} = \hat{w}_{h(ab)}{}^h$, and $w^* = \hat{w}_{hf}{}^{hf}$. Using the representation (7.30) and equations (7.31) we obtain:

(7.36a)
$$3\not{P}w^*{}_{rsef} + \not{D}_{(rs}w^*{}_{ef)} = -i\tfrac{3}{2}\nabla_{gh}w_{abcd(ef}\varepsilon^{ghabcd}{}_{rs)},$$

(7.36b)
$$\begin{aligned} &\tfrac{4}{3}\not{P}w^*{}_{sf} - \not{D}^{hk}w^*{}_{hksf} + \tfrac{1}{2}\not{D}_{sf}w^* \\ &\quad = -i\nabla_{gh}w_{abcdk(f}\varepsilon^{ghabcdk}{}_{s)} + \tfrac{i}{4}\nabla_{gh}w_{abcd(sf)}\varepsilon^{ghabcdk}{}_k, \end{aligned}$$

(7.36c)
$$\tfrac{3}{8}\not{P}w^* - \tfrac{1}{2}\not{D}^{hk}w^*{}_{hk} = -i\tfrac{3}{8}\nabla_{gh}w_{abcdef}\varepsilon^{ghabcdef}.$$

Using on the other hand the definition of $w_{ahcdef}$ by (7.27), one obtains

$$\begin{aligned} &\nabla_{hh'}w_{aa'bb'ef}\varepsilon^{hh'aa'bb'}{}_{rr'} \\ &\quad = \{2\nabla_{hh'}\nabla_{aa'}\psi_{hh'ef} - \nabla_{hh'}(2\varphi_{k(f}R^k{}_{e)aa'bb'} \\ &\qquad\qquad + [F_{aa'bb'}, \varphi_{ef}])\}\varepsilon^{hh'aa'bb'}{}_{rr'}. \end{aligned}$$

Adding to this the identity

$$0 = \{[\mathbb{\nabla}_{hh'}\mathbb{\nabla}_{aa'} - \mathbb{\nabla}_{aa'}\mathbb{\nabla}_{hh'}]\nabla_{bb'}\varphi_{ef} - \nabla_{hh'}[\nabla_{aa'}\mathbb{\nabla}_{bb'} - \mathbb{\nabla}_{bb'}\mathbb{\nabla}_{aa'}]\varphi_{ef}\}\varepsilon^{hh'aa'bb'}{}_{rr'},$$

where the square brackets indicate the two different ways in which the three-fold covariant derivatives are to be evaluated, yields

$$\begin{aligned}
&\mathbb{\nabla}_{hh'}w_{aa'bb'ef}\varepsilon^{hh'aa'bb'}{}_{rr'} \\
&\quad = \{-2\nabla_{hh'}\nabla_{aa'}\omega_{cc'ef} - [l_{hh'aa'}\omega_{cc'ef}] + [F_{hh'aa'}, \omega_{cc'ef}] \\
(7.37)\qquad &\qquad + 2\mathbb{\nabla}_{gh'}\varphi_{h(f}K^{g}{}_{e)aa'cc'} - \nabla_{hh'}S_{aa'}{}^{pp'}{}_{cc'}\nabla_{pp'}\varphi_{ef} \\
&\qquad + 2\varphi_{k(f}\nabla_{|hh'|}r^{k}{}_{e)aa'cc'} - 2\varphi_{k(f}\nabla_{|hh'|}R^{k}{}_{e)aa'cc'} \\
&\qquad - S_{aa'}{}^{pp'}{}_{cc'}\mathbb{\nabla}_{hh'}\nabla_{pp'}\varphi_{ef}\}\varepsilon^{hh'aa'cc'}{}_{rr'}.
\end{aligned}$$

We have

$$(7.38)\quad \nabla_{hh'}R_{efaa'cc'}\varepsilon^{hh'aa'cc'}{}_{rr'} = 2i\{\Omega H_{efrr'} - U_{r}{}^{h'}{}_{er'fh'} + Q^{h}{}_{r'}\varphi_{efrh}\}.$$

Using the Bianchi identities (6.23), (6.24) and Ricci identity to get rid of the second-order derivatives of $\omega_{cc'ef}$ and the derivatives of the torsion and curvature spinors in (7.37) we can with (7.38) write the right-hand sides of equations (7.36) as linear functions of the zero-quantities.

The equations (7.32), (7.35), (7.36) constitute the desired subsidiary system. For the matter fields which we have studied here, we assume that in the propagation equations (5.16), (5.17) the spinor field $t^{*}{}_{abcc'}$ is given by (7.21). We may then complete the discussion of the subsidiary system for the geometrical quantities. In equations (5.19), (5.20) the terms on the right-hand side in curly brackets may obviously be expressed as linear functions of the zero-quantities. For the terms in curly brackets on the right of (5.28), (5.30), this can also be shown by a straightforward though somewhat lengthy calculations, in which the zero-quantity $w_{abcdef}$ is used to substitute the derivatives of $\psi_{abcd}$ occurring in the expressions. Since the final result is quite lengthy and not particularly illuminating, it is omitted.

## 8. Some general properties of the propagation and subsidiary equations

It will be convenient to give a solution of the conformal Einstein-Maxwell-Yang-Mills (EMYM) equations in the form $(P, M, \pi, A, g, \Omega)$, where $P$ is the bundle space of a principal bundle with structure group $G$ (as specified in §7), projection $\pi$, and four-dimensional base space $M$, $A$ is a connection form on $P$, $g$ is a Lorentz metric, and $\Omega$ is the conformal factor on $M$. This notation is somewhat redundant since $\Omega$ is determined for a solution of the conformal EMYM equations to a large extent by the metric $g$. A solution of the EMYM equations ($\Omega \equiv 1$) will be denoted by $(\widetilde{P}, \widetilde{M}, \tilde{\pi}, \widetilde{A}, \tilde{g})$. From the information given above we can calculate all the fields which appear in the conformal equations (2.14)–(2.21), (7.1)–(7.3). To cast the equations into a form which will allow us to derive existence results we impose gauge conditions. The role played here by the conformal factor is somewhat special—it is a scalar and its evolution is fixed by the choice of the Ricci scalar—which is the reason for our notation above. Beside the conformal factor we need to choose locally a section of $P$, an orthonormal frame for $g$, and a coordinate system. As explained in §6 part of the information about the last two choices is contained in the frame $c_A = c^\mu{}_A \partial/\partial x^\mu$. Finally we choose a time-like vector field $\tau$. Then we extract from the conformal EMYM equations the coupled system of equations (5.13)–(5.17), (6.13), (6.14), (6.21), (6.22), (7.17)–(7.19), (7.31) of propagation equations. These constitute a quasilinear first-order system of partial differential equations for the unknown

$$u = (\tau^A, e^A{}_{ab}, \Gamma_{abdc}, \Omega, \Sigma_{ab}, s, \Phi_{abcd}, A_{ab}, \varphi_{ab}, \psi_{abcd}). \tag{8.1}$$

The time-like vector field $\tau$ used to derive the propagation equations so far was not subject to any conditions besides those stated in (4.1). We require $\tau$ now to be related to the chosen frame $e_{aa'}$ such that it has components

(8.2)
$$\tau^{aa'} = \varepsilon_0{}^a \varepsilon_{0'}{}^{a'} + \varepsilon_1{}^a \varepsilon_{1'}{}^{a'} \quad \text{and thus} \quad e^A{}_{aa'} = -\tau^b{}_{a'}(e^A{}_{ab} + \tfrac{1}{2}\varepsilon_{ab}\tau^A).$$

The reality condition (3.2) is then a consequence of the reality requirements $\tau^{A+} = \tau^A$, $e^{A+}{}_{ab} = -e^A{}_{ab}$, and the differential operator $P$, $D_{ab}$ occurring in the propagation equations satisfy (4.17) and are given in fact by (4.22). It follows furthermore that we have by (8.2)

$$\chi_{abcd} = -\Gamma_{abcd} - \Gamma^+{}_{badc}. \tag{8.3}$$

Multiplying now the various subsystems occurring in the propagation equations suitably by binomial coefficients, as indicated for equation (4.25), resp. (4.27), we find the *the system of propagation equations is symmetric hyperbolic.* We have again a system of the form (4.28) with hermitian matrices $A^\mu$ for which $A^\mu \tau_\mu$ is positive definite and (4.29) holds with certain positive integers $k, j, j_0, \cdots, j_k$. When we want to solve the system, a subtlety is to be observed. Since $e^A{}_{ab}, \tau^A$ are themselves part of the solution, we have to take care that the reality requirements on $e^A{}_{ab'} \tau^A$ be satisfied to ensure the hermiticity of the matrices $A^\mu$. This can be done by writing in an obvious way the whole system as a real symmetric hyperbolic system for real unknowns or in other ways (see, e.g., [14]). If in the following we speak of the "symmetric hyperbolic propagation equations" implied by the conformal EMYM equations, it will always be meant in the sense indicated above.

It will be convenient for us to derive still another formulation of the field equations. Suppose that $(P, M, \pi)$ is a principal fiber bundle with structure group $G$ as in §7, that $\widehat{A}$ is the connection form of a principal connection on $P$, that $\widehat{F}$ is the associated curvature form, and that on $M$ a Lorentz metric $\hat{g}$ and a real-valued function $\widehat{\Omega}$ are given. Assume $V'$ is an open subset of $M$ on which we have local coordinates $x^\mu$, $\mu = 0, \cdots, 3$, and over which $P$ is trivializable. We denote again by $\widehat{A}_\mu$, resp. $\widehat{F}_{\mu\nu}$, the pullback to $V'$ of the connection, resp. curvature, form with respect to a suitably chosen local section of $P$. Let $\{c_A\}_{A=0,\cdots,3}$ be a frame, defined on some open set $V$ containing $V'$, which is orthonormal for $\hat{g}$,

$$\hat{g}(c_A, c_B) = \eta_{AB} = \mathrm{diag}(1, -1, -1, -1),$$

and set $\hat{e}_{aa'} = \hat{e}^A_{aa'} c_A$ with constant Levi-Civita symbols $\hat{e}^A_{aa'}$, given by

$$\begin{aligned} \sqrt{2}\hat{e}^A_{00'} &= \delta^A_0 + \delta^A_3, & \sqrt{2}\hat{e}^A_{01'} &= \delta^A_1 + i\delta^A_2, \\ \sqrt{2}\hat{e}^A_{10'} &= \delta^A_1 - i\delta^A_2, & \sqrt{2}\hat{e}^A_{11'} &= \delta^A_0 - \delta^A_3. \end{aligned} \tag{8.4}$$

All these structures are assumed to be of class $C^\infty$.

Using $\tau^{aa'}$ given in (8.2) we may now derive from $\hat{e}^A_{aa'}$ and $\widehat{A}_{ab}$ fields $\hat{\tau}^A$, $\hat{e}^A{}_{ab}$, and $\widehat{A}_{ab}$ as described in §§6 and 7. These together with $\Omega$ may be complemented (in a somewhat arbitrary way) to obtain a collection

$$\hat{u} = (\hat{\tau}^A, \hat{e}^A{}_{ab}, \widehat{\Gamma}_{adbc}, \widehat{\Omega}, \widehat{\Sigma}_{ab}, \hat{s}, \widehat{\Phi}_{abcd}, \hat{\varphi}_{abcd}, \widehat{A}_{ab}, \hat{\varphi}_{ab}, \hat{\psi}_{abcd}) \tag{8.5}$$

of smooth fields which satisfy the same symmetry and reality conditions as the corresponding fields given by (8.1). Though this is not necessary for our purpose, we require for definiteness that with the exception of $\hat{\varphi}_{abcd}$ the additional fields are derived from $\widehat{A}$, $\hat{g}$, $\widehat{\Omega}$, $\hat{e}^A_{aa'}$, and $\tau^{aa'}$ in the same way as the corresponding fields in (8.1) were obtained from $A$, $g$, $\Omega$, $e^A_{aa'}$, and $\tau^{aa'}$. However it may be emphasized that so far the fields given by

(8.5) need not satisfy any particular field equations. By $24\widehat{\Lambda}$, $\widehat{F}^A$, $\widehat{F}^{ab}$, $\widehat{B}$ we denote the Ricci scalar and the gauge source functions derived from $\widehat{A}$, $\hat{g}$ with respect to $\hat{e}_{aa'}$, $c_A$ and the local section of $P$ over $V'$.

We shall assume now that the fields $(A, g, \Omega)$ of the solution of the conformal EMYM equations which we seek to construct live also on the bundle $(P, M, \pi)$. Since we will formulate the field equations and in particular fix the gauge freedom with respect to $(\widehat{A}, \hat{g}, \widehat{\Omega})$, we will call the latter together with (8.5) the "reference field".

There is the freedom to perform gauge transformations $(A, g, \Omega) \to (\Phi_* A, \varphi_* g, \varphi_* \Omega)$, where $\Phi: P \to P$ is a bundle isomorphism which implies the diffeomorphism $\varphi: M^* \to M$. It will be assumed in the following that after some such gauge transformation coupled with a suitable conformal rescaling of the metric $g$ the following gauge conditions are satisfied on $P$ and $M$:

(8.6) $$\Lambda = \widehat{\Lambda},$$

(8.7) $\mathrm{id}_M$ is a harmonic map from $(M, g)$ onto $(M, \hat{g})$,

(8.8) $*\nabla * (A - \widehat{A}) = 0$, where $\nabla$ denotes covariant derivation with respect to the connection defined by $A$ and $*$ is defined with respect to $g$.

The discussion in [14] shows that this can always be achieved near some suitable initial hypersurface $S$ by solving a system of hyperbolic equations. It leaves the freedom to choose the conformal factor and its differential on $S$ as well as the tangent map of the bundle isomorphism $\Phi$ over points of $S$, which now is supposed to map the bundle $\pi^{-1}(S)$ induced over the initial hypersurface into itself.

As described in §6 we expand now on $V'$ the frame $e_{aa'}$ satisfying (3.3) in terms of the frame $c_A$. We will assume, possibly after shrinking $V'$, that the further gauge condition

(8.9) $$F^{ab} = \widehat{F}^{ab}$$

holds on $V'$. As shown in [14] this can be achieved locally near $S$ and leaves the freedom to specify the frame up to first order on $S$. The way the remaining gauge freedom on $S$, resp. $\pi^{-1}(S)$, is taken care of is reflected in the construction of initial data for the propagation equation and will be dealt with later.

In the following we will assume that the pullback of $A$, resp. $\widehat{A}$, to $V$ is always performed with the same local section of $P$ over $V$. The harmonicity condition (8.7), $F^A = g^{\mu\nu}\widehat{\nabla}_\mu \alpha^A{}_\nu$, can now be written as

$$F^A - \widehat{F}^A = \varepsilon^{ab}\varepsilon^{a'b'}(e^F_{aa'}e^H_{bb'} - \hat{e}^F_{aa'}\hat{e}^H_{bb'})\widehat{\Gamma}_F{}^A{}_H \tag{8.7$'$}$$

and condition (8.8) translates into

$$\begin{aligned} B - \widehat{B} = -\varepsilon^{ab}\varepsilon^{a'b'}\{&[A_{aa'}, (A_{bb'} - \widehat{A}_{bb'})] \\ &+ (e^F_{aa'}e^H_{bb'} - \hat{e}^F_{aa'}\hat{e}^H_{bb'})\widehat{\nabla}_F \widehat{A}_H\}. \end{aligned} \tag{8.8$'$}$$

Denote by $v$ the collection of "difference fields"

$$v = u - \hat{u} = (\tau^A - \hat{\tau}^A, e^A{}_{ab} - \hat{e}^A{}_{ab}, \cdots, \varphi_{ab} - \hat{\varphi}_{ab}, \psi_{abcd} - \hat{\psi}_{abcd}). \tag{8.10}$$

We will consider $v$ as a function which takes values in a suitably chosen finite-dimensional Hilbert space $H$ (either $\mathbb{R}^N$ or $\mathbb{C}^N$ for some $N \in \mathbb{N}$, indices of matrices, etc. refer in the following to the natural basis in either of these spaces). Taking into account the gauge conditions (8.6)–(8.9), the symmetric hyperbolic propagation equations for $u$, rewritten as an equivalent system for the difference field $v$, takes on $V$ the form

$$\mathbf{C}^\mu[\hat{u} + v]v_{,\mu} + h[\hat{u}, v] = f. \tag{8.11}$$

The matrices which define the principal part are given by

$$\mathbf{C}^\mu[\hat{u} + v] = (\hat{\tau}^A + (\tau^A - \hat{\tau}^A))c^\mu_A\mathbf{D} + (\hat{e}^A_{ab} + (e^A_{ab} - \hat{e}^A_{ab}))c^\mu_A\mathbf{B}^{ab} \tag{8.12}$$

with a constant positive definite diagonal matrix $\mathbf{D}$ and constant matrices $\mathbf{B}^{ab}$ such that the matrices (8.12) are hermitian. The functions $h[\hat{u}, v]$ and $f$ take values in $H$. The components of $h[\hat{u}, v]$ are polynomials in the components $v^i$ of $v$. The coefficients of these polynomials as well as the components of $f$ are smooth on $V$ and are determined by $\hat{u}$ and its first derivatives. The function $f$ is chosen here such that

$$h[\hat{u}, 0] = 0, \tag{8.13}$$

(8.14) $f$ vanishes at points of $V$ where $\hat{u}$ satisfies the symmetric hyperbolic propagation equations.

For later reference we state the following consequence of the previous discussion. A solution $v$ of (8.11), resp. $u$ of the original propagation equations, provides a frame $\tau = \tau^A c_A$, $e_{ab} = e^A{}_{ab}c_A$ and thus a metric $g$ which satisfies

$$g(\tau, \tau) = 2, \quad g(\tau, e_{ab}) = 0, \quad g(e_{ab}, e_{cd}) = h_{abcd} \tag{8.15}$$

and we have:

(8.16) For a given solution $v$ the characteristics of the system (8.11) are time-like or null hypersurfaces with respect to the metric $g$.

The symmetric hyperbolic propagation equations will be used later in the form (8.11) with suitably chosen reference field $\hat{u}$ and possibly another choice of the function on the right-hand side.

By the general theory of symmetric hyperbolic systems we know that *Cauchy problems for the propagation equations are well posed*. Let us assume for a moment that for suitably chosen smooth Cauchy data on some hypersurface $S$ in $V'$ which is space-like for the Cauchy data we have obtained a solution $v$ of the propagation equations (8.11) and thus a solution $u$, resp. $(A, g, \Omega)$, of the original symmetric hyperbolic propagation equations. We may assume that this solution is determined uniquely by the data and up to gauge transformations independent of the chosen reference field $\hat{u}$. By (4.29) domains of dependence defined by the characteristic ray cones of the propagation equation coincide for the solution $u$ with the domains of dependence defined by $g$. Assume now that the data have been given such that the constraint equations and consequently the complete set of conformal EMYM equations are satisfied on the initial surface $S$. We want to show that the constraints are satisfied in the domain of dependence of $S$ with respect to $g$. For this purpose we use the system of subsidiary equations (5.18)–(5.20), (5.27), (5.29), (6.25), (6.26), (7.32), (7.35), (7.36). Since we have (8.2), we find that after multiplying again the various subsystems by suitable binomial coefficients, the *subsidiary system is also symmetric hyperbolic*. It has the form (4.28) with hermitian matrices $A^\mu$, such that $A^\mu \tau_\mu$ is positive definite and a term $B(u)$, which, as explained in the preceding sections, may be written (somewhat arbitrarily) as a linear function $B(u) = B(x) \cdot u$ with a matrix-valued function $B(x)$ on the solution space-time. Again we have the relations (4.29), (4.30), (4.31) with some nonnegative integers $k, j, j_0, \cdots, j_k$. In the case where the matter fields vanish, such that we deal with only equations (5.17)–(5.16), (5.26), (5.28), (6.25), (6.26), we find that $j_0 = 0$, or in other words, that all characteristic rays of the subsidiary system are time-like with respect to the metric $g$. Though the exact location of the characteristic ray cones of the subsidiary equations is dependent on the choice of $\tau$, the fact that all characteristics are strictly time-like in the source-free case does not depend on $\tau$. In any case we find that the domains of dependence of the subsidiary system contain the corresponding domains of dependence with respect to $g$. The standard argument for showing uniqueness for solutions to symmetric hyperbolic systems implies with the notion of "domain of dependence" given in [25] the

**Reduction Theorem (8.1).** *Any (sufficiently smooth) solution* $u$, *resp,.* $(A, g, \Omega)$, *of the symmetric hyperbolic propagation equations (implied by the conformal Einstein-Maxwell-Yang-Mills equations) which satisfies the constraint equations on a space-like hypersurface* $S$ *defines in the domain of dependence of* $S$ *with respect to* $g$ *a solution to the conformal Einstein-Maxwell-Yang-Mills equations* (2.14)–(2.21), (7.1)–(7.3).

The (rather mild) smoothness conditions required on $u$ for Theorem (8.1) to be true may be inferred from Kato's result on symmetric hyperbolic systems [26, Theorem I]. These conditions will be satisfied in all of the following applications of the symmetric hyperbolic propagation equations.

The reduction theorem was one of the prime motivations for the lengthy discussion of the field equations in the preceding sections. It allows us to derive various global and semiglobal existence for solutions of the Einstein-Maxwell-Yang-Mills equations.

## 9. Existence results: the case $\tau^2\lambda < 0$

In this section we will show the existence of a large class of semiglobal, resp. global, solutions to the Einstein-Maxwell-Yang-Mills equations with cosmological constant $\lambda < 0$ (in our signature), and prove a general stability theorem for global solutions with smooth asymptotic behavior. The notion of (semi) globality used here is that of null (past) completeness. The asymptotic behavior is that described by the notion of (past) asymptotic simplicity ([28], [29], [30]) which will be discussed now. For the causal notions like "Cauchy hypersurface", "global hyperbolicity", etc. we refer to [25].

**Definition (9.1).** A principal bundle $(\widetilde{P}, \widetilde{M}, \tilde{\pi})$ with structure group $G$, a four-dimensional connected manifold $\widetilde{M}$, a Lorentz metric $\tilde{g}$ on $\widetilde{M}$ (such that the space-time is orientable and time-orientable), and a connection form $\widetilde{A}$ on $\widetilde{P}$ is called "asymptotically simple" if, possibly after a transition $(\widetilde{A}, \tilde{g}) \to (\Phi_* \widetilde{A}, \varphi_* \tilde{g})$ with a bundle isomorphism $(\Phi, \varphi)$ of $\widetilde{P}, \widetilde{M}, \tilde{\pi})$, there is another principal bundle $(P, M, \pi)$ with structure group $G$, a Lorentz metric $g$ and a function $\Omega$ on $M$, and a principal connection form $A$ on $P$ such that the following conditions are satisfied:

(i) $M$ is a manifold with boundary $\mathscr{J}$ and interior $\widetilde{M}$, i.e., $M = \widetilde{M} \cup \mathscr{J}$, and $\pi^{-1}(\widetilde{M}) = \widetilde{P}$.

(ii) $\Omega = 0$, $d\Omega \neq 0$ at all points of $\mathscr{J}$, $\Omega > 0$ on $\widetilde{M}$,

(iii) $g = \Omega^2 \tilde{g}$ on $\widetilde{M}$,

(iv) each null geodesic in $(\widetilde{M}, \tilde{g})$ acquires a past and a future endpoint on $\mathcal{J}$,

(v) $A = \widetilde{A}$ on $\pi^{-1}(\widetilde{M})$.

(vi) there exists a smooth, smoothly embedded compact Cauchy hypersurface $S$ for $(\widetilde{M}, \tilde{g})$.

We call $(\widetilde{P}, \widetilde{M}, \tilde{\pi}, \tilde{g}, \widetilde{A})$ "asymptotically simple in the past" if the conditions above are satisfied with the exception of (iv) which is replaced by

(iv′) each null geodesic in $(\widetilde{M}, \tilde{g})$ acquires a past (and no future) endpoint on $\mathcal{J}$.

We will refer to $(\widetilde{P}, \widetilde{M}, \tilde{\pi})$ as the "physical bundle" and to $(\widetilde{A}, \tilde{g})$ as the "physical fields", whereas $(P, M, \pi)$ will be called the "conformal bundle" and $(A, g, \Omega)$ the "conformal fields". The hypersurface $\mathcal{J}$ is called the "conformal boundary" or "conformal infinity". Later we will construct spaces $(\widetilde{P}, \widetilde{M}, \tilde{\pi}, \widetilde{A}, \tilde{g})$ as above by solving the EMYM equations. This motivates assumption (v) which is not part of the usual requirements of asymptotic simplicity. The compactness of $S$ has been assumed because there appear to exist no natural boundary or fall-off conditions for solutions of the EMYM equations with cosmological constant $\lambda < 0$. For convenience we require that all fields be of class $C^\infty$ but in the course of the existence proof we shall see that such strong smoothness requirements are not forced on us by the problem.

To obtain (past) asymptotically simple solutions $(\widetilde{P}, \widetilde{M}, \tilde{\pi}, \widetilde{A}, \tilde{g})$ of the EMYM equations we shall try to find solutions of the conformal EMYM equations which have the properties of the conformal bundle and the conformal fields discussed above. For this it is important that we have the following result which shows in particular that the globality property mentioned above is a consequence of asymptotic simplicity.

**Lemma (9.2)** [29]. *Suppose $(\widetilde{P}, \widetilde{M}, \tilde{\pi}, \widetilde{A}, \tilde{g})$ is a (past) asymptotically simple solution of the EMYM equations with cosmological constant $\lambda < 0$, and $(P, M, \pi, A, g, \Omega)$ is conformally related to it as described in Definition* (9.1). *Then*

(i) $-3\nabla_\mu \Omega \nabla^\mu \Omega = \lambda$ *on $\mathcal{J}$, i.e., $\mathcal{J}$ is space-like,*

(ii) *The rescaled Weyl-tensor $d^\mu{}_{\nu\rho\sigma} = \Omega^{-1} \widetilde{C}^\mu{}_{\nu\rho\sigma} = \Omega^{-1} C^\mu{}_{\nu\rho\sigma}$ on $\widetilde{M}$ extends to a smooth tensor field on $M$.*

(iii) *The null geodesics of $(\widetilde{M}, \tilde{g})$ are (past) complete.*

**Remark.** For the source-free case it has been shown in [21] that the physical time-like geodesics, which also necessarily acquire endpoints on

$\mathscr{J}$ (cf. Lemma (9.3)), satisfy similar completeness conditions. It is likely that this property persists under the more general conditions considered here but the arguments used in [21] do not apply immediately to the present case.

Property (i) follows from (2.20). Equation (2.18), which holds on $\widetilde{M}$, can be rewritten as $\nabla_\mu \Omega C^\mu{}_{\nu\rho\sigma} = \Omega \nabla_\mu C^\mu{}_{\nu\rho\sigma} - \kappa\Omega t_{\rho\sigma\nu}$. By our smoothness assumptions on $g$ and $A$ this implies $\nabla_\mu \Omega C^\mu{}_{\nu\rho\sigma} = 0$ on $\mathscr{J}$. Since by (i) $\nabla_\mu \Omega$ is time-like it follows that $C^\mu{}_{\nu\rho\sigma} = 0$ on $\mathscr{J}$, which entails (ii). Since by condition (iii) of Definition (9.1) null geodesics with respect to the metric $g$ coincide up to parametrization with null geodesics with respect to $\tilde{g}$ on $\widetilde{M}$ it follows from condition (iv) (resp. (iv$'$)) that all null geodesics with respect to $g$ which pass through a point of $\widetilde{M}$ end in the past and in the future (resp. in the past) on $\mathscr{J}$ for finite values of an affine parameter. Condition (ii) entails that all null geodesics with respect to the metric $\tilde{g}$ are complete (resp. past complete).

The simply connected, geodesically complete, conformally flat standard example with vanishing gauge field of an asymptotically simple solution of the EMYM equations is provided by the De-Sitter solution. In terms of the conformal structures it is given by

$$
\begin{aligned}
&(P, M, \pi, A, g, \Omega) \text{ with } P = M \times G, \\
&M = \{(t, x) \in \mathbb{R} \times S^3 / |t| \le \pi/2\}, \\
(9.1) \quad &\pi \text{ the natural projection onto } M,\ A \text{ such that the section} \\
&\qquad M \ni x \to (x, s) \in P,\ \ s \in G,\ \text{ is horizontal}, \\
&g = dt^2 - d\omega^2,\ \ \Omega = \cos t,
\end{aligned}
$$

where $d\omega^2$ denotes the standard line element on the three-dimensional unit sphere and the conformal boundary consists of two components, "past time-like and null infinity" $\mathscr{J}^- = \{t = -\pi/2\}$ and "future time-like and null infinity" $\mathscr{J}^+ = \{t = \pi/2\}$. The solution (9.1) of the conformal EMYM equations can be extended analytically to all values $t \in \mathbb{R}$ to yield an analytic solution of the conformal EMYM equations on $\mathbb{R} \times S^3$.

There are known explicit solutions to the EMYM equations, e.g., the Schwarzschild-De-Sitter solution [23], which satisfy all conditions of asymptotic simplicity with the exception of (iv). Some of their null geodesics end on the smooth hypersurface $\mathscr{J}$ of the conformal extension, while others run into a singularity. Such are called "weakly asymptotically simple" [30].

The example of the De-Sitter solution shows many of the features of the general case. Since $(\widetilde{M}, \tilde{g})$ is orientable and time-orientable the same is true for $(M, g)$. The conformal boundary splits into two components, the set $\mathscr{I}^-$ ("past time-like and null infinity") and the set $\mathscr{I}^+$ ("future time-like and null infinity") of past and future, respectively, endpoints of futures directed null geodesics. In the case of past asymptotic simplicity the conformaly boundary will be denoted by $\mathscr{I}^-$.

**Lemma (9.3).** *Under the assumptions of Lemma* (9.2) *the following hold*:

(i) $M$ *is diffeomorphic to* $[-1, 1] \times S$ (*resp. to* $[-1, 0[\times S)$.

(ii) *The hypersurface* $\mathscr{I}^-$, *resp.* $\mathscr{I}^+$ (*resp.* $\mathscr{I}^-$), *is diffeomorphic to* $S$.

(iii) *There exists a time function* $t \in C^\infty(M, [-1, 1])$ (*resp.* $t \in C^\infty(M, [-1, 0[)$ *which has future directed time-like gradient with respect to* $g$, *its level surfaces are Cauchy hypersurfaces for* $(M, g)$, *and it takes value* $\pm 1$ *on* $\mathscr{I}^\pm$ (*resp.* $-1$ *on* $\mathscr{I}^-$).

Over the Cauchy hypersurface $S$ we consider the subset of the tangent bundle $TM$ formed by the future directed null vectors which have scalar product equal to 1with the future directed unit normal of $S$. Pushing forward and backward this set with the geodesic flow on $TM$ (everything here is done with respect to the conformal metric $g$) one obtains a subset of $TM$ which is compact and which projects onto $M$ by (iv) and (vi) of Definition (9.1). Thus $M$, $\mathscr{I}^-$, and $\mathscr{I}^+$ are compact. It also follows from (vi) that there exists a time function $\tilde{\tau}$ on $\widetilde{M}$ whose level sets are Cauchy hypersurfaces for $(\widetilde{M}, \tilde{g})$ and which goes to $-\infty$, resp. $+\infty$ along any inextendible non-space like curve in $\widetilde{M}$ if we follow it into the past, resp. into the future ([22], [25]). From Lemma (9.2i) it follows that any such curve approaches a unique past, resp. future, endpoint on $\mathscr{I}^-$, resp. $\mathscr{I}^+$. Thus, $\mathscr{I}^-$ and $\mathscr{I}^+$ are each Cauchy hypersurfaces for $(M, g)$. We may assume that $\tilde{\tau}$ is smooth with future directed time-like gradient (with respect to $g$ or $\tilde{g}$) [33] and after composition with a suitable smooth, strictly increasing real-valued function yields a time function on $\widetilde{M}$, denoted again by $\tilde{\tau}$, such that $\tilde{\tau}(p) \to \pm 1$ if $p \in \widetilde{M}$ approaches $\mathscr{I}^\pm$. Now let $\tau^-$ be a time function with future directed time-like gradient for $g$ on the set $\mathscr{I}^- \cup \{\tilde{\tau} \le c\}$ for some constant $c$ with $-1 < c < 0$ such that $\tau^- = -1$ on $\mathscr{I}^-$. Choose real numbers $a$, $b$ with $-1 < a < b < c$ and a monotonous function $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ with $\varphi(s) = 0$ for $s \le a$, and $\varphi(s) = 1$ for $s \ge b$. Then a new smooth time function $t$ is obtained on $\mathscr{I}^- \cup \widetilde{M}$ by setting $t = \tau^-$ on $\mathscr{I}^- \cup \{\tilde{\tau} \le a\}$, $t = \tau^-(1 - \varphi(\tilde{\tau})) + \alpha\tilde{\tau}\varphi(\tilde{\tau})$ on $\{a < \tilde{\tau} < b\}$, and $t = \tilde{\tau}$ on $\{\tilde{\tau} \ge b\}$, with

$\alpha = \frac{1}{a} \sup \tau^-$, where the supremum is taken on $\{a \le \tilde{\tau} \le b\}$. Proceeding in a similar way near $\mathcal{J}^+$, we obtain a smooth time function $t$ on $M$ which has everywhere future directed time-like gradient with respect to $g$ which takes values $-1$ on $\mathcal{J}^-$ and $1$ on $\mathcal{J}^+$, and for which the level surfaces $S_\tau = \{t = \tau\}$ for given $\tau \in [-1, 1]$ are Cauchy hypersurfaces of $(M, g)$ diffeomorphic to $S$ (cf. [8]). By similar arguments one finds in the case of a past asymptotically simple bundle that there exists a smooth time function $t$ which takes values in $[-1, 0[$ on $M$, $t = -1$ on $\mathcal{J}^-$, which has future directed time-like gradient with respect to $g$, and which is such that all its level surfaces are Cauchy hypersurfaces diffeomorphic to $S$.

If $(P, M, \pi, A, g, \Omega)$ is conformally related to a (past) asymptotically simple solution $(\widetilde{P}, \widetilde{M}, \pi, \widetilde{A}, \tilde{g})$ of the EMYM equations as described in Definition (9.1), it is a solution of the conformal EMYM equations. To obtain such solutions we are by the previous discussion free to describe data for solutions to the conformal EMYM equations either on a Cauchy hypersurface $S \subset \widetilde{M}$ or on $\mathcal{J}^-$. An initial data set for the conformal EMYM equations which is implied on the hypersurface $\mathcal{J}^-$ of an asymptotically simple solution of the EMYM equations and which comprises the fields given by (8.1) is called an "asymptotic initial data set". Its explicit form depends on the way the gauge freedom described in §8 is fixed on $\mathcal{J}^-$. This has been discussed for the case of a trivial gauge field in [16]. The present case is very similar. Observing that $\Omega = 0$ one finds that by a suitable choice of the conformal factor, of the frame, and of the local section of the principal bundle $P$ with respect to which we express the gauge potential and the gauge field on $\mathcal{J}^-$ one may obtain on $\mathcal{J}^-$

$$\tau^A = \sqrt{2}\delta_0^A, \quad \Sigma_{ab} = \varepsilon_{ab}\Sigma, \quad \Sigma > 0,$$
$$s = 0, \quad \Lambda = \Lambda_0, \quad \Gamma^{ab} = 0, \quad \alpha = 0, \tag{9.2}$$

where $\Lambda_0$ is an arbitrarily given smooth real-valued function on $\mathcal{J}^-$. The essential properties of an asymptotic initial data set are easily found by an inspection of the constraint equations (2.20), (5.8)–(5.12), (6.12), (6.20), (7.15), (7.16) and are listed in the following lemma.

**Lemma (9.4)** (Structure of asymptotic initial data). (i) $\Sigma^2 = (-\lambda/6)$, $\chi_{abcd} = 0$, *and* $\Phi^*_{ab} = 0$ *hold in the gauge* (9.2).

(ii) *The interior metric* $h$ *on* $\mathcal{J}^-$ *is not subject to any constraints and may be given arbitrarily. Since* $\mathcal{J}^-$ *is three-dimensional, orientable, and compact we may choose a global frame* $\{c_A\}_{A=1,2,3}$ *on* $\mathcal{J}^-$ *with* $h(c_A, c_B) = -\delta_{AB}$ *and set* $e_{ab} = \hat{e}^A_{ab} c_A = \tau_{(a}{}^{a'} \hat{e}^A_{b)a'} c_A$ *with the constant*

*Levi-Civita symbols* (8.4) *such that* $h(e_{ab}, e_{cd}) = h_{abcd}$. *Because of* (i) *the operator* $D_{ab}$, *which is the covariant Levi-Civita differential operator with respect to* $h$, *and the connection coefficients* $\Gamma_{abcd}$, *which satisfy* $\Gamma_{abcd} = \Gamma_{(ab)(cd)} = -\Gamma^{+}_{abcd}$ (*cf.* (8.3)), *are then determined completely in terms of* $h$.

(iii) *The symmetric spinor fields* $\alpha_{ab}$, $\varphi_{ab}$ *are subject to the constraint equations*

$$
\begin{aligned}
&2D^{f}{}_{(ab}\alpha_{b)f} - [\alpha_{af}, \alpha_{b}{}^{f}] - \varphi_{ab} - \varphi^{+}{}_{ab} = 0, \\
&D^{hf}\varphi_{hf} + [\alpha^{hf}, \varphi_{hf}] = 0.
\end{aligned} \tag{9.3}
$$

*Denote by* $A'$, *resp.* ${}^{-}F'$, *the pullback of the connection form* $A$, *resp. of the horizontal, complex form* ${}^{-}F = \frac{1}{2}(F + i * F)$, *where* $*$ *is defined with respect to* $g$, *to the bundle* $\pi^{-1}(\mathcal{J}^{-})$ *over* $\mathcal{J}^{-}$. *Then* $A'$ *is a connection form on* $\pi^{-1}(\mathcal{J}^{-})$ *with curvature form* $F' = {}^{-}F' + ({}^{-}F')^{+}$, *as follows from the first of equations (9.3). If the forms* $A'$, $F'$, *resp.* ${}^{-}F'$, *are pulled back to* $M$ *by the restriction to* $\mathcal{J}^{-}$ *of the local section which has been used to obtain equations (9.3), they are given by* $\alpha_{ab}\beta^{ab}$, $(\varphi_{ab} + \varphi^{+}_{ab})\beta^{a}{}_{c} \wedge \beta^{bc}$, *resp.* $\varphi_{ab}\beta^{a}{}_{c} \wedge \beta^{bc}$, *where the 1-forms* $\beta^{ab}$ *on* $\mathcal{J}^{-}$ *are dual to* $e_{ab}$ *in the sense that* $\langle \beta^{ab}, e_{cd} \rangle = h^{ab}{}_{cd}$. *By the second of equations* (9.3) *the form* ${}^{-}F'$ *satisfies the equation* $*' \not{D}\, {}^{-}F' = 0$ *on* $\pi^{-1}(\mathcal{J}^{-})$, *where* $*'$ *is defined with respect to* $h$ *and* $\not{D}$ *denotes the covariant derivative with respect to the connection* $A'$.

(iv) *The electric part* $\eta_{abcd}$ *of* $\varphi_{abcd} = \eta_{abcd} + i\mu_{abcd}$ *represents a covariant, symmetric, traceless tensor* $\eta$ *of valence* 2 *on* $\mathcal{J}^{-}$ *which is subject to the constraint equation*

$$
D^{hf}\eta_{abhf} = 2\kappa\Sigma(\varphi_{f(a}/\varphi^{+}{}_{b)}{}^{f}). \tag{9.4}
$$

(v) *The remaining curvature fields are then given by*

$$
\Phi^{*} = \tfrac{1}{4}\,{}^{3}R - 3\Lambda_{0}, \quad \Phi^{*}_{abcd} = {}^{3}R_{(abcd)}, \quad \mu_{abcd} = \tfrac{i}{\Sigma}D^{f}{}_{(a}\Phi^{*}_{bcd)f},
$$

*where* ${}^{3}R_{abcd}$ *denotes the Ricci tensor of* $h$ *and* ${}^{3}R$ *its Ricci scalar.*

(vi) *Finally* $\psi_{abcd}$ *can be derived from the fields obtained so far according to* (7.24) *by using* (7.19).

To provide asymptotic initial data sets we are thus free to prescribe the metric $h$, give a solution of equations (9.3), which decouple from the equations for the other fields, and give a solution of equation (9.4), which can be dealt with by the techniques discussed in [9]. The remarkable fact that the interior metric of an asymptotic initial data set is not subject to any constraints may give the impression that there is "too much freedom" to prescribe data. This is not true, however, since families of asymptotic initial data which are conformally related in a certain sense give rise to the same physical fields (cf. [16]). By giving the gauge conditions (9.2) we did not restrict this remaining conformal freedom which could be done, e.g., by imposing conditions on the Ricci scalar of the metric $h$ [32].

**Theorem (9.5).** *Let $\lambda$ be a negative real number. Suppose $(P', S, \pi')$ is a principal bundle with structure group $G$ (as specified in §7) over a three-dimensional connected compact, orientable manifold $S$. Let $h$ be a negative definite metric and $\eta$ a real, covariant, symmetric, traceless tensor of valence 2 on $S$, and let $A'$ a connection form and ${}^{-}F'$ a complex horizontal two-form on $P'$. Suppose that these structures are of class $C^\infty$, that the fields satisfy equations* (9.3), (9.4), *and imply together with the conditions* (9.2) *an asymptotic initial data set*

$$u = (\tau^A, e^A{}_{ab}, \Gamma_{adbc}, \Omega = 0, \Sigma_{ab}, s, \Phi_{abcd}, \varphi_{abcd}, A_{ab}, \varphi_{ab}, \psi_{abcd})$$

*as described in Lemma* (9.2). *Then there exists a smooth past asymptotically simple solution of the EMYM equations with cosmological constant $\lambda$ such that its asymptotic initial data implied on $(\pi^{-1}(\mathscr{I}^-), \mathscr{I})$ in a suitable gauge can be identified by a bundle isomorphism with the data $u$ on $(P', S, \pi')$. The solution is unique up to the possibility to construct extensions into the future.*

**Remarks.** (i) We shall see in the following that the requirement that $S$ be compact is not necessary to obtain local solutions "near $\mathscr{I}$". In the case of weakly asymptotically simple solution such as the Schwarzschild-De-Sitter space-time, the parts of the solution near the "smooth pieces of $\mathscr{I}$" can be obtained by the method discussed below, however, the complete space-time cannot be obtained from asymptotic data.

(ii) The following discussion will also show that the smoothness requirements could be relaxed considerably.

To prove the theorem we first construct a suitable reference field as discussed in §8.. Let $(P, M, \pi)$ denote the principal fiber bundle with structure group $G$, bundle space $P = I \times P'$, and base $M = I \times S$, with $I$ the open interval $]-1, 1[$, such that $s \in G$ acts on $(t, p') \in P$ according to $(t, p')s = (t, p's)$, which implies that the projection $\pi$ maps $(t, p')$ into $(t, \pi'(p'))$. The projection of $M$ onto $I$ defines a smooth function on $M$ which we denote by $t$. For $\tau \in \mathbb{R}$ we set $S_\tau = \{x \in M | t(x) = \tau\}$ and identify $S$ with $S_0$ and $P'$ with $\pi^{-1}(S_0)$. Let $T$ be the vector field on $P$ which has the curves $I \ni t \mapsto (t, p') \in P$, $p' \in P'$, as integral curves. It projects under $\pi$ onto a vector field on $M$ denoted by $c_0$. We define the reference connection form $\widehat{A}$ on $P$ by the requirements that its pull-back to $P'$ coincide with $A'$, that $T$ be horizontal for $\widehat{A}$, and that $\widehat{A}$ be Lie-propagated in the direction of $T$. The reference metric $\hat{g}$ on $M$ is defined by the requirements that it imply the interior metric $h$ on $S_0$, that $c_0$ coincide with the (future directed time-like) unit normal of $S_0$ with respect to $\hat{g}$, and that $\hat{g}$ be Lie-propagated in the direction of $c_0$. The space-time $(M, \hat{g})$ is then globally hyperbolic, $t$ is a time function, and $c_0$ is a future directed time-like geodesic Killing field for it. This latter fact could be used to simplify the following discussion which we keep general, however, since we want to apply it later to other situations as well. We choose now a smooth real-valued function $\widehat{\Omega}$ on $M$ with $\widehat{\Omega} \neq 0$ on $M \backslash S_0$, $\widehat{\Omega} = 0$ while $\widehat{\Sigma} = \sqrt{2} c_0(\widehat{\Omega}) = \Sigma$ on $S_0$, and $4\hat{s} = \widehat{\nabla}_\mu \widehat{\nabla}^\mu \widehat{\Omega} = 0$ on $S$.

As described in Lemma (9.4ii) we choose a frame field $c_A$, $A = 1, 2, 3$, on $S_0$ which is orthonormal for the interior metric implied by $\hat{g}$ on $S_0$. The vector fields $c_A$ are extended to $M$ as solutions of the Fermi-transport equation

$$(9.5) \quad \nabla'_{c_0} c_A + g'(\nabla'_{c_0} c, c_A) c_0 - g'(c_0, c_A) \nabla'_{c_0} c_0 = 0, \qquad A = 1, 2, 3,$$

to yield together with $c_0$ an orthonormal frame for $\hat{g}$ on $M$.

In the following our goal is to construct local solutions of the conformal EMYM equations which will be patched together to yield the solution we are looking for. We first describe the type of sets on which the local solutions will be obtained. To allow for further applications the following discussion will be slightly more general than required for our immediate purpose.

Choose a real number $\theta > 1$ and define $M$ a Lorentz metric $g^*$ by the conditions $g^*(c_0, c_0) = \theta^2$ and $g^*(c_A, c_B) = -\delta_{AB}$ for $A, B = 1, 2, 3$. The space-time $(M, g^*)$ is then also globally hyperbolic.

Fix another real number $r > 1$. For given $x_0 \in S_\tau$, $\tau \in I$, choose an open neighborhood $U''$ of $x_0$ in $S_\tau$ on which exist a smooth local section of $P$ and a smooth coordinate system $x^\alpha$, $\alpha = 1, 2, 3$, whose image contains the closed ball $\overline{B}_r(0)$ and which maps $x_0$ onto the origin of $\mathbb{R}^3$. Let $U'$ with $\overline{U}' \subset U''$ be the open neighborhood of $x_0$ in $S_\tau$ which is the preimage of $B_r(0)$ under the coordinate map. We extend the coordinates $x^\alpha$ off $U''$ by requiring $c_0(x^\alpha) = 0$ and we extend the local section off $U''$ by requiring the extended section to be tangent to $T$. Let $D^+(\overline{U}')$ be the future domain of dependence of $\overline{U}'$ in $M$ with respect to the metric $g^*$. There exists a number $\eta > 0$ (with $\tau + \eta < 1$)

such that the set $D_\eta(\overline{U}') = D^+(\overline{U}') \cap \{x \in M' | \tau \leq t(x) \leq \tau + \eta\}$ has the following properties. $D_\eta(\overline{U}')$ has an open neighborhood in $M$ on which the $x^\mu$ with $x^0 = t$ and the $x^\alpha$ given above define a smooth coordinate system, where the smooth local section of $P$ defined above exists, and the vector fields $c_A = c_A{}^\nu(x^\mu)\partial/\partial x^\nu$ are defined and smooth. The boundary of $D_\eta(\overline{U}')$ consists of $\overline{U}'$, of a smooth null hypersurface with respect to $g^*$, and of a set of the form $\overline{V}$ with $V$ open in $S_{\tau+\eta}$ and such that the projection of $V$ into $S_\tau$ along the flow lines of $c_0$ yields an open neighborhood $U \subset U'$ of $x_0$. We shall call an open subset of $S_\tau$ with the properties of $U'$ a "standard neighborhood in $S_\tau$ with evolution $D_\eta(U')$, evolution time $\eta$, and associated neighborhood $U'$".

Since $S_\tau$ is compact, there exist a finite number $U'_1, \cdots, U'_p$ of standard neighborhoods in $S_\tau$ with evolution times $\eta_1, \cdots, \eta_p$ such that the associated neighborhoods $U_1, \cdots, U_p$ form an open covering of $S_\tau$. The number $\eta = \min \eta_i$ is positive and the sets $\operatorname{int}(D_\eta(\overline{U}'_i))$, $i = 1, \cdots, p$, form an open covering of $E_{\tau,\eta} = \{x \in M | t < t(x) < \tau + \eta\}$, while the sets $\operatorname{int}(D_\eta(\overline{U}'_i) \cap S_{\tau+\eta})$ cover $S_{\tau+\eta}$. We shall call the set $\overline{E}_{\tau,\eta}$ under these circumstances an "evolution of $S_\tau$ with evolution time $\eta$".

Let $U'$ be a standard neighborhood in $S_\tau$ with evolution time $\eta$. Following the discussion in §8 we derive from the fields $\widehat{A}$, $\hat{g}$, and $\widehat{\Omega}$ in an open neighborhood of $D_\eta(\overline{U}')$ a smooth reference field $\hat{u} = \hat{u}(x^\mu)$ as given in (8.5). By the coordinate map we identify $U'$ with its image $B_r(0)$ in $\mathbb{R}^3$. We denote by $H^s(U', H)$ the standard $L^2$-type Sobolev space with Sobolev index $s \in \mathbb{N}$ of $H$-valued functions on $U'$ and the Sobolev norm on $H^s(U', H)$ by $\|\cdot\|_{U',s}$. Let $U'_1, \cdots, U'_p$, $p \in \mathbb{N}$, be standard neighborhoods in $S_\tau$ which define an open covering of $S_\tau$. Any solution of the conformal EMYM equations or any reference field defined near, resp. any initial data set for the conformal EMYM equations given on $(\pi^{-1}(S_\tau), S_\tau, \pi|_{S_\tau})$, supplies a collection $u = \{u_i = u_i(\tau, x^\alpha) | i = 1, \cdots p\}$ of $H$-valued functions of the type (8.1) which are given on the $U'_i$ with respect to the gauge chosen above near the $U'_i$. For two such collections $u^1 = \{u^1_i | i = 1, 2, \cdots, p\}$ and $u^2 = \{u^2_i | i = 1, 2, \cdots, p\}$ we set

$$\|u^1 - u^2\| = \sum_{1 \leq i \leq p} \|u^1_i - u^2_i\|_{U'_i,s}. \tag{9.6}$$

The distance function thus defined depends of course on the choice of the $U'_i$, of the coordinates, of the local section of $P$, and of the frame field $c_A$.

However, the topology defined by this distance function is independent of these choices.

In the following we identify $D_\eta(\overline{U}')$ with its image in $\mathbb{R}^4$ by the coordinate map and consider the reference field $\hat{u}$ and the vector fields $c_A = c_A{}^\nu(x^\mu)\partial/\partial x^\nu$ to be defined near $D_\eta(\overline{U}')$ in $\mathbb{R}^4$. We extend the frame $c_A$ and the field $u$ smoothly to a neighborhood of $\mathbb{R}^3 \times [\tau, \tau+\eta]$ in $\mathbb{R}^4$ such that the $c_A$ remain linearly independent, $c_0{}^0 \geq \kappa'$ everywhere for some $\kappa' > 0$, $\hat{\tau}^A$ and $\hat{e}^A_{ab}$ remain constant, and such that $c^\nu{}_A(x^\mu) = \pm\delta^\nu{}_A$ (the sign depending on the orientation of the frame $c_A$ on $U'$), while the remaining fields provided by $\hat{u}$ vanish for sufficiently large $\sum(x^\mu)^2$. With the frame and the reference field so extended we consider equation (8.11) as an equation on $\mathbb{R}^3 \times [\tau, \tau+\eta]$, where the functions $h$ and $f$ are calculated from the extended $\hat{u}$ by the same calculation as on $D_\eta(\overline{U}')$.

Assume now that we are given on $(\pi^{-1}(S_\tau), S_\tau, \pi)$ smooth initial data for the conformal EMYM equations with cosmological constant $\lambda$ which satisfy the conformal constraint equations. Let $U'$ be one of the standard neighborhoods used to define (9.6). We think of the data as being given in the neighborhood $U''$ of $U'$ by a collection of fields $u$ as in (8.1) which has the meaning described in §8 with respect to the frame $c_A$ and the local section we have chosen on $U''$. On $U''$ we write $u = u(x^\alpha)$, set $v(x^\alpha) = u(x^\alpha) - \hat{u}(\tau, x^\alpha)$ as in (8.10), and consider $v$ again as a function defined on an open neighborhood of $U' \equiv \overline{B}_r(0) \times \{\tau\}$ in $\mathbb{R}^3 \times \{\tau\}$. Fix $s \in \mathbb{N}$, $s > \frac{3}{2} + 1$. There exists an extension operator which maps $v \in H^s(U', H)$ into $v_0 \in H^s(\mathbb{R}^3 \times \{\tau\}, H)$ such that $v_0$ vanishes for $\sum(x^\alpha)^2 > R$, $R$ some large number $> r$, and for any component $v^i$, resp. $v_0^i$, the inequality

$$\|v_0^i\|_s \leq \kappa_e \|v^i\|_{U', s}, \qquad v \in H^s(U', H) \tag{9.7}$$

holds, where on the left the standard Sobolev norm on $\mathbb{R}^3 \times \{\tau\}$ is used and $\kappa_e$ is a positive constant which possibly depends on $R$, $r$, and $s$ but not on $v$ [1]. The extended data $v_0$ are now considered as initial data for the symmetric hyperbolic system of type (8.11), (8.12) prepared in the preceding paragraph. We refer to the initial value problem thus prepared in the following as IVP. To apply Kato's result on quasilinear symmetric hyperbolic systems [26, Theorem II], we adapt our notation to Kato's for this problem. We again write $t$ for $x^0$ and regard the unknown

$v = v(t, x^\alpha)$ as the map

$$[\tau, \tau + \eta] \ni t \mapsto v(t) \in H^s(\mathbb{R}^3, H).$$

We also use the notation (8.10) for the unknown and the extended data such that the components are written $\tau^A - \hat{\tau}^A$, $e^A_{ab} - \hat{e}^A{}_{ab}$, etc. Denote by $\kappa_s$ the Sobolev imbedding constant such that $|v^i(x^\alpha)| \leq \kappa_s \|v^i\|_s$ for $x^\alpha$ in $\mathbb{R}^3$ and any component $v^i$ of some $v \in H^s(\mathbb{R}^3, H)$. Let $\delta$ be a real number, $0 < \delta < 1$, and choose $\kappa > 0$ such that $\kappa^{-1} < \|D\| < \kappa$, $\|B^{ab}\| < \kappa$, where the norm used here is the operator norm in $H$.

We are now able to derive a local existence result for solutions of the conformal EMYM equations which will allow us to prove Theorem (9.5).

**Lemma (9.6).** *Suppose $s > \frac{3}{2} + 1$ and, with the notation introduced above,*

*$X$ is a bounded open subset of $H^s(\mathbb{R}^3, H)$ such that for $v \in X$ the following conditions hold:*

$$\text{(i) } \|\tau^0 - \hat{\tau}^0\|_s,\ \|e^0{}_{ab} - \hat{e}^0{}_{eb}\|_s \text{ are smaller than } \frac{\hat{\tau}^0(1-\delta)}{5\kappa^2\kappa_s}, \tag{9.8}$$

$$\text{(ii) } \|\tau^A - \hat{\tau}^A\|_s,\ \|e^A{}_{ab} - \hat{e}^A{}_{ab}\|_s \text{ are smaller than } \frac{15\kappa_s(\theta - 1)}{\theta + 1}.$$

*Then the following hold*:

(i) *For initial data $v_0$ in $X$ there is a unique solution $v$ of IVP defined on $[\tau, \tau + \eta']$, where $0 < \eta' \leq \eta$, such that*

$$v \in C[\tau, \tau + \eta'; X] \cap C^1[\tau, \tau + \eta'; H^{s-1}(\mathbb{R}^3, H)].$$

(ii) *The restriction of the solution $v$, which can be identified with a function $v = v(t, x^\alpha)$ of class $C^1$ defined on $\mathbb{R}^3 \times [\tau, \tau + \eta']$, to $D_{\eta'}(\overline{U}')$ is smooth with bounded uniformly continuous derivatives of all orders. On $D_{\eta'}(\overline{U}')$, $u(t, x^\alpha) = v(t, x^\alpha) + \hat{u}(t, x^\alpha)$ solves the conformal EMYM equations and is determined uniquely by the data on $\overline{U}'$.*

The statement on the existence and uniqueness of the solution follows from Theorem II of [26]. That conditions (4.2)–(4.7) of that theorem are satisfied follows from the discussion of the structure of equation (8.11) and from (8.12). Condition (9.8i) implies that

$$\|C^0[\hat{u} + v]\| \geq \kappa^{-1}\hat{\tau}^0\delta\kappa' \quad \text{for } v \in X,$$

where again the operator norm in $H$ is used. Comparing this with condition (4.9) of [26, Theorem II] and observing remark (3.1b) of [26], we obtain our result (i).

Condition (9.8ii) has been given to ensure that the vector fields $\tau = \tau^A c_A$ and $e_{ab} = e^A{}_{ab} c_A$ supplied by $u$ are linearly independent and that the metric $g$ given by (8.15) is such that for tangent vectors $Z \neq 0$ on $D_{\eta'}(\overline{U}')$ for which $g(Z, Z) \geq 0$ one has $g^*(Z, Z) > 0$. This implies that the boundary of $D_{\eta'}(\overline{U}')$ is space-like for $g$ and that the future domain of dependence of $\overline{U}'$ in $D_{\eta'}(\overline{U}')$ with respect to $g$ coincides with $D_{\eta'}(\overline{U}')$. Since we have (8.16) it follows by a standard argument for symmetric hyperbolic systems that the restriction of $u$ to $D_{\eta'}(\overline{U}')$ depends only on the given data $\overline{U}'$. The smoothness of this restriction follows from the smoothness of the original data near $\overline{U}'$ and from the smoothness of the coefficients of our equation (8.11) by a standard result on smoothness of symmetric hyperbolic systems. That $u$ defines on $D_{\eta'}(\overline{U}')$ a solution of the conformal EMYM equations is a consequence of the Reduction Theorem (8.1).

We are now in a position to prove Theorem (9.5). We assume that $\tau = 0$ in the considerations above and observe that by our construction the reference fields $\hat{u}$ on $D_\eta(\overline{U}')$ satisfy on $\overline{U}' \subset S_0$ the gauge condition (9.2). We may assume that the asymptotic initial data $u$ are given on $S_0$ with respect to the function $\Lambda_0$ which is supplied by the reference field. On $\overline{U}'$ we obtain for the difference field $v = u - \hat{u}$ that $\tau^A - \hat{\tau}^A = 0$, $e^A{}_{ab} - \hat{e}^A{}_{ab} = 0$ and this holds by (9.7) also for the extended data. Thus, conditions (9.8) will always be satisfied by the extended data for suitably chosen $X$ and we may assume for our data on $\overline{U}'$ by Lemma (9.6) the existence of a solution of the conformal EMYM equation on a set $D_{\eta'}(\overline{U}')$ for some $\eta' > 0$. If two of these sets have nonempty intersection the corresponding solutions are related on the intersection to each other by a known transformation of the local coordinates $x^\mu$ and by a known transition between the local sections which have been constructed. Therefore, by the compactness argument above, there exists an evolution $M^* = \overline{E}_{0,\delta}$ of $S_0$ with some evolution time $\delta > 0$ such that on $(P^*, M^*, \pi^*)$ with $P^* = \pi^{-1}(M^*)$ and $\pi^*$ the restriction on $\pi$ to $M^*$ we obtain a smooth solution of the conformal EMYM equations which is uniquely determined by the given conformal data. We may assume, possibly after choosing $\delta$ small enough, that the conformal factor $\Omega$ supplied by this solution is positive in the future of $S_0$. Using the conformal factor to rescale the fields appropriately on $M^* \backslash \{S_0 \cup S_\delta\}$ and restricting to this set the bundle $(P^*, M^*, \pi^*)$, we obtain a smooth asymptotically simple solution of the EMYM equations for which the sets $S_\tau$, $0 < \tau < \delta$, are Cauchy hypersurfaces.

We considered above only the solution of the Cauchy problem in the future direction. Since we are dealing with symmetric hyperbolic systems there also exists backward evolutions. Thus we obtain for Theorem (9.6) the

**Corollary (9.7).** *Suppose $(\widetilde{P}_0, \widetilde{M}_0, \tilde{\pi}_0, \widetilde{A}_0, \tilde{g}_0)$ is a smooth asymptotically simple solution with structure group $G$ (as in* §7*) of the EMYM equations with cosmological constant $\lambda_0 < 0$. Let $(P_0, M_0, \pi_0, A_0, g_0, \Omega)$ be a smooth conformal extension. Then there exists a solution $(P, M, \pi, A, g)$ with structure group $G$ of the conformal EMYM equations and a morphism $(\Phi, \varphi)$ of the bundle $(P_0, M_0, \pi_0)$ into $(P, M, \pi)$ such that:*

(i) *$M$ is a manifold without boundary and $(M, g)$ is globally hyperbolic.*

(ii) *$\varphi$ is an embedding of $M_0$ into $M$.*

(iii) *$(\Phi, \varphi)$ is a bundle isomorphism onto $(\pi^{-1}(\varphi(M_0)), \varphi(M_0), \pi)$.*

(iv) *The pullbacks of $A$, $g$, and $\Omega$ by $\varphi$ and $\Phi$ coincide with $A_0$, $g_0$, and $\Omega_0$, respectively.*

In other words, $(P, M, \pi, A, g, \Omega)$ extends $(P_0, M_0, \pi_0, A_0, g_0, \Omega_0)$ as a solution of the conformal EMYM equations "beyond the conformal boundary $\mathscr{I}^- \cup \mathscr{I}^+$".

The example of the Schwarzschild-De-Sitter solution also shows that from arbitrary standard Cauchy data we cannot expect to obtain an asymptotically simple solution. However, in the following we will specify conditions under which we can prove a stability theorem for asymptotically simple space-times, which shows that not only the null geodesic completeness is preserved under sufficiently small though finite perturbations of the solution, but that even the smoothness of the asymptotic structure is retained.

**Theorem (9.8)** (Stability of asymptotic simplicity). *Suppose $(\widetilde{P}_0, \widetilde{M}_0, \tilde{\pi}_0, \widetilde{A}_0, \tilde{g}_0)$ is a smooth asymptotically simple solution with structure group $G$ (as specified in* §7*) of the EMYM equations with cosmological constant $\lambda_0 < 0$. Let $(P_0, M_0, \pi_0, A_0, g_0, \Omega_0)$ be a smooth conformal extension of this solution which implies asymptotic initial data $u_0$ on $(\pi_0{}^{-1}(S_0), S_0, \pi_0)$ with $S_0 = \mathscr{I}_0^-$ (the past part of the conformal boundary of the solution). Then there exists a neighborhood $W$ of $u_0$ in the topology defined by any of the distance functions* (9.6) *on the sets of smooth asymptotic initial data with cosmological constant $\lambda_0$ on $(\pi_0{}^{-1}(S_0, S_0, \pi_0)$ such that any such initial data set in $W$ determines a unique smooth asymptotically simple* *solution of the EMYM equations with cosmological constant* $\lambda_0$.

**Remarks.** (i) The stability theorem has been formulated here in terms of asymptotic initial data since we already discussed these data in detail above. It could also be stated (and proven in a similar way) in terms of standard initial data given with respect to some Cauchy hypersurface $S$ in $\widetilde{M}$.

(ii) Since the De-Sitter solution satisfies the conditions of the theorem, this stability result for asymptotically simple space-time implies the global existence and asymptotic simplicity of solutions for data which are close to De-Sitter data.

(iii) Again it can be seen from the proof and from [26] that the smoothness assumptions made in the theorem on the given solution and on the solution whose existence is shown could be relaxed.

We shall use in the following a solution $(P, M, \pi, A, g, \Omega)$ of the conformal EMYM equations which is related to our given solution as described in Corollary (9.7) as a reference field. We identify $(P_0, M_0, \pi_0, A_0, g_0, \Omega_0)$ with its image under the morphism $(\Phi, \varphi)$ and extend the time orientation given on $M_0$ to $M$. We make a definite choice of the conformal factor $\Omega$ which on $S_0$ is consistent with the gauge condition (9.2). Restricting $M$ if necessary, we may assume that $\Omega < 0$ on $M \backslash M_0$. Let $S_1$ be a Cauchy hypersurface for $(M, g)$ in the future of $\mathscr{J}_0^+$ (the future part of the conformal boundary of our solution) such that $\Omega < 0$ on $S_1$. By the arguments used in the proof of Lemma (9.3) there exists a smooth time function $t$ on $M$ with future directed time-like gradient such that $t = 0$ on $S_0 = \mathscr{J}_0^-$ and $t = 1$ on $S$ (whence $0 \leq t \leq 1 - \varepsilon'$ on $M_0$ for some $\varepsilon' > 0$) and such that for some $\gamma > 0$ any set $S_\tau = \{x \in M | t(x) = \tau\}$ for $\tau \in I_\gamma = [-\gamma, 1+\gamma]$ is a Cauchy hypersurface for $(M, g)$. On $M$ we define a future directed time-like unit vector field by setting $c_0 = f \operatorname{grad}_g t$ with a suitable positive function $f$. We may assume that the time function $t$ has been chosen such that $\nabla_{c_0} c_0 = 0$ on $S_0$ (which happens, e.g., if $t$ coincides near $S_0$ with the time function arising from a Gauss coordinate system based on $S_0$).

As described in Lemma (9.4ii) we choose a frame field $c_A$, $A = 1, 2, 3$, on $S_0$ which is orthonormal for the interior metric implied on $S_0$ and extend the vector fields $c_A$ to $M$ as solutions of the Fermi-transport equation (9.5) in the direction of $c_0$ to obtain together with $c_0$ a smooth orthonormal frame for $g$ on $M$. Let $T$ be the horizontal lift of $c_0$ to $P$ with respect to $A$.

For given number $\theta > 1$ we can define on $M$ with respect to the frame $c_A$ a metric $g^*$ as described in the proof of Theorem (9.5). Since $(M, g)$

is globally hyperbolic we can choose $\theta$ such that the space-time obtained by restriction of $g^*$ to $M'_\gamma = \{x \in M | t(x) \in I_\gamma\}$ is still globally hyperbolic [22].

With respect to the structures introduced above we can now consider for given $\tau \in I_\gamma$ on $S_\tau$ standard neighborhoods $U'$ with evolution $D_\eta(\overline{U}')$ and associated neighborhood $U$ as described in the proof of Theorem (9.5). As seen there $S_\tau$ can be covered by a finite number of such standard neighborhoods such that from the union of their corresponding evolutions we obtain an evolution of $S_\tau$ with some positive evolution time. Since $M'_\gamma$ is compact, it can be covered by a finite number of such evolutions. Taking suitable intersections of these evolutions and of the evolutions with $M'_\gamma$ one finds the following to be true. There are numbers $\tau_0, \cdots, \tau_p$, $p \in \mathbb{N}$, with $0 = \tau_0 < \tau_1 < \cdots < \tau_p = 1$ such that for $i = 0, \cdots, p-1$ there is an evolution $\overline{E}_{\tau_i, \eta_i}$ of $S_{\tau_i}$ with evolution time $\eta_i = \tau_{i+1} - \tau_i$ which is generated by evolutions $D_{\eta_i}(\overline{U}'_{i,j})$ of standard neighborhoods $U'_{i,j}$ in $S_{\tau_i}$, where $j = 1, \cdots, q_i$ with a certain $q_i \in \mathbb{N}$.

As described in the proof of Theorem (9.5) we construct for any $i, j$ with $0 \le i \le p-1$, $1 \le j \le q_i$ coordinates $x^\mu$ and a local section of $P$ in an open neighborhood of the set $D_{\eta_i}(\overline{U}'_{i,j})$ and identify this neighborhood with its image in $\mathbb{R}^4$ under the coordinate map. In the given gauge the reference solution $(P, M, \pi, g, A, \Omega)$ determines in a neighborhood of $D_{\eta_i}(\overline{U}'_{i,j})$ a reference field $\hat{u}_{i,j} = \hat{u}_{i,j}(x^\mu)$ as given in (8.5).

Suppose that for certain data on $S_0$ close to $u_0$ we have been able to show for some $i < p$ the existence of a solution of the conformal EMYM equations on $(\pi^{-1}(M_i^*), M_i^*, \pi|_{M_i^*})$, where $M_i^* = \{x \in M | 0 \le t(x) \le \tau_i\}$. Denote by $u_{i,j} = u_{i,j}(x^\alpha)$ the initial data of the form (8.1) implied in a neighborhood of $U'_{i,j}$ in $S_{\tau_i}$ in the gauge associated with the neighborhood $U'_{i,j}$ (of course in the case $i = 0$ these arise from $u_0$). We want to show that under suitable conditions these data determine a solution to the conformal EMYM equations on all of $D_{\eta_i}(\overline{U}'_{i,j})$. To prepare the data for the initial value problem which we want to investigate for this purpose, set $v(x^\alpha) = u_{i,j}(x^\alpha) - \hat{u}_{i,j}(\tau_i, x^\alpha)$ on $\overline{U}'_{i,j} \subset \mathbb{R}^3 \times \{\tau_i\}$ and extend $v$ to $v_0 \in H^s(\mathbb{R}^3 \times \{\tau_i\}, H)$ in the same way as described in the proof of Theorem (9.5), such that we have in particular relation (9.7).

We extend the fields $c_A$ and the reference field $\hat{u}_{i,j}$ on $D_{\eta_i}(\overline{U}'_{i,j})$ smoothly to a neighborhood of $\mathbb{R}^3 \times [\tau_i, \tau_{i+1}]$ in $\mathbb{R}^4$ such the extensions have the same properties as described earlier. With these extended fields we again write the equation (8.11) on $\mathbb{R}^3 \times [\tau_i, \tau_{i+1}]$. Since the reference field $\hat{u}_{i,j}$ is derived from a solution of the conformal EMYM equation, the function $f$ obtained by this procedure vanishes on $D_{\eta_i}(\overline{U}'_{i,j})$ by (8.14). We shall now set $f = 0$ everywhere on $\mathbb{R}^3 \times [\tau_i, \tau_{i+1}]$. While the equation so obtained from (8.11) is still equivalent to the symmetric hyperbolic propagation equations on $D_{\eta_i}(\overline{U}'_{i,j})$, outside this set this is in general not the case. However, we have gained by this procedure a symmetric hyperbolic system on $\mathbb{R}^3 \times [\tau_i, \tau_{i+1}]$ which is homogeneous everywhere by (8.13).

The initial value problem which we have formulated now will be referred to as IVPH in the following. We state now those properties of IVPH which are of interest to us, where again Kato's notation is employed.

**Lemma (9.9).** (i) *Suppose $s > \frac{3}{2} + 1$. There exists a positive number $\rho$ such that for any initial data $v_0 \in H^s(\mathbb{R}^3, H)$ with $\|v_0\|_s \leq \rho$ there is a unique solution $v(t)$ of IVPH defined on $[\tau_i, \tau_{i+1}]$. If $v_0^k$, $k \in \mathbb{N}$, are initial data with $\|v_0^k\|_s \leq \rho$ such that $\|v_0^k - v_0\|_s \to 0$ as $k \to \infty$ then the solutions $v^k(t)$ determined by $v_0^k$ satisfy $\|v^k(t) - v(t)\|_s \to 0$ uniformly in $t \in [\tau_i, \tau_{i+1}]$.*

(ii) *The solution $v(t)$ above can be identified on $D_{\eta_i}(\overline{U}'_{i,j})$ with a smooth function $v = v(t, x^\alpha)$ which has bounded and uniformly continuous derivatives of all order. The function $u(t, x^\alpha) = v(t, x^\alpha) + \hat{u}_{i,j}(t, x^\alpha)$ solves the conformal EMYM equations on $D_{\eta_i}(\overline{U}'_{i,j})$ and is determined uniquely by the data $u_{i,j}(x^\alpha)$ on $\overline{U}'_{i,j}$.*

If $\rho$ is chosen sufficiently small conditions (9.8) of Lemma (9.6) will be satisfied and thus the existence of a solution with a certain positive life time follows as in Lemma (9.6). Since the equation (8.11) with $f \equiv 0$ admits the trivial solution $v \equiv 0$, the fact that the solution $v(t)$ exists on all of $[\tau_i, \tau_{i+1}]$ and the stability property stated in (i) are a consequence of Theorem IIIb in [26]. That condition (4.11) of this theorem is satisfied in the present case follows immediately from (8.11), (8.12) and the smoothness of the extension of $\hat{u}_{i,j}$ in a neighborhood of $\mathbb{R}^3 \times [\tau_i, \tau_{i+1}]$ in $\mathbb{R}^4$. The second part of the lemma follows by the same reasoning as Lemma (9.8ii).

It follows from Lemma (9.9) that if the solution of the conformal EMYM equations, whose existence up to $(\pi^{-1}(S_{\tau_i}), S_{\tau_i}, \pi)$ is assumed close enough to the given solution in the sense that the distance

$$d_i = \sum_{1 \leq j \leq q_i} \|(u_{i,j} - \hat{u}_{i,j}) \mid \overline{U}'_{i,j}\|,$$

is sufficiently small, then the solution extends in fact as a smooth solution to $(\pi^{-1}(M^*_{i+1}), M^*_{i+1}, \pi|_{M^*_{i+1}})$. Moreover, it follows that the corresponding distance $d_{i+1}$ on $S_{\tau_i+1}$ will become as small as well like if we let $d_i$ become sufficiently small. (This conclusion requires the transition between two different distance functions of the type (9.6) on $S_{\tau_i+1}$, but by the remark following (9.6) this causes no difficulty.)

Since only $p$ evolutions are required to reach $S_1$, it follows that there is a number $\varepsilon > 0$ such that for initial data on $S_0$ with $d_0 < \varepsilon$ the corresponding solution will exist on $(\pi^{-1}(M_1), M_1, \pi|_{M_1})$, where $M_1 = \{x \in M | 0 \leq t(x) \leq 1\}$. Furthermore, by choosing $\varepsilon$ small enough we can ensure by Lemma (9.9i) that the conformal factor $\Omega$ supplied by any of these solutions on $M_1$ will be negative everywhere on $S_1$, since this is true for $\Omega_0$. We choose to take as neighborhood $W$ of $u_0$ the open ball of data for which $d_0 < \varepsilon$.

Given any solution of the conformal EMYM equations on $M_1$ for data in $W$, the desired solutions of the EMYM equations are obtained in the following way. We set $\mathscr{J}^- = S_0$ since $\Omega = 0$ on this set. Along any integral curve of the vector field $c_0$, which starts from $S_0$ into the future, $\Omega$ will become positive in the immediate future of $S_0$ because of equations (5.13) and (9.2). Then there will be on the integral curve a first point in the future of $S_0$ at which $\Omega = 0$. We denote the set of such points by $\mathscr{J}^+$ and the set of points between $\mathscr{J}^-$ and $\mathscr{J}^+$ where $\Omega > 0$ is denoted $\widetilde{M}$. It follows from (2.20) that $\mathscr{J}^+$ is a smooth spacelike hypersurface in $M$. After a suitable rescaling of the fields on $\widetilde{M}$ the asymptotically simple solution whose existence has been asserted is obtained on $(\pi^{-1}(\widetilde{M}), \widetilde{M}, \pi|_{\widetilde{M}})$.

## 10. Existence results: the case $\lambda = 0$

To construct global or semiglobal solutions to the EMYM equations with cosmological constant $\lambda = 0$ one may prescribe, as suggested by the procedure and the results of §9, conformal standard Cauchy data, now on a manifold diffeomorphic to $\mathbb{R}^3$ (say), which satisfy fall-off conditions at infinity, and try to study the evolution of the corresponding solution of the conformal EMYM equations into a region which comprises a hypersurface

$\mathscr{I} = \{\Omega = 0, d\Omega \equiv 0\}$. Again this surface could be considered as representing endpoints at infinity of physical null geodesics (and now only of these (cf. [21])). However, the geometrical structure of "conformal infinity" in the case $\lambda = 0$ is radically different from that observed previously, as may be surmised already from the fact that equation (2.20) forces $\mathscr{I}$ to be a null hypersurface. This will be illustrated now by analyzing the conformal structure at infinity for Minkowski space, which represents the simply connected, geodesically complete, conformally flat standard situation of the case $\lambda = 0$.

An analytic solution of the source-free conformal Einstein equations with $\lambda = 0$ is provided by

(10.1)
$(P_0, M_0, \pi_0, A_0, g_0, \Omega_0)$ with $P_0 = M_0 \times G$, $M_0 = \mathbb{R} \times S^3$,
$\pi_0$ the natural projection onto $M_0$, $A_0$ such that the section
$M_0 \ni x \to (x, s) \in P_0$, $s \in G$, is horizontal,
$g_0 = dt'^2 - d\omega^2$, $\Omega = \cos t' + \cos\chi$,

where $t' \in \mathbb{R}$ and $d\omega^2$ is the standard line element on the three-dimensional unit sphere given in standard spherical coordinates $\varphi, \theta, \chi$, with $0 \le \varphi < 2\pi$, $0 \le \theta \le \pi$, $0 \le \chi \le \pi$, by

$$d\omega^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{10.2}$$

The solution has Ricci scalar $\Lambda_0 = \Lambda_0 = \text{const} > 0$. A maximal connected subset of $M_0$, where $\Omega_0$ is positive, is given for example by $\widetilde{M} = \{0 \le \chi < \pi, \chi - \pi < t < \pi - \chi\}$ and the restriction of $\tilde{g} = \Omega_0{}^{-2} g_0$ to this submanifold yields a space-time $(\widetilde{M}, \tilde{g})$ isomorphic to Minkowski space.

Denoting the point $\{\chi = 0\}$ of $S^3$ by $N$ and $\{\chi = \pi\}$ by $S$, the set $\widetilde{M}$ can also be described in the following way. The null geodesics of $g_0$ which start from $i^- = (-\pi, N) \in M_0$ into the future, sweep out a smooth null hypersurface $\mathscr{I}^-$, on which $\Omega_0 = 0$ and $d\Omega_0 \ne 0$, until they reconverge again at the point $i^0 = (0, S)$. After passing $i^0$ they generate another smooth null hypersurface $\mathscr{I}^+$, on which again $\Omega_0 = 0$ and $d\Omega_0 \ne 0$, until they reconverge at the point $i^+ = (\pi, N)$. The "physical" space $\widetilde{M}$ is the intersection of the time-like future of $i^-$ with the time-like past of $i^+$. The surface $\mathscr{I}^-$ $(\mathscr{I}^+)$ is past (future) conformal infinity for the physical space-time $(\widetilde{M}, \tilde{g})$. While the physical null geodesics have past and future endpoints on $\mathscr{I}^-$ and $\mathscr{I}^+$, respectively, physical geodesics which are time-like have past (future) endpoints at "past (future) time-like infinity" $i^-$ (resp. $i^+$). The point $i^0$, "space-like infinity", is the endpoint (in both directions) of space-like geodesics. The points $i^0$, $i^{\pm}$ are nondegenerate critical points of the function $\Omega_0$ which also vanishes at these points. Any Cauchy hypersurface of the physical space-time "touches" $i^0$ and is compactified to a topological three-dimensional sphere by the inclusion of $i^0$. We will again call the manifold $M = \widetilde{M} \cup \mathscr{I}^- \cup \{i^-, i^0, i^+\}$ together with the fields obtained as restrictions of $\Omega_0$, $g_0$ to $M$ a conformal extension of $(\widetilde{M}, \tilde{g})$.

Generalizing this situation one would expect to find solutions of the EMYM equations which satisfy conditions (i)–(iv) and (vi) of definition (9.1). Such space-times are called "asymptotically flat" ([28], [29], [30]). The basic open question we are interested in is whether suitable though not too special standard Cauchy data on $\mathbb{R}^3$ evolve into solutions of the EMYM equations, which allow the construction of a conformal extension which resembles that of Minkowski space even more completely. The somewhat vague formulation of this question is due to the following fact. If we prescribe standard Cauchy data on $\mathbb{R}^3$ with the "usual" fall-off conditions at infinity (cf. [9]) and try to compactify $\mathbb{R}^3$ by adjoining a point $i^0$ that represents space-like infinity in such a way that the conformal geometry on $\mathbb{R}^3$ extends to $i^0$ as smoothly as possible, we find that after the corresponding rescaling some of the nonphysical fields given by (8.1) are in general not smooth and some become even unbounded near $i^0$ (cf. [18], for examples). This may not appear to be particularly surprising, since the point $i^0$, i.e., spatial infinity, is the place where the mass of the gravitational field manifests itself and it is well known that fields with mass are not well behaved under conformal rescalings. In fact, the conformal fields are bounded at $i^0$ only if the mass vanishes. On the other hand, the concept of the mass of a gravitational field as well as the rescaling behavior of the gravitational field equations are very special as compared with other fields. An understanding of the structure of the "singularity at $i^0$" should give much deeper insight into the meaning of "mass", the asymptotic structure of the fields, and the concepts associated with "null infinity" like "Bondi-mass", "radiation field", etc. ([29], [30], [31]). Moreover, it may be expected that a thorough analysis of the consequences of the conformal rescaling behavior of the field equations for the evolution of the fields near $i^0$ will shed considerable light onto the conformal, respectively causal, structure of "physical" space-time singularities and the related idea of cosmic censorship. Because of the singularity of the conformal fields at $i^0$ it is clear that we cannot expect in the case of nonvanishing mass to have a conformal extension of the evolution of the given data which is diffeomorphic to the conformal extension of Minkowski space described above. However, we may still ask if the maximal development has a smooth structure at null infinity which allows the construction of a conformal extension of the fields with smooth hypersurfaces $\mathcal{J}^-$, $\mathcal{J}^+$, to which the fields given by (8.1) extend smoothly, such that $\mathcal{J}^-$, $\mathcal{J}^+$ are diffeomorphic to $\mathbb{R} \times S^2$ and are complete in the sense that for a choice of conformal factor which makes these hypersurfaces expansion free the generating null geodesics on $\mathcal{J}^-$, $\mathcal{J}^+$ are complete (as it is the case for the conformal completion of Minkowski space given above), and finally, whether the solution is regular at past and future time-like infinity in the sense that in suitable conformal extensions there exist points $i^-$ $(i^+)$ such that $\mathcal{J}^-$ $(\mathcal{J}^+)$ is swept out by the future (past) directed null geodesics through $i^-$ $(i^+)$.

It is therefore reasonable to split the investigation of the conformal structure at infinity into two subproblems. There is the "$i^0$-problem", which asks for the asymptotic behavior of the solution near spatial infinity, in particular, whether "near" (in the conformal picture) $i^0$ it is possible to construct conformal extensions which contain smooth "pieces of surfaces $\mathcal{J}^-$, $\mathcal{J}^+$" diffeomorphic to $\mathbb{R} \times S^2$, to which the fields given by (8.1) extend smoothly. If the answer is positive, one has to investigate the "hyperboloidal initial value problem". Here Cauchy data for the conformal EMYM equations are prescribed on a space-like hypersurface $S$ with boundary $\partial S$ (diffeomorphic to $S^2$), which is thought to have intersection $\partial S$ with $\mathcal{J}^+$ (say). The basic question then is whether these data evolve into a solution of the EMYM equations which has a smooth, future complete structure at future null infinity and which is regular at future time-like infinity in the sense described above.

So far the $i^0$-problem, which from the point of view of differential geometry as well as of PDE theory poses an exceedingly difficult question (cf. [18] for further discussion), is still unsolved for data of any generality. The hyperboloidal initial value problem has been discussed in [13], [17] in the source-free case and will be analyzed here for the EMYM equations. By the topology of the space-times over which we want to construct a solution and because of the triviality of the solution (10.1) of the conformal EMYM equations which we shall use to construct a reference field we can assume that our solution lives on a trivial bundle and express the field equations with respect to one globally defined section.

**Definition (10.1).** A "hyperboloidal initial data set for the conformal EMYM equations" consists of a pair $(\overline{S}, u_0)$ such that:

(i) $\overline{S} = S \cup \partial S$ is a smooth manifold with boundary $\partial S$ and interior $S$ which is mapped by a certain map $\varphi : \overline{S} \mapsto \mathbb{R}^3$ diffeomorphically onto the closed unit ball in $\mathbb{R}^3$. We denote by $x^\alpha$, $\alpha = 1, 2, 3$, the coordinates obtained on $\overline{S}$ as pullbacks under $\varphi$ of the natural coordinates on $\mathbb{R}^3$.

(ii) $u_0 = (\tau^A, e_{ab}, \Gamma_{abdc}, \Omega, \Sigma_{ab}, \Phi_{abcd}, \varphi_{abcd}, A_{ab}, \varphi_{ab}, \psi_{abcd})$ is a collection of fields $\overline{S}$ which have a similar meaning as those given in (8.1) (with respect to the Lie algebra $\mathfrak{g}$) and which satisfy the gauge conditions: $\tau^A = \sqrt{2}\delta_0^A$, $e_{ab}$ is a frame field on $\overline{S}$ which can be written in the form $e_{ab} = \hat{e}^A{}_{ab} c'_A$ with a certain real frame field $c'_A = c'^\alpha{}_A \partial/\partial x^\alpha$, $A = 1, 2, 3$, on $\overline{S}$, $\Gamma^{ab} = 0$, $\alpha = 0$.

(iii) The coefficients $c'^\mu{}_A$ and the fields $\Gamma_{abdc}, \Omega, \Sigma_{ab}, s, \Phi_{abcd}, \varphi_{abcd}, A_{ab}, \varphi_{ab}, \psi_{abcd}$ have with respect to the coordinate system $x^\alpha$ bounded and uniformly continuous derivatives of all orders.

(iv) On $S : \Omega > 0$, on $\partial S : \Omega = 0$, $\Sigma^{ab}\Sigma_{ab} = 0$, $\Sigma^f{}_f < 0$.

(v) The fields provided by $u$ satisfy the conformal constraint equations (5.8)–(5.12), (6.12), (6.20), (7.15), (7.16), and (7.24) on $\overline{S}$, where (7.12) is used to calculate $\psi_{abcd}$. (As a consequence of these conditions and of (iv), (2.20) is also satisfied with $\lambda = 0$.)

**Remark.** The strong smoothness requirements are made again for convenience. The gauge conditions can always be satisfied and impose no restriction on the space-times we want to construct. The properties which are specific for hyperboloidal data are stated by conditions (iv). They comprise the requirement that $\partial S$ be part of a null hypersurface which represents future null infinity $\mathscr{I}^+$ for some space-time. If the data fields provided by $u$ are suitably rescaled on $S$ by using the conformal factor $\Omega$ one obtains a standard initial data set for the EMYM equations which comprise in particular a "physical" interior metric on $S$. The Riemannian space so obtained behaves near "infinity", which in the present case is represented by the set $\partial S$, like a space of constant negative curvature as opposed to the situation considered in the standard Cauchy problem for the EMYM equations where the data behave near infinity similar to a Euclidean space [9].

We shall describe now a particular class of hyperboloidal initial data and a reference field, derived from (10.1), which will be used in the proof of the existence of solutions of the conformal EMYM equations for general hyperboloidal initial data. Let $S^*_M$ be a smoothly embedded space-like Cauchy hypersurface (necessarily diffeomorphic to $S^3$) of the solution (10.1) of the conformal EMYM equations, which intersects each null generator of the null hypersurface $\mathscr{I}^+$ associated to Minkowski space as described above. By the arguments given in §9 there exists a smooth time-function $t$ on $M_0$ with future directed time-like gradient with respect to $g_0$ such that $t = 0$ on $S_M^*$ and such that $t = t'$ in the part of $M_0$ where $t' \geq \frac{3}{2}\pi$. We define a smooth time-like unit vector field by setting $c_0 = f \operatorname{grad}_{g_0} t$ with a suitable positive function $f$ on $M_0$. Again we can assume that the $t$ has been chosen such that $\nabla^0_{c_0} c_0 = 0$ on $S_M^*$, where $\nabla^0$ denotes the Levi-Civita connection associated with $g_0$. On $S_M^*$ choose a frame $c_A$, $A = 1, 2, 3$, which is orthonormal for the interior metric induced on $S_M^*$ by $g_0$ and extend the fields $c_A$ to $M_0$ by solving the Fermi-transport equations (9.5) with respect to $\nabla^0$ to obtain together with $c_0$ a smooth orthonormal frame $c_A$, $A = 0, 1, 2, 3$, for $g_0$ on $M_0$. As discussed in §8 we use $\tau^{aa'}$ given in (8.2) and the frame $\hat{e}_{aa'} = \hat{e}^A_{aa'} c_A$ to describe the solution (10.1) by a collection of fields on $M_0$ which we denote by

(10.3)
$$\hat{u} = (\hat{\tau}^A, \hat{e}^A{}_{ab}, \widehat{\Gamma}_{adbc}, \widehat{\Omega}, \widehat{\Sigma}_{ab}, \hat{s}, \widehat{\Phi}_{abcd}, \hat{\varphi}_{abcd}, \widehat{A}_{ab}, \hat{\varphi}_{ab}, \hat{\psi}_{abcd}),$$

where in particular $\widehat{\Omega} = \Omega_0$, $\hat{\varphi}_{abcd} = 0$, and, by our choice of section of $P_0$, $\widehat{A}_{ab} = 0$, $\hat{\varphi}_{ab} = 0$, $\hat{\psi}_{abcd} = 0$.

Denoting by $S_M$ that part of $S_M^*$ on which $\Omega_0 \geq 0$, a hyperboloidal initial data set is given by $(S_M, \hat{u}_0)$, where $\hat{u}_0 = \hat{u}|_{S_M}$. We call an initial data set for the conformal EMYM equations which is obtained in this way a "Minkowskian hyperboloidal initial data set". We shall say that $(S_M, \hat{u}_0)$ is obtained by restriction from the Cauchy data set $(S_M^*, \hat{u}|_{S_M^*})$ for the solution (10.1) of the conformal EMYM equations. A given Minkowskian hyperboloidal initial data set may of course be obtained by restriction from different Cauchy data sets for (10.1).

Let $(\overline{S}, u_0)$ be a hyperboloidal initial data set for the conformal EMYM equations. To compare this set with some Minkowskian hyperboloidal initial data set $(S_M, \hat{u}_0)$ (such that both sets are defined with respect to the same Lie group $G$, resp. Lie algebra $\mathfrak{g}$) assume that the latter is obtained by restriction from an initial data set $(S_M^*, \hat{u}|_{S_M^*})$. Let $\chi: \overline{S} \to S_M^*$ be a smooth embedding which maps $\overline{S}$ onto a closed "ball" in $S_M^*$ that may but need not coincide with $S_M$. We use $\chi$ to identify $\overline{S}$ with its image and to carry the fields provided by $u_0$ to $S_M^*$. On $\overline{S}$ we use the frame $c_A$ to expand the frame $e_{ab} = e^A{}_{ab} c_A$, where $e^0{}_{ab} = 0$, we write $u_0 = (\tau^A, e^A{}_{ab}, \Gamma_{abdc}, \Omega, \Sigma_{ab}, s, \Phi_{abcd}, \varphi_{abcd}, A_{ab}, \varphi_{ab}, \psi_{abcd})$, and we define the difference field $v_0 = u_0 - \hat{u}_0$ as in (8.10). Let $h$ be any Riemannian metric on $S_M^*$. Fix now $s, j \in \mathbb{N}$, $s > \frac{3}{2} + j$, $j > 2$. With respect to $h$ we define Sobolev spaces $H^s(\overline{S}, H)$ and $H^s(S_M^*, H)$ with norms $\|\cdot\|_s$ and $\|\cdot\|_s^*$, respectively, where $H$ denotes a Hilbert space as in §8. There is an extension operator which maps any function $v_0 \in H^s(\chi(\overline{S}), H)$ onto a function $v \in H^s(S_M^*, H)$ such that (with an obvious meaning of the norms) we have for their components $v_0^i$, resp. $v^i$,

$$\|v^i\|_s^* \leq \kappa_e \|v_0^i\|_s, \qquad v_0 \in H^s(\chi(\overline{S}), H), \tag{10.4}$$

with a positive constant $\kappa_e$ not depending on $v_0$ [1]. We choose the extension operator such that the extensions of the fields $\tau^A - \hat{\tau}^A$, $e^0{}_{ab} - \hat{e}^0{}_{ab}$, $\alpha - \hat{\alpha}$, $\Gamma^{ab} - \hat{\Gamma}^{ab}$, which vanish on $\chi(\overline{S})$, vanish everywhere on $S_M^*$. We remark that the embedding $\chi$ must identify $(\overline{S}, u_0)$ with $(S_M, \hat{u}_0)$ if $\|v_0\|_s$ and thus $\|v\|_s^*$ vanishes. Let $\varepsilon$ be a positive real number. We shall say that "the hyperboloidal initial data set $(\overline{S}, u_0)$ is in an $\varepsilon$-neighborhood of the Minkowskian hyperboloidal initial data set $(S_M, \hat{u}_0)$" if the initial data set $(S_M^*, \hat{u}|_{S_M^*})$, the embedding $\chi$, and the extension $v$ of the difference field $v_0$ can be chosen such that $\|v\|_s^* < \varepsilon$.

**Theorem (10.2).** *Suppose $(\overline{S}, u_0)$ is a hyperboloidal initial data set for the conformal EMYM equations. Then the following hold*:

(i) *There exists a smooth solution $(P, M, \pi, A, g, \Omega)$ of the conformal EMYM equations with $\lambda = 0$, unique up to the possibility to construct extensions into the future, which has the following properties. $P$ is a trivial bundle. In $M$ there is a smooth space-like Cauchy hypersurface $\overline{S}'$ on which the solution implies, with respect to a suitably chosen section of $P$ and a suitable choice of the other gauge conditions, hyperboloidal Cauchy data $u'_0$, which can be identified diffeomorphically with $(\overline{S}, u_0)$. The manifold $M$ has boundary $\mathscr{J}^+ = \{\Omega = 0, d\Omega \neq 0\}$ diffeomorphic to $[0, 1[ \times \partial\overline{S}$ whose edge is identical with the boundary $\partial\overline{S}'$ of $\overline{S}'$, $\mathscr{J}^+ \backslash \partial\overline{S}'$ is in the future of $\overline{S}'$, and $\Omega > 0$ on $\widetilde{M} = M \backslash \mathscr{J}^+$.*

(ii) *Let $(S_M, \hat{u}_0)$ be a Minkowskian hyperboloidal Cauchy data set. There is an $\varepsilon > 0$ such that the solution of the conformal EMYM equations considered in* (i) *has the following additional properties if the hyperboloidal initial data $(\overline{S}, u_0)$ are in an $\varepsilon$-neighborhood of $(S_M, \hat{u}_0)$. There is a smooth extension $(M', g')$ of the space-time $(M, g)$ (and of the given time orientation), a smooth extension $\Omega'$ of $\Omega$ to $M'$, and a point $i^+ \in M'$ such that the null geodesics through $\partial\overline{S}'$ which generate $\mathscr{J}^+$ meet at the point $i^+$ such that $N = \mathscr{J}^+ \cup \{i^+\}$ coincides with the set of points in $M'$ swept out by the past directed null geodesics through $i^+$. At $i^+$ the* *function* $\Omega'$ *has a nondegenerate critical point. The "physical" space-time* $(\widetilde{M}, \tilde{g} = \Omega^{-2} g)$ *is null geodesically future complete and* $\mathscr{I}^+$ *represents future null infinity, while* $i^+$ *represents future time-like infinity for that space-time.*

**Remarks.** (i) The second part may be expected to reduce the investigation of the global existence and the asymptotic structure of gravitational fields arising from asymptotically flat (at spatial infinity) standard Cauchy data to the investigation of the $i^0$-problem. In fact, Cutler and Wald [10] recently gave a construction of a class of hyperboloidal initial data for the Einstein-Maxwell equations, which satisfy the conditions of part (ii) of the theorem and which arise from the time development of time-symmetric standard Cauchy data on $\mathbb{R}^3$, which outside some compact set coincide with Schwarzschild data for positive mass. The development of such standard Cauchy data is known explicitly near $i^0$ and allows the construction of smooth pieces of $\mathscr{I}^-$, $\mathscr{I}^+$. This allows us for the first time since the introduction of the notion of a conformal boundary [28] to show the existence of nontrivial solutions of the Einstein-Maxwell equations, which have complete $\mathscr{I}^-$, $\mathscr{I}^+$ and which are regular at past and future time-like infinity $i^-$, $i^+$.

(ii) A general investigation of the constraint equations on hyperboloidal hypersurfaces, following, e.g., the known procedures in the case of standard Cauchy data [9] is still missing. Only recently have results been obtained which are related to this problem ([2], [3], [4]). The critical problem is of course whether the smoothness of the data at infinity can be demonstrated.

For the source-free case, the existence of a certain class of (analytic) hyperboloidal data has been shown recently [18]. Also, the fact that in the source-free case the characteristic of the subsidiary equations are time-like with respect to the metric, implies that in this case domains of dependence of the subsidiary system can be strictly larger than domains of dependence defined by the metric. This can be made into an argument to show the existence of hyperboloidal data, but the meaning of these data is somewhat obscure.

(iii) Our theorem may be generalized to show the existence of solutions with a regular point $i^+$ near comparison solutions different from Minkowski space. Suitable candidates of comparison solutions with regular $i^+$ are provided in [18].

(iv) It is most remarkable that the existence of the point $i^+$ is enforced by the conformal field equations, essentially equations (2.15), (2.16), once the lifetime of the solution is long enough. It is likely that a more systematic analysis of these equations and of equation (2.17) will bring out to a further extent the geometric content of the field equations.

(v) Since the proof of the first part of Theorem (10.2) rests on local existence results for the conformal EMYM equations, the assumption that the hyperboloidal data be defined essentially on a closed unit ball in $\mathbb{R}^3$ can be weakened to deal with more general topologies of the initial surface. We could consider also surfaces similar to or more complicated than those space-like hypersurfaces (diffeomorphic to $[0, 1] \times S^2$) in the conformal extension of the maximal analytic extension of the Schwarzschild solution with positive mass, which connect a given space-like spherical section of one $\mathscr{J}^+$ with a similar such section on the second $\mathscr{J}^+$. From the following proof of Theorem (10.2) it becomes clear, however, that such data must lead to a solution with some kind of "conformal singularity". Either the $\mathscr{J}^+$'s cannot be made complete, which would represent already a singularity of the conformal structure. Or they can be shown to be complete, but then the topological structure of the solution will be in conflict with the topological requirements (cf. (10.7)) implied by the field equations in the case where the solutions may be conformally extended sufficiently smooth through future time-like infinity. It should be most interesting to se whether an argument can be given which would show that this conflict leads necessarily to a curvature singularity.

In analogy to the procedure in the proof of Theorem (9.5) we use the data to construct a reference field on $M' = I \times \mathbb{R}^3$ with $I = ]-1, 1[$. We identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\} \subset M'$. By the map $\varphi$ given in (i) of Definition (10.1) we identify $\overline{S}$ with the closed unit ball $\overline{U}'$ in $\mathbb{R}^3$ and extend $\tau^A$, the vector fields $c'_A = c'^{\alpha}{}_A \partial/\partial x^{\alpha}$, $A = 1, 2, 3$, $e_{ab}$, $\Omega$, and $A_{ab}$ smoothly to all of $\mathbb{R}^3$ such that $\tau^A = \sqrt{2}\delta^A_0$, the vector fields $c'_A$ are linearly independent, $e_{ab} = \hat{e}^A{}_{ab} c'_A$, $\alpha = 0$ on $\mathbb{R}^3$, and such that $c'^{\alpha}{}_A(x^{\beta}) = \pm\delta^{\alpha}_A$ (the sign depending on the orientation of the frame $c'_A$), $\Omega = 1$, and $A_{ab} = 0$ for large $\sum(x^{\alpha})^2$. In analogy to the procedure in the proof of Theorem (9.5) we use the extended data to construct a reference field on $M' = I \times \mathbb{R}^3$ with $I = ]-1, 1[$. The projection of $M'$ onto $I$ defines a smooth function which we denote either by $t$ or by $x^0$. Together with the $x^{\alpha}$ it defines a smooth coordinate system on $M'$. On $M'$ we set $c_0 = \partial/\partial x^0$, define for $A = 1, 2, 3$ the vector fields $c_A = c^{\alpha}{}_A \partial/\partial x^{\alpha}$ by setting $c^{\beta}{}_A(t, x^{\alpha}) = c'^{\beta}{}_A(x^{\alpha})$, set $\hat{\tau} = \sqrt{2}c_0$ and $\hat{e}_{ab} = \hat{e}^A{}_{ab} c_A$, and finally require $\widehat{A}_{ab}$ and $\widehat{\Omega}$ to be Lie-transported in the direction of $c_0$ and to coincide with $A_{ab}$ and $\Omega$, respectively, on $\mathbb{R}^3$.

We use these field to obtain a reference field $\hat{u}$ on $M'$ as described in §8 and to derive the corresponding system (8.11). For this system we define initial data $v_0$ as follows. On $U'$ we set $v(x^\alpha) = u(x^\alpha) - \hat{u}(0, x^\alpha)$. Fix $s, j \in \mathbb{N}$, $s > \frac{3}{2} + j$, $j \geq 2$. As described in §9 we can extend $v$, which is in $H^s(U', H)$, to a function $v_0 \in H^s(\mathbb{R}^3, H)$ such that (9.7) holds. We require the extension of the fields $\tau^A - \hat{\tau}^A$ and $e^A{}_{ab} - \hat{e}^A{}_{ab}$, which vanish on $U'$, to vanish everywhere on $\mathbb{R}^3$. We apply part (i) of Lemma (9.6) to the initial value problem which we have formulated here, which we shall refer to as IVP′ (with some $\theta$, $\delta$ as in §9). Since we are dealing with a symmetric hyperbolic system, there exists a backward as well as a forward (meaning in the direction of $c_0$) evolution and we conclude that there is a $t_0$, $0 < t_0 < 1$, such that on $M'' = ]-t, t_0[\times\mathbb{R}^3$ there is a unique solution $v = v(t, x^\alpha)$ of class $C^j$ of IVP′, which yields a solution $u = v + \hat{u}$ of the symmetric hyperbolic propagation equations such that the vector fields $\tau, e_{ab}$ are linearly independent on $M''$ and define a metric $g$ by (8.15) for which $t$ is a time function with level surfaces which are Cauchy hypersurfaces for $(M, g)$. Denote by $D^+(\overline{U}')$, resp. $D^-(\overline{U}')$, the future, resp. past, domain of dependence and by $H^+(\overline{U}')$, resp. $H^-(\overline{U}')$, the future, resp. past, Cauchy horizon of $\overline{U}'$ in $M''$ with respect to the metric $g$. Because of (8.16) and the way we set up IVP′, the solution $u$ is determined on the sets $D^\pm(\overline{U}')$ uniquely by the data $u$ given on $\overline{S}$, i.e., independent of the chosen extensions, and by the Reduction Theorem (8.1) and condition (v) of Definition (10.1) we conclude that $u$ satisfies the conformal EMYM equations on the sets $D^\pm(\overline{U}')$. Since the boundary $\partial S$ of $S$ is smooth, $H^+(\overline{U}')$ represents near $\partial S$ a smooth null hypersurface with respect to $g$. By choosing $t_0$ sufficiently small we can assume hat $H^+(\overline{U}')$ is smooth and null everywhere. In the following we restrict attention to $D^+(\overline{U}')$ and to the restriction of the metric $g$ to this set. Condition (iv) of Definition (10.1) implies that the null vector field $l_\mu = \nabla_\mu \Omega \neq 0$ is tangent to $H^+(\overline{U}')$ on $\partial S$. Extend $l_\mu$ to $H^+(\overline{U}')$ such that $l^\mu \nabla_\mu l^\nu = 0$. From equation (2.15) it follows that $\Omega$ satisfies the equation

$$l^\mu \nabla_\mu (l^\nu \nabla_\nu \Omega) = \Omega l^\mu l^\nu (-s_{\mu\nu} + \tfrac{1}{2}\kappa \Omega^2 T_{\mu\nu}) \tag{10.5}$$

on $H^+(\overline{U}')$, which implies with condition (iv) of Definition (10.1) that $\Omega \equiv 0$ on $H^+(\overline{U}')$. It also follows from condition (iv) that $d\Omega \neq 0$ near $\partial S$. We shall see below that $d\Omega$ cannot vanish on $H^+(\overline{U}')$ unless the null generators of $H^+(\overline{U}')$ develop a caustic, thus by our smoothness assumption we must have $\nabla_\mu \Omega \neq 0$ on $H^+(\overline{U}')$. If there were a point $p$ in $\mathrm{int}(D^+(\overline{U}'))$ at which $\Omega = 0$, then there would be a past directed null vector $k^\mu$ at $p$ such that $k^\mu \nabla_\mu \Omega = 0$. This is obvious if $p$ is a critical point for $\Omega$ and otherwise it follows from equation (2.20) that we can choose $k^\mu$ proportional to $\nabla^\mu \Omega$. Replacing in equation (10.5) the vector $l^\mu$ by the tangent vector of the geodesic $\gamma$ in $D^+(\overline{U}')$, we conclude that $\Omega$ must vanish on $\gamma$. This leads to a contradiction since $\gamma$ must have an endpoint on $S$, where $\Omega > 0$ by condition (iv) of Definition (10.1). A similar conclusion can be drawn for $\mathrm{int}(D^-(\overline{U}'))$. If we set $\mathcal{J}^+ = H^+(\overline{U}')$, $M = D^+(\overline{U}') \cup \mathrm{int}(D^-(\overline{U}'))$, and restrict the solution $u$ to $M$, part (i) of Theorem (10.2) follows.

To prove part (ii) of the theorem, we take (10.3) as a reference field to set up the system (8.11) on the manifold $M' = \{x \in M_0 | t(x) \geq 0\}$, where $t$ is the time function we used to define (10.3), and construct solutions for the extended data $v$ on $S^*_M = \{t = 0\}$ for which (10.4) holds. Using the same arguments as in the proof of Theorem (9.8), we find that for sufficiently small $\varepsilon > 0$ any extended initial data set which satisfies $\|v\|^*_s < \varepsilon$ determines a unique solution on the set $M'' = \{x \in M' | 0 \leq t(x) \leq \frac{3}{2}\pi\}$, denoted again by $v$, of class $C^j$ of equation (8.11) such that the vector fields $\tau$, $e_{ab}$ supplied by $u = v + \hat{u}$ on $M''$ (in the notation (8.1), (8.10) are linearly independent, thus define by (8.15) a metric $g$ of class $C^j$ on $M''$ for which $t$ is a time function with level sets $S_\tau = \{x \in M'' | t(x) = \tau\}$, $0 \leq \tau \leq \frac{3}{2}\pi$, which are Cauchy hypersurfaces for $(M'', g)$. Since $c_0(\Omega_0) = 1$ on $S_{3\pi/2}$ we conclude from Lemma (9.9i) that for $\varepsilon > 0$ chosen small enough the solution $v$ will be such that $\Sigma_f{}^f > 0$ on $S_{3\pi/2}$. We denote by $D^+(\overline{S})$ the future domain of dependence and by $H^+(\overline{S})$ the future Cauchy horizon in $M''$ of the set $\overline{S}$ with respect to $g$ and conclude again from condition (v) of Definition (10.1) and from Theorem (8.1) that $u$ represents a solution of the conformal EMYM equations on $D^+(\overline{S})$.

It is well known that $H^+(\overline{S})$ is generated by null geodesic segments which end on $\partial S$ [25]. Though we have seen already that near $\partial S$ the set $H^+(\overline{S})$ represents a smooth null hypersurface on which $\Omega = 0$ but $d\Omega \neq 0$ we may expect that further into the future the null generators will tend to form complicated caustics and self-intersections [20] or simply arrive at the future boundary $S_{3\pi/2}$ of $M''$. We have to show that the caustic coincides with the self-intersection set and consists of exactly one point $i^+$ which is met by any null generator of $H^+(\overline{S})$. Assume that for a number $\tau$ with $0 < \tau < \frac{3}{2}\pi$ it has been shown that the set $H_\tau = H^+(\overline{S}) \cap \{x \in M'' | 0 \le t(x) < \tau\}$ is a smooth null hypersurface on which $\Omega = 0$ by equation (10.5) and on which $d\Omega \neq 0$. For a given point $x \in \partial S$ denote by $\gamma$ the null geodesic in $M''$ through $x$ whose tangent vector $l^\mu$ coincides at $x$ with $\nabla^\mu \Omega$ such that part of $\gamma$ lies on $H_\tau$. Let $n^\mu$ be the null vector field which is parallely transported along $\gamma$ and which at $x$ is orthogonal to $\partial S$ and satisfies $n^\mu l_\mu = 1$. Equations (2.15), (2.16) then imply on $\gamma \cap H_\tau$ the system of ordinary differential equations

$$\left\{ \begin{array}{l} l^\mu \nabla_\mu (n^\nu \nabla_\nu \Omega) = s, \\ l^\mu \nabla_\mu s = -(n^\nu \nabla_\nu \Omega) l^\mu l^\nu s_{\mu\nu} \end{array} \right. \tag{10.6}$$

for $n^\nu \nabla_\nu \Omega$ and $s$ since we have an expansion $\nabla_\nu \Omega = (n^\nu \nabla_\nu \Omega) l_\mu$ along $\gamma \cap H_\tau$. By the definition of hyperboloidal data, $n^\mu \nabla_\mu \Omega \neq 0$ on $\partial S$ and we conclude from (10.6) that $n^\nu \nabla_\nu \Omega$ and $s$ cannot vanish simultaneously at any point of the closure of $\gamma \cap H_\tau$. If $n^\mu \nabla_\mu \Omega \neq 0$ or, equivalently, $d\Omega \neq 0$ on $\gamma \cap S_\tau$ for all the null geodesics $\gamma$ which start from $\partial S$, the set $H^+(\overline{S})$ extends for some time as a smooth null hypersurface into the future of $S_\tau$ and we may replace $\tau$ by a larger number than that chosen above. However, the value of $\tau$ cannot approach $\frac{3}{2}\pi$. Since by our assumptions on hyperboloidal data we have $\Sigma_f{}^f < 0$ on $\partial S$ whereas we have $\Sigma_f{}^f > 0$ on $S_{3\pi/2}$ by our choice of $\varepsilon$, there is a smallest value $\tau^*$, $0 < \tau^* < \frac{3}{2}\pi$, such that for at least one null geodesic $\gamma$ of the type considered above we have $\Sigma_f{}^f = 0$ at $\gamma \cap S_{\tau^*} =: i^+$. It follows that $n^\mu \nabla_\mu \Omega \neq 0$ on $H_{\tau^*}$ but $n^\mu \nabla_\mu \Omega = 0$ and thus $d\Omega = 0$ at $i^+$. Equations (2.15) and (10.6) imply

$$\nabla_\mu \nabla_\nu \Omega = s g_{\mu\nu}, \qquad s \neq 0 \text{ at } i^+. \tag{10.7}$$

Thus, the function $\Omega$ has an isolated critical point at $i^+$ and it follows that all null geodesic generators of $H^+(\overline{S})$ must run into that point. All other statements in part (ii) of Theorem (10.2) now follow immediately.

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