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HOMOTOPY K3 SURFACES CONTAINING $\Sigma(2, 3, 7)$

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1. Introduction

An interesting question in 4-dimensional topology is whether each irreducible simply connected smooth 4-manifold other than S^4 must admit a complex structure. One technique which has been suggested for answering this question is to try to produce examples which have all of their Donaldson polynomials [3] vanishing, and then use Donaldson's theorem that complex algebraic surfaces have nontrivial polynomial invariants. A natural place to begin study is among smooth manifolds with the homotopy type of a K3 surface; we refer to such manifolds as homotopy K3 surfaces. Kodaira [7] has produced a family of homotopy K3 surfaces by performing logarithmic transforms on the fibers of elliptic K3's but these manifolds all have complex structures. (Their diffeomorphism types have been studied recently by Friedman and Morgan.) Also, there were many (unpublished) examples of homotopy K3 surfaces constructed about a decade ago by Kirby calculus pictures.

A common aspect of many of these latter examples is that they admit an embedding of the Brieskorn homology 3-sphere $\Sigma(2, 3, 7)$, which may be described as the link of a complex algebraic singularity $\{(x, y, z) \in \mathbf{C}^3 : x^2 + y^3 + z^7 = 0\} \cap S^5$, or, equivalently, as the result of -1 surgery on the right-handed trefoil knot. (The Poincaré homology sphere, $\Sigma(2, 3, 5)$, is the result of $+1$ surgery on the right-handed trefoil.) In this article we shall show

Theorem 1.1. *Any homotopy* K3 *surface which admits an embedding of* $\Sigma(2, 3, 7)$ *has a nontrivial Donaldson polynomial invariant of degree* 10.

We will give a fairly elementary proof of this fact based on Donaldson's study of 4-manifolds whose intersection form has one or two positive parts [2] and on our study of the representation space of $\Sigma(2, 3, 7)$ [4]. In particular, our calculations of Donaldson's invariant use no algebraic geometry.

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The philosophy of our proof is closely related to forthcoming work of Donaldson which gives his relative polynomial invariants in terms of Floer's instanton homology theory. However, our result neither uses the generality of that theory, nor does it admit any obvious generalizations in that direction. The proof uses an analysis of the effect on anti-self-dual moduli spaces of letting a metric on the homotopy K3 surface degenerate along our homology sphere. It then applies rather specific knowledge about the representation space of $\pi_1(\Sigma(2, 3, 7))$ into $SU(2)$.

Our result is also a key step in Akbulut's construction of a fake relative smooth structure on a compact contractible 4-manifold [1]. Akbulut's construction produces a homotopy K3 surface M containing $\Sigma(2, 3, 7)$ and contractible 4-manifold W together with a self-diffeomorphism f of ∂W which extends to a self-homeomorphism of W such that either

(1) M has all its Donaldson polynomial invariants trivial, or

(2) there is no self-diffeomorphism of W extending f.

Our theorem then implies that (1) is false; so Akbulut's construction gives (2).

2. Donaldson's polynomial invariant

Let us begin by quickly reviewing the construction of Donaldson's polynomial invariant [3]. Suppose we have a simply connected closed 4-manifold M^4, where $b_2^+(M)$ is odd and > 1. The moduli space $\mathscr{M}_k(M)$ of anti-self-dual connections on the $SU(2)$ bundle over M with $c_2 = k$ has formal dimension $8k - 3(1 + b_2^+) = 2d$. For a generic choice of Riemannian metric on M, the moduli space, $\mathscr{M}_k(M)$, if nonempty, will be a manifold of this dimension [6], [2]. (Here, as usual, we must use Sobolev spaces of connections; see [2] for details.) If $k > \frac{3}{4}(1+b_2^+)$, Donaldson [3] defines a polynomial $q_k(M)$ in the polynomial algebra $P[H_2(M; \mathbf{Z})]$ as follows. Let $\mathscr{B}(M)$ denote the space of gauge equivalence classes of connections on $SU(2)$ bundles over M. For $z \in H_2(M; \mathbf{Z})$ choose an embedded oriented surface S representing z with an open neighborhood N such that $H_2(N; \mathbf{Z}) = \mathbf{Z}[S]$. If $r_S: \mathscr{B}(M) \to \mathscr{B}(N)$ denotes the restriction map and $\mathscr{B}_N^\dagger$ denotes the space of gauge equivalence classes of $SU(2)$ connections over N which are either irreducible or trivial, then Donaldson shows in an appendix to [3] that the surface S representing z can be chosen so that $\coprod_{l=0}^k \mathscr{M}_l(M)$ is contained in $r_i^{-1}(B_N^\dagger)$. Furthermore, there is a complex line bundle L_S over $\mathscr{B}_N^\dagger$ with a section so that when pulled back over $r_s^{-1}(B_N^\dagger)$ it gives a section σ_S of $r_S^*(L_S)$ whose zero set V_S is a codimension-2 submanifold of $r_S^{-1}(B_N^\dagger)$ which meets all of the moduli spaces $\mathscr{M}_l$ $(l \leq k)$ transversely. In particular, since $\mathscr{M}_0(M) = \{\Theta\}$, where Θ is the trivial SU(2) connection, transversality means that $\Theta \notin V_S$. (The map $z \mapsto c_1(r_S^*(L_S))$ is the map $\mu\colon H_2(M; \mathbf{Z}) \to H^2(\mathscr{B}(M); \mathbf{Z})$ constructed in [2].)

Given homology classes $z_1, \cdots, z_d \in H_2(M; \mathbf{Z})$, represent them by surfaces $S_1, \cdots, S_d$ in M chosen as above. It is possible to pick the surfaces to be in general position and have their open neighborhoods N_i so that all triple intersections $N_i \cap N_j \cap N_k$ are empty. The intersection $V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{M}_k(M)$ will then be discrete, and the condition $k > \frac{3}{4}(1+b_2^+)$ will imply that it is compact. (The V_{S_i} are also chosen to have transverse multiple intersections.) The Donaldson polynomial invariant is defined to be

$$q_k(M)(z_1, \cdots, z_d) = \#(V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{M}_k(M)),$$

where # denotes a count with signs. Donaldson proves that $q_k(M)$ depends only on the smooth structure of M. (See [3].)

Now let M denote a smooth oriented homotopy K3 surface. This means that $\pi_1(M) = 0$ and that the intersection form of M is $2E_8 \oplus 3H$ of rank 22 and signature -16.

Lemma 2.1. $\Sigma(2, 3, 7)$ *embeds in a* K3 *surface, splitting it into submanifolds with intersection forms* $E_8 \oplus H$ *and* $E_8 \oplus 2H$.

Proof. This follows from [8] where it is shown that the triangle singularity $D_{2,3,7} = \Sigma(2, 3, 7)$ embeds in a K3 surface with smoothing given by the E_{10} diagram.

Lemma 2.2. *If* $\Sigma(2, 3, 7)$ *embeds in a homotopy* K3 *surface* M, *it splits* M *into smooth oriented submanifolds* $M = X \cup Y$, *where the intersection form of* X *is* $E_8 \oplus H$, *and the intersection form of* Y *is* $E_8 \oplus 2H$.

Proof. The μ-invariant of $\Sigma(2, 3, 7)$ is nonzero; so the intersection forms of X and Y must each have an E_8 summand. Now by Lemma 2.1, $\pm\Sigma(2, 3, 7)$ bounds manifolds which have intersection forms $E_8 \oplus H$ and $E_8 \oplus 2H$. Thus, if either X or Y had the intersection form E_8, we could construct a simply connected 4-manifold with intersection form $2E_8 \oplus kH$ for $k = 1$ or 2, in contradiction to Donaldson's Theorems B and C of [2]. q.e.d.

Next we present information about the representations of the fundamental group of $\Sigma = \Sigma(2, 3, 7)$. We shall use the same notation for (the conjugacy class of) a representation $\pi(\Sigma) \to \mathrm{SU}(2)$ as for the gauge equivalence class of flat connections over Σ which it induces. Consider an SU(2) connection A over $\pm\Sigma \times \mathbf{R}$ with finite action. It must be asymptotically flat with limiting connections ρ, σ as $t \to \pm\infty$. Let $\mathscr{M}_{\pm\Sigma}(\rho, \sigma)$ denote the moduli space of anti-self-dual SU(2) connections over $\pm\Sigma \times \mathbf{R}$ with these asymptotic conditions. (It is imperative in this situation to use connections with exponential decay and a corresponding group of gauge transformations. See [5].) Let θ denote the trivial representation of $\pi_1(\Sigma)$. The next proposition follows directly from computations in [4].

Proposition 2.3. *Let* $\Sigma = \Sigma(2, 3, 7)$. *Then up to conjugacy there are two nontrivial representations* $\alpha, \beta \colon \pi_1(\Sigma) \to \mathrm{SU}(2)$, *and the* (mod 8) *dimensions of the corresponding moduli spaces of anti-self-dual connections on* $\pm\Sigma \times \mathbf{R}$ *are*:

$$\dim \mathscr{M}_{\Sigma}(\alpha, \theta) \equiv 2, \qquad \dim \mathscr{M}_{-\Sigma}(\alpha, \theta) \equiv 3,$$
$$\dim \mathscr{M}_{\Sigma}(\beta, \theta) \equiv 6, \qquad \dim \mathscr{M}_{-\Sigma}(\beta, \theta) \equiv 7.$$

Now let $M = X \cup Y$ as in Lemma 2.2. Choose homology classes $z_1, z_2, z_3, z_4 \in H_2(X; \mathbf{Z}) = E_8 \oplus H$ such that the pair $z_1, z_2 \in E_8$ satisfies $z_1^2 = z_2^2 = 2$ and $z_1 \cdot z_2 = 1$, and the pair $z_3, z_4 \in H$ satisfies $z_3^2 = z_4^2 = 0$ and $z_3 \cdot z_4 = 1$. Similarly, choose $z_5, \cdots, z_{10} \in H_2(Y; \mathbf{Z}) = E_8 \oplus 2H$ so that z_5, z_6 form a pair in E_8 and z_7, z_8 and z_9, z_{10} form pairs in the two copies of H. For each z_i choose an oriented surface S_i in X or Y as in the definition of Donaldson's polynomial. We may suppose that the S_i are in general position. Note that $\dim \mathscr{M}_4(M) = 20$.

Consider now the result of collapsing the homology sphere Σ to a point. After removing the singular point, we are left with a disjoint union $X_+ \amalg Y_-$, where $X_+ = X \cup (\partial X \times [0, \infty))$ and $Y_- = (\partial Y \times (-\infty, 0]) \cup Y$. According to [11, §8], for generic metrics g_X and g_Y any moduli spaces of anti-self-dual connections $\mathscr{M}_{X_+}(\rho)$ and $\mathscr{M}_{Y_-}(\rho)$ are manifolds. Since we are dealing with finite action solutions of the anti-self-duality equations, these solutions must be asymptotically flat. Here "ρ" denotes the representation corresponding to the flat asymptotic connection.

Fix a generic metric g on M. We can then choose codimension-2 submanifolds $V_{S_1} \cdots V_{S_{10}}$ whose multiple intersections with the moduli spaces $\mathscr{M}_l(M, g)$, $l \leq 4$, are transverse. Furthermore, the V_{S_i} can also be chosen to have transverse multiple intersections with any components of $\mathscr{M}_{X_+}(g_X, \rho) \amalg \mathscr{M}_{Y_-}(g_Y, \rho)$ of total formal dimension ≤ 20. (We can view the V_{S_i} as living in $\mathscr{B}^*(X_+ \amalg Y_-)$ because the restriction maps $\mathscr{B}(M) \to \mathscr{B}(N_i)$ factor through $\mathscr{B}(M \backslash \Sigma)$.)

Now consider a sequence $\{g_n\}$ of generic metrics on M with $g_0 = g$ which converge to the (necessarily singular) metric $g_X \vee g_Y$ on $M/\sim$, where $\sim$ collapses Σ to a point. Since the cone on Σ minus the cone point is conformally equivalent to a cylinder $\Sigma \times \mathbf{R}$, we get our limiting metrics g_X and g_Y on X_+ and Y_-. The V_{S_i}, being submanifolds of the Hilbert manifold $\mathscr{B}^*(M)$, have tubular neighborhoods. For each i, let $\mathscr{V}_{S_i}^{(n)}$ be such a sequence of tubular neighborhoods with radii $\to 0$ as $n \to \infty$. Then, using [3], for each $1 \leq n \leq \infty$ we can perturb the sections σ_i to obtain new codimension-2 zero sets, $V_{S_i}^{(n)}$, contained in $\mathscr{V}_{S_i}^{(n)}$, so that the intersection of any p of the $V_{S_i}^{(n)}$ have empty intersection with any moduli space $\mathscr{M}_l(M, g_n)$ of formal dimension less than $2p$. Donaldson also shows that the polynomial invariant can be computed from these $V_{S_i}^{(n)}$ after further, arbitrarily small, perturbation making all the intersections transverse.

Suppose now that $q_4(M)(z_1, \cdots, z_{10}) \equiv 1 \pmod 2$. In particular, for each n, $V_{S_1}^{(n)} \cap \cdots \cap V_{S_{10}}^{(n)} \cap \mathscr{M}_4(M, g_n)$ is nonempty. Choose an anti-self-dual connection $A_n \in V_{S_1}^{(n)} \cap \cdots \cap V_{S_{10}}^{(n)} \cap \mathscr{M}_4(M, g_n)$ for each n. By Uhlenbeck's Compactness Theorem [12], there are limiting anti-self-dual connections A_X over $X_+ \backslash \{x_1, \cdots, x_r\}$, A_Y over $Y_- \backslash \{y_1, \cdots, y_s\}$, and limiting instantons at the points x_i and y_j. Both A_X and A_Y are asymptotically flat with limiting flat connections ρ_X and ρ_Y. In addition there is the possibility that, in the limit, curvature is lost at the neck between X_+ and Y_-. In that case one also has flat connections $\rho_X = \rho_0, \rho_1, \cdots, \rho_m = \rho_Y$ and limiting nontrivial anti-self-dual connections $B_i \in \mathscr{M}_{\pm\Sigma}(\rho_{i-1}, \rho_i)$ for $i = 1, \cdots, m$. Let n_X and n_Y denote the formal dimensions (i.e., the dimensions given by the index theorem) of the components of the moduli spaces containing A_X and A_Y, and for each $i = 1, \cdots, m$ let $n_i > 0$ be the formal dimension of the component of $\mathscr{M}_{\pm\Sigma}(\rho_{i-1}, \rho_i)$ containing B_i. Also, let $T \geq 0$ be the number of $\rho_i = \theta$. Then, counting dimensions, we have:

$$20 = n_X + n_Y + 8(r+s) + \sum n_i + 3T.$$

Since the surfaces S_i are in general position in M, no point lies on more than two surfaces. So on X_+ the points $x_1, \cdots, x_r$ lie on at most $2r$ of the S_i. Recall that a connection lives in V_{S_i} if and only if when restricted to the open neighborhood N_i of S_i it lies in the zero set of the section σ_i. Each $A_n \in V_{S_i}^{(n)}$, which converge as sets to V_{S_i}; so if no $x_j \in S_i$, then $A_X \in V_{S_i}$. Similarly, for $k = 5, \cdots, 10$, $A_Y \in V_{S_k}$ if no $y_l \in S_k$. Suppose first that A_X is the trivial connection, Θ_X, on X_+; so $n_X = -6$ and $T \geq 1$. But Θ_X does not lie in any V_{S_i}; so for $i = 1, \cdots, 4$, and $j = 1, \cdots, r$ each S_i contains some point x_j. Thus

$r \geq 2$. If A_Y is also trivial then $n_Y = -9$, and similarly $s \geq 3$. The above count of formal dimensions,

$$20 = -6 + (-9) + 8(r+s) + \sum n_i + 3T \geq 25 + \sum n_i + 3,$$

then gives a contradiction. If A_X is trivial but A_Y is nontrivial then $n_Y \geq 0$ and A_Y lies in at least $6 - 2s$ of the V_{S_k}'s. Since each V_{S_i} is codimension-2 this means that $2(6-2s) \leq n_Y$. The formal dimension count then gives the contradiction

$$20 \geq -6 + 2(6-2s) + 8(2+s) + \sum n_i + 3 \geq 25 - 4s + \sum n_i.$$

A similar formal dimension count shows that we cannot have $A_Y = \Theta_Y$ and A_X nontrivial so that neither A_X nor A_Y is trivial. Applying our formal dimension count once more we get

$$20 \geq 2(4-2r) + 2(6-2s) + 8(r+s) + \sum n_i + 3T.$$

Thus, $r = s = 0$, $T = 0$, and $\sum n_i = 0$; so $m = 0$ and $\rho_X = \rho_Y = \rho$, say. Also $n_X = 8$ and $n_Y = 12$. Arguing further we get:

Proposition 2.4. *If $q_4(M)(z_1, \cdots, z_{10}) \equiv 1 \pmod 2$, then there are connections $A_X \in V_{S_1} \cap \cdots \cap V_{S_4} \cap \mathscr{M}_{X_+}(\alpha)$ and $A_Y \in V_{S_5} \cap \cdots \cap V_{S_{10}} \cap \mathscr{M}_{Y_-}(\alpha)$, where $\dim \mathscr{M}_{X_+}(\alpha) = 8$ and $\dim \mathscr{M}_{Y_-}(\alpha) = 12$. Furthermore, $\partial X = \Sigma$, and for any sequence of connections $\{A_n\}$ as above we have $\rho = \alpha$.*

Proof. If we get the asymptotic condition ρ, then $\dim \mathscr{M}_{X_+}(\rho) = 8$. Let B be any connection over $\partial X \times \mathbf{R}$ (where $\partial X = \pm\Sigma$), which tends asymptotically to ρ as $t \to -\infty$ and to θ as $t \to +\infty$. Grafting A_X to B as in [5] we obtain a connection $A_X \# B$ over X_+, which is asymptotically trivial, and so the index of the anti-self-duality operator $D_{A_X \# B}$ is $8k - 3(1 + b_2^+(X)) \equiv -6 \pmod 8$. But also $\operatorname{Ind} D_{A_X \# B} = \operatorname{Ind} D_{A_X} + \operatorname{Ind} D_B = 8 + \operatorname{Ind} D_B \equiv \operatorname{Ind} D_B \pmod 8$ since β is irreducible. It now follows from Proposition 2.3 that $\partial X = +\Sigma$ and that $\rho = \alpha$. q.e.d.

Conversely, we have

Proposition 2.5. *Let $\mathscr{M}_{X_+}(\alpha)$ and $\mathscr{M}_{Y_-}(\alpha)$ be moduli spaces of anti-self-dual connections of dimensions equal to 8 and 12 respectively. Let z_i, S_i, $i = 1, \cdots, 10$, be as above, and let $m_\alpha = \#(V_{S_1} \cap \cdots \cap V_{S_4} \cap \mathscr{M}_{X_+}(\alpha))$ and $n_\alpha = \#(V_{S_5} \cap \cdots \cap V_{S_{10}} \cap \mathscr{M}_{Y_-}(\alpha))$. Then $q_4(z_1, \cdots, z_{10}) \equiv m_\alpha n_\alpha \pmod 2$.*

Proof. If $A_X \in V_{S_1} \cap \cdots \cap V_{S_4} \cap \mathscr{M}_{X_+}(\alpha)$ and $A_Y \in V_{S_5} \cap \cdots \cap V_{S_{10}} \cap \mathscr{M}_{Y_-}(\alpha)$, then for metrics g on M close enough to $g_X \vee g_Y$ there is a grafted anti-self-dual connection $A_X \#_g A_Y$ in $V_{S_1} \cap \cdots \cap V_{S_{10}} \cap \mathscr{M}_k(M, g)$, where

$8k - 3(1 + b_2^+(M)) = 8 + 12$; so $k = 4$. A study of this grafting process shows that for g close to $g_X \vee g_Y$ each $A \in \mathcal{M}_4(M, g)$ can be uniquely written as $A = A_X \#_g A_Y$ (see [9]).

3. Proof of Theorem 1.1

Let $M = X \cup Y$ be as in Lemma 2.2 and let $S_1, \cdots, S_{10}$ be the surfaces in X described in §2. The goal of this section is to build moduli spaces $\mathcal{M}_{X_+}(\alpha)$ and $\mathcal{M}_{Y_-}(\alpha)$ as described in Proposition 2.5 such that m_α and n_α are odd, so that $q_4(M)(S_1, \cdots, S_{10}) \equiv 1 \pmod 2$. The basic idea is to apply ideas of the proofs of Donaldson's Theorems B and C in [2]. In this case, instead of obtaining contradictions, we obtain information about the ends of the moduli space, which correspond to the asymptotics of X_+ (or Y_-). First we work with X_+. It follows from the work of Taubes [10], [11] that the moduli space $\mathcal{M}_{X_+,2}(\theta)$ of $c_2 = 2$ asymptotically trivial anti-self-dual connections over X_+ is nonempty, and is therefore a 10-dimensional manifold when X_+ is given a generic metric. We want to study the 2-manifold $N^2 = V_{S_1} \cap \cdots \cap V_{S_4} \cap \mathcal{M}_{X_+,2}(\theta)$, where $S_1, \cdots, S_4$ are the surfaces described in §2. We need to examine the ends of N^2.

The ends of $\mathcal{M}_{X_+,2}(\theta)$ correspond to the ways that sequences of anti-self-dual connections in $\mathcal{M}_{X_+,2}(\theta)$ can converge to an anti-self-dual connection with a different c_2 or asymptotic condition. For example, such a sequence could converge to

(1) a $c_2 = 1$ anti-self-dual connection $A_\infty \in \mathcal{M}_{X_+,1}(\theta)$ together with an instanton at a point $x \in X$,

(2) the trivial connection Θ_X together with a pair of instantons at points x and y in X, or

(3) an anti-self-dual connection $A_\rho \in \mathcal{M}_{X_+,1}(\rho)$, ρ a nontrivial asymptotic condition, together with an instanton over $\partial X \times \mathbf{R}$ (where $\partial X = \pm\Sigma$) which tends asymptotically to ρ as $t \to -\infty$ and to θ as $t \to +\infty$.

This description also indicates how the moduli space $\mathcal{M}_{X_+,2}(\theta)$ is compactified. For details see [2].

If a sequence $\{A_n\}$ in N^2 converges to an $A_\infty \in \mathcal{M}_{X_+,1}(\theta)$ together with an instanton at a point $x \in X$, then as in §2 the point x lies on at most two of the surfaces S_i, and so there are $i_1 \neq i_2$ with A_∞ lying in the transverse intersection $V_{S_{i_1}} \cap V_{S_{i_2}} \cap \mathcal{M}_{X_+,1}(\theta)$. But $V_{S_{i_1}} \cap V_{S_{i_2}}$ is codimension-4 and $\dim \mathcal{M}_{X_+,1}(\theta) = 2$; so this situation cannot occur.

Suppose there is an end of N^2 corresponding to a sequence $\{A_n\}$ converging to Θ_X together with instantons at x and y in X. If there is an S_i containing neither x nor y, then $\Theta_X \in V_{S_i}$. However as noted in §2, $\Theta_X \notin V_{S_i}$ for any i. Thus the ends of N^2 coming from sequences converging to Θ_X corresponding to pairs $\{x, y\} \in S^2(X)$ such that each S_i contains x or y. It is shown in [2] that this correspondence is 1-1. By our choice of surfaces there are

$$(S_{i_1} \cdot S_{i_2})(S_{i_3} \cdot S_{i_4}) + (S_{i_1} \cdot S_{i_3}) + (S_{i_1} \cdot S_{i_4})(S_{i_2} \cdot S_{i_3}) \equiv 1 \pmod 2$$

such pairs of points.

Donaldson shows that the end of N^2 corresponding to $(\Theta_X, \{x, y\})$ is the cone on a circle $L_{x,y}$ and further that $[L_{x,y}] \neq 0$ in $H_1(\mathscr{B}_X^\dagger; \mathbf{Z}_2)$. This is proved by producing a class $u_1 \in H^1(\mathscr{B}_X^\dagger; \mathbf{Z}_2)$ which evaluates nontrivially on each $[L_{x,y}]$. Since there is an odd number of these links, there must be other ends of N^2, and each of these other ends must correspond to a sequence of "instantons travelling down the tube" $\Sigma \times [0, \infty)$ in X_+ together with instantons at some points of X as above. This means that there is a corresponding sequence $\{A_n\}$ in N^2 such that for large enough n, A_n is close to a grafted connection $I_1 \# \cdots \# I_k \# A_X \# B_1 \# \cdots \# B_l$, where each I_i is an instanton at $x_i \in X$, $A_X \in \mathscr{M}_{X_+}(\rho)$ for some flat connection ρ on Σ, and there are flat connections ρ_j on Σ, $j = 0, 1, \cdots, l$, such that $\rho_0 = \rho$, $\rho_l = \theta$, and $B_j \in \mathscr{M}_{\pm\Sigma}(\rho_{j-1}, \rho_j)$, where $\dim \mathscr{M}_{\pm\Sigma}(\rho_{j-1}, \rho_j) \geq 1$. (See [5, 1.c.2].)

A dimension count quickly clarifies this situation. The sum of the dimensions of the moduli spaces containing the I_i, A_X, and B_j must be less than or equal to $10 = \dim \mathscr{M}_{X_+,2}(\theta)$. It is clear that $k \leq 2$. If $k = 2$ then we are left as above with limit the trivial connection, and there is no energy left for instantons to travel down the tube giving $B_1, \cdots, B_l$. If $k = 1$ then as before $A_X \in V_{S_{i_1}} \cap V_{S_{i_2}} \cap \mathscr{M}_{X_+,1}(\rho)$. This is also impossible since by hypothesis B_1 is nontrivial; so $\dim \mathscr{M}_{X_+,1}(\rho) \leq \mathscr{M}_{X_+,1}(\theta) = 2$. Hence, there are no point instantons in the limit. Thus, A_X lies in the transverse intersection $V_{S_1} \cap \cdots \cap V_{S_4} \cap \mathscr{M}_{X_+}(\rho)$. This means that $\dim \mathscr{M}_{X_+}(\rho) \geq 8$. Since B_1 is nontrivial, it follows from Proposition 2.3 that $\pm\Sigma = \Sigma$, $\rho = \alpha$, $\rho_1 = \theta$, and $l = 1$. Thus this end of $\mathscr{M}_{X_+,2}(\theta)$ is related to a local diffeomorphism

$$\mathscr{M}_{X_+}(\alpha) \times \mathscr{M}_\Sigma(\alpha, \theta) \to \mathscr{M}_{X_+,2}(\theta),$$

where $\dim \mathscr{M}_{X_+}(\alpha) = 8$ and $\dim \mathscr{M}_\Sigma(\alpha, \theta) = 2$.

It follows that $\mathscr{M}_{X_+}(\alpha) \cap V_{S_1} \cap \cdots \cap V_{S_4}$ is a 0-dimensional submanifold, and is compact by another codimension argument. Hence, $\mathscr{M}_{X_+}(\alpha) \cap V_{S_1} \cap \cdots \cap V_{S_4}$ is a finite set, say equal to $\{A_i : i = 1, \cdots, r\}$. Furthermore the existence of a temporal gauge shows that $\mathscr{M}_\Sigma(\alpha, \theta) = \widehat{\mathscr{M}_\Sigma}(\alpha, \theta) \times \mathbf{R}$, where $\widehat{\mathscr{M}_\Sigma}(\alpha, \theta)$ is a 1-manifold. Any sequence of connections $\{B_n\}$ in $\widehat{\mathscr{M}_\Sigma}(\alpha, \theta)$ which fails to converge will correspond to a local diffeomorphism

$$\widehat{\mathscr{M}_\Sigma}(\alpha, \sigma_1) \times \mathbf{R} \times \widehat{\mathscr{M}_\Sigma}(\sigma_1, \sigma_2) \times \mathbf{R} \times \cdots \times \widehat{\mathscr{M}_\Sigma}(\sigma_k, \theta) \to \widehat{\mathscr{M}_\Sigma}(\alpha, \theta).$$

Proposition 2.3 together with a simple dimension count implies that this does not occur, and so $\widehat{\mathscr{M}_\Sigma}(\alpha, \theta)$ is compact. Say that $\widehat{\mathscr{M}_\Sigma}(\alpha, \theta)$ is the disjoint union of components $\{S_i^1\}_{i=1}^t$. The end of N^2 corresponding to $\{A_i\} \times S_j^1$ is $A_i \# S_j^1 = \{A_k \# B : B \in S_j^1\}$, and the characteristic class u_1 evaluates nontrivially on an odd number of these.

Donaldson defines the class u_1 as follows. For each connection $A \in \mathscr{B}^*_{X_+}$ we can twist the Dirac operator over X_+ to get a family of operators $\not{D}_A$ over $\mathscr{B}^*_{X_+}$. Since the bundles in question (i.e., the bundle supporting A and the $\pm$spin bundles) have structure group $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$, it follows that the index bundle $\operatorname{Ind}\not{D}_A$ has a real structure. The class u_1 is defined to be $w_1(\det(\operatorname{Ind}\not{D}_A))$. (The descent of the real line bundle to $\mathscr{B}^*_{X_+}$ follows from the fact that the numerical index $\operatorname{ind}\not{D}_A$ is even; see [2].) To evaluate u_1 on an end $A_i \# S_j^1$ of N^2, first restrict the real line bundle $\det(\operatorname{Ind}\not{D}_A)$ over $A_i \# S_j^1$. Since A_i is fixed, as an element of $K(A_i \# S_j^1)$, the index $\operatorname{Ind}\not{D}_A$ has a constant contribution (the numerical index) coming from A_i and a perhaps twisted contribution from S_j^1. (This can be seen by an excision argument exactly as in [2, Lemma 3.24].) Thus there is a real line bundle over S_j^1 whose first Stiefel-Whitney class $v(S_j^1) \in H^1(S_j^1; \mathbf{Z}_2)$ satisfies $u_1|_{A_i \# S_j^1} = w_1(\det(\operatorname{Ind}\not{D}_A|_{A_i \# S_j^1})) = v(S_j^1)$ for each $i = 1, \cdots, r$ where $H^1(A_i \# S_j^1; \mathbf{Z}_2)$ is identified with $H^1(S_j^1; \mathbf{Z}_2)$. If $v(S_j^1) \neq 0$ for $j = 1, \cdots, s$, and $v(S_j^1) = 0$ for $j = s+1, \cdots, t$, then rs is the number of ends $A_i \# S_j^1$ of N^2 on which u_1 evaluates nontrivially. So rs is odd and therefore r is odd. Hence we have:

Proposition 3.1. *There is a nonempty moduli space $\mathscr{M}_{X_+}(\alpha)$ of* SU(2) *anti-self-dual connections over X_+ such that $\dim \mathscr{M}_{X_+}(\alpha) = 8$ and $m_\alpha = \#(V_{S_1} \cap \cdots \cap V_{S_4} \cap \mathscr{M}_{X_+}(\alpha)) \equiv 1 \pmod 2$.*

As we mentioned earlier, the key idea in the proof of Proposition 3.1 is that Donaldson's proof of his Theorem B [2], which shows that there are no *closed* simply connected 4-manifolds with intersection form $pE_8 \oplus H$ for $p > 0$, does not contradict the existence of $E_8 \oplus H$ on X_+ but rather gives information about the ends of $\mathcal{M}_{X_+,2}(\theta)$ which correspond to the end of X_+. Similarly, by applying the proof of Donaldson's Theorem C [2], which shows that there are no closed simply connected 4-manifolds with intersection form $pE_8 \oplus 2H$ for $p > 0$, we get information about the ends of $\mathcal{M}_{Y_-,3}(\theta)$ corresponding to the end of Y_-. Since Proposition 3.1 has determined that $\partial X = \Sigma$, it follows that $\partial Y = -\Sigma$. Now using the proof of Donaldson's Theorem C in an argument completely analogous to that of Proposition 3.1 we obtain:

Proposition 3.2. *There is a nonempty moduli space* $\mathcal{M}_{Y_-}(\alpha)$ *of* SU(2) *anti-self-dual connections over* Y_- *such that* $\dim \mathcal{M}_{Y_-}(\alpha) = 12$ *and* $n_\alpha = \#(V_{S_5} \cap \cdots \cap V_{S_{10}} \cap \mathcal{M}_{Y_-}(\alpha)) \equiv 1 \pmod 2$.

Thus Proposition 2.5 now implies

Theorem 3.3. *Suppose* M *is a homotopy* K3 *surface containing* $\Sigma(2, 3, 7)$. *Then its Donaldson polynomial invariant* $q_4(M)(S_1, \cdots, S_{10}) \equiv 1 \pmod 2$.

As a corollary we have Theorem 1.1.

It is natural to ask whether $q_4(M)(S_1, \cdots, S_{10}) \equiv 1 \pmod 2$ in *any* homotopy K3 surface.

References

[1] S. Akbulut, *A fake compact contractible* 4-*manifold*, J. Differential Geometry **33** (1991).

[2] S. Donaldson, *Connections, cohomology, and the intersection form of* 4-*manifolds*, J. Differential Geometry **24** (1986) 275–341.

[3] ——, *Polynomial invariants for smooth* 4-*manifolds*, Topology **29** (1990) 257–315.

[4] R. Fintushel & R. Stern, *Instanton homology of Seifert-fibered homology three spheres*, Proc. London Math. Soc. (3) **61** (1990) 109–137.

[5] A. Floer, *An instanton invariant for* 3-*manifolds*, Comm. Math. Phys. **118** (1988) 215–240.

[6] D. Freed & K. Uhlenbeck, *Instantons and four-manifolds*, Math. Sci. Research Inst. Series, vol. 1, Springer, Berlin, 1984.

[7] K. Kodaira, *On homotopy* K3 *surfaces*, Essays on Topology and Related Topics, Mémoires dédiés a Georges deRham, Springer, 1970, 58–69.

[8] E. Looijenga, *The smoothing components of a triangle singularity*. I, Proc. Sympos. Pure Math., Vol. 40, Amer. Math. Soc., Providence, RI, 1983, 173–184.

[9] T. Mrówka, *A local Mayer-Vietoris principle for Yang-Mills moduli spaces*, Ph.D. Thesis, Berkeley, 1988.

[10] C. Taubes, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geometry **19** (1984) 517–560.
[11] ——, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geometry **25** (1987) 363–430.
[12] K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982) 31–42.

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