J. DIFFERENTIAL GEOMETRY
34 (1991) 93–114

SUMS OF ELLIPTIC SURFACES

ROBERT E. GOMPF

Introduction

The classification of smooth, simply connected 4-manifolds remains one of the main unsolved problems of topology. A rich collection of examples is provided by the simply connected algebraic surfaces. A well-known folk conjecture states that perhaps all simply connected, smooth, closed 4-manifolds are connected sums of algebraic surfaces (possibly with some orientations reserved). This suggests the importance of studying the topology of algebraic surfaces—particularly, their behavior under connected sums and related constructions.

An important subclass of algebraic surfaces is comprised by the *elliptic surfaces.* These achieved fame in topology via Donaldson's invariants [4], [5], which showed that there are infinitely many diffeomorphism types of simply connected elliptic surfaces within each homeomorphism type [8], [9], [23], even though there is no classical smoothing uniqueness obstruction. In particular, these are infinite families of counterexamples to the smooth h-Cobordism conjecture for 4-manifolds. In some sense, "most" simply connected algebraic surfaces are elliptic. For any fixed integer $b \geq 10$, there are infinitely many diffeomorphism types of simply connected elliptic surfaces with $b_2 = b$, but only finitely many diffeomorphism types of simply connected, nonelliptic algebraic surfaces with $b_2 \leq b$.

Some progress has been made on the question of how algebraic surfaces behave under connected sum. Donaldson's invariants are "stable" under connected sum with $\overline{CP^2}$, where the bar denotes reversed orientation. Thus, this operation (the algebraic geometer's "blowup") tends not to collapse diffeomorphism types. However, Mandelbaum [18], [20] and Moishezon [22] showed that for any M in a large class of algebraic surfaces (containing all simply connected elliptic surfaces and "complete intersections") the connected sum $M \# CP^2$ (with the standard orientation

Received September 11, 1989 and, in revised form, January 29, 1990. The author was supported by National Science Foundation Grant DMS 8801135.

on each summand) decomposes as $\sharp \pm \mathbf{C}P^2$, a connected sum of copies of $\mathbf{C}P^2$ and $\overline{\mathbf{C}P^2}$. Using this result, it can be shown [11] that for M, N in a large class of algebraic surfaces (containing all simply connected elliptic surfaces and complete intersections except for $\mathbf{C}P^2$) $M \sharp \overline{N}$ decomposes as $\sharp \pm \mathbf{C}P^2$, provided that either M or N does not admit a spin structure. Clearly, this cannot hold if M and N are both spin (since $\mathbf{C}P^2$ is not spin), but one might hope for an analogous decomposition in the spin case. Mandelbaum [18] showed that for V a spin, simply connected elliptic surface, $V \sharp S^2 \times S^2$ decomposes as $\sharp_k K3 \sharp_l S^2 \times S^2$, a sum of copies of $S^2 \times S^2$ and the $K3$ *surface*, the quartic hypersurface in $\mathbf{C}P^3$. In the present paper, Corollary 10 of the first main theorem shows that for V, W spin, simply connected elliptic surfaces $V \sharp \overline{W} \approx \pm(\sharp_k K3 \sharp_l S^2 \times S^2)$.

To state our results conveniently, we make a definition.

Definition. A manifold M *dissolves* if it is diffeomorphic to either $\sharp_k \mathbf{C}P^2 \sharp_l \overline{\mathbf{C}P^2}$ or $\pm(\sharp_k K3 \sharp_l S^2 \times S^2)$ for some $k, l \geq 0$.

Note that only one of the two possibilities can occur for a given M, and this is determined by the *type* of the intersection form (odd or even). The numbers k and l of each summand are determined by the rank and the signature of the form, as is the sign $(\pm)$ denoting the choice of orientation in the even case. We put the sign outside the parentheses, since it is well known that $K3 \sharp \overline{K3} \approx \sharp_{22} S^2 \times S^2$ (Corollary 2). A standard trick of Wall [24] shows that if M is simply connected and nonspin, then $M \sharp S^2 \times S^2 \approx M \sharp S^2 \tilde{\times} S^2$, where the last summand is the twisted S^2-bundle over S^2 and is diffeomorphic to $\mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}$. It follows immediately from this and the work of Mandelbaum and Moishezon that if M dissolves, so do $M \sharp S^2 \times S^2$, $M \sharp \mathbf{C}P^2$, and $M \sharp \overline{\mathbf{C}P^2}$ (with the exception of $\pm(\sharp_k K3 \sharp \overline{\mathbf{C}P^2})$, $k > 0$).

Corollary 10 is now easily stated:

Corollary 10. *If V and W are simply connected elliptic surfaces, then $V \sharp \overline{W}$ dissolves.*

We obtain similar results with other constructions. As we will see in the next section, large elliptic surfaces are made from smaller ones (in the smooth category) by a process called *fiber sum*. This process normally preserves the natural orientations on the elliptic surfaces. However, we may ask what manifolds are obtained from elliptic surfaces by fiber sums which do not respect orientation. (We might even hope to construct manifolds which are not connected sums of algebraic surfaces!) Theorem 13 tells us that if P is a nontrivial fiber sum of elliptic surfaces, with both orientations present, and if P is simply connected, then P dissolves.

Theorem 13 graphically illustrates the importance of orientations in these results: A nontrivial fiber sum of elliptic surfaces with their usual orientations can never dissolve (except for the $K3$ surface, by definition), since it will again be an elliptic surface. Similarly, no information is available about connected sums of nonrational algebraic surfaces with their usual orientations. This phenomenon is also discussed in [11] (in the nonspin connected sum case). In practice, it occurs because nonrational algebraic surfaces tend to contain embedded spheres of negative square (under the intersection pairing) but no spheres of positive square. When we perform either type of sum with an algebraic surface with reversed orientation, it tends to introduce spheres of positive square. (Note that this fails for connected sum with $\overline{\mathbf{C}P^2}$, but it applies to sum with $\mathbf{C}P^2$). The interaction of spheres of positive and negative square is what allows us to prove our theorems, by creating an $S^2 \times S^2$ summand. For example, Corollary 10 follows from Theorem 7, which shows that a simply connected elliptic surface decomposes after connected sum with any manifold containing a sphere of square 2 or 4. (Other positive numbers could also be used.) This main theorem of [11] is in the same spirit.

As another example of this phenomenon we have Theorem 11, which states that a simply connected elliptic surface decomposes after connected sum with any nonorientable 4-manifold. Of course, "positive square" and "negative square" become interchangeable in this setting. As an application, we analyze what happens if the "subtle" exoticness of smooth structures from elliptic surfaces is mixed with the "crude" exoticness of the Cappell-Shaneson fake $\mathbf{R}P^4$ [3], an exotic smooth structure on $\mathbf{R}P^4$ which is detected by classical smoothing theory. If we take a collection of homeomorphic simply connected elliptic surfaces and sum members of this with $\mathbf{R}P^4$ (with either the standard or the Cappell-Shaneson structure), we obtain exactly one or two diffeomorphism types (depending on whether the elliptic surfaces are nonspin or spin), distinguished by classical theory. This prompts a question: Do compact, nonorientable 4-manifolds admit exotic smooth structures which are not detected by classical theory?

We consider one more construction, which seems related to Donaldson-Floer theory. The structure of elliptic surfaces provides us with embedded E_8 plumbings. The boundary of an E_8 plumbing is the Poincaré homology sphere, which also bounds a smaller manifold (with the homotopy type of a 2-sphere) obtained by gluing a 2-handle to a 4-ball along a left-handed trefoil knot with framing -1 (for example, [16]). We may form a manifold M from a simply connected elliptic surface by cutting out some E_8 plumbings and gluing in copies of the smaller manifold. M will be simply connected, with the same b_+ as the original elliptic surface, but smaller b_-. Since we are replacing negative definite pieces by other negative definite pieces along homology spheres with simple Floer theory, we might hope to use Donaldson-Floer theory to derive the Donaldson invariants of M from those of the original elliptic surface. In particular, if we apply the procedure to an infinite family of elliptic surfaces which are homeomorphic but not diffeomorphic, we might hope to distinguish an infinite family of homeomorphic but nondiffeomorphic M's. This could imply that some M's cannot be written as connected sums of algebraic surfaces (by the finiteness result for nonelliptic surfaces mentioned above). Unfortunately, this program fails, since Theorem 14 states that M necessarily dissolves. (Viro has also proved this in the case $b_2 = 10$.) However, this allows us to compute the Donaldson invariants of M directly. Presumably, we obtain constraints on the Donaldson-Floer invariants of the various pieces involved. We prove that M dissolves by relating the construction to another one, in which we replace some singular fibers by other singular fibers with reversed orientation. Manifolds produced by the latter construction also dissolve (Theorem 16). This should seem plausible from our discussion of constructions which reverse orientations, although our method of proof differs somewhat from that of previous proofs.

It should be noted that Matsumoto [21] studied singular torus fibrations over S^2 with fibers which look locally like elliptic fibers (possibly with reversed orientation). Theorems 13 and 16 of the present paper also follow from his work, together with Mandelbaum's theorem that $V \sharp S^2 \times S^2$ dissolves for V elliptic and simply connected.

In the sequel, we will continue to work in the smooth category. Except where otherwise specified, all 4-manifolds will be oriented (canonically, in the algebraic case), and diffeomorphisms and codimension zero embeddings will be assumed to preserve orientation. We will use νF to denote a closed tubular neighborhood of an embedded surface F, which will implicitly be identified with the normal disk bundle to F. A circle (for example, $\mathring{\nu} F$) will be used to denote the interior of a subset.

2. Preliminaries

We begin with a discussion of the topology of elliptic surfaces. Other references for this are [13], [18], [22]. An *elliptic surface* is a compact, complex surface V which admits an *elliptic fibration*, a holomorphic map

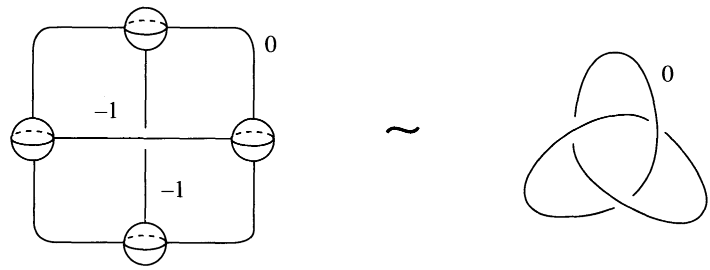

FIGURE 1

$\pi: V \to C$ onto a compact, connected complex curve, such that the generic fibers of π are elliptic curves. Topologically, V is a closed, oriented 4-manifold, mapping onto a closed surface, with generic fibers diffeomorphic to a torus T^2. We will deal primarily with simply connected elliptic surfaces. In this case, C must be a 2-sphere, since any nontrivial loop in C would lift to a nontrivial loop in V. Note that the Euler characteristic of a closed, simply connected 4-manifold is always positive. We will frequently restrict attention to *minimal elliptic surfaces*, i.e., those which are not blowups of other elliptic surfaces. This causes no loss of generality, since an arbitrary elliptic surface has the form $V \sharp_k \overline{\mathbf{C}P^2}$, where V is minimal elliptic.

An elliptic fibration has well-understood structure. It has only finitely many critical values, and away from these it is a bundle projection with torus fibers (called *regular* fibers). The *singular* fibers, or preimages of critical values, have been classified by Kodaira [17]. These fall into two general types: nonmultiple singular fibers (on which π has finitely many critical points) and multiple fibers (on which π is singular everywhere). We will be concerned with two types of nonmultiple singular fibers, namely *cusp* fibers and $\widetilde{E}_8$ fibers, and with *smooth* multiple fibers.

A cusp fiber is a PL-embedded sphere with a unique nonlocally flat point, which is a cone on a right-handed trefoil knot. Thus, a regular neighborhood of a cusp fiber is diffeomorphic to the handlebody obtained by gluing a 2-handle to B^4 along a 0-framed right trefoil. This neighborhood can be obtained from a tubular neighborhood νF of a regular fiber F by ambiently adding a pair of 2-handles along a basis for $H_1(F)$. Figure 1 shows a Kirby calculus picture of this. The 1-handles and 0-framed 2-handle represent $\nu F \approx T^2 \times D^2$; the other two 2-handles cancel the 1-handles to yield the neighborhood of the cusp fiber. (See [13].) Note that if V is an elliptic surface which contains a cusp fiber and $F \subset V$ is a regular fiber, then the inclusion map $F \hookrightarrow V$ is trivial on π_1 (since the 2-handles kill $\pi_1 F$).

An $\widetilde{E}_8$ fiber is described as follows: Begin with the Dynkin diagram for E_8. Add an extra vertex to the long arm to obtain an $\widetilde{E}_8$ graph. Take nine D^2-bundles over S^2 with Euler class -2, and plumb them together according to the $\widetilde{E}_8$ graph. This yields a regular neighborhood of an $\widetilde{E}_8$ fiber (with the fiber itself being the union of the nine 0-sections). Note that by construction, our neighborhood contains an E_8 plumbing. This is the starting point of the last construction described in the introduction.

Simply connected minimal elliptic surfaces without multiple fibers are classified up to diffeomorphism by a positive integer n [14], [22]. Each such manifold V_n has Euler characteristic $12n$ and signature $-8n$, and admits an elliptic fibration with both a cusp fiber and an $\widetilde{E}_8$ fiber. V_1 is the rational elliptic surface $\mathbf{C}P^2 \sharp_9 \overline{\mathbf{C}P^2}$ obtained from $\mathbf{C}P^2$ by blowing up the base locus of a suitable pencil of cubic curves. V_2 is diffeomorphic to the $K3$ surface. In general, V_n is obtained from V_1 by *fiber sum*, as will be described next.

Let M and N be 4-manifolds (not necessarily orientable), and let $K \subset M$ be a compact, codimension zero submanifold. Let $\varphi \colon K \hookrightarrow N$ be an embedding (possibly reversing orientation).

Definition. $M \sharp_\varphi N$ denotes the manifold obtained by gluing $N - \varphi(\mathring{K})$ to $M - \mathring{K}$ along ∂K by the map $\varphi | \partial K$.

For example, suppose that K is diffeomorphic to B^4, and if M and N are both oriented, suppose that φ reverses orientation. Then $M \sharp_\varphi N$ is just the ordinary connected sum $M \sharp N$. In general, if M and N are oriented and φ reverses orientation, then $M \sharp_\varphi N$ will inherit an orientation.

Now suppose that V and W are oriented 4-manifolds with singular torus fibrations over connected, oriented surfaces. (In particular, suppose that they have elliptic fibrations, after possibly reversing orientations.) Let νF be a tubular neighborhood of a regular fiber F in V, which is the preimage of a disk under the fibration. Let $\varphi \colon \nu F \to W$ be an orientation-reversing, fiber-preserving map onto a similar neighborhood in W, such that the induced map between disks reverses orientation.

Definition. The *fiber sum* of V and W is the manifold $V \sharp_\varphi W$.

Note that this inherits a singular torus fibration. The fiber sum of V and W is clearly independent of all choices involving F and φ, except possibly the isotopy class of the map $\varphi | F$ onto its image. This last choice may also be eliminated, provided that either V or W has a cusp fiber (possibly with reversed orientation). (If we perturb the cusp fiber into two "fishtail" fibers, the monodromy of the bundle will realize all orientation-preserving self-diffeomorphisms of F.) V_n is now easily described, as the fiber sum of n copies of V_1. (The orientation on the base does not matter here, since complex conjugation $V_1 \to V_1$ may be assumed compatible with the fibration but it reverses orientation on the base.)

Next, we consider multiple fibers. After a small perturbation of the fibration, we may assume that all multiple fibers are *smooth*, i.e., they are smoothly embedded tori which are multiply covered by nearby regular fibers. (Essentially, they are Seifert multiple fibers crossed with S^1.) Any elliptic fibration can be made from one without multiple fibers by a process called *logarithmic transform*. In this procedure, we remove a neighborhood $\mathring{\nu}F$ of a regular fiber, and glue it back in by some diffeomorphism of $\partial\nu F \approx T^3$. Any time we glue $T^2 \times D^2$ into a 4-manifold along a boundary T^3, the resulting diffeomorphism type is determined by the image μ of $\{\text{point}\} \times S^1$. (In fact, we are adding a 2-handle, two 3-handles, and a 4-handle, so the diffeomorphism type is determined by the framed circle along which the 2-handle is attached. The circle is μ, and the framing is uniquely determined: If we lift to the cover of T^3 corresponding to the $\mathbf{Z}$ subgroup of $\pi_1 T^3$ determined by μ, the framing is the one determined by two distinct lifts of μ.) In the case of logarithmic transforms, μ is determined by its projections into the base (by the fibration) and into the fiber F. The former is essentially a nonnegative integer called the *multiplicity*, and the latter is an element of $H_1(F)$, which is a multiple of a primitive class called the *direction*. Logarithmic transforms of multiplicity zero have no algebrogeometric interpretation (they destroy complex and elliptic structure), but we will still find them useful. It can be shown (for example, [12]) that in the presence of a cusp fiber, the diffeomorphism type resulting from logarithmic transforms depends only on the multiplicities involved.

We form manifolds $V_n(p_1, \cdots, p_k)$ by applying logarithmic transforms of multiplicities $p_1, \cdots, p_k$ to V_n. By the above remarks, these are completely determined up to diffeomorphism by the nonnegative integers $p_1, \cdots, p_k$ (unordered) and $n \geq 1$. Since the trivial logarithmic transform has multiplicity one, we can add or delete p_i's equal to one without affecting diffeomorphism type. $V_n(p_1, \cdots, p_k)$ is elliptic if all p_i's are nonzero. (In fact, this yields all minimal elliptic surfaces over S^2 with nonzero Euler characteristic.)

$V_n(p_1, \cdots, p_k)$ will be simply connected if and only if it can be written (by adding or deleting p_i's equal to one) as $V_n(p, q)$ with p, q relatively prime (including $(p, q) = (0, 1)$). (To understand π_1, it should be noted that for $F \subset V_n$ a regular fiber, $\pi_1(V_n - F) = 1$. When $n = 1$, this follows because $\pi_1 V_1 = 1$ and an exceptional curve provides a nullhomotopy for a meridian of F in $V_1 - F$. The general case follows from the $n = 1$ case.) A complete list of simply connected elliptic surfaces (up to diffeomorphism) is given by $V_n(p, q)\#_k\overline{\mathbf{C}P^2}$, $p, q \neq 0$ [22]. (The minimal ones are precisely those with $k = 0$.) The remaining case $V_n(0)$ dissolves (into $\# \pm \mathbf{C}P^2$) [12], so for $n > 1$ it is neither elliptic nor complex.

Since logarithmic transform preserves Euler characteristic and signature, the intersection form of a simply connected $V_n(p, q)$ will have rank $12n - 2$ and signature $-8n$. By Freedman's classification [6], two such manifolds, $V_n(p, q)$ and $V_n(p', q')$, will be homeomorphic if and only if their forms have the same type (even or odd). But the form of $V_n(p, q)$ is even (i.e., $V_n(p, q)$ is spin) if and only if n is even and p, q are both odd. (Note, in particular, that the answer depends on n only through its residue $\bmod 2$.) This provides a homeomorphism classification: one type for each odd n, and two for each even n. The diffeomorphism classification is much more complex and only partially understood. A summary appears in [12].

We will need some general facts about 4-manifolds. In particular, we will use the following mathematical folklore.

Proposition 1. *Let H^4 be obtained from a 4-ball by attaching 2-handles. Let $2H$ denote the double of H. Then $2H$ dissolves.*

Proof. $2H$ is the boundary of $H \times I$, which is a 5-manifold built from B^5 by attaching 2-handles. The attaching circles cannot be knotted or linked in $\partial B^5 = S^4$, since homotopic embeddings of a 1-manifold in a 4-manifold are isotopic. Thus, $H \times I$ is a boundary sum of D^3-bundles over S^2. In particular, $2H = \partial(H \times I)$ is a connected sum of $S^2 \times S^2$'s and (if H is nonspin) $S^2 \tilde{\times} S^2$'s. Since $S^2 \tilde{\times} S^2 \approx \mathbf{C}P^2\#\overline{\mathbf{C}P^2}$, we are done.

Corollary 2. *Let M be a closed 4-manifold which admits a handle decomposition with no 1- or 3-handles. Then $M\#\overline{M}$ dissolves. In particular, $K3\#\overline{K3} \approx \#_{22}S^2 \times S^2$.*

Proof. The manifold $H = M - \mathring{B}^4$ admits a handle decomposition as in Proposition 1, and $2H = M\#\overline{M}$. The $K3$ surface admits a handle decomposition without 1- or 3-handles, by [13]. q.e.d.

Other examples of closed 4-manifolds with handle decompositions as in Corollary 2 include V_n [13] and $V_n(p)$ [12].

We also need the following lemma of Moishezon.

Lemma 3. (Moishezon [22, Lemma 13]). *Let L be a lens space (possibly S^3 or $S^2 \times S^1$). Let M be a manifold obtained from $L \times S^1$ by surgery on two circles, one of which is* $\{\text{point}\} \times S^1$. *Suppose* $\pi_1 M = 1$. *Then M is diffeomorphic to $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.*

Proof. Details appear in [22], so we merely sketch the argument. The main trick is as follows: Suppose $T^2 \times D^2$ is embedded in a 4-manifold N. If we perform surgery on the pair $(N, T^2 \times \{0\})$ along an essential circle μ in T^2 with the framing induced by the $T^2 \times D^2$ product structure, we obtain a 4-manifold N^* with an embedded copy of $S^2 \times D^2$. Performing surgery on N^* along this S^2 results in a manifold $\widehat{N}$. Clearly, N^* may be reconstructed from $\widehat{N}$ by surgery along a certain circle. Moishezon computed that $\widehat{N}$ is obtained from N by deleting $T^2 \times \mathring{D}^2$ and gluing it back in by a map sending $\{\text{point}\} \times S^1$ to the circle $\mu \times \{\text{point}\}$ in $T^2 \times S^1$. This is easily seen by Kirby calculus: If $T^2 \times D^2$ is drawn as Borromean rings with two dotted circles and a 0-framed 2-handle, the first surgery turns a dotted circle to a 0-framed one, and surgery on the 2-sphere turns the original 0-framed circle to a dotted one.

Now let N be $L \times S^1$. Let N^* be obtained from N by surgery on $\{\text{point}\} \times S^1$. There are two possible framings on this circle, but by a self-diffeomorphism of $L \times S^1$ (preserving the S^1 coordinate) we may change one to the other. The genus 1 Heegaard splitting of L decomposes N into two copies of $T^2 \times D^2$. Applying the previous paragraph to one of these, we see that N^* is obtained from a manifold $\widehat{N}$ by surgery on a circle, and $\widehat{N}$ is obtained by gluing together two copies of $S^1 \times S^1 \times D^2$ by a map sending $1 \times 1 \times S^1$ to $1 \times S^1 \times 1$. Thus, $\widehat{N}$ is $S^1 \times S^3$. N^* is made from $S^1 \times S^3$ by a surgery, so M is made from $S^1 \times S^3$ by two surgeries on circles. Since M is simply connected, we may slide the surgery circles over each other until one is trivial and the other is $S^1 \times \{\text{point}\}$. The result of these surgeries is either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.

3. The main results

Mandelbaum and Moishezon proved that many algebraic surfaces (including simply connected elliptic surfaces) dissolve after connected sum with $\mathbf{C}P^2$. Their work is based on an "irrational connected sum" lemma of Mandelbaum [19], which allows objects such as fiber sums to be decomposed into ordinary connected sums in the presence of an additional

$S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ connected summand. Our Lemma 4 is an adaptation of this lemma to the spin case, which (presumably) was known to Mandelbaum since he asserted [18] that $V \sharp S^2 \times S^2$ dissolves when V is a spin, simply connected elliptic surface. The proof of Lemma 4 and its application to dissolving elliptic surfaces are essentially taken from the work of Mandelbaum and Moishezon. The new ideas of the present paper center around constructing the required $S^2 \times S^2$ from pieces of a more complicated ambient space (cf. [11]).

Lemma 4. (Mandelbaum). *Let M and N be oriented 4-manifolds. Let $F \subset M$ be a closed, connected, orientable, surface of genus g, and let $\varphi\colon \nu F \hookrightarrow N$ be an orientation reversing embedding. Let $P = M \sharp_\varphi N$. Suppose that the inclusion map $\partial \nu F \hookrightarrow M - \mathring{\nu} F$ is trivial on π_1. Suppose that either* (1) *M is spin, or* (2) *P is not spin and $\pi_1 P = 1$. Then $P \sharp S^2 \times S^2$ is diffeomorphic to $M \sharp N^*$, where N^* is a manifold obtained from N by surgery on $2g$ circles in $\varphi(\nu F)$ forming a (preassigned) symplectic basis for $H_1(\varphi(F))$. If M or N has boundary, then we may assume the diffeomorphism is the identity on $\partial P = \partial M \cup \partial N$.*

Corollary 5. (Mandelbaum). *Under the above hypotheses, if $\pi_1 N = 1$, then $P \sharp S^2 \times S^2$ is diffeomorphic to $M \sharp N \sharp_{2g} S^2 \times S^2$ (if P is spin) or $M \sharp N \sharp_{2g} \mathbf{C}P^2 \sharp_{2g} \overline{\mathbf{C}P^2}$ (if P is not spin).*

Proof of Corollary 5. This follows immediately, by the techniques of Wall [24]. Since N is a simply connected 4-manifold, it contains (up to isotopy) only one collection of $2g$ circles. Thus, each surgery is a connected sum with either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. On a simply connected nonspin manifold, these operations are the same.

Remark. The missing case of Lemma 4 (P is spin and M is not spin) is clearly false, since $P \sharp S^2 \times S^2$ will be spin and $M \sharp N^*$ will not be. It becomes true, however, if $S^2 \times S^2$ is replaced by $S^2 \tilde{\times} S^2$. This case arises, for example, when P is the $K3$ surface, since it is the fiber sum of two copies of the nonspin manifold V_1.

Proof of Lemma 4. Let $\mu \subset \partial \nu F$ be a meridian to F. Then μ inherits a canonical normal framing in $\partial \nu F$ from the local product structure induced by the normal bundle to F. Extend this to a normal framing of μ in P, and let P^* be the result of surgery on this framed circle in P. We verify that P^* is diffeomorphic to $P \sharp S^2 \times S^2$: Since inclusion $\partial \nu F \hookrightarrow M - \mathring{\nu} F$ is π_1-trivial, μ is nullhomotopic in $M - \mathring{\nu} F$. Thus, P^* is either $P \sharp S^2 \times S^2$ or $P \sharp S^2 \tilde{\times} S^2$. In Case 1 M is spin, and the framing on μ is the one corresponding to spin surgery on M. Restricting to $M - F$, we see that the surgery is the same as connected sum with

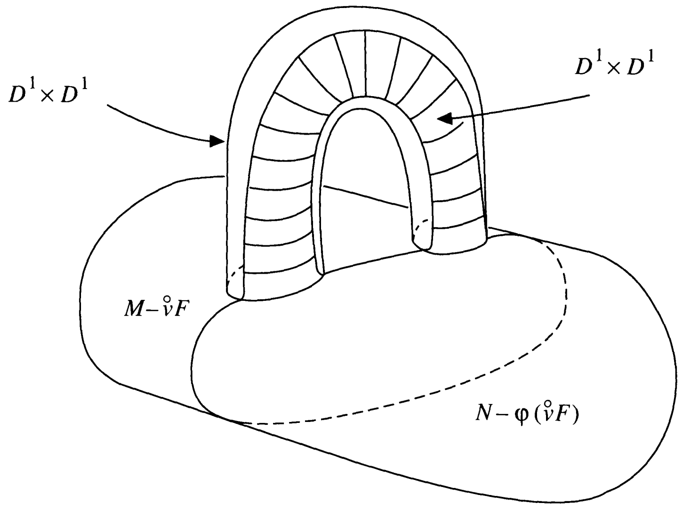

FIGURE 2

$S^2 \times S^2$. In Case 2, P is nonspin and simply connected, so by Wall, $P \sharp S^2 \tilde{\times} S^2 \approx P \sharp S^2 \times S^2$.

Now let $F_0 = F - \mathring{D}^2$, and let φ_0 be the restriction of φ to νF_0. We show that P^* is diffeomorphic to $M \sharp_{\varphi_0} N$. P^* is obtained from P by deleting a tubular neighborhood of μ to obtain a manifold P_0, and then gluing in $S^2 \times D^2$. P_0 is obtained by gluing $M - \mathring{\nu}F$ to $N - \varphi(\mathring{\nu}F)$ everywhere along the boundary except near μ. If we split $S^2 \times D^2$ into two copies of $D^2 \times D^2$, we may think of one $D^2 \times D^2$ as a 2-handle attached to $M - \mathring{\nu}F$ along μ, and the other $D^2 \times D^2$ as a 2-handle attached to $N - \varphi(\mathring{\nu}F)$ along $\varphi(\mu)$. Figure 2 shows the analogous phenomenon in dimension 2. The 2-handle attached to $M - \mathring{\nu}F$ is glued along μ with the canonical framing, so it has the effect of filling in a normal disk to F, resulting in $M - \mathring{\nu}F_0$. Similarly, the other 2-handle gives $N - \varphi(\mathring{\nu}F_0)$. P^* is now seen to be $M \sharp_{\varphi_0} N$.

Finally, we show that P^* is $M \sharp N^*$ as required. Observe that νF_0 is a regular neighborhood of a 1-complex. Let B be a regular neighborhood of the 0-cell, and let $\gamma_1, \cdots, \gamma_{2g}$ be arcs in M attached to B to determine νF_0. Then $M \sharp_{\varphi|B} N$ is the connected sum $M \sharp N$, and $P^* \approx M \sharp_{\varphi_0} N$ is obtained from this by deleting neighborhoods of the $2g$ circles $\gamma_i \cup \varphi(\gamma_i)$ and gluing each resulting boundary component to itself by φ. This is the same as performing surgeries on the circles $\gamma_i \cup \varphi(\gamma_i)$ in $M \sharp N$. But $\partial \nu F$ is π_1-trivial in $M - \mathring{\nu}F$, so the arcs γ_i may be isotoped into ∂B, and we see that P^* is made from $M\sharp N$ by surgery on $2g$ circles in N representing a basis for $H_1(\varphi(\nu F))$. q.e.d.

We may weaken the hypotheses of Lemma 4 to obtain the following lemma (which we need for Theorem 11, and for Theorem 7 if $\pi_1 W \neq 1$).

Lemma 6. *Let M, N, F, φ, and P be as in Lemma* 4, *but with M and N possibly nonorientable.* (*If either is nonorientable, drop the requirement that φ reverse orientation.*) *In place of the last hypothesis, assume that either* (1) $\widetilde{M}$ *is spin or* (2) $\widetilde{P}$ *is not spin* (*where the tilde denotes universal cover*). *Then $P\sharp S^2 \times S^2 \approx M\sharp N^*(\mathrm{rel}\,\partial)$ as in Lemma* 4.

Proof. Note that νF is an orientable manifold, since $\pi_1 F \to \pi_1 M$ is trivial. The hypotheses which we have weakened are only used in the first paragraph of the proof of Lemma 4, where we proved $P^* \approx P\sharp S^2 \times S^2$. Here, Case 1 follows as before, lifting to the universal cover. In Case 2, $\widetilde{P}$ is nonspin and simply connected, so it contains an immersed 2-sphere with odd normal Euler class. We push this down into P, where it allows us to apply Wall's trick to obtain $P\sharp S^2 \times S^2 \approx P\sharp S^2 \tilde{\times} S^2$.

Theorem 7. *Let V be a simply connected minimal elliptic surface. Let $12n$ be its Euler characteristic. Let W be an oriented 4-manifold containing an embedded 2-sphere of square -2 or -4, and let $\widetilde{W}$ denote its universal cover.*

(A) *If V and $\widetilde{W}$ are spin, then $V\sharp\overline{W}$ is diffeomorphic* $(\mathrm{rel}\,\partial\overline{W})$ *to* $\sharp_{n/2}K3\sharp_{(n/2)-1}S^2\times S^2\sharp\overline{W}$. *In particular, if $W\sharp S^2\times S^2$ dissolves and $n>2$, then $V\sharp\overline{W}$ dissolves.*

(B) *If V or $\widetilde{W}$ is not spin, then $V\sharp\overline{W}$ is diffeomorphic* $(\mathrm{rel}\,\partial\overline{W})$ *to* $\sharp_{2n-1}\mathbf{C}P^2\sharp_{10n-1}\overline{\mathbf{C}P^2}\sharp\overline{W}$. *If $W\sharp\mathbf{C}P^2$ dissolves, then $V\sharp\overline{W}$ dissolves.*

Corollary 8 (Mandelbaum [18]). *If V is a simply connected elliptic surface, then $V\sharp S^2\times S^2$ dissolves.*

Corollary 9 (Mandelbaum [18] and Moishezon [22]). *If V is a simply connected elliptic surface, then $V\sharp\mathbf{C}P^2$ dissolves.*

Corollary 10. *If V and W are simply connected elliptic surfaces, then $V\sharp\overline{W}$ dissolves.*

Proof. Corollary 10 is immediate from Theorem 7 unless V and W are both spin with Euler characteristic 24. In this case, $V\sharp\overline{W}\approx K3\sharp\overline{W}\approx K3\sharp\overline{K3}$ (by Theorem 7, reversing orientation in one case). But $K3\sharp\overline{K3}$ dissolves by Corollary 2.

Proof of Theorem 7. We begin with Part A. Write V as $V_n(p,q)$, where n is even since V is spin. Suppose $n>2$. Then V is the fiber sum $V_2\sharp_\varphi V_{n-2}(p,q)$, where φ maps a neighborhood νF of a regular fiber in V_2 onto a neighborhood of a regular fiber $\varphi(F)$ in $V_{n-2}(p,q)$.

By changing the elliptic fibration on $V_{n-2}(p, q)$, we may assume it has an $\widetilde{E}_8$ fiber. In particular, we have a pair of embedded spheres S_1, S_2 with square -2, intersecting (transversely) at a single point. In $V_{n-2}(p, q)\sharp\overline{W}$ we are given a sphere with square 2 or 4. Tube this into S_2 to obtain a sphere with square 0 or 2. In the latter case, tube S_2 into one of the remaining spheres of square -2 in the $\widetilde{E}_8$ fiber, to drop the square to 0. We are left with a wedge of two 2-spheres, with squares -2 and 0. A regular neighborhood of this is a punctured $S^2 \times S^2$, so we have decomposed $V_{n-2}(p, q)\sharp\overline{W}$ as $N\sharp S^2 \times S^2$, where the image of φ lies in N. Write $V\sharp\overline{W} = V_2\sharp_\varphi V_{n-2}(p, q)\sharp\overline{W} = V_2\sharp_\varphi N\sharp S^2 \times S^2$. We may apply Lemma 4 to $V_2\sharp_\varphi N$ (since $\pi_1(V_2 - \mathring{\nu}F) = 1$ and V_2 is spin) to obtain $V\sharp\overline{W} \approx V_2\sharp N^*$. But N^* is made from N by surgery on two homotopically trivial loops (namely, two generators of $H_1(\varphi(\nu F))$), so $N^* \approx N\sharp_2 S^2 \times S^2$. (There can be no $S^2 \widetilde{\times} S^2$ summand since N^* has spin universal cover. This is because the universal cover of $V_2\sharp N^* \approx V\sharp\overline{W}$ is spin.) Thus, by the definition of N, $N^* = V_{n-2}(p, q)\sharp\overline{W}\sharp S^2 \times S^2$, and $V\sharp\overline{W} \approx V_2\sharp S^2 \times S^2\sharp V_{n-2}(p, q)\sharp\overline{W}$. But V_2 is the $K3$ surface, so by induction on n we reduce to the $n = 2$ case.

When $V = V_2(p, q)$, it is the fiber sum $V_2\sharp_\varphi Q$, where Q is obtained from $S^2 \times T^2$ by logarithmic transforms of multiplicities p and q. Since the diffeomorphism type of $V_2(p, q)$ depends only on p and q, we may assume that the directions of the log transforms are along the first factor of $T^2 = S^1 \times S^1$. It follows that $Q = L \times S^1$, where L is a Seifert fibered space with two multiple fibers, i.e., a lens space. The elliptic fibers are Seifert fibers crossed with S^1.

The proof in the $n = 2$ case now proceeds as before. $V_2\sharp\overline{W}$ decomposes as $S^2 \times S^2\sharp M$, with a regular fiber F lying in M. Thus, $V\sharp\overline{W} = \overline{W}\sharp V_2\sharp_\varphi Q = S^2 \times S^2\sharp M\sharp_\varphi Q$. We may apply Lemma 6 to $M\sharp_\varphi Q$, because $\widetilde{M}$ is spin and $\partial\nu F$ is π_1-trivial in $M - \mathring{\nu}F$ (since it is trivial in $V_2 - \mathring{\nu}F$). We get $V\sharp\overline{W} \approx M\sharp Q^*$, where Q^* is obtained from Q by surgery on a pair of generators of a regular fiber of Q. If we choose $\{\text{point}\} \times S^1$ in $L \times S^1 = Q$ as one generator, Lemma 3 implies that Q^* is $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$. (To see that $\pi_1 Q^* = 1$, note that this group is obtained from $\pi_1 Q$ by killing $\varphi_*\pi_1 F$. But $1 = \pi_1 V = \pi_1(V_2\sharp_\varphi Q)$ is obtained in the same way, since $\pi_1(V_2 - \mathring{\nu}F)$ is trivial.) Q^* cannot be $S^2 \widetilde{\times} S^2$ since the universal cover of $V\sharp\overline{W}$ is spin, so $V\sharp\overline{W} \approx M\sharp Q^* \approx M\sharp S^2 \times S^2 = V_2\sharp\overline{W} = K3\sharp\overline{W}$.

Part B of the theorem follows from a similar argument. The only differences are that we reduce to the $n = 1$ case by splitting off copies of $V_1 = \mathbf{C}P^2 \sharp_9 \overline{\mathbf{C}P^2}$, and we apply Case 2 of Lemma 6 in place of Case of Lemmas 4 and 6. (Note that after the first step of the induction, we may find that $V_{n-1}(p, q)$ and $\widetilde{W}$ are spin. If this occurs, we can force $\widetilde{W}$ to be nonspin by adding one of the extra $\mathbf{C}P^2$'s to W. Also recall that $M \sharp S^2 \tilde{\times} S^2 \approx M \sharp S^2 \times S^2$ if $\widetilde{M}$ is nonspin; see the proof of Lemma 6.)

Theorem 11. *Let V be a simply connected minimal elliptic surface with Euler characteristic $12n$. Let W be a nonorientable 4-manifold, with universal cover $\widetilde{W}$.*

(A) *If V and $\widetilde{W}$ are spin, then $V \sharp W$ is diffeomorphic* $(\mathrm{rel}\, \partial W)$ *to either $\sharp_{6n-1} S^2 \times S^2 \sharp W$ (if $\frac{1}{2}n$ is even) or $K3 \sharp_{6n-12} S^2 \times S^2 \sharp W$ (if $\frac{1}{2}n$ is odd).*

(B) *If V or $\widetilde{W}$ is not spin, then $V \sharp W$ is diffeomorphic* $(\mathrm{rel}\, \partial W)$ *to $\sharp_{12n-2} \mathbf{C}P^2 \sharp W$.*

Proof. The proof is similar to that of Theorem 7. We prove Part A; Part B follows by the same method. If $n > 2$, write $V = V_n(p, q) = V_2 \sharp_\varphi V_{n-2}(p, q)$ as before (or $V \approx V_1 \sharp_\varphi V_{n-1}(p, q)$ for Part B, $n > 1$). To decompose $V_{n-2}(p, q) \sharp W$ as $N \sharp S^2 \times S^2$, locate an $\widetilde{E}_8$ fiber in $V_{n-2}(p, q)$. Find a pair of intersecting spheres of square -2 in this fiber, along with a disjoint sphere of square -2. Tube the latter into one of the former spheres in $V_{n-2}(p, q) \sharp W$, using a tube which runs along an orientation-reversing loop in W. This produces a sphere with trivial normal bundle, transversely intersecting a sphere of normal Euler class -2. A regular neighborhood of these two spheres is the required punctured $S^2 \times S^2$.

Now Lemma 6 applies in place of Lemma 4, to show that $V \sharp W = V_2 \sharp_\varphi N \sharp S^2 \times S^2 \approx V_2 \sharp N^*$. (For Part B, observe that $V_1 \sharp_\varphi N$ has a nonspin universal cover since $V \sharp W$ does.) As before, $V \sharp W \approx V_2 \sharp S^2 \times S^2 \sharp V_{n-2}(p, q) \sharp W$. By Corollary 8, this is the same as $\sharp_{n/2} K3 \sharp_{(n/2)-1} \times S^2 \times S^2 \sharp W$. Since W is nonorientable, $K3 \sharp K3 \sharp W \approx K3 \sharp \overline{K3} \sharp W \approx \sharp_{22} S^2 \times S^2 \sharp W$ by Corollary 2. Thus, we are done unless $n = 2$ (or $n = 1$ in Case B).

Assuming $n = 2$, write $V = V_2 \sharp_\varphi Q$ as before. $V_2 \sharp W = S^2 \times S^2 \sharp M$, as above. By Lemma 6, $V \sharp W = S^2 \times S^2 \sharp M \sharp_\varphi Q \approx M \sharp Q^*$. (Note that in Part A $\widetilde{M}$ is spin, and in Part B $M \sharp_\varphi Q$ has nonspin universal cover.) As before, $V \sharp W \approx M \sharp S^2 \times S^2 = V_2 \sharp W$, and we are done.

Remarks. It follows easily from Freedman's classification of simply connected topological 4-manifolds [6] that $K3 \sharp W$ is homeomorphic to

$\sharp_{11}S^2 \times S^2 \sharp W$. ($K3 \approx E_8 \sharp E_8 \sharp_3 S^2 \times S^2$ and $E_8 \sharp \overline{E}_8 \approx \sharp_8 S^2 \times S^2$.) These cannot be diffeomorphic if W is the twisted S^3 bundle over S^1. (Otherwise, the methods of [10] would contradict the stable exoticness of the Cappell-Shaneson fake $\mathbf{R}P^4$.) Parts A and B of Theorem 11 are distinguished by whether the universal cover of $V \sharp W$ is spin. However, if $\widetilde{W}$ is not spin, then $V \sharp W$ can be written as $\sharp_{6n-1} S^2 \times S^2 \sharp W$ by Wall's trick.

Theorem 11 has an amusing application. Let W_1 denote $\mathbf{R}P^4$, and let W_2 denote the (simplest) fake $\mathbf{R}P^4$ of Cappell and Shaneson [3]. These are homeomorphic (by Freedman's s-cobordism theorem with $\pi_1 = \mathbf{Z}_2$ [7]) but not diffeomorphic. W_1 and W_2 exhibit a "crude" sort of exotic smoothing, distinguished by high-dimensional smoothing theory, in contrast with the subtle exotic structures given by elliptic surfaces. What happens if we combine the two types of examples by connected sum? If $\{V_i | i = 1, 2, 3, \cdots\}$ is a family of nonspin, simply connected elliptic surfaces with fixed Euler characteristic k, then the manifolds $V_i \sharp W_j$ are all diffeomorphic to $\sharp_{k-2} \mathbf{C}P^2 \sharp \mathbf{R}P^4$ by Theorem 11 and Akbulut's result [1] that $W_2 \sharp \mathbf{C}P^2 \approx \mathbf{R}P^4 \sharp \mathbf{C}P^2$. If the manifolds V_i are spin, we get that $V_i \sharp W_j$ decomposes as in Part A, so it is independent of i. In this case, there are exactly two diffeomorphism types, distinguished by j. (This follows because $W_1 \sharp_l S^2 \times S^2$ and $W_2 \sharp_l S^2 \times S^2$ are not diffeomorphic for any l [3]. Note that if a $K3$ also appears in the decomposition, we can eliminate it by sum with $\overline{K3}$.) The same remarks apply if $W_1 = K3 \sharp S^3 \tilde{\times} S^1$ and $W_2 = \sharp_{11} S^2 \times S^2 \sharp S^3 \tilde{\times} S^1$: In the nonspin case $V_i \sharp W_j$ is $\sharp_{k+20} \mathbf{C}P^2 \sharp S^3 \tilde{\times} S^1$, and in the spin case we get two distinct diffeomorphism types $\sharp_{(k/2)+10} S^2 \times S^2 \sharp S^3 \tilde{\times} S^1$ and $K3 \sharp_{(k/2)-1} S^2 \times S^2 \sharp S^3 \tilde{\times} S^1$.

Next, we consider fiber sums of elliptic surfaces with incompatible orientations. Let $P = V_m(p_1, \cdots, p_k) \sharp_\varphi \overline{V}_n(p_{k+1}, \cdots, p_l)$ be the fiber sum of two minimal elliptic surfaces, where we have reversed orientation on the second summand. By reversing the orientation of P if necessary, we may assume $m \geq n$. Since we may isotope fibers from one summand to the other, we may write P as $V_m(p_1, \cdots, p_l) \sharp_\varphi \overline{V}_n$. We need to understand the algebraic topology of P. Clearly, the Euler characteristic is $12(m+n)$ and the signature $8(n-m)$. Since $\pi_1(\overline{V}_n - \varphi(F)) = 1$ and inclusion of a fiber into $V_m(p_1, \cdots, p_l)$ is π_1-trivial, we have $\pi_1 P \cong \pi_1 V_m(p_1, \cdots, p_l)$. In particular, P is simply connected if and only if it can be written (by adding or deleting p_i's equal to 1) as $V_m(p, q) \sharp_\varphi \overline{V}_n$, with p, q relatively prime. This applies even when some p_i's equal zero, so we will include the nonelliptic situation $(p, q) = (0, 1)$ in further discussions.

It remains to consider when P is spin. The homeomorphism type of P will then be completely determined in the simply connected case. Let P' denote the manifold $V_{m+n}(p_1, \cdots, p_l)$.

Proposition 12. *P has an even intersection form if and only if P' does. P is spin if and only if P' is. In particular, if $\pi_1 P = 1$, P is spin if and only if $m+n$ is even and p, q are both odd.*

Proof. (Compare with [12].) The elliptic surface V_n contains a section, which is an embedded 2-sphere with square $-n$, hitting each fiber exactly once. Let $N_n \subset V_n$ denote a closed regular neighborhood of this section union a cusp fiber. N_n has the homotopy type of $S^2 \vee S^2$ and unimodular intersection form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -n \end{smallmatrix}\right]$, so ∂N_n is a homology sphere. Performing logarithmic transforms in $\mathring{N}_n$, we obtain a manifold $N_n(p_1, \cdots, p_k) \subset V_n(p_1, \cdots, p_k)$, called the *nucleus* of the elliptic surface [12], with boundary the homology sphere ∂N_n. Note that if we change n, the nucleus is only changed by modifying the framing on one 2-handle (namely, the one coming from the section). This allows us to abstractly define a manifold $N_0(p_1, \cdots, p_k)$, and shows that n affects whether $N_n(p_1, \cdots, p_k)$ is spin or has an even intersection form only through its mod 2 residue. Now observe that $V_m \sharp_\varphi \overline{V}_n$ admits a section with square $n - m$, obtained by summing together sections of V_m and $\overline{V}_n$. This allows us to construct an embedding $N = N_{m-n}(p_1, \cdots, p_l) \subset P$, by analogy with the above procedure. Note that since $m - n \equiv m + n \pmod 2$, the nucleus N' of P' is spin if and only if N is, and the intersection forms of N and N' have the same type.

We show that N and P have intersection forms of the same type. Since ∂N is a homology sphere, it induces an orthogonal sum decomposition of the form of P, so it suffices to see that $P - \mathring{N}$ has an even form. But this lies in $V_m \sharp_\varphi \overline{V}_n - F$ (F being a regular fiber), which embeds in $V_m \sharp_\varphi \overline{V}_m$. The latter has an even form, since it is the double of the spin manifold $V_m - \mathring{\nu} F$.

To finish the argument, apply similar reasoning to N' and P'. We see that the intersection forms of P, N, N', and P' all have the same type. To see that all four of these are spin if any one is, note that $\partial N \times \mathbf{R}$ admits a unique spin structure and apply the same technique.

Theorem 13. (Compare with Matsumoto [21].). *Let P be the fiber sum of two elliptic surfaces with nonzero Euler characteristics, one of which has the nonstandard orientation. If P is simply connected, then P dissolves. The same result holds if we allow logarithmic transforms of multiplicity zero.*

Proof. It suffices to deal with minimal elliptic surfaces and $V_n(0)$. Since P is simply connected, both elliptic surfaces must have 2-sphere base, so (reversing orientation if necessary) we can put P in the form $V_m(p, q)\sharp_\varphi \overline{V}_n$, $m \geq n > 0$, p, q relatively prime (including $(p, q) = (0, 1)$).

We begin with the case where P is spin. Then $m-n$ is even. Suppose $m > n$. Write $P = V_2\sharp_\psi V_{m-2}(p, q)\sharp_\varphi \overline{V}_n$, $m-2 \geq n > 0$. Using $\widetilde{E}_8$ fibers, we decompose $V_{m-2}(p, q)\sharp_\varphi \overline{V}_n$ as $N\sharp S^2 \times S^2$, with the image of ψ in N. By Lemma 4, $P = V_2\sharp_\psi N\sharp S^2 \times S^2 \approx V_2\sharp N^*$. Since N is simply connected, $P \approx V_2\sharp N\sharp_2 S^2 \times S^2 = V_2\sharp S^2 \times S^2\sharp V_{m-2}(p, q)\sharp_\varphi \overline{V}_n$. By induction, we reduce to the case where $m = n$. To simplify $V_n(p, q)\sharp_\varphi \overline{V}_n$, write it as $V_n\sharp_\varphi \overline{V}_n\sharp_\psi Q$, which is $V_n\sharp_\varphi \overline{V}_n$ by Lemma 4 (as in the proof of Theorem 7), since $V_n\sharp_\varphi \overline{V}_n - F$ is simply connected. Now observe that $V_n\sharp_\varphi \overline{V}_n$ is the double of $V_n - \mathring{\nu} F$, and the latter has a handle decomposition without 1- or 3-handles [13]. (Begin with a neighborhood of a cusp fiber = 0-handle $\cup$ 2-handle; the remaining singular fibers add more 2-handles.) Thus $V_n\sharp_\varphi \overline{V}_n$ dissolves, by Proposition 1.

Now suppose P is not spin and $m > 1$. Write $P = V_1\sharp_\psi V_{m-1}(p, q)\sharp_\varphi \overline{V}_n$. As above, Lemma 4 splits this as $V_1\sharp S^2 \times S^2\sharp V_{m-1}(p, q)\sharp_\varphi \overline{V}_n$. If p and q are odd, then $V_{m-1}(p, q)\sharp_\varphi \overline{V}_n$ is spin, and we have reduced to the previous case. Otherwise, we continue by induction (reversing orientation when necessary) to reduce to the case of $V_1(p, q)\sharp_\varphi \overline{V}_1$, where p or q is even.

This last case must be handled separately. As before, we can split the manifold as $V_1\sharp_\varphi \overline{V}_1\sharp_\psi Q$, but Lemma 4 yields $V_1(p, q)\sharp_\varphi \overline{V}_1 \approx M\sharp S^2 \widetilde{\times} S^2$, where $M\sharp S^2 \times S^2 = V_1\sharp_\varphi \overline{V}_1$ is spin. (If we started with $m > 1$, we are now done, since the extra $\mathbf{C}P^2$'s allow us to change $S^2 \widetilde{\times} S^2$ to $S^2 \times S^2$.) What we see from this is that the diffeomorphism type of $V_1(p, q)\sharp_\varphi \overline{V}_1$ is independent of p, q (provided that p or q is even and p, q are relatively prime). Thus, it suffices to show that $V_1(0)\sharp_\varphi \overline{V}_1$ dissolves.

Figure 3 (next page) shows a Kirby calculus picture of $V_1 - \mathring{\nu} F$ as constructed by Harer, Kas, and Kirby [13]. The 1-handles and 0-framed 2-handle are a tubular neighborhood of a regular fiber (cf. Figure 1). The two ribbons represent twelve 2-handles which are interleaved where the ribbons cross. These 2-handles come from the singular fibers of V_1. We can draw $V_1\sharp_\varphi \overline{V}_1 - \mathring{\nu} F$ by the same method, since $V_1\sharp_\varphi \overline{V}_1$ has a singular fibration over S^2. (This looks just like an elliptic fibration, except that

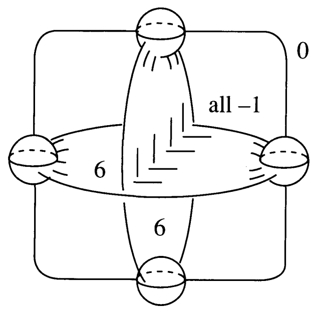

FIGURE 3

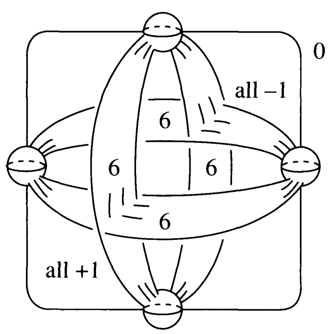

FIGURE 4

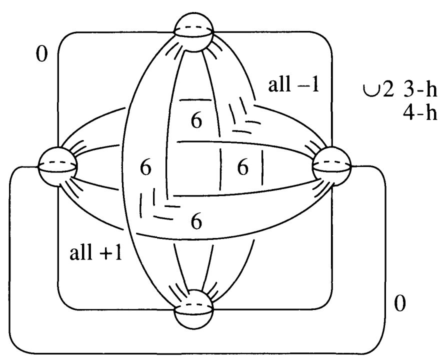

FIGURE 5

half of the singular fibers are backward.) The result is Figure 4. To obtain $V_1(0)\sharp_\varphi \overline{V}_1$, we add a $T^2 \times D^2$, i.e., a 2-handle, two 3-handles, and a 4-handle. The 2-handle goes along an essential curve in a single fiber, with framing determined by the $F \times S^1$ product structure. Figure 5 shows $V_1(0)\sharp_\varphi \overline{V}_1$. (Compare with the picture of $V_n(0)$ in [12].)

We show that $V_1(0)\sharp_\varphi \overline{V}_1$ is diffeomorphic to $\sharp_{11}\mathbf{C}P^2\sharp_{11}\overline{\mathbf{C}P^2}$ by blowing down Figure 5 to obtain S^4. Consider the parallel pair of horizontal curves in the center of Figure 5, with framings 1 and -1. Slide the -1 over the $+1$, so that it becomes a 0-framed meridian to the $+1$. The pair can now be removed by sliding the $+1$ off of the 1-handle using any convenient cancelling 2-handle, and then unlinking the pair from all 2-handles by sliding 2-handles over the 0-framed meridian. This splits off an $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ (depending on whether the cancelling 2-handle had odd or even framing). We have now exposed a parallel pair of vertical curves. Repeating the procedure eventually yields Figure 6. This is easily seen to

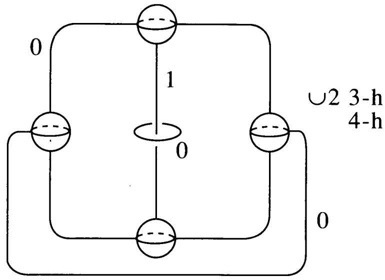

FIGURE 6

be S^4, by cancelling handles. Note that this argument also shows that $V_n(0)\sharp_\varphi \overline{V}_n$ dissolves. (Replace 6 by $6n$ everywhere.) q.e.d.

Now we return to the construction described in the introduction, replacing E_8 plumbings by smaller handlebodies. The manifold V_1 admits an elliptic fibration with exactly two singular fibers: one cusp fiber and one $\widetilde{E}_8$ fiber. By taking fiber sums, we obtain an elliptic fibration on V_n with n cusp fibers and n $\widetilde{E}_8$ fibers. This locates n canonical $\widetilde{E}_8$ plumbings (and hence, n canonical E_8 plumbings) in the manifold $V_n(p, q)$. But the boundary of an E_8 plumbing is the Poincaré homology sphere, which is also the boundary of the handlebody H obtained by gluing a 2-handle to B^4 along a left trefoil with framing -1 [16]. Thus, we may construct a manifold $M_{n,k}(p, q)$ $(1 \leq k \leq n)$ by removing k E_8 plumbings from $V_n(p, q)$ and replacing them by copies of H. (The gluing map is uniquely determined, by [2]. Thus, the diffeomorphism type of $M_{n,k}(p, q)$ depends only on n, k, p, q by [12, Proposition 2.1].) We take p, q relatively prime, so that $M_{n,k}(p, q)$ is simply connected. We wish to prove:

Theorem 14. *The manifold* $M_{n,k}(p, q)$ *dissolves* (*into* $\sharp \pm \mathbf{C}P^2$).

The above construction is closely related to the following procedure: Let C denote B^4 union a 2-handle attached to a right trefoil with framing 0. Thus, a regular neighborhood of a cusp fiber is diffeomorphic to C. Because V_1 admits an elliptic fibration with one cusp fiber and one $\widetilde{E}_8$ fiber, the boundary of a regular neighborhood of the $\widetilde{E}_8$ fiber is $\partial\overline{C}$. (This is also seen by Kirby calculus, using the methods of [15], [16].) Thus, we may remove k $\widetilde{E}_8$ plumbings from $V_n(p, q)$ and replace them by copies of $\overline{C}$, to obtain a simply connected manifold $P_{n,k}(p, q)$ depending only on n, k, p, q. (It is routine to check that this is independent of the gluing map. From the structure of ∂C as a torus bundle over S^1, one

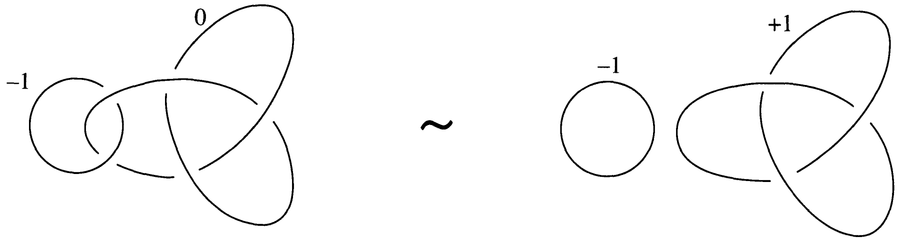

FIGURE 7

may compute that ∂C admits only one nontrivial self-diffeomorphism, and this extends over C.)

We compare the two constructions, beginning with the elliptic surface V_1. Since V_1 is formed by gluing together C and an $\widetilde{E}_8$ plumbing, our latter construction yields $P_{1,1}(1, 1) = C \bigcup_\partial \overline{C}$ which is the double of C. In Kirby calculus, this is obtained from C by adding a 2-handle along a 0-framed meridian to the knot, as well as a 4-handle. To analyze the first construction, consider the nucleus $N_1 \subset V_1$, a regular neighborhood of a cusp fiber union a section (with square -1). Then $V_1 = N_1 \bigcup_\partial E_8$ (for example, [12]). N_1 is obtained from C by gluing a 2-handle along a -1-framed meridian, as in Figure 7. Observe that $N_1 \approx \overline{H} \sharp \overline{\mathbf{C}P^2}$ (by sliding the knot off the -1 as shown). Thus $M_{1,1}(1, 1) = N_1 \bigcup_\partial H$ is obtained by doubling the $\overline{H}$, i.e., adding a 2-handle along a 0-framed meridian to the trefoil, and a 4-handle. If we do this to the first picture in Figure 7, we can slide the -1 over the 0-framed meridian to obtain $C \bigcup_\partial \overline{C} \sharp \overline{\mathbf{C}P^2}$. This shows that $M_{1,1}(1, 1) \approx P_{1,1}(1, 1) \sharp \overline{\mathbf{C}P^2}$. Now observe that the diffeomorphism may be taken to be the identity on C. (We slid one meridian over the other without disturbing the trefoil.) It follows that the two constructions are always related in this manner:

Proposition 15. *If we remove the E_8 plumbing from an $\widetilde{E}_8$ plumbing and replace it by H, the result is $\overline{C} \sharp \overline{\mathbf{C}P^2}$. In particular, $M_{n,k}(p, q) \approx P_{n,k}(p, q) \sharp_k \overline{\mathbf{C}P^2}$.*

(This can also be seen by applying Kirby calculus directly to $\widetilde{E}_8$, using the method of [15], [16].)

Theorem 14 now follows immediately from:

Theorem 16. *$P_{n,k}(p, q)$ dissolves $(1 \leq k \leq n)$.*

Proof. Split $P_{n,k}(p, q)$ as a "fiber sum" $P_{n-1,k-1}(p, q) \sharp_\varphi P_{1,1}(1, 1)$. (If $k = 1$, $n > 1$, we define the first summand to be $V_{n-1}(p, q)$. If $n = 1$, then $k = 1$ and we define the first term to be the elliptic surface $Q = L \times S^1$ obtained by logarithmic transforms on $S^2 \times T^2$ as in the proof of Theorem 7.) We have already seen that $P_{1,1}(1, 1)$ is the double of a neighborhood C of a cusp fiber. C is obtained from the neighborhood $\varphi(\nu F)$ of a regular fiber by adding a pair of 2-handles along a basis for $H_1(F)$ (as in Figure 1). We double C by adding a third 2-handle h along a meridian to the fiber (with the 0-framing in Figure 1, i.e., the canonical framing induced by the product structure on νF), and adding a 4-handle. Since the "fiber sum" given above essentially identifies $\partial \varphi(\nu F)$ with $\partial \nu F$, we have described $P_{n,k}(p, q)$ as $P_{n-1,k-1}(p, q) - \mathring{\nu} F$ union three 2-handles and a 4-handle. Since the 2-handle h is attached to $\partial \nu F$ along a canonically framed meridian, it fills in a normal disk to F. (Compare with the proof of Lemma 4.) Thus, $P_{n,k}(p, q)$ is given by $P_{n-1,k-1}(p, q) - \mathring{\nu}(F - \mathring{D}^2)$ union two 2-handles and a 4-handle. But $\nu(F - \mathring{D}^2)$ is a regular neighborhood of $S^1 \vee S^1$, and the 2-handles are attached along a longitude of each circle. It is now easily verified that $P_{n,k}(p, q)$ is obtained from $P_{n-1,k-1}(p, q)$ by surgery on a pair of circles representing a basis for $H_1(F)$. In the $n = 1$ case, $P_{0,0}(p, q) = Q$, and $P_{1,1}(p, q)$ is $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ by Lemma 3. If $n > 1$, then $P_{n-1,k-1}(p, q)$ is simply connected, so $P_{n,k}(p, q)$ is $P_{n-1,k-1}(p, q) \sharp_2 S^2 \times S^2$ or $P_{n-1,k-1}(p, q) \sharp_2 S^2 \tilde{\times} S^2$, and the result follows by indication on k and Corollary 8.

References

[1] S. Akbulut, *A fake 4-manifold*, Four Manifold Theory, Contemp. Math., Vol. 35, Amer. Math. Soc., Providence, RI, 1984, 75–142.

[2] M. Boileau & J. Otal, *Groupe des difféotopies de certaines variétés de Seifert*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986) 19–22.

[3] S. Cappell & J.Shaneson, *Some new 4-manifolds*, Ann. of Math. (2) **104** (1976) 61–72.

[4] S. Donaldson, *Irrationality and the h-Cobordism conjecture*, J. Differential Geometry **26** (1987) 141–168.

[5] ——, *Polynomial invariants of smooth four-manifolds*, Topology, to appear.

[6] M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982) 357–453.

[7] M. Freedman & F. Quinn, *Topology of 4-manifolds*, Princeton University Press, to appear.

[8] R. Friedman & J. Morgan, *On the diffeomorphism types of certain algebraic surfaces*, I, II, J. Differential Geometry **27** (1988) 297–398.

[9] ——, to appear.

[10] R. Gompf, *Stable diffeomorphism of compact 4-manifolds*, Topology Appl. **18** (1984) 115–120.

[11] ——, *On sums of algebraic surfaces*, Invent. Math. **94** (1988) 171–174.

[12] ——, *Nuclei of elliptic surfaces*, Topology, to appear.

[13] J. Harer, A. Kas & R. Kirby, *Handlebody decompositions of complex surfaces*, Mem. Amer. Math. Soc., Vol. 62, No. 350, 1986.

[14] A. Kas, *On the deformation types of regular elliptic surfaces*, Complex Analysis and Algebraic Geometry, Cambridge University Press, Cambridge, 1977, 107–111.

[15] R. Kirby, *A calculus for framed links in* S^3, Invent. Math. **45** (1978) 35–56.

[16] R. Kirby & M. Scharlemann, *Eight faces of the Poincaré homology* 3-*sphere*, Geometric Topology, Proc. 1977 Georgia Topology Conference, 1979, 113–146.

[17] K. Kodaira, *On compact analytic surfaces* II, Ann. of Math. (2) **77** (1963) 563–626.

[18] R. Mandelbaum, *Decomposing analytic surfaces*, Geometric Topology, Proc. 1977 Georgia Topology Conference, 1979, 147–218.

[19] ——, *Irrational connected sums*, Trans. Amer. Math. Soc. **247** (1979) 137–156.

[20] ——, *Four-dimensional topology: an introduction*, Bull. Amer. Math. Soc. **2** (1980) 1–159.

[21] Y. Matsumoto, *Torus fibrations over the* 2-*sphere with the simplest singular fibers*, J. Math. Soc. Japan **37** (1985) 605–636.

[22] B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Math., **603**, Springer, Berlin, 1977.

[23] C. Okonek & A. Van de Ven, *Stable bundles and differentiable structures on certain elliptic surfaces*, Invent. Math. **86** (1986) 357–370.

[24] C. T. C. Wall, *Diffeomorphisms of* 4-*manifolds*, J. London Math. Soc. **39** (1964) 131–140.

University of Texas, Austin