J. DIFFERENTIAL GEOMETRY
33 (1991) 743-747

# DIAMETER, VOLUME, AND TOPOLOGY FOR POSITIVE RICCI CURVATURE

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*Dedicated to Wilhelm Klingenberg on the occasion of his 65th birthday*

## 1. Introduction

A compact Riemannian $n$-manifold with (normed) Ricci curvature $\mathrm{ric} := \mathrm{Ric}/(n-1) \geq 1$ has diameter $\leq \pi$, and equality holds if and only if $M$ is isometric to the unit $n$-sphere (Cheng's rigidity theorem, cf. [4], [12], [5]). The aim of the present paper is to prove the following theorem.

**Theorem 1.** *Let $M^n$ be a compact Riemannian manifold with Ricci curvature $\geq 1$. Let $-k^2$ be a lower bound of the sectional curvature of $M^n$, and $\rho$ a lower bound of the injectivity radius. Then we may compute a number $\varepsilon = \varepsilon(n, \rho, k) > 0$ such that $M$ is homeomorphic to the $n$-sphere whenever* $\mathrm{diam}(M) > \pi - \varepsilon$.

More precisely, $\varepsilon = v(\delta)/\mathrm{vol}(S^{n-1})$, where $v(r)$ denotes the volume of a ball of radius $r$ in the unit $n$-sphere and

$$\delta = \begin{cases} \rho - \cosh^{-1}(\cosh(k\rho)^2)/(2k) & \text{for } k > 0, \\ (1 - \sqrt{2}/2)\rho & \text{for } k = 0. \end{cases}$$

For sectional curvature, a much stronger result is known:

**Theorem 2** (Berger [3], Grove-Shiohama [8], [9]). *Let $M^n$ be a compact Riemannian manifold with sectional curvature $K \geq 1$ and diameter $D > \pi/2$. Then $M$ is homeomorphic to a sphere.*

One may not expect such a theorem for Ricci curvature since, e.g., for $M = S^m \times S^m$ with $\mathrm{ric} = 1$ we have $\mathrm{diam}(M) = (1 - 1/(2m-1))^{1/2} \cdot \pi$. So the bound on the diameter must depend at least on the dimension. A diameter pinching theorem for Ricci curvature in the diffeomorphism category was first stated by Brittain [2] (whose proof used an incorrect version of Gromov's compactness theorem) and proved by Katsuda [11, p. 13] using a result of Kasue [10]. However, the proof needs also an upper curvature bound, and it would be hard to compute the $\varepsilon$. We give a

Received March 27, 1989 and, in revised form, January 22, 1990.

direct proof combining the methods of Grove and Shiohama [12], [8] with the idea that large diameter implies small excess in the sense of Abresch and Gromoll [1]. After this work was finished, Grove and Petersen [7] investigated the excess in more generality and proved our theorem in this context. In fact, using earlier work [6], they could replace the injectivity radius bound with a lower volume bound.

## 2. Proofs

To prove the theorems, we consider points $p, q \in M$ of maximal distance in $M$ and the functions $r_p(x) = d(p, x)$, $r_q(x) = d(q, x)$. For any $x \in M$ let $\Gamma_p(x)$ and $\Gamma_q(x)$ be the sets containing the final tangent vectors of all shortest unitary geodesics from $p$ and $q$ (resp.) to $x$. A point $x$ is called a *regular point* (in the sense of Grove-Shiohama and Gromov) for the function $r_p - r_q$ if there exists $v \in T_xM$ with

$$\langle v, a - b\rangle > 0$$

for all $a \in \Gamma_p(x)$ and $b \in \Gamma_q(x)$. Such a vector $v$ is called *admissible.* The admissible vectors at $x$ form an open convex cone. If an admissible vector $v$ at $x$ is extended to a smooth vector field $V$, then $V(y)$ is admissible for all $y$ close to $x$. Otherwise, there would be sequence $y_j \to x$ and $a_j \in \Gamma_p(y_j)$, $b_j \in \Gamma_q(y_j)$ with $\langle a_j - b_j, V(y_j)\rangle \leq 0$. But subsequences of $(a_j)$ and $(b_j)$ would converge to some $a \in \Gamma_p(x)$, $b \in \Gamma_q(x)$ for which we would also get $\langle a - b, V(x)\rangle \leq 0$. This is a contradiction.

The following lemma (cf. [8], [9]) is basic for our proof.

**Lemma.** *If all points of $M \setminus \{p, q\}$ are regular points for $r_p - r_q$, then $M$ is homeomorphic to the n-sphere.*

*Proof.* Any $x \in M\setminus\{p, q\}$ has a neighborhood $U_x$ and a smooth vector field $V_x$ on $U_x$ which is admissible. Further, we let $U_p$ and $U_q$ be open balls centered at $p$ and $q$ where the exponential maps have smooth inverse maps, and put $V_p = \nabla(r_p^2)$ and $V_q = -\nabla(r_q^2)$. Then $V_p$ is admissible outside $p$ since $\Gamma_p(x) = \{\nabla r_p(x)\}$ for any $x \in U_p$ and $\nabla r_p(x) \notin \Gamma_q(x)$; likewise, $V_q$ is admissible. By compactness, finitely many of the open sets $U_x$, $x \in M$, cover $M$, say $U_1, \cdots, U_N$ with corresponding vector fields $V_1, \cdots, V_N$. If $\{\phi_j; j = 1, \cdots, N\}$ is a corresponding decomposition of unity, then $V = \sum \phi_j V_j$ is admissible outside $\{p, q\}$ and extends the vector fields $\nabla(r_p^2)$ and $-\nabla(r_q^2)$ near $p$ and $q$. In particular, $p$ and $q$ are the only zeros of $V$. Thus by the flow of $V$ we get a diffeomorphism of $B_r(p)$ onto $M \setminus B_r(q)$ for small enough $r$, which shows that $M$ is homeomorphic to a sphere. q.e.d.

Thus it suffices to prove that $r_p - r_q$ has only regular points if the diameter bound $\varepsilon$ is small enough. In the proof of the previous proposition we saw that $B_r(p)\setminus\{p\}$ and $B_r(q)\setminus\{q\}$ contain only regular points if $r$ is smaller than the injectivity radius at $p$ and $q$, in particular if $r<\rho$.

Now suppose that $x\in M$ is a nonregular (*critical*) point of $r_p - r_q$. We claim that there exist $a\in\Gamma_p(x)$ and $b\in\Gamma_q(x)$ with

$$(*) \qquad \langle a, b\rangle \geq 0.$$

Otherwise, $\Gamma_p(x)$ would be contained in the open convex cone $C=\{v\in T_xM;\ \langle v, b\rangle<0 \text{ for any } b\in\Gamma_q(x)\}$, and $C$ would contain a vector $c$ with $\langle c, v\rangle>0$ for all $v\in C$. Hence $\langle c, a-b\rangle>0$ for any $a\in\Gamma_p(x)$, $b\in\Gamma_q(x)$, and $c$ would be an admissible vector, which is impossible.

It is now easy to finish the proof of Theorem 2. Namely, we find a geodesic triangle with vertices $x, p, q$ and angle $\leq\pi/2$ at $x$. By Toponogov's comparison theorem, a triangle $(x_0, p_0, q_0)$ with the same side lengths in the unit sphere $S^2$ also has angle $\leq\pi/2$ at $x_0$. But such a triangle cannot exist if the largest side $p_0q_0$ has length $>\pi/2$. Namely, either the length $a$ of, say, $q_0x_0$ also exceeds $\pi/2$, in which case $p_0$ lies in the convex ball $B_{\pi-a}(-q_0)$ whose boundary intersects $q_0x_0$ orthogonally at $x_0$, so the angle at $x_0$ is larger than $\pi/2$, or both sides $p_0x_0$ and $q_0x_0$ have lengths $\leq\pi/2$. The length of $p_0q_0$ is certainly not larger than the diameter of the triangle $(p_0, q_0, x_0)$. If the angle at $x_0$ is $\leq\pi/2$, this triangle is contained in a triangle of side lengths and angles equal to $\pi/2$, i.e., a quarter half-sphere. This has diameter $\pi/2$, so the length of $p_0q_0$ cannot exceed $\pi/2$. Thus there are no such triangles and hence $M\setminus\{p, q\}$ contains only regular points, which proves Theorem 2. q.e.d.

To prove Theorem 1, let $\alpha$ and $\beta$ be the shortest geodesics from $p$ and $q$ to $x$ with final vectors $a$ and $b$ satisfying $(*)$. Now we consider the *excess function* (cf. [1])

$$e = r_p + r_q - D,$$

where $D=d(p, q)=\operatorname{diam}(M)$. By the triangle comparison theorem, $e(x)$ is bounded from below by the excess

$$e_0 = d(p_0, x_0) + d(q_0, x_0) - d(p_0, q_0)$$

of a triangle $(p_0, q_0, x_0)$ in the hyperbolic plane of curvature $-k^2$ with $d(x_0, p_0)=r_p(x)$, $d(x_0, q_0)=r_q(x)$ where the angle at $x_0$ equals the angle between $\alpha$ and $\beta$, which by $(*)$ is at most $\pi/2$. This hyperbolic excess is decreasing if we make the angle at $x_0$ larger and the side lengths

$d(x_0, p_0)$ and $d(x_0, q_0)$ shorter. Since $x$ is a critical point, we have

$$r_p(x) \geq \rho, \qquad r_q(x) \geq \rho$$

and, therefore, $e(x)$ is bounded from below by the excess $e_1$ of a hyperbolic triangle $(p_1, q_1, x_1)$ with angle $\pi/2$ at $x_1$ and sides lengths $d(x_1, p_1) = d(x_1, q_1) = \rho$. By the cosine law we have

$$e(x) \geq e_1 = 2\rho - \cosh^{-1}(\cosh(k\rho)^2)/k.$$

Let us put $\delta = e_1/2$. Then we have

$$r_p(x) \geq r + \delta, \qquad r_q(x) \geq D - r + \delta$$

for some $r > 0$. In other words,

$$B_\delta(x) \subset P := M \setminus (B_r(p) \cup B_{D-r}(q)).$$

If $v(t)$ denotes the volume of a ball of radius $t$ in the unit $n$-sphere $S^n$, we have the Bishop-Gromov inequality (e.g., cf. [5, 4.3]), applied to balls with radii $\delta$ and $D$,

$$\text{(1)} \qquad \operatorname{vol}(P) \geq \operatorname{vol}(B_\delta(x)) \geq v(\delta) \cdot \operatorname{vol}(M)/\operatorname{vol}(S^n).$$

On the other hand, the Bishop-Gromov inequality also gives an upper bound for $\operatorname{vol}(P)$. Namely,

$$\operatorname{vol}(B_r(p)) + \operatorname{vol}(B_{D-r}(q)) \geq (v(r) + v(D-r)) \cdot \operatorname{vol}(M)/\operatorname{vol}(S^n),$$

and $\operatorname{vol}(S^n) - (v(r) + v(D-r))$ is the volume of a tubular neighborhood of radius $(\pi - D)/2$ around a small sphere of spherical radius $r + \frac{1}{2}(\pi - D)$. By Cavallieri's principle, this volume gets larger if we replace the small sphere by a great sphere, and therefore

$$\operatorname{vol}(S^n) - (v(r) + v(D-r)) \leq (\pi - D) \cdot \operatorname{vol}(S^{n-1}).$$

Hence

$$\operatorname{vol}(B_r(p) \cup B_{D-r}(q)) \geq \operatorname{vol}(M) - (\pi - D) \cdot \operatorname{vol}(S^{n-1}) \cdot \operatorname{vol}(M)/\operatorname{vol}(S^n),$$

which shows

$$\text{(2)} \qquad \operatorname{vol}(P) \leq (\pi - D) \cdot \operatorname{vol}(S^{n-1}) \cdot \operatorname{vol}(M)/\operatorname{vol}(S^n).$$

Now (1) and (2) cannot hold together if

$$\pi - D < \varepsilon := v(\delta) \cdot \operatorname{vol}(S^{n-1}).$$

So, in this case, the function $r_p - r_q$ has only regular points, which finishes the proof.

**Acknowledgements**

We wish to thank the referee for several useful hints.

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