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ON THE FORMATION OF SINGULARITIES IN THE CURVE SHORTENING FLOW

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Abstract

In this paper the asymptotic behavior of solutions of the so called "curve shortening equation" for locally convex plane curves is studied. This is done by looking at the blow-up of solutions of the quasilinear parabolic PDE

$$\frac{\partial k}{\partial t} = k^2 \frac{\partial^2 k}{\partial \theta^2} + k^3$$

with periodic boundary conditions.

The main results (Theorems A, B, D, and D) may be summarized as follows: "small convex nooses get tightened by the Grim Reaper." In other words, a convex plane curve $C(t)$ $(0 \leq t < T)$ which evolves according to its curvature will either shrink to a point in an asymptotically self similar manner (as described by Abresch & Langer, and Epstein & Weinstein), or else its maximal curvature will blow up faster than $(T-t)^{-1/2}$. In the second case, there is a sequence of times $t_n \uparrow T$ such that the curve obtained by magnifying $C(t_n)$ so that its maximal curvature becomes 1 will converge to the graph of $y = -\log\cos x$.

If the total curvature which disappears into the singularity is less than 2π, then it must actually be π. Moreover, the last statement of the previous paragraph is true for any sequence $t_n \uparrow T$, instead of just for some sequence. In this situation we also have an upper bound for the rate at which the maximal curvature $\kappa(t)$ of $C(t)$ blows up:

$$\kappa(t) \leq \frac{C_\varepsilon}{(T-t)^{1/2+\varepsilon}}$$

for any $\varepsilon > 0$.

1. Introduction

In this paper we take a look at the way in which a plane immersed curve becomes singular, as it evolves according to its curvature.

Let S^1 be a unit circle, and $\mathbf{R}^2$ a Euclidean plane. A family of immersed curves $X\colon S^1 \times [0, T) \to \mathbf{R}^2$ evolves according to its curvature, if

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at any $(u, t) \in S^1 \times (0, T)$ one has

$$\frac{\partial X}{\partial t} = kN, \tag{1}$$

in which k and N denote respectively the curvature and unit normal of the immersed curve $u \to X(u, t)$.

For any C^1 immersed curve $X_0 \colon S^1 \to \mathbf{R}^2$, one can construct a family of immersed curves $X \colon S^1 \times [0, T) \to \mathbf{R}^2$, for some small $T > 0$, which satisfies (1), and has $X(u, 0) = X_0(u)$; i.e., the family X is a solution to the initial value problem (1) with X_0 as initial data. Assuming that the initial curve is C^∞ smooth, M. Gage and R. S. Hamilton showed in [8] that this is a special case of a much more general local existence theorem, the proof of which uses the Nash-Moser implicit function theorem. In [2, part 1] another proof for locally Lipschitz and even worse initial data was given, using the more specialized theory of parabolic partial differential equations.

A special solution of the curve shortening equation is given by a circle centered at the origin, whose radius at time t is given by $R(t) = [2(T-t)]^{1/2}$, for some constant $T > 0$.

This example is typical in the sense that any solution of (1) must become singular in finite time. In [8] Gage and Hamilton showed that if the initial curve is convex, then the corresponding solution of the initial value problem will shrink to a point in finite time, and will asymptotically behave like the shrinking circle.

M. Grayson [9] subsequently showed that for X_0 any *simple closed* curve (i.e., $X_0 \colon S^1 \to \mathbf{R}^2$ is a diffeomorphism onto its image), the solution will become convex in finite time, and hence, by the result of Gage and Hamilton, it will shrink to a "round point."

If the initial curve has self-intersections, then the corresponding solution can become singular without shrinking to a point. Consider as an example the closed cardioidlike curve $C(0)$ with one self-intersection and index $+2$ depicted in Figure 1. Let $C(t)$ be the family of curves which evolves according to its curvature and has $C(0)$ as its initial value. We showed in [2, part 2] that the number of self-intersections of an evolving curve cannot increase with time, so that the curve $C(t)$ will always have exactly one self-intersection. Let $C_1(t)$ and $C_2(t)$ denote the outer and inner loops of $C(t)$, respectively, and let $A_1(t)$, $A_2(t)$ denote the areas which they enclose. If one computes the rates at which these areas decrease with

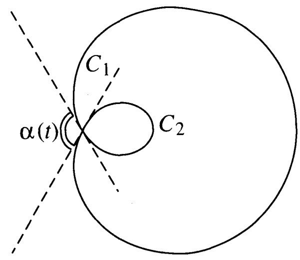

FIGURE 1. A CARDIOIDLIKE CURVE.

time, then one finds

$$A_j'(t) = \int_{C_j(t)} k\,ds = \begin{cases} 3\pi - \alpha(t), & \text{if } j = 1, \\ \pi + \alpha(t), & \text{if } j = 2, \end{cases}$$

where $0 < \alpha(t) < \pi$ is the angle of the self-intersection.

At any time one has $\pi < -A_2'(t) < 2\pi < -A_1'(t) < 3\pi$, so if $A_1(0) > 3A_2(0)$, then the area in the smaller loop must disappear before $t = A_2(0)/\pi$, while the area in the larger loop $C_1(t)$ stays positive, and bounded from below. Obviously, the curve becomes singular when this happens; it becomes singular before it shrinks to a point.

In this paper we shall give a more detailed description of the way in which the loop $C_2(t)$ "contracts." We shall restrict our attention to *convex immersed curves*, i.e., immersed curves without inflection points—in general, such curves can have self-intersections, and the curve of Figure 1 falls in this class.

The reason for restricting our attention to convex immersed curves is that they admit a parametrisation in which (1) turns out to be equivalent to a scalar quasilinear parabolic PDE.

Let $X: S^1 \to \mathbf{R}^2$ be a convex immersed curve, and for each $u \in S^1$ denote the unit tangent to X by $\vec{t}u \in S^1$. Since the curve is convex, $\vec{t}: S^1 \to S^1$ is locally one-to-one, and $\vec{t}$ is a covering. The degree ν of the covering is of course the index of the curve ("number of times its tangent winds around as one goes along the curve").

Let $\mathbf{T}_\nu = (\mathbf{R}/2\nu\pi)\mathbf{Z}$. For any $u \in S^1$ we can write $\vec{t}(u)$ as $(\cos\theta(u), \sin\theta(u))$ for some $\theta(u) \in \mathbf{T}_\nu$, and we can choose $\theta(u)$ so that it depends continuously (C^{n-1} if the curve is C^n) on $u \in S^1$. Then $\theta: S^1 \to \mathbf{T}_\nu$ is one-to-one and onto, so that $X \circ \theta^{-1}: \mathbf{T}_\nu \to \mathbf{R}^2$ is a parametrisation of the curve for which the tangent $\widehat{X}(\theta)$ is $(\cos\theta, \sin\theta)$.

If the curvature k of the curve is known as a function of the angle θ, then the curve is completely determined, up to a translation. The inversion is quite easy; the arc length and the angle are related by $d\theta = k(\theta)ds$, so that $X(\theta)$ is given by

$$X(\theta_0) = X(0) + \int_{\theta=0}^{\theta_o} \vec{t}(\theta)\, ds = X(0) + \int_{\theta=0}^{\theta_o} \frac{e^{\sqrt{-1}\theta}}{k(\theta)}\, d\theta, \tag{2}$$

where we have identified $\mathbf{R}^2$ and $\mathbf{C}$.

This formula shows that any positive function $k \in C(\mathbf{T}_\nu)$, for which

$$\int_{\theta=0}^{2\nu\pi} \frac{e^{\sqrt{-1}\theta}}{k(\theta)}\, d\theta = 0 \tag{3}$$

holds, defines a C^2 convex immersed curve. Thus, instead of considering curves, we can look at functions $k \in C(\mathbf{T}_\nu)$, which satisfy (3). It turns out that the curve shortening equation for convex immersed curves is equivalent to the following PDE for the curvature k, as a function of (θ, t),

$$k_t = k^2(k_{\theta\theta} + k) \qquad (\theta \in \mathbf{T}_\nu, 0 \le t < T). \tag{4}$$

The special solution given by the shrinking circle corresponds to the solution $k(\theta, t) = [2(T-t)]^{-1/2}$ of (4). The form of this solution suggests that one might separate variables, and find other solutions of (4) of the form $k^* = K(\theta)[2(T-t)]^{-1/2}$. A short computation shows that k^* satisfies (4) if and only if K is a $2\nu\pi$ periodic solution of

$$K_{\theta\theta} + K - \frac{1}{2K} = 0. \tag{5}$$

In [1] U. Abresch and J. Langer classified all solutions of this equation (note that, since $1/K$ is of the form $f_{\theta\theta} + f$, it automatically satisfies (3), and hence represents a closed curve).

The solutions of (5) represent convex curves which do not change their shape while they evolve under the curve shortening flow; i.e., they represent curves which shrink to a point simply by dilation. We refer to these functions as "Abresch-Langer functions."

Now let $k: S^1 \times [0, T) \to \mathbf{R}_+$ be a maximal classical solution of (4) and define

$$\kappa(t) = \|k(\cdot, t)\|_\infty \qquad (0 \le t < T).$$

Since our solution k blows up at $t = T$, and its maximum satisfies

$$\kappa'(t) \le \kappa(t)^3,$$

we know that

$$\kappa(t) \ge [2(T-t)]^{-1/2}. \tag{6}$$

Our first result states that, if the maximal curvature blows up like $(T-t)^{-1/2}$, then it must shrink to a point in an asymptotically self-similar manner.

Theorem A. *If* $(T-t)^{-1/2}\kappa(t)$ *remains bounded as* $t \uparrow T$, *then the rescaled curvature* $(T-t)^{1/2}k(\theta, t)$ *converges in* $C^\infty(\mathbf{T}_\nu)$ *to one of the Abresch-Langer functions.*

This shows, for example, that for the cardioidlike curve of Figure 1, the maximal curvature blows up faster than $(T-t)^{-1/2}$.

Although the precise rate at which the curvature must blow up still seems to be unknown, we can prove the following rough upper bound.

Theorem B. *For any solution* $k(\theta, t)$ *of* (4) *which blows up at* $t = T$, *one has*

$$\lim_{t\to T}(T-t)\kappa(t) = 0.$$

If the family of curves becomes singular, without being asymptotically self-similar, then one expects that "one of its loops contracts." The following theorem is the most precise version of this statement which we can prove at the moment.

Theorem C. *If* $(T-t)^{-1/2}\kappa(t)$ *is unbounded when* $t \uparrow T$, *then there exist sequences* $t_n \uparrow T$ *and* $\theta_n \in \mathbf{T}_\nu$ *such that*

$$\lim_{n\to\infty} \frac{k(\theta_n + \theta, t_n)}{\kappa(t_n)} = \cos\theta \qquad (|\theta| < \pi/2),$$

where the convergence is uniform for $|\theta| \leq \pi/2$. *Furthermore, the limit also exists in* $C^\infty([-\pi/2+\delta, \pi/2-\delta])$ *for any* $\delta > 0$.

For any $t \in (0, T)$ choose a point $P(t) \in C(t)$ at which the curvature is maximal, and let $\widehat{C}(t)$ be the curve which is obtained by translating $C(t)$ so that $P(t)$ becomes the origin, rotating $C(t)$ so that the unit tangent at $P(t)$ becomes the vector $(1, 0)$, and finally dilating the curve so that its maximal curvature becomes 1.

If $(T-t)^{1/2}\kappa(t)$ is not bounded, then Theorem B states that there is a sequence of instances in time $t_n \uparrow T$, such that the curve $\widehat{C}(t_n)$ will converge to what M. Grayson has called the "Grim Reaper," i.e., the graph of $y = -\log\cos x$.

Define the *blowup set* to be the set

$$\Sigma = \{\theta \in \mathbf{T}_\nu \mid \lim_{t\to T} k(\theta, t) = \infty\}.$$

We shall show that Σ is the union of a finite number of intervals, whose lengths all are at least π (in view of the earlier results of Gage and Hamilton, Grayson, and others, this is not at all surprising). Each component of Σ corresponds to a singularity of the limit curve $C(T)$.

Theorem D. *If the blowup set consists of exactly one interval, and the length of* Σ *is less than* 2π, *then for some* $\alpha \in \mathbf{T}_\nu$ *one has* $\Sigma = [\alpha - \pi/2, \alpha + \pi/2]$, *and*

$$\lim_{t\to T} \frac{k(\alpha+\varphi, t)}{\kappa(t)} = \cos(\varphi)$$

uniformly in $\varphi \in [-\pi/2, \pi/2]$. *Moreover,*

$$\lim_{t\to T} (T-t)^{1/2+\epsilon}\kappa(t) = 0$$

for any $\epsilon > 0$.

The organization of this paper is as follows.

In §2 and 3, we recall the "Sturmian theorem" and some of its consequences for solutions of the curve shortening flow. These results are an essential ingredient in the proof of Theorems A, B, and D.

In §4 we observe that solutions of (4) eventually become monotone increasing, or, what is the same, *subsolutions.*

The proof of Theorem A is contained in §5,6 and 7. The main part of the proof is concerned with showing that if $(T-t)^{1/2}\kappa(t)$ is bounded from above, the function $(T-t)^{-1/2}k(\theta, t)$ is also uniformly bounded away from zero. This fact is proven in §7, by comparing a rescaled version of $k(\theta, t)$ with certain special solutions of the curve shortening equation, which correspond to "shrinking spirals."

Theorems C and D are then proven in §§8 through 10.

Finally, in the last section we give an example of a class of curves to which Theorem D can be applied (the curve of Figure 1 belongs to this class).

In their paper [7] A. Friedman and B. McLeod also studied the blowup of solutions of (4) (and various generalizations, with more independent variables). As M. Gage pointed out, the proof of their Theorem 4.1 contains an error; the fourth line on page 72 should read $c \le \sqrt{-t}w_\lambda \le C$, instead of $c \le w_\lambda \le C$. Indeed, Friedman and McLeod never use the fact that their solution of (4) satisfies Dirichlet boundary conditions, while the Abresch-Langer functions would provide counterexamples for their Theorem 4.1, if those boundary conditions are dropped. Nevertheless, some of the ideal in [7] have proved to be useful, and our proof of Theorem D was inspired by the discussion in [7, p. 73].

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2. The Sturmian theorem

Let $u(x, t)$ be a classical solution of

$$u_t = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u \tag{7}$$

on the rectangle $Q_T = [x_0, x_1] \times [0, T]$, where the coefficients are assumed to be smooth (i.e., C^∞) functions on Q_T, and a must also be positive.

The (classical) maximum principle states that the solution u either vanishes everywhere in Q_T or else is strictly positive in the interior of Q_T, if it is nonnegative on the parabolic boundary $S_T = \{x_0, x_1\} \times [0, T] \cup [0, 1] \times \{0\}$.

The Sturmian theorem is a refinement of this principle, and unlike the maximum principle it describes a strictly one-dimensional phenomenon. There is no known generalization to parabolic equations in more than one space dimension.

The Sturmian theorem gives a precise description of the zero set of a solution of (7). In its formulation we shall use the following terminology. If $f(x)$ is a C^∞ function of one variable, which vanishes at some $x_0 \in \mathbf{R}$, then x_0 will be called a *simple* zero if $f'(x_0) \neq 0$. The *order* of the zero is the smallest integer k for which $f^{(k)}(x_0) \neq 0$; if all derivatives of f vanish at x_0, then x_0 is a zero of infinite order.

The *number of zeroes of* f *in* (a, b) *counted with multiplicity* is, by definition, the sum of the orders of all zeroes of f in the interval (a, b).

Theorem 2.1. *If $u \in C^\infty(Q_T)$ is a solution of (7), and*

$$u(x_j, t) \neq 0 \quad (0 \le t \le T; j = 0, 1),$$

then at any time $t \in (0, T]$ the zero set of $x \to u(x, t)$ will be finite, even when counted with multiplicity.

The number of zeroes of $x \to u(x, t)$ counted with multiplicity is a nonincreasing function of t; at any time t when $x \to u(x, t)$ has a zero of order $k > 1$, $z(t)$ drops by at least $k - 1$.

The first proof of this result was given by C. Sturm in 1836 [14]; he assumed that the equation had the form $u_t = (k(x)u_x)_x + q(x)u$, but his analysis of the zero set by looking at the Taylor series of the solution near its zero set would work without any substantial change for solutions of (7) (see [4]).

Sturm also assumed (implicitly) that the solution u is a real analytic function; i.e., he assumed that solutions (7) cannot have zeroes of infinite order, without vanishing identically. That this is actually true under the assumption that u does not vanish on $\{x_0, x_1\} \times [0, T]$ is in fact a conclusion of the theorem.

Since 1836 the result has been rediscovered several times in several forms (see the references in [4]), and in the form in which we use it here it was proved in [3], under even weaker assumptions on the coefficients a, b, and c in the equation.

The Sturmian theorem can also be applied to solutions of nonlinear equations. Let $f = f(x, t, u, p, q)$ be a smooth function of its five variables or, alternatively, consider an $f \in C^\infty([0, T] \times J^2[x_0, x_1])$, where $J^2[x_0, x_1]$ is the space of two-jets on the interval $[x_0, x_1]$.

Let u and v be two smooth solutions of the nonlinear equation

$$\frac{\partial u}{\partial t} = f(x, t, u, u_x, u_{xx}). \tag{8}$$

Assuming that

$$\frac{\partial f}{\partial q} > 0,$$

so that (8) is parabolic, the difference $w = v - u$ will satisfy a linear parabolic equation like (7) in which the coefficient $a(x, t)$ is given by

$$a(x, t) = \int_0^1 \frac{\partial f}{\partial q}(x, t, u^\theta(x, t), u_x^\theta(x, t), u_{xx}^\theta(x, t))\, d\theta,$$

and b and c are given by similar expressions. Here we have written u^θ for $(1-\theta)v + \theta u$.

The Sturmian theorem now implies that if $u(x_j, t) \neq v(x_j, t)$ for $0 \leq t \leq T$ and $j = 0, 1$, the number of zeroes of $x \to v(x, t) - u(x, t)$, when counted with multiplicity, is finite for any $t \in (0, T]$, and does not increase with time.

3. Vertices

As a simple application of the Sturmian theorem, we prove the following statement about the vertices of a family of curves which evolve according to their curvature. Recall that a *vertex* of an immersed curve is a point on the curve at which the curvature is stationary. If $k_{\theta\theta} \neq 0$ at a vertex, so that the curvature attains a strict local maximum or minimum at the vertex, then the vertex is by definition *nondegenerate.*

Theorem 3.1. *Let $\{C(t), 0 \leq t < T\}$ be a family of convex immersed curves which evolve according to their curvature. Then, for any $t > 0$, the* *curve $C(t)$ has at most a finite number of vertices, and this number is a nonincreasing function of time. In fact, it decreases whenever the curve $C(t)$ has a degenerate vector.*

This theorem follows immediately from the Sturmian theorem, if one observes that vertices correspond to zeroes of $h = k_\theta(\theta, t)$, and that h satisfies

$$h_t = k^2 h_{\theta\theta} + (2kk_{\theta\theta} + 3k^2)h,$$

a linear parabolic PDE like (7). (Just differentiate (4) with respect to θ.)

In [2] we already proved this, without assuming that the curve is convex.

We also proved the following statements about the way in which the solutions of the curve shortening equation (and various generalizations thereof) intersect. See Theorem 1.4 of [2, part 2].

Theorem 3.2. *If $C_1(t)$ and $C_2(t)$ $(0 \le t < T)$ are two families of immersed plane curves which evolve according to their curvature, and their initial values $C_1(0)$ and $C_2(0)$ are different, then for any $t > 0$, the two curves $C_1(t)$, $C_2(t)$ have a finite number of intersections, even when counted with multiplicity. This number of intersections does not increase with time, and must decrease whenever the two curves have a nontransversal intersection.*

If one considers only one family $C(t)$ of curves, evolving according to its curvature, then the same statements are true for the self-intersections of $C(t)$.

Just as with the theorem on vertices, this theorem is true for general solutions of the curve shortening problem; convexity plays no role in the proof.

4. Eventual monotonicity

Let $k(\theta, t)$ be a $2\nu\pi$ periodic solution of

$$k_t = k^2(k_{\theta\theta} + k) \quad (\theta \in \mathbf{R}, 0 \le t < T),$$

which blows up at time $t = T$, and assume that the initial data satisfies

$$k(\theta, 0) \ge \delta, \tag{9}$$

$$k(\theta, 0)^2 + k_\theta(\theta, 0)^2 \le A^2, \tag{10}$$

for certain constants δ and A.

Lemma 4.1. *At each point $(\theta_0, t_0) \in \mathbf{R} \times (0, T)$, one either has*

$$(k + k_{\theta\theta}) > 0$$

or

$$k^2 + k_\theta^2 \le A^2.$$

This lemma implies that whenever and wherever the solution k becomes larger than A, it automatically becomes a subsolution.

Proof. Put $B = \{k(\theta_0, t_0)^2 + k_\theta(\theta_0, t_0)^2\}^{1/2}$, and assume that $B > A$. Choose a $\varphi \in (-\pi/2, \pi/2)$ such that

$$k(\theta_0, t_0) = B\cos(\varphi),$$
$$k_\theta(\theta_0, t_0) = B\sin(\varphi),$$

and consider the functions

$$k^*(\theta) = B\cos(\theta_0 - \theta + \varphi),$$
$$w(\theta, t) = k(\theta, t) - k^*(\theta).$$

Then k^* is a time-independent solution of (4) on the region $\Omega \times [0, T)$, where $\Omega = (\theta_0 - \varphi - \pi/2, \theta_0 - \varphi + \pi/2)$.

Since $B > A$, one has $w_\theta < 0$ at each zero of $w(\theta, 0)$ in the interval $(\theta_0 - \varphi - \pi/2, \theta_0 - \varphi)$, and $w_\theta > 0$ at each zero in $(\theta_0 - \varphi, \theta_0 - \varphi + \pi/2)$. Moreover, $w(\theta_0 - \varphi, 0) \le A - B < 0$ and $w(\theta_0 - \varphi \pm \pi/2, 0) > 0$, from which we conclude that the function $w(\theta, 0)$ has exactly two zeroes in Ω. By the Sturmian theorem this number cannot increase with time.

The function k^* was chosen so that at $t = t_0$, the function $w(\theta, t)$ has a multiple zero at $\theta = \theta_0$. Since $\theta \to w(\theta, t)$ has at most two zeroes when counted with multiplicity, $w(\theta, t_0)$ has no other zeroes besides θ_0. On $\partial\Omega$, $w(\theta, t_0)$ is strictly positive, so $w_{\theta\theta}(\theta_0, t_0) \ge 0$, which implies that $k + k_{\theta\theta} > 0$ at (θ_0, t_0).

Corollary 4.2. *The set*

$$\Omega(t) = \{\theta \in \mathbf{R} | k(\theta, t) > A\}$$

is increasing with time, and for any $\theta \in \Omega(t_0)$ the function $t \to k(\theta, t)$ is strictly increasing for $t \ge t_0$.

Proof. The second statement clearly implies the first. Let $\theta \in \Omega(t)$ be given, and assume that there is a $t' > t$ such that $k(\theta, t') \le k(\theta, t)$. Since Lemma 4.1 implies that $k_t(\theta, t) = k^2(k_{\theta\theta} + k) > 0$, there is a minimal $t' > t$ with $k(\theta, t') \le k(\theta, t)$. But then $k(\theta, t'') \ge A$ for all $t'' \in (t, t')$, and we can apply the lemma again to conclude that $k_t(\theta, t'') > 0$ for all $t'' \in (t, t')$. This is a clear contradiction. q.e.d.

The eventual monotonicity of the solution also allows us to estimate the derivative k_θ in terms of $\kappa(t)$.

Lemma 4.3. *For any $(\theta, t) \in \mathbf{R} \times (0, T)$ one has $|k_\theta(\theta, t)| \le A + 2\nu\pi\kappa(t)$.*

Proof. Let θ_1 and t_1 be given, and assume that $|k_\theta(\theta_1, t_1)| > A$. Without losing generality, we may assume that $k_\theta > 0$.

Let θ_2 be the smallest $\theta_2 > \theta_1$ for which either $k_\theta(\theta_2, t_1) = 0$ or $k^2 + k_\theta^2 \le A^2$ holds. Such a θ_2 must exist, because $\theta \to k(\theta, t_1)$ is periodic so that k_θ vanishes somewhere in the interval $(\theta_1, \theta_1 + 2\nu\pi)$. In both cases we have $k + k_{\theta\theta} > 0$ on (θ_1, θ_2) and $|k_\theta(\theta_2, t_1)| \le A$. Hence

$$\begin{aligned} k_\theta(\theta_1, t_1) &= k_\theta(\theta_2, t_1) - \int_{\theta_1}^{\theta_2} k_{\theta\theta}\, d\theta \\ &\le k_\theta(\theta_2, t_1) + \int_{\theta_1}^{\theta_2} k\, d\theta \\ &\le A + \kappa(t_1)|\theta_1 - \theta_2|. \end{aligned}$$

Since k is $2\nu\pi$ periodic, we have $\theta_2 < \theta_1$, and therefore $k_\theta < A + 2\nu\pi\kappa(t_1)$, as claimed.

Corollary 4.4. *The family of functions*

$$k_t^*(\theta) = \frac{k(\theta, t)}{\kappa(t)} (0 \le t < T)$$

is uniformly Lipschitz continuous, and hence precompact in $W^1_\infty(\mathbf{T}_\nu)$ with the weak topology.*

5. The rescaled flow

To analyze the way in which the solution blows up, we consider the rescaled curvature

$$K(\theta, t) = e^{-t/2} k(\theta, T(1 - e^{-t})), \tag{11}$$

which satisfies

$$K_t = K^2 K_{\theta\theta} + K^3 - \frac{K}{2} \tag{12}$$

for $\theta \in \mathbf{R}$ and all $t > 0$. This substitution is motivated by the fact that $k(\theta, t) \le \text{Const} \times (T - t)^{-1/2}$ is equivalent to the uniform boundedness of $K(\theta, t)$ as t tends to infinity.

A straightforward computation shows that

$$\mathscr{J}(K) = \int_0^{2\nu\pi} \{K_\theta^2 - K^2 + \log K\}\, d\theta$$

is a Lyapunov function for the PDE (12). Indeed, one has

$$\frac{d}{dt}\mathcal{J}(K(\cdot\,,t)) = -\int_0^{2\nu\pi}\left(\frac{K_t}{K}\right)^2 d\theta.$$

The equilibria of (12) are exactly the Abresch-Langer functions, which we mentioned in the introduction. Let $AL_\nu \subset C^\infty(\mathbf{T}_\nu)$ denote the set of Abresch-Langer functions of period $2\nu\pi$. Then AL_ν is the union of a finite number of disjoint circles $AL_{\nu,N}$, where $AL_{\nu,N}$ consists of those Abresch-Langer functions whose minimal period is $2\nu\pi N$; the N's which can occur are exactly those for which $\sqrt{2}\nu < N < 2\nu$ holds. Each circle in $AL_{\nu,N}$ contains all translates of one particular $2\nu\pi/N$-periodic solution of $K_{\theta\theta} + K = \frac{1}{2}K$, which is why it is a circle.

Lemma 5.1. *If* $\sup_{\theta\in R,t>0} K < \infty$ *and* $\inf_{\theta\in R,t>0} K > 0$, *then as* $t \to \infty$, *the function* $K(\cdot\,,t)$ *converges in the* C^∞ *topology to one of the Abresch-Langer functions.*

Proof. Let $\epsilon \le K \le \epsilon^{-1}$ for some $\epsilon > 0$. Then (12) is a quasilinear uniformly parabolic equation, and the standard results on such equations (as in the book of Ladyžhenskaya, Ural′ceva and Solonnikov [11]) imply that all derivatives of $k(\cdot\,,t)$ remain uniformly bounded. Therefore $\mathcal{J}(K(\cdot\,,t))$ is bounded from below, and

$$\int_0^\infty\int_0^{2\nu\pi}\left(\frac{K_t}{K}\right)^2 d\theta\, dt < \infty$$

so that

$$\lim_{n\to\infty}\int_n^{n+1}\int_0^{2\nu\pi}(K_t)^2\, d\theta\, dt = 0,$$

i.e., K_t tends to zero in the L_2 norm. The derivative bounds then guarantee that $K_t(\cdot\,,t) \to 0$ in $C^\infty(\mathbf{T}_\nu)$.

Any limit of a subsequence $K(\cdot\,,t_n)$, with $t_n \uparrow \infty$, must satisfy $K_t = 0$, or rather $K_{\theta\theta} + K = 1/2K$, so the only limits which can occur are the Abresch-Langer functions. Since the Abresch-Langer functions appear on disjoint circles in $C^\infty(\mathbf{T}_\nu)$, the $K(\cdot\,,t)$ must converge to one of these circles, and by the result of C. Epstein and M. Weinstein [6] they must converge to exactly one function on such a circle.

6. Shrinking spirals

We shall prove Theorem A by comparing the given solution of K of (12) with special solutions of the form $K(\theta\,,t) = K(\theta - ct)$. Clearly, a function $K(\theta)$ will generate a solution of this form if and only if it satisfies the equation

$$K^2K_{\theta\theta} + cK_\theta + K^3 - \frac{K}{2} = 0. \tag{13}$$

The function $\mathscr{E}$ given by

$$\mathscr{E}(\theta) = K_\theta^2 + K^2 - \log K$$

satisfies

$$\mathscr{E}'(\theta) = -2c\frac{K_\theta^2}{K}, \tag{14}$$

so that $\mathscr{E}(\theta)$ is monotone if $c \neq 0$, and the only periodic solution of (13) with $c \neq 0$ is the constant $K = \frac{1}{2}\sqrt{2}$.

For $c = 0$, the periodic solutions of (13) are, of course, the Abresch-Langer functions.

There is one particular solution of (12) which we shall need; its existence and its relative properties are given below.

Theorem 6.1. *For any $c > 0$, there is a unique solution $K_c \in C^\infty((-\infty, 0])$ of (13) with the following properties:*

$$\begin{aligned} &K_c'(\theta) > 0 \quad \textit{for } \theta \in (-\infty, 0), \\ &K_c'(0) = 0, \\ &K_c'(\theta) \le \lambda_c K_c(\theta) \quad \textit{for } \theta \le 0. \end{aligned}$$

As a function of $c > 0$, $K_c(0)$ is strictly decreasing, and given any $\delta > 0$ and $A > 0$, one can choose $c = c(\delta, A) > 0$ so small that

$$K_c(0) > \delta^{-1}, \tag{15}$$

$$K_c'(\theta) > A \quad \textit{whenever } \delta \le K_c(\theta) \le \delta^{-1}. \tag{16}$$

Proof. It will be convenient to deal with $H = \sqrt{K}$ instead of K itself. A short computation shows that (13) is equivalent to

$$HH_{\theta\theta} - \frac{1}{2}H_\theta^2 + cH_\theta + 2H^2 - H = 0. \tag{17}$$

This ordinary differential equation is equivalent to the first order systems of ODE's

$$HH_\theta = HG, \tag{18}$$

$$HG_\theta = \frac{1}{2}G^2 - cG + H - 2H^2. \tag{19}$$

Thus up to a reparametrisation, the solutions of (13) are in one-to-one correspondence with the orbits of the vector field

$$X_c(H, G) = HG\frac{\partial}{\partial H} + \left(\frac{1}{2}G^2 - cG + H - 2H^2\right)\frac{\partial}{\partial G},$$

which has three zeroes, $(0, 0)$, $(\frac{1}{2}, 0)$, and $(0, 2c)$. When one computes the linearization of X_c, one discovers that $(0, 2c)$ is a source, and also that $(\frac{1}{2}, 0)$ is a spiraling sink. The origin is a degenerate zero of X_c, so that we have to take a closer look at the vector field near $(0, 0)$, to see what its unstable set is.

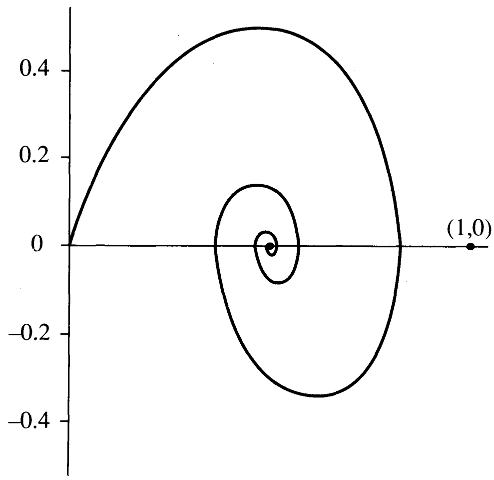

FIGURE 2. THE UNSTABLE MANIFOLD OF THE ORIGIN. $(c = 0.3)$.

By definition, the unstable set $W^u(O)$ of the origin consists of all orbits of X_c, which tend to O as $\theta \downarrow -\infty$. Since the origin is not a hyperbolic fixed point of the vector field X_c, it is *a priori* not clear what the unstable set will look like.

If K_c is the solution for which we are looking, then

$$(H_c(\theta), G_c(\theta)) = \left(\sqrt{K_c(\theta)}, \frac{K_c'(\theta)}{\sqrt{K_c(\theta)}}\right)$$

parametrizes a trajectory of X_c in the unstable set of the origin $W^u(O)$.

Existence of a trajectory in $W^u(O)$. Let l_λ be the half line $G = \lambda H (H \geq 0)$. On this half line we have

$$H\frac{dG - \lambda H}{d\theta} = \left\{1 - \lambda c - \left(2 + \frac{1}{2}\lambda^2\right)H\right\}H. \tag{20}$$

Choose two λ's such that $\lambda_1 < c^{-1} < \lambda_2$, put $h^* = (1 - c\lambda_1)/(2 + \lambda_1^2/2)$, and define the points

$$A = (h^*, \lambda_1 h^*), \; B = (h^*, \lambda_2 h^*), \text{ and } O = (0, 0).$$

From (20) one sees that the trajectories of X_c enter the triangle OAB through the sides OA and OB, while they leave OAB through the remaining vertical side AB. Since $G_\theta > 0$ in the triangle, it follows from Ważewski's principle that at least one of the trajectories through AB tends to the origin, as $\theta \to -\infty$.

Uniqueness of the trajectory in $W^u(O)$. First of all, we observe that near the origin, one has $G_\theta < 0$, if $G \geq \lambda H$ for sufficiently large λ (i.e., $\lambda > c^{-1}$). This implies that any trajectory (H, G) in $W^u(O)$ satisfies $G \leq \lambda H$ for some $\lambda \gg 0$, as $\theta \to -\infty$.

Let (H_1, G_1) and (H_2, G_2) be two different orbits in $W^u(O)$; then near the origin they can be represented as graphs $G_i = g_i(H)$, where the g_i are solutions of

$$g'(H) = \frac{H \cdot G_\theta}{H \cdot H_\theta} = \frac{g(H)/2 - c}{H} + \frac{1 - 2H}{g(H)}.$$

Orbits cannot intersect, so we may assume that $g_1(H) < g_2(H)$. Their difference $w(H) = g_2(H) - g_1(H)$ then satisfies

$$w'(H) = \frac{w}{2H} - \frac{1 - 2H}{g_1(H)g_2(H)} w,$$

which implies that $w'(H) \leq 0$ for sufficiently small $H > 0$ in view of the fact that $g_i(H) \leq \lambda H$ for some λ; but then $w(H) \equiv 0$, because $\lim_{H \downarrow 0} w(H) = 0$. So the two solutions were equal after all.

Let (H_c, G_c) denote the trajectory whose existence and uniqueness have been just established, and let K_c be the corresponding function of θ.

It follows from (14) that the quantity

$$\mathscr{E}(H, G) = \frac{G^2}{4H} + H - \frac{1}{2}\log H$$

is strictly decreasing on orbits of X_c, except when $G = 0$. Thus $H - \frac{1}{2}\log H$, and therefore H, are bounded from above on any orbit of X_c. Using the fact that $(\frac{1}{2}, 0)$ is an attracting spiral point, one easily shows that any orbit converges to $(\frac{1}{2}, 0)$, and winds around this point infinitely often. In particular, any orbit will intersect the H-axis. For the function K_c, this means that it will converge to $\frac{1}{2}\sqrt{2}$, as $\theta \to \infty$, and that it will oscillate infinitely often around its limit value. Its derivative $K_c'(\theta)$ must therefore vanish infinitely often; by replacing $K_c(\theta)$ with $K_c(\theta - \theta_0)$ for some $\theta_0 \in \mathbf{R}$, we can arrange that the first zero of $K_c'(\theta)$ is $\theta = 0$.

This completes our construction of K_c. To finish the proof, we have to show that K_c has the properties (15) and (16).

To verify (15), we observe that the segment of $W^u(O)$ which lies in the first quadrant is the graph of some function $G = g_c(H)$ for $0 \le H \le h_c$, where $(h_c, 0)$ is the first point of intersection of $W^u(O)$ with the H-axis. Since $K_c(0) = \sqrt{h_c}$, we have to show that h_c is monotone decreasing in c.

Let $c' < c$ be given, and suppose that $h_{c'} \le h_c$. Then, keeping in mind that the left side of (19) is strictly decreasing in c, one finds that the backwards orbit of $X_{c'}$ through $(h_{c'}, 0)$ cannot pass through the graph of g_c. As a consequence, it would have to hit the H-axis before it reaches the origin, but that is a contradiction, because the orbit of $X_{c'}$ through $(h_{c'}, 0)$ is contained in the unstable set of O (relative to the vector field $X_{c'}$). Thus we see that $h_{c'} > h_c$ if $c' < c$.

A similar argument also shows that $g_c(H)$ is a strictly decreasing function of c for fixed H; i.e., as $c \downarrow 0$ to the unstable set $W^u(O)_c$ moves upwards.

Assume that $K_c(0)$ were bounded, as $c \downarrow 0$. Then the h_c's would converge to some $h_0 > 0$. The vector field X_c is well defined and smooth for all $c \in \mathbf{R}$, so the unstable set $W^u(O)_c$, being the orbit of X_c through $(h_c, 0)$, would converge to the orbit of X_0 through $(h_0, 0)$. But X_0 has $\mathscr{E}(H, G)$ as a conserved quantity, and all its orbits are periodic. In particular, for $c = 0$, the orbit through $(h_c, 0)$ will intersect the H-axis, when followed backwards in time. By continuous dependence on parameters, the same will be true for some small $c > 0$–a clear contradiction! This shows that (15) is indeed true. A similar argument shows that $g_c(H) \uparrow \infty$ as $c \downarrow 0$, and uniformly so on intervals $\delta \le H \le \delta^{-1}$. Therefore (16) also holds.

7. Proof of Theorem A

In view of Lemma 5.1, we only have to show that K is bounded from below if we know that it is bounded from above. So we shall assume that K is bounded from above, and we choose a constant A so large that

$$K(\theta, t) \le A \quad (\theta \in \mathbf{R}, t \ge 0),$$
$$|K_\theta(\theta, 0)| \le A \quad (\theta \in \mathbf{R}),$$
$$K(\theta, 0) \ge A^{-1} (\theta \in \mathbf{R}).$$

Choose $c > 0$ so small that the shrinking spiral of K_c of the last section satisfies $K_c(0) > A$, and $K_c'(\theta) > A$ whenever $A^{-1} \le K_c(\theta) \le A$. By Theorem 6.1 such a c exists, as well as a $\lambda > 0$ for which $0 < K_c'(\theta) \le \lambda K_c(\theta)$ holds for all $\theta \le 0$.

Lemma 7.1. *At any* $(\theta, t) \in \mathbf{T}_\nu \times (0, \infty)$ *one has* $|K_\theta(\theta, t)| \le \lambda K(\theta, t)$.

This implies that $(\log K)_\theta \le \lambda$, which, after integration, yields a Harnack-type inequality:

$$\sup_{\theta \in \mathbf{T}_\nu} K(\theta, t) \le e^{2\lambda\nu\pi} \inf_{\theta \in \mathbf{T}_\nu} K(\theta, t)$$

for all $t > 0$. Combined with

$$\sup_{\theta \in \mathbf{T}_\nu} K(\theta, t) = e^{-t/2} \kappa(T(1 - e^{-t})) \ge \frac{1}{2}\sqrt{2},$$

this implies that K is bounded from below, and therefore converges to one of the Abresch-Langer functions.

Proof of the Lemma. Let $t_0 > 0$, $\theta_0 \in \mathbf{R}$ be given, and choose c, λ as above. Since $0 < K(\theta_0, t_0) < K_c(0)$, and K_c is strictly monotone on $(-\infty, 0)$, there is a unique $\theta_1 < 0$ for which $K_c(\theta_1) = K(\theta_0, t_0)$. Consider the function

$$K^*(\theta, t) = K_c(\theta - \theta_0 + \theta_1 - ct).$$

Then K^* is a solution of (12) on the region

$$Q = \{(\theta, t) | \theta < \theta_0 - \theta_1 + ct, t > 0\},$$

and the difference $w = K^* - K$ satisfies a linear parabolic equation like (7).

On $\partial Q \cap \{t > 0\}$, i.e., when $\theta = \theta_0 - \theta_1 + ct$, one has $w(\theta, t) = K_c(0) - K(\theta, t) \ge K_c(0) - A > 0$. On the other part of ∂Q, i.e., when $\theta < \theta_0 - \theta_1$ and $t = 0$, $w(\theta, 0)$ has exactly one zero. Indeed, $w(\theta_0 - \theta_1, 0) > 0$, while $\limsup_{\theta \downarrow -\infty} w(\theta, 0) \le -A^{-1}$, so that w must have at least one zero. Moreover, at any zero one has $K = K^*$, so that $A^{-1} \le K^* \le A$, and hence $K_\theta^* > A \ge K_\theta$; i.e., at any zero of $w(\cdot, 0)$, one has $w_\theta > 0$ so there cannot be more than one zero.

By the Sturmian theorem, the number of zeroes of $\theta \to w(\theta, t)$, counted with multiplicity, cannot increase with time. But by our construction we have $w(\theta_0, t_0) = 0$, so that θ_0 is the only zero of $w(\cdot, t_0)$. The signs of the boundary values of w, i.e., of $w(\theta_0 - \theta_1 + ct_0, t_0)$ and $w(-\infty, t_0)$, are such that $w_\theta(\theta_0, t_0)$ must be positive. Thus

$$K_\theta(\theta_0, t_0) < K^*_\theta(\theta_0, t_0) \le \lambda K^*(\theta_0, t_0) = \lambda K(\theta_0, t_0)$$

holds at any prescribed point (θ_0, t_0).

By applying the same argument to $K(-\theta, t)$, we also get $-K_\theta < \lambda K$, so that $|K_\theta| < \lambda K$, which is what we had to prove.

8. The noose (proof of Theorem C)

Let $k(\theta, t)$ be a solution of (4) which blows up at time $T > 0$. From the Sturmian theorem we know that the number of local maxima of $\theta \to k(\theta, t)$ (i.e., the number of vertices of the curve) does not increase as $t \to T$, so that it must be eventually constant. Since we are only interested in the asymptotic behavior of k as $t \to T$, we may as well assume that the number of vertices of the curve is constant. Since the number of local maxima and minima must drop whenever a stationary point becomes degenerate, it follows that $\theta \to k(\theta, t)$ is a Morse function for all t.

In the same way we may assume that the number of self-intersections of the curve $C(t)$ is constant, so that all self-intersections of $C(t)$ are transversal for all $t \in [0, T)$.

Define a *noose* of the curve $C(t)$ to be an interval $(\alpha, \beta) \subset \mathbf{R}$ such that $C(t)|_{(\alpha, \beta)}$ is injective, $C(t, \alpha) = C(t, \beta)$, and $\pi < \beta - \alpha < 2\pi$. The area of the noose is the area enclosed by the closed curve $C(t)|_{[\alpha, \beta]}$.

Lemma 8.1. *If $C(t_0)$ has a noose with area A_0, and the curve $C(t)$ becomes singular at time $t = T$, then*

$$A_0 \ge \pi(T - t_0).$$

Proof. Since all self-intersections are transversal, one can follow a noose (α_0, β_0) through a family of nooses $(\alpha(t), \beta(t))$ $(t_0 \le t < T)$ of $C(t)$. A short computation shows that the area $A(t)$ of the noose at time t satisfies

$$A'(t) = \int_{\alpha(t)}^{\beta(t)} k(\theta, t)\, ds(\theta) = \beta(t) - \alpha(t),$$

so that $\pi < -A'(t) < 2\pi$. Therefore, $0 \le \lim_{t \uparrow T} A(t) \le A_0 - \pi(T - t_0)$. q.e.d.

For any given $t \in (0, T)$, we choose a $\theta(t)$ for which $k(\theta(t), t) = \kappa(t)$. As before, we let A be the constant given by

$$A = \sup_{\theta \in \mathbf{T}_\nu} [k(\theta, 0)^2 + k_\theta(\theta, 0)^2]^{1/2}.$$

Lemma 8.2. *If $\kappa(t) > A$, then $k(\theta, t) > \kappa(t)\cos(\theta - \theta(t))$ for all θ with $|\theta - \theta(t)| < \arccos(A/\kappa(t))$.*

Proof. At $\theta = \theta(t)$ one has $k(\theta, t) = \kappa(t)$ and $k_\theta(\theta, t) = 0$, so that

$$k(\theta, t) = \kappa(t)\cos(\theta - \theta(t)) + \int_{\theta(t)}^{\theta} \sin(\theta - \vartheta)\{k_{\theta\theta}(\vartheta, t) + k(\vartheta, t)\}\, d\vartheta.$$

If there were a $\theta \in (\theta(t), \theta(t) + \arccos(A/\kappa(t)))$ for which $k(\theta, t) \le \kappa(t)\cos(\theta - \theta(t))$, then we could choose a minimal θ with this property. Since $k > A$ at $(\theta(t), t)$, we would have $k_{\theta\theta} + k > 0$ at $(\theta(t), t)$ and hence $\theta > \theta(t)$. Between $\theta(t)$ and θ we would also have $k > \kappa(t)\cos(\theta - \theta(t)) > A$ and thus $k_{\theta\theta} + k > 0$, which would imply that $k(\theta, t) > \kappa(t)\cos(\theta - \theta(t))$, a contradiction. q.e.d.

For each $t < T$ choose a $\theta(t) \in \mathbf{R}$ which maximizes $k(\theta, t)$, and define the function

$$k^{(t)}(\theta) = \frac{k(\theta(t) + \theta, t)}{\kappa(t)},$$

as well as

$$\epsilon(t) = \max \frac{k(\theta \pm \pi/2, t)}{\kappa(t)}.$$

Lemma 8.3. *If for some sequence $t_n \uparrow T$, one has $\epsilon(t_n) \to 0$, then*

$$\lim_{n\to\infty} k^{(t_n)}(\theta) = \cos\theta,$$

uniformly on $[-\pi/2, \pi/2]$.

Proof. The sequence $k^{(t_n)}$ is uniformly Lipschitz, and we may assume, after passing to a subsequence, that it converges uniformly. Denote the limit by $K^*(\theta)$.

At all points where $k^{(t_n)} > A/\kappa(t_n)$, one has $k_{\theta\theta}^{(t_n)} + k^{(t_n)} > 0$, so that in the limit we have $K^*_{\theta\theta} + K^* \ge 0$ in the sense of distributions, on the *open* set where $K^* > 0$. Thus $k^{(t_n)}(0) = \max_\theta k^{(t_n)}(\theta) = 1$, so that $K^*(0) = 1 \ge K^*(\theta)$ for all θ.

Let $J \subset \mathbf{R}$ be the largest interval containing $\theta = 0$, on which K^* is positive. Then $J \subset (-\pi/2, \pi/2)$, since

$$K^*(\pm\pi/2) = \lim_{n\to\infty} k^{(t_n)}(\pm\pi/2) \le \lim_{n\to\infty} \epsilon(t_n) = 0.$$

On the interval J we have

$$K^*(\theta) = \cos(\theta) + \int_0^\theta \sin(\theta - \vartheta)\{K^*_{\theta\theta}(\vartheta) + K^*(\vartheta)\}\, d\vartheta,$$

which implies that $K^*(\theta) \geq \cos\theta$ on J, and hence that $J = (-\pi/2, \pi/2)$. Moreover the inequality is an equality when $\theta = \pm\pi/2$, so that the integral must vanish for those values of θ. Since the integrand has one sign on J, this can only be true if $K^*_{\theta\theta} + K^* = 0$ on J, i.e., if $K^*(\theta) = \cos\theta$. q.e.d.

In addition to uniform convergence the hypothesis $\epsilon(t_n) \to 0$ also implies that $k^{(t_n)}$ converges in $C^\infty(I_\delta)$ for any $\delta > 0$, where $I_\delta = [-\pi/2 + \delta, \pi/2 - \delta]$. Since we already know that the $k^{(t_n)}$ converge uniformly, we only have to prove that they remain bounded in $C^\infty(I_\delta)$; convergence in the same space then follows immediately.

To bound the derivatives of $k^{(t_n)}$, we consider

$$\mathfrak{k}(\theta, t) = \frac{k(\theta_n + \theta, t_n + \kappa(t_n)^{-2} t)}{\kappa(t_n)},$$

which is defined on $Q_n = \mathbf{T}_\nu \times [-\kappa(t_n)^2, 0]$ (it is also defined for some positive t's, but we would not be interested in what happens to $\mathfrak{k}_n$ for $t > 0$). On Q_n the function $\mathfrak{k}_n$ satisfies $|\mathfrak{k}_{n,\theta}| \leq M$, $\mathfrak{k}_{n,t} \geq 0$ wherever $\mathfrak{k}_n(\theta, t) \geq A/\kappa(t_n)$, $0 < \mathfrak{k}_n(\theta, t) \leq 1$ and $\mathfrak{k}_n(0, 0) = 1$ for some finite constant M.

Lemma 8.4. *There is a constant $C < \infty$ such that*

$$\mathfrak{k}_n(\theta, t) - \mathfrak{k}_n(\theta, t') \leq C\sqrt{t - t'}$$

holds for all $\theta \in \mathbf{T}_\nu$ and $-\kappa(t_n)^2 < t' < t < 0$.

Assume for the moment that this lemma is true. Then, given a $\delta > 0$, there exist $\tau > 0$ and $n_\delta < \infty$ such that for all $n \geq n_\delta$ one has $\mathfrak{k}_n(\theta, t) \geq \delta/4$ on $I_{\delta/2} \times [-\tau, 0]$. In fact, this follows from the lemma and the fact that $\mathfrak{k}_n(\theta, 0)$ converges uniformly to $\cos\theta$, which is strictly larger than $\delta/4$ on $I_{\delta/2}$.

Since the $\mathfrak{k}_n$'s are bounded away from zero on $I_{\delta/2} \times [-\tau, 0]$, the equation $k_t = k^2(k_{\theta\theta} + k)$ which they satisfy is uniformly parabolic on this domain. The theory of quasilinear parabolic equations in [11] tells us that each derivative of $\mathfrak{k}_n$ is uniformly bounded on the smaller rectangle $I_\delta \times [-\tau/2, 0]$. In particular, all derivatives of $k^{(t_n)}$ are uniformly bounded on any interval I_δ.

Proof of Lemma 8.4. Let $u(\theta, t)$ be the solution of

$$\begin{aligned} &u_t = u_{\theta\theta} + u \qquad (|\theta| \leq 1, 0 \leq < \infty), \\ &u(\theta, 0) = M|\theta|, \\ &u(\pm 1, t) = M. \end{aligned}$$

One could solve this initial value problem explicitly, if one wanted to. The solution has the following two relevant properties.

First, there is a constant C such that $u(0, t) \leq C\sqrt{t}$ for $0 < t < 1$. In addition, u is strictly increasing for $t > 0$, i.e., $u_{\theta\theta} + u > 0$ holds for all $\theta \in [-1, 1]$ and $t > 0$ (the initial value is a subsolution).

If $\theta_0 \in \mathbf{T}_\nu$ and $t_0 < t_1 < 0$ are given, since $|\mathfrak{k}_{n,\theta}| \leq M$, the function $w(\theta, t) = k(\theta_0, t_0) + u(\theta - \theta_0, t - t_0)$ satisfies $w(\theta, t_0) \geq \mathfrak{k}_n(\theta, t_0)$ for all $|\theta - \theta_0| \leq 1$. In addition, whenever $w(\theta, t) \leq 1$ one has

$$w_t \geq w^2 w_t = w^2(w_{\theta\theta} + w),$$

since $w_t \geq 0$. So w is a supersolution for (4) on the set $\mathscr{O} = \{(\theta, t) | w(\theta, t) < 1\}$. By the maximum principle this implies that $\mathfrak{k}_n \leq w$ for all $t \geq t_0$ and $\theta \in (\theta_0 - 1, \theta_0 + 1)$ (recall that $\mathfrak{k}_n \leq 1$, so that we only have to compare the two functions on $\mathscr{O}$).

If $t_0 < t_1 < t_0 + 1$, then this immediately shows us that $\mathfrak{k}_n(\theta, t_1) \leq \mathfrak{k}(\theta, t_1) + C(t_1 - t_0)^{1/2}$; if $t_1 > t_0 + 1$, then this inequality also holds if we choose $C > 1$, since $0 < \mathfrak{k}_n \leq 1$. q.e.d.

In order to prove Theorem C, we may therefore assume that $\epsilon(t) \geq \epsilon$ for some constant $\epsilon > 0$, and try to reach a contradiction. We shall show that if $\epsilon = \inf_{0<t<T} \epsilon(t) > 0$, then for all t close to T the curve $C(t)$ will have a noose whose area $A(t)$ is bounded by $\text{Const} \times \kappa(t)^{-2}$. Since the area of noose must satisfy $A(t) \geq \pi(T - t)$, this would imply that $\kappa(t) \leq \text{Const} \times (T - t)^{-1/2}$, contrary to the assumption of Theorem C.

Construction of the noose. Recall that for each t we had chosen a $\theta(t)$ which maximizes $k(\theta, t)$, and defined

$$k^{(t)}(\theta) = \frac{k(\theta(t) + \theta, t)}{\kappa(t)}.$$

We have $|k_\theta^{(t)}| \leq 2\nu\pi + A/\kappa(t)$, and since $\kappa(t) \to \infty$ as $t \to T$, we may assume that $|k_\theta^{(t)}| \leq 10\nu$ (we have used $2\pi \approx 6.28\ldots < 10$).

By assumption we have either $k^{(t)}(\pi/2) \geq \epsilon$, or $k^{(t)}(-\pi/2) \geq \epsilon$; we shall assume that the first inequality holds. The Lipschitz estimate for $k^{(t)}$

then implies that

$$k^{(t)}(\theta) \geq \epsilon - 10\nu|\theta - \pi/2|$$

for $|\theta - \pi/2| \leq \epsilon/10\nu$.

Introduce the function

$$w(\theta) = \max(\cos\theta, \epsilon - 10\nu|\theta - \pi/2|),$$

which is defined on the interval $-\pi/2 \leq \theta \leq \pi$. Then $w(\theta)$ is strictly positive on $[0, \pi/2]$, and therefore its minimum $\rho = \min_{0\leq\theta\leq\pi/2} w(\theta)$ is also strictly positive.

If t is close enough to T, then $\kappa(t)$ will be so large that $A/\kappa(t) < \rho$. Since $k^{(t)}_{\theta\theta} + k^{(t)} \geq 0$ at any θ for which $k^{(t)}(\theta) \geq A/\kappa(t)$, this implies that $k^{(t)}(\theta) \geq w(\theta)$ for $0 \leq \theta \leq \pi/2 + \epsilon/10\nu$. Applying Lemma 8.2 again, we get

$$k^{(t)}(\theta) \geq w(\theta) \text{ for } -\arccos(A/\kappa(t)) \leq \theta \leq \pi/2 + \epsilon/(10\nu).$$

Let Γ be the convex curve which corresponds to the function w, i.e., the curve parametrised by

$$\Gamma(\vartheta) = \int_0^{\vartheta} \frac{e^{i\theta}}{w(\theta)}\,d\theta, \qquad -\pi/2 < \vartheta < \pi/2 + \epsilon/(10\nu).$$

Then Γ is an unbounded curve with two asymptotes, and since its total curvature $(= \pi + \epsilon/(10\nu))$ exceeds π, it must have a self-intersection. Let the ϑ angles corresponding to this intersection be $-\pi/2 < \alpha < 0$ and $\pi/2 < \beta < \pi/2 + \epsilon/(10\nu)$, so that $\Gamma([\alpha, \beta])$ is a noose. Denote the area of this noose by A_ϵ.

For all t sufficiently close to T, we shall have $-\arccos(A/\kappa(t)) < \alpha$, so that $k^{(t)}(\theta) \geq w(\theta)$ on $[\alpha, \beta]$. Thus the curve corresponding to $k^{(t)}$ will have a noose which is contained in the noose of Γ, and whose area is therefore bounded by A_ϵ. Since our original curve $C(t)$ is obtained from the curve corresponding to $k^{(t)}$ by a Euclidean motion, and by shrinking it by a factor $\kappa(t)$, the curve $C(t)$ must have a convex noose whose area is at most $A_\epsilon/\kappa(t)^2$.

As we argued in the beginning of the proof, this contradicts the hypothesis of Theorem C, since it implies $A_\epsilon\kappa(t)^{-2} \geq \pi(T-t)$, i.e., $\kappa(t) \leq [A_\epsilon/\pi(T-t)]^{-1/2}$.

9. An upper bound for the rate of blowup

In this section we shall prove Theorem B. The length of a convex curve C is given by $L = \int_C d\theta/k$; if $C(t)$ evolves according to its curvature, then the length of $C(t)$ will change according to

$$L'(t) = -\int_{\mathbf{T}_\nu} k(\theta, t)\, d\theta. \tag{21}$$

But if t is close enough to T, then for some $\theta_0 \in \mathbf{T}_\nu$ we know that $k(\theta, t) \geq \kappa(t)\cos(\theta - \theta_0)$ holds on the interval $|\theta - \theta_0| \leq \arccos(A/\kappa(t))$ for some constant $A > 0$. Thus $-L'(t) \geq c\kappa(t)$ for some constant $c > 0$, and therefore

$$c\int_0^T \kappa(t)\, dt \leq L(0) - \lim_{t\to T} L(t) \leq L(0) < \infty.$$

On the other hand $\kappa(t)$ is eventually increasing, so that for sufficiently large t one has

$$(T-t)\kappa(t) \leq \int_t^T \kappa(\tau)\, d\tau,$$

and since we have just shown that $\int_0^T \kappa(t) dt$ converges, the integral on the right side will vanish if t tends to T. So Theorem B is indeed true.

To obtain the more precise blowup rate of Theorem D, we consider

$$l(t) = \int_{\mathbf{T}_\nu} \log k(\theta, t)\, d\theta.$$

Differentiation under the integral, and integration by parts show that

$$l'(t) = \int_{\mathbf{T}_\nu} \frac{k_t}{k}\, d\theta,$$

$$l''(t) = 2\int_{\mathbf{T}_\nu} \left(\frac{k_t}{k}\right)^2 d\theta.$$

In particular, one sees that $l(t)$ is a convex function of time.

If the blowup set consists of an interval of length less than 2π, then it follows from the first part of Theorem D (which we shall prove in the next section), that the length of Σ actually is π. Thus $\Sigma \subset [\alpha, \alpha + \pi]$ for some $\alpha \in \mathbf{T}_\nu$, and as we shall see in the next section, $k(\theta, t)$ remains bounded outside of every interval $\Sigma_\epsilon = [\alpha - \epsilon, \alpha + \pi + \epsilon]$. By standard parabolic theory the same will then be true for all derivatives of k, and therefore

$$M_\epsilon = \sup\left\{\frac{k_t(\theta, t)}{k(\theta, t)} | \theta \notin \Sigma_\epsilon, 0 \le t < T\right\}$$

is finite.

Using Cauchy's inequality we then get

$$\begin{aligned} l'(t) &\le M_\epsilon \left(2\nu\pi - |\Sigma_\epsilon|\right) + \int_{\Sigma_\epsilon} \frac{k_t}{k} k\theta \\ &\le (2\nu - 1)\pi M_\epsilon + |\Sigma_\epsilon|^{1/2} \left(\frac{l''(t)}{2}\right)^{1/2}, \end{aligned}$$

i.e.,

$$l''(t) \ge \frac{2}{\pi + \epsilon}(l'(t) - (2\nu - 1)\pi M_\epsilon)^2,$$

which, after integration yields

$$l'(t) \le (2\nu - 1)\pi M_\epsilon + \frac{\pi + \epsilon}{2(T - t)},$$

and integrating one more time, we find

$$l(t) \le l(0) + (2\nu - 1)\pi M_\epsilon T - \frac{\pi + \epsilon}{2} \log(T - t).$$

On the other hand Lemma 8.2 implies that $k(\theta, t) \ge \kappa(t)\cos(\theta - \theta(t))$ for some $\theta(t) \in \mathbf{T}_\nu$, and thus

$$\begin{aligned} l(t) &\ge \int_{-\arccos(A/\kappa(t))}^{\arccos(A/\kappa(t))} \log(\kappa(t)\cos\varphi)\, d\varphi \\ &= 2\arccos(A/\kappa(t)) \log\kappa(t) + O(1) \qquad (t \uparrow T) \\ &= \pi \log\kappa(t) + O(1) \qquad (t \uparrow T). \end{aligned}$$

Combining the two inequalities for $l(t)$ then will give us an upper bound for $\kappa(t)$:

$$\kappa(t) \le \frac{C_\epsilon}{(T - t)^{1/2+\epsilon/\pi}}.$$

Since $\epsilon > 0$ was arbitrary, we have shown that the second part of Theorem D follows from the first part of the same theorem.

10. The blowup set (proof of Theorem D)

Since our solution $k(\theta, t)$ of (4) becomes strictly increasing when and wherever it becomes larger than A, the set $\Omega_M(t) = \{\theta \in \mathbf{T}_\nu | k(\theta, t) > M\}$ is strictly increasing in time, for any $M \ge A$. The blowup set

$\Sigma = \{\theta \mid \lim_{t\to T} k(\theta, t) = \infty\}$ may therefore be written as

$$\Sigma = \bigcap_{M\geq A} \bigcup_{0<t<T} \Omega_M(t).$$

Consider an $M > A$, and let (α, β) be a component of $\Omega_M(t)$. Then, for any $\theta_0 \in (\alpha, \beta)$ which maximizes $k(\theta, t)$ on (α, β) one has

$$k(\theta, t) = k(\theta_0, t)\cos(\theta - \theta_0) + \int_{\theta_0}^{\theta} \sin(\theta - \varphi)\{k_{\theta\theta} + k\}(\varphi, t)\,dt,$$

so that, just as in Lemma 8.2, we find $k(\theta, t) \geq k(\theta_0, t)\cos(\theta - \theta_0)$, for $|\theta-\theta_0| \leq \arccos(A/k(\theta_0, t))$, and, as a consequence, that (α, β) contains all θ with $|\theta - \theta_0| \leq \arccos(A/M)$. Thus the length of any component of $\bigcup_{0<t<T}\Omega_M(t)$ is at least $2\arccos(A/M)$.

As M goes up, the set $\bigcup_{0<t<T}\Omega_M(t)$ shrinks, while $2\arccos(A/M)$ becomes larger. This implies that the length of any component of $\bigcup_{0<t<T}\Omega_M(t)$ or Σ is at least π.

From here on we shall assume that Σ is an interval. By rotating our coordinate system, we can arrange that the closure of this interval is given by

$$\overline{\Sigma} = [-\alpha, \alpha]$$

for some $\alpha \geq \pi/2$. In order to prove Theorem D, we shall assume in this section that $\alpha \in (\pi/2, \pi)$, and show that this assumption leads to a contradiction.

Since $\Sigma \subset \bigcup_{t<T}\Omega_M(t)$ for any $M > 0$, $\{\Omega_M(t)\}_{t<T}$ is an open covering of any compact interval $[-\beta, \beta] \subset \Sigma$. Therefore one of the $\Omega_M(t)$'s contains $[-\beta, \beta]$, or in other words we have

Lemma 10.1. *For any $\beta < \alpha$ and $M > 0$, there exists a $t_{M,B} < T$ such that $k(\theta, t) \geq M$ when $|\theta| \leq \beta$ and $t \geq T_{M,B}$.*

Lemma 10.2. *For any sufficiently small $\epsilon > 0$, there is a $t_\epsilon < T$ such that for all $t \in (t_\epsilon, T)$ one has*

$$k_\theta(\theta, t) > 0 \ (< 0)$$

on the interval $\theta \in [-\alpha + \epsilon, -\alpha + \pi/2 - \epsilon]$ (or on the interval $[\alpha - \pi/2 + \epsilon, \alpha + \epsilon]$).

Proof. By assumption the quantity $M_{\epsilon/2} = \sup_{t<T} k(\alpha - \epsilon/2, t)$ is finite. We choose t_ϵ so close to T that

$$k(\theta, t) \geq \max\left(A, \frac{M_{\epsilon/2}}{\sin(\epsilon/2)}\right)$$

for $t \geq t_\epsilon$ and $|\theta| \leq \alpha - \epsilon$.

Suppose that for some $\theta \in [-\alpha+\epsilon, -\alpha+\pi/2-\epsilon]$ and $t \in [t_\epsilon, T)$ one would have $k_\theta(\theta, t) \leq 0$. Arguing as in Lemma 8.2, it would follow from

$$\begin{aligned} k(\vartheta, t) &= k(\theta, t)\cos(\vartheta-\theta) + k_\theta(\theta, t)\sin(\vartheta-\theta) + \int_\theta^\vartheta \sin(\vartheta-\varphi)p(\varphi)\,d\varphi \\ &\geq k(\theta, t)\cos(\vartheta-\theta) + \int_\theta^\vartheta \sin(\vartheta-\varphi)p(\varphi)\,d\varphi, \end{aligned}$$

where $p(\varphi) = k_{\theta\theta}(\varphi, t) + k(\varphi, t)$, that $k(\vartheta, t) \geq k(\theta, t)\cos(\vartheta-\theta)$ for all $\vartheta \leq \theta$ with $\theta - \vartheta \leq \arccos(A/k(\theta, t))$. But then one would have

$$k(-\alpha-\epsilon/2, t) \geq k(\theta, t)\cos(\theta+\alpha+\epsilon/2) \geq k(\theta, t)\sin(\epsilon/2) \geq M_{\epsilon/2},$$

which is inconsistent with the definition of $M_{\epsilon/2}$. Hence the lemma is true. $\square$

Our proof of Theorem D is based on an analysis of the horizontal distance $X_\beta(t)$ between the two points on the curve which correspond to the angles $\pm\beta$, where $\pi/2 < \beta < \alpha$. Thus we consider

$$X_\beta(t) = \int_{-\beta}^{\beta} \frac{\cos\theta}{k(\theta, t)}\,d\theta.$$

The dominated convergence theorem implies that

$$\lim_{t\uparrow T} X_\beta(t) = 0,$$

while, on the other hand, we also have

$$\begin{aligned} X_\beta'(t) &= -\int_{-\beta}^{\beta} \cos\theta(k_{\theta\theta}+k)\,d\theta \\ &= \{k_\theta(-\beta, t) - k_\theta(\beta, t)\}\cos\beta - \{k(\beta, t) + (-\beta, t)\}\sin\beta. \end{aligned}$$

Thus for $\beta \in (\pi/2, \alpha)$, when $\cos\beta < 0 < \sin\beta$, it follows from Lemma 10.2 that $X_\beta'(t) < 0$ for all t which are sufficiently close to T.

Since $\lim_{t\uparrow T} X_\beta(t) = 0$, this implies:

Proposition 10.3. *If* $\pi/2 < \alpha < \pi$, *then for any* $\beta \in (\pi/2, \alpha)$ *there exists a* $t_\beta < T$ *such that* $X_\beta(t) > 0$ *for* $t_\beta \leq t < T$.

Next, we choose sequences $t_n \uparrow T$, and $\theta_n \in [-\alpha, \alpha]$ such that $k(\theta_n, t_n) = \kappa(t_n)$, and for which

$$k(\theta_n \pm \pi/2, t_n) = o(\kappa(t_n)) \qquad (n \to \infty). \tag{22}$$

The existence of these sequences follows from Theorems A and C.

We shall complete the proof of Theorem D by showing that $X_\beta(t_n) < 0$ for large enough n and some suitably chosen β so that the hypothesis $\pi/2 < \alpha < \pi$ of Lemma 10.3 can never be fulfilled.

After passing to a subsequence, if necessary, we may assume that all the θ_n's have the same sign, and without loss of generality, we may also assume that they are all positive. In addition, we may assume that $\lim_{n\to\infty}\theta_n = \overline{\theta}$ exists. Then $0 \le \overline{\theta} \le \alpha - \pi/2$, because $k(\theta, t_n) \ge \kappa(t_n)\cos(\theta - \theta_n)$ when $|\theta - \theta_n| \le \arccos(A/\kappa(t_n))$.

To estimate $X_\beta(t_n)$, we introduce a rescaled version of $X_\beta(t_n)$. First consider

$$u_n(\theta) = \lambda_n k(\theta, t_n) \text{ with } \lambda_n = \frac{1}{k(-\pi/2, t_n)}.$$

Our assumptions imply that $\lambda_n \to 0$ as $n \to \infty$, and from the lemma on eventual monotonicity (i.e., Lemma 4.1) it follows that $u_n''(\theta) + u_n(\theta) \ge 0$ whenever $u_n(\theta) \ge \lambda_n A$.

Since $\cos\theta \le 0$ for $\pi/2 \le \theta \le \beta$, we have

$$\begin{aligned} X_\beta(t_n)/\lambda_n &= \int_{-\beta}^{\beta} \frac{\cos\theta}{u_n(\theta)}\,d\theta \\ &\le \int_{-\beta}^{\pi/2} \frac{\cos\theta}{u_n(\theta)}\,d\theta \\ &= \int_{-\beta}^{\beta-\pi} \frac{\cos\theta}{u_n(\theta)}\,d\theta + \int_{\beta-\pi}^{\beta} \frac{\cos\theta}{u_n(\theta)}\,d\theta \\ &= I_n(\beta) + II_n(\beta). \end{aligned}$$

We shall show that for large n and for β close enough to α, the first term $I_n(\beta)$ dominates the other term, and also that $I_N(\beta)$ becomes negative; this is what we are looking for since it implies $X_\beta(t_n) < 0$.

To estimate the two terms I_n, II_n, we first find an upper bound for u_n to the left of $-\pi/2$.

Let $\epsilon > 0$ be so small that $\pi/2 < \alpha + \epsilon < \pi$, and define

$$\begin{aligned} U_n(\theta) &= \delta_n \frac{\cos\theta}{\cos(\alpha+\epsilon)} + \frac{\sin(\theta+\alpha+\epsilon)}{\sin(-\pi/2+\alpha+\epsilon)} \\ &= \frac{-\delta_n\cos\theta + \sin(\theta+\alpha+\epsilon)}{-\cos(\alpha+\epsilon)}, \end{aligned}$$

where $\delta_n = \lambda_n \max(A, M_\epsilon)$ and $M_\epsilon = \sup_{0<t<T} k(-\alpha-\epsilon, t)$. Then U_n satisfies $U_n'' + U_n = 0$, $U_n(-\alpha-\epsilon) \ge u_n(-\alpha-\epsilon)$, and $U_n(-\pi/2) = u_n(-\pi/2)(=1)$. We also know that $u_n'' + u_n \ge 0$ wherever $u_n \ge \lambda_n A$, so that the maximum principle implies

$$u_n(\theta) \le U_n(\theta)(-\alpha-\epsilon \le \theta \le \pi/2),$$

which gives a lower bound for u_n' at $-\pi/2$, namely, $u_n'(-\pi/2) \ge U_n'(-\pi/2)$.

Combining this with $U_n(-\pi/2) = u_n(-\pi/2) = 1$, we get for $|\theta| \le \pi/2$ (i.e., when $\cos\theta \ge 0$)

$$\begin{aligned} u_n(\theta) &= -u_n(-\pi/2)\sin\theta + u_n'(-\pi/2)\cos\theta + \int_{-\pi/2}^{\theta} \sin(\theta-\varphi)v_n(\varphi)\,d\varphi \\ &\ge U_n(\theta) + \int_{-\pi/2}^{\theta} \sin(\theta-\varphi)v_n(\varphi)\,d\varphi, \end{aligned}$$

where $v_n = u_n'' + u_n$.

On the interval $[-\pi/2, 0]$ the function $U_n(\theta)$ is bounded from below by $U_n(0) = (-\delta_n + \sin(\alpha+\epsilon))/(-\cos(\alpha+\epsilon))$. For large enough $n \in \mathbf{N}$ one therefore has $U_n(\theta) \ge \lambda_n A$ on $[-\pi/2, 0]$, and hence, arguing as in 8.2 we obtain the following lower bound for $u_n(\theta)$

$$u_n(\theta) \ge U_n(\theta) \qquad (-\pi/2 \le \theta \le 0),$$

which holds for sufficiently large n. This allows us to estimate $I_n(\beta)$ from above, for large n:

$$I_n(\beta) \le \int_0^{\beta-\pi/2} \sin(\varphi)\left[\frac{1}{U_n(-\pi/2+\varphi)} - \frac{1}{U_n(-\pi/2-\varphi)}\right] d\varphi.$$

As $n \to \infty$ the δ_n's tend to zero, so that

$$U_n(-\pi/2+\varphi) \to \frac{\sin(-\pi/2+\varphi+\alpha+\epsilon)}{-\cos(\alpha+\epsilon)} = \frac{\cos(\varphi+\alpha+\epsilon)}{\cos(\alpha+\epsilon)},$$

and hence

$$\limsup_{n\to\infty} I_n(\beta) \le \cos(\alpha+\epsilon)\int_0^{\beta-\pi/2} \sin\varphi\{\sec(\alpha+\epsilon+\varphi) - \sec(\alpha+\epsilon-\varphi)\}\,d\varphi.$$

This holds for any positive ϵ, so by putting $\epsilon = 0$

$$\limsup_{n\to\infty} I_n(\beta) \le \cos(\alpha)\int_0^{\beta-\pi/2} \sin\varphi\{\sec(\alpha+\varphi) - \sec(\alpha-\varphi)\}\,d\varphi < 0.$$

To estimate the other term, we split it into two parts:

$$\begin{aligned} II_n(\beta) &= \int_{\beta-\pi}^{\alpha-\pi} \frac{\cos\theta}{u_n(\theta)}\,d\theta + \int_{\alpha-\pi}^{-\pi/2} \frac{\cos\theta}{u_n(\theta)}\,d\theta \\ &= III_n(\beta) + IV_n(\beta). \end{aligned}$$

On the interval $[\beta-\pi, \alpha-\pi]$ we already have the inequality $u_n \ge U_n$; since the minimal value of U_n on this interval is $U_n(\alpha-\pi)$, we get

$$III_n(\beta) \le \frac{\alpha-\beta}{U_n(\alpha-\pi)}.$$

A short computation shows that

$$\lim_{n\to\infty, \epsilon\downarrow 0} U_n(\alpha - \pi) = \frac{\sin(2\alpha - \pi)}{-\cos\alpha} = 2\sin\alpha ,$$

so

$$\limsup_{n\to\infty} III_n(\beta) \leq \frac{(\alpha - \beta)}{2\sin\alpha}.$$

We can make this quantity as small as we like, by choosing β close enough to α.

Concerning the last term, we have $\lim_{n\to\infty} IV_n(\beta) = 0$. Indeed, we have $\theta_n - \pi/2 \in [-\pi/2, 0]$, so that $\lambda_n k(\theta_n - \pi/2, t_n)$ is bounded from below by $U_n(0)$. The sequences θ_n, t_n were chosen so that (22) holds, and thus

$$\lim_{n\to\infty} \lambda_n \kappa(t_n) = \lim_{n\to\infty} \lambda_n k(\theta_n - \pi/2, t_n) \times \frac{\kappa(t_n)}{k(\theta_n - \pi/2, t_n)} = \infty.$$

Using Lemma 8.2 we then conclude that $u_n(\theta) \to \infty$ on the interval $\alpha - \pi < \theta < \pi/2$, so that $IV_n(\beta)$ tends to zero, by the dominated convergence theorem, as claimed.

Combining our estimates for I_n, II_n, III_n, and IV_n implies

$$\begin{aligned} 0 &= \lim_{n\to\infty} X_\beta(t_n)/\lambda_n \\ &= \limsup_{n\to\infty} I_n(\beta) + III_n(\beta) + IV_n(\beta) \\ &\leq \text{Const} \times (\alpha - \beta) + \limsup_{n\to\infty} I_n(\beta) \\ &< 0, \end{aligned}$$

if β is close enough to α. This contradiction shows that our initial assumption $\pi/2 < \alpha < \pi$ must have been wrong, so that Theorem D holds.

11. Blowup for symmetric cardioids

In this section we shall point out a class of curves to which Theorem D is applicable.

Let our initial curve C_0 have index 2, and assume that its curvature function satisfies

$$(\heartsuit) \qquad \begin{aligned} k_0(-\theta) &= k_0(\theta) \qquad (\forall \theta \in \mathbf{T}_2), \\ k_0'(\theta) &< 0 \qquad (0 < \theta < 2\pi). \end{aligned}$$

Such curves are cardioids, and if we assume that the point corresponding to $\theta = 0$ lies on the y-axis, then they are invariant under reflection in the

y-axis. An example of such a curve is the curve corresponding to

$$k_0(\theta) = \frac{1}{a + \cos(\theta/2)},$$

where $a > 1$ is a constant.

The first condition implies $k_0'(0) = k_0'(2\pi) = 0$, so that C_0 has exactly two vertices. If $k(\theta, t)$, $0 < t < T$, is the maximal classical solution of (4) with $k_0(\theta)$ as initial data, then for each $t > 0$, the solution $k(\cdot, t)$ will also satisfy ($\heartsuit$).

Theorem. *The blowup set of* Σ *of the solution* $k(\theta, t)$ *to the curve shortening equation, whose initial data satisfies* ($\heartsuit$) *is the interval* $[3\pi/2, 5\pi/2]$.

Proof. Since $k_\theta < 0$ for $0 < \theta < 2\pi$ and all $t > 0$, by symmetry that closure of the blowup set must have the form $[2\pi - \alpha, 2\pi + \alpha]$ for some $\alpha \geq \pi$. Therefore we have to prove that $\alpha = \pi/2$, and that the blowup set is closed, i.e., that it contains its endpoints.

If $|\Sigma| = \pi$, then the arguments of Gage and Hamilton [8] imply that Σ must contain its endpoints, so that we really only need to show that $|\Sigma| = \pi$.

Consider the horizontal distance between the two points corresponding to $\theta = 0$ and $\theta = \pi$:

$$D(t) = \int_0^\pi \frac{\cos\theta}{k(\theta, t)}\, d\theta.$$

Monotonicity of $\theta \to k(\theta, t)$ on the interval $(0, 2\pi)$ implies that $D(t) < 0$ for all $t > 0$, i.e., that the point with $\theta = \pi$ lies to the left of the y-axis.

On the other hand,

$$\begin{aligned} D'(t) &= -\int_0^\pi \cos\theta\{k_{\theta\theta}(\theta, t) + k(\theta, t)\}\, d\theta \\ &= [-k_\theta(\theta, t)\cos\theta - k(\theta, t)\sin\theta]_0^\pi \\ &= k_\theta(\pi, t) \\ &< 0. \end{aligned}$$

So $D(t)$ is decreasing, and it is bounded away from zero.

The symmetry of k implies that

$$\int_0^{2\pi} \frac{\cos\theta}{k(\theta, t)}\, d\theta = 0,$$

so that the point with $\theta = 2\pi$ also lies on the y-axis. If $\alpha \geq \pi$, then by the monotone convergence theorem we have

$$\lim_{t\uparrow T} D(t) = -\lim_{t\uparrow T} \int_\pi^{2\pi} \frac{\cos\theta}{k(\theta, t)}\, d\theta = 0.$$

The contradiction shows that α is less than π, so that $|\Sigma| < 2\pi$ and we can apply Theorem D. Hence the blowup set has length π.

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Appendix: The local semiflow, and stable and unstable manifolds.

Consider the initial value problem

$$(IVP)\qquad \begin{aligned} &k_t = k^2 k_{\theta\theta} + k^3 \quad (\theta \in \mathbf{T}_\nu,\ 0 < t < T), \\ &k(\theta, 0) = k_0(\theta) \quad (\theta \in \mathbf{T}_\nu). \end{aligned}$$

This is a quasilinear parabolic initial value problem, and if the initial value is a strictly positive function, the equation will be nondegenerate, so that the theory in Eidelman's book [5] implies the existence of a short-term solution to (IVP), assuming that the initial function is a smooth function. The *a priori* estimates of Ladyženskaya et al. [11] then allow one to prove local existence for arbitrary continuous initial functions k_0.

In this Appendix, we recall how one can use the theory of analytic semigroups to establish a similar local existence result for (IVP). The advantage of this approach is that, except for some facts about Sturm-Liouville operators of the form $a(x)(d/dx)^2 + b(x)$ with Hölder continuous coefficients, the whole discussion stays within the realm of "calculus on Banach spaces."

The only disadvantage, perhaps, is that we have to introduce the following two function spaces:

$$h^{\alpha}(\mathbf{T}_{\nu}) = \left\{ u \in C(\mathbf{T}_{\nu}) \,\middle|\, \lim_{\epsilon \downarrow 0} \sup_{|\theta-\theta'|<\epsilon} \frac{|u(\theta) - u(\theta')|}{\epsilon^{\alpha}} = 0 \right\}$$

and

$$h^{2,\alpha}(\mathbf{T}_{\nu}) = \{u \in C^2(\mathbf{T}_{\nu}) | u'' \in h^{\alpha}(\mathbf{T}_{\nu})\}.$$

The elements of $h^{\alpha}(\mathbf{T}_{\nu})$ are called "little-Hölder continuous functions." Equipped with the norm

$$\|u\|_{h^{\alpha}(\mathbf{T}_{\nu})} = \sup_{\theta \in \mathbf{T}_{\nu}} |u(\theta)| + \sup_{\theta \neq \theta'} \frac{|u(\theta) - u(\theta')|}{|\theta - \theta'|^{\alpha}},$$

the vector space $h^{\alpha}(\mathbf{T}_{\nu})$ is a Banach space; it is the closure of $C^{\infty}(\mathbf{T}_{\nu})$ in the usual space of Hölder continuous functions, $C^{\alpha}(\mathbf{T}_{\nu})$. Likewise, $h^{2,\alpha}(\mathbf{T}_{\nu})$ is the closure of $C^{\infty}(\mathbf{T}_{\nu})$ in $C^{2,\alpha}(\mathbf{T}_{\nu})$.

Theorem. *The initial value problem (IVP) generates a real analytic local semiflow on the open subset $\mathscr{O} \subset h^{\alpha}(\mathbf{T}_{\nu})$ consisting of all strictly positive functions.*

The theorem means the following. If we denote the local semiflow by $\varphi^t (t \geq 0)$, so that the maximal solution $k(\theta, t)$ of (IVP) is given by

$$k(\theta, t) = (\varphi^t(k_0))(\theta),$$

then φ is defined on some open subset $\mathscr{D} \subset [0, \infty) \times \mathscr{O}$, which contains $\{0\} \times \mathscr{O}$. The semiflow is a continuous map $\varphi: \mathscr{D} \to \mathscr{O}$, which is real analytic on $\mathscr{D}_+ = \{(t, k_0) \in \mathscr{D} | t > 0\}$. It also satisfies the familiar semigroup properties, namely:

(i) $\varphi^0(k_0) = k_0$ for all $k_0 \in \mathscr{O}$.

(ii) If for some $t, s > 0$, and $k_0 \in \mathscr{O}$, both $(t, k_0) \in \mathscr{D}$ and $(s, \varphi^t(k_0)) \in \mathscr{D}$, then $(t+s, k_0) \in \mathscr{D}$, and $\varphi^{t+s}(k_0) = \varphi^s(\varphi^t(k_0))$.

Once one has proved these statements, one can use the existing proofs in the theory of dynamical systems to prove (as in [F]) that hyperbolic fixed points have smooth stable and unstable manifolds.

Indeed, if $k_0 \in \mathscr{O}$ is a fixed point of φ^t, then its stable and unstable manifolds coincide with the same manifolds for the time-one map φ^1, which is defined and real analytic in a neighborhood of k_0.

If the semiflow has a compact and hence finite dimensional invariant manifold, which consists of fixed points and is normally hyperbolic, then one can again use the existing proofs of the analogous statement in finite dimensions, to construct smooth stable and unstable manifolds of the invariant manifold.

The point we wish to make is that, in order to construct stable and unstable manifolds for fixed points or sets, one only has to prove smooth dependence of the solution on the initial data, i.e., smoothness of the time-one map. Once one has a real analytic semiflow on a Banach space, the particular (PDE or functional analytic) techniques which were used to construct this semiflow are no longer important.

The reader may object that our semiflow φ^t has no fixed points or invariant sets, since all solutions to (IVP) blow up in finite time, but the previous discussion is also applicable to the rescaled version of the curve shortening equation,

$$K_t = K^2 K_{\theta\theta} + K^3 - \frac{1}{2K},$$

which has the Abresch-Langer functions as fixed points. Thus the analytic semigroup approach could be used to give an alternative construction of the invariant manifolds associated to the Abresch-Langer functions.

Instead of proving the theorem here, we merely state that it follows from one of the various existing theories on "Abstract parabolic initial value problems," which one can find in the publications listed below (the list is surely not complete).

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