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ON TWISTOR SPACES OF THE CLASS $\mathscr{C}$

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0. Introduction

Let M^{2n} be a $2n$-dimensional compact and connected oriented Riemannian manifold, and $Z(M)$ be its twistor space. The M^{2n} for which $Z(M)$ is Kähler are classified, up to conformal equivalence, in [16], [13] for $n = 2$, in [24] for $n \geq 4$ and even, and in [3] for $n \geq 3$. The proofs are mainly differential-geometric.

Y. S. Poon has, however, constructed self-dual metrics on $\mathbb{P}_2(\mathbb{C}) \neq \mathbb{P}_2(\mathbb{C}) = M^4$ for which $Z(M)$ is in Fujiki's class $\mathscr{C}$ (i.e., bimeromorphic to a compact Kähler manifold), but *not* Kähler.

We show here that:

(1) for $n \geq 3$, $Z(M)$ is in $\mathscr{C}$ iff it is Kähler, iff $M^{2n} = S^{2n}$;

(2) for $n = 2$, if $Z(M)$ is in $\mathscr{C}$, then M is either S^4, or *homeomorphic* to the connected sum of $\tau(M) > 0$ copies of $\mathbb{P}_2(\mathbb{C})$.

Apart from well-known facts, the proof consists in showing that if $Z(M)$ is in $\mathscr{C}$, then $\pi_1(M) = \pi_1(Z(M)) = 0$ where π_1 denotes the fundamental group.

This last equality is obtained by purely complex-geometric methods, using the simple-connectedness of the twistor fibers, and the compactness of the Chow scheme of manifolds in $\mathscr{C}$. More precisely, it is possible (see Theorem 2.2) to evaluate $\pi_1(Z)$, for Z in $\mathscr{C}$, from $\pi_1(Y)$ and $\pi_1(A)$ if A and Y are compact connected submanifolds of Z, such that Y has enough "deformations" meeting A in Z. When Y is a smooth rational curve with ample normal bundle in Z (for example, a twistor fiber in $Z(M^4)$), and A is a point on Y, we get, in particular, $\pi_1(Z) = 0$. This extends a former result of J. P. Serre on the fundamental group of a unirational variety.

1. Preliminaries

1.1 Notation. Let X be any irreducible complex analytic space. Then $\pi_1(X) := \pi_1(X, a)$ for some unspecified a in X.

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Let $f : X \to Y$ be a morphism of irreducible analytic spaces. Then $f_* : \pi_1(X) := \pi_1(X, a) \to \pi_1(Y) := \pi_1(Y, f(a))$ denotes the morphism of groups induced by f. If no confusion arises, we denote also by f_* the morphism induced by the restriction of f to any subspace of X.

Let A and B be two irreducible subspaces of X, and let $\alpha : A \to X$ and $\beta : B \to X$, respectively, be the natural inclusions. Let $\mu : B' \to B$ be any modification of B (for example, its normalization or its desingularization). We shall denote by $\langle \pi_1(A), \pi_1(B') \rangle$ the subgroup of $\pi_1(Z)$, generated in $\pi_1(Z)$ by $\alpha_*(\pi_1(A))$ and $(\beta \circ \mu)_*(\pi_1(B'))$.

1.2 Remarks. Let $d : X'' \to X'$ be a desingularization of the normal analytic space X'. Then d_* is surjective, since all fibers of d are connected. However, d_* is not always injective: blow-up the vertex of the cone over an elliptic curve.

Let $n : X' \to x$ be the normalization of X. Then n_* is not always surjective: identify two points in $X' = \mathbb{P}_1(\mathbb{C})$ to obtain X.

1.3 Proposition. *Let $f : X \to Y$ be a proper surjective morphism of irreducible analytic spaces. Assume Y is normal. Then $(f_* \cdot \pi_1(X))$ has finite index in $\pi_1(Y)$.*

Proof. Let $f := h \circ g$, where $g : X \to Y_0$ has connected fibers so that (g_*) is surjective, and $h : Y_0 \to Y$ is finite surjective. We can thus assume that $f = h$ and $Y_0 = X$.

Let Y^* be a dense Zariski open subset of Y over which f is an unramified covering. Let $X^* := f^{-1}(Y^*)$; then $f_*(\pi_1(X^*))$ has finite index in $\pi_1(Y^*)$. The assertion now follows from the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(X^*) & \longrightarrow & \pi_1(X) & & \\ \downarrow & & \downarrow & & \\ \pi_1(Y^*) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1 \end{array}$$

in which the exactness of the bottom line follows from the normality of Y, since any $y \in Y$ has a fundamental basis of (contractible) neighborhoods U in Y such that $U^* := (U \cap X^*)$ is pathwise connected.

1.4 Proposition. *Let $f : X \to S$ be a surjective analytic map between irreducible compact analytic spaces, with S normal and X smooth. Let X_s be a connected component of a smooth fiber $f^{-1}(s)$ of f, and let Y be a compact irreducible analytic subset of X such that $f(Y) = S$. Then $\langle \pi_1(Y), \pi_1(X_s) \rangle$ is a subgroup of finite index in $\pi_1(X)$.*

Proof. Let S^* be a dense Zariski open subset of S over which f is smooth. Let X^* be $f^{-1}(S^*)$, and let $Y^* := (X^* \cap Y)$. The following homotopy sequence provides an exact sequence of groups:

$$\pi_1(X_s) \to \pi_1(X^*) \to \pi_1(S^*).$$

(We may assume by Stein reduction, as in Proposition 1.3, that the fibers of f are connected.) Thus, $\langle \pi_1(X_s), \pi_1(Y^*) \rangle$ has finite index in $\pi_1(X^*)$, and hence in $\pi_1(X)$ since X is smooth. Hence $\langle \pi_1(X_s), \pi_1(Y) \rangle$ has finite index in $\pi_1(X)$, by the functoriality of π_1.

2. The main result

2.0 Notation. All analytic spaces here are reduced. Let Z, A, and S be compact irreducible analytic spaces, where A is a subspace of Z, and S is a subspace of $C(Z)$, the analytic space of compact, pure dimensional, analytic cycles of Z constructed in [2].

Let $G_S \subset S \times Z$ be the *graph* of the universal analytic family (Y_s), $s \in S$, of cycles of Z parametrized by S, and let $p : G_S \to S$ and $q : G_S \to Z$ be the restriction of the natural projections of $S \times Z$. Recall that, set-theoretically, $G_S : \{(s, z)$ s.t. $z \in Y_s\}$. We call $(Y_s)_{s \in S}$ simply the "family S". We say that S is *Z-covering* if q is surjective. Equivalently, this means that any z of Z belongs to at least one member of the family S. Because S is compact and Z is irreducible, it is sufficient to check this condition for z in some open nonempty subset of Z.

Finally, $C(Z)_A$ denotes the closed analytic subset of $C(Z)$ consisting of cycles of Z meeting A. Thus, S is contained in $C(Z)_A$ iff, for any s in S, Y_s meets A.

2.1 Definition. Let (Z, A, S) be as in Notation 2.0. Then Z is said to be *(A, S)-connected* if:

(1) Z is normal,
(2) Y_s is irreducible for s generic in S,
(3) S is contained in $C(Z)_A$,
(4) S is Z-covering.

2.2 Theorem. *Let Z be (A, S)-connected. Let s be generic in S, and $n : Y'_s \to Y_s$ be the normalization of Y_s. Then $\langle \pi_1(A), \pi_1(Y'_s) \rangle$ is of finite index in $\pi_1(Z)$.*

2.3 Remark. In particular, $\langle \pi_1(A), \pi_1(Y_s) \rangle$ and $\langle \pi_1(A), \pi_1(Y''_s) \rangle$ are of finite index in $\pi_1(Z)$ if $d : Y''_s \to Y_s$ is a desingularization of Y_s.

2.4 Corollary. *Let Z be (A, S)-connected. Then the following hold:*

(i) *If $\pi_1(A) = 0$ (in particular, if $A = \{a\}$ is a single point of Z), then $\pi_1(Y'_s)$ is of finite index in $\pi_1(Z)$.*

(ii) *If* $\pi_1(Y'_s) = 0$, *then* $\pi_1(A)$ *is of finite index in* $\pi_1(Z)$.

(iii) *If* $\pi_1(A) = \pi_1(Y'_s) = 0$, *then* $\pi_1(Z)$ *is finite.*

Proof of Theorem 2.2. Let $G \subset S' \times Z$ be the graph of the family S', where $\nu : S' \to S$ is the normalization of S. Let $p_0 : G \to S'$ and $q_0 : G \to Z$ be the natural projections. Let $d : G' \to G$ be a desingularization of G and $p' := (p_0 \circ d)$ (resp. $q' := (q_0 \circ d)$). Remark that G' is connected. Let H be an irreducible component of $(q')^{-1}(A)$ such that $p'(H) = S'$. The existence of H follows from Definition 2.1(3).

By Proposition 1.4, we get that $\langle \pi_1(G'_s), \pi_1(H)\rangle$ has finite index in $\pi_1(G')$ if $G'_s := (q')^{-1}(s)$ is smooth.

Since Z is normal, $(q')_*(\pi_1(G'))$ has finite index in $\pi_1(Z)$ (Proposition (1.4)). Hence $q'_*(\langle \pi_1(G'_s), \pi_1(H)\rangle) = \langle q'_* \cdot \pi_1(G'_s), q'_* \cdot \pi_1(H)\rangle$ has finite index in $\pi_1(Z)$. However, $(q'_* \cdot \pi_1(G'_s)) = (\pi_1(Y'_s))$ in $\pi_1(Z)$, and $(q'_* \cdot \pi_1(H))$ is contained in $\pi_1(A)$. Hence the assertion.

2.5 Remark. Even when $A = (a)$ is a point of Z, and Y_s is smooth for generic s in S, it may happen that $\pi_1(Y_s) \neq \pi_1(Z)$.

Let, for example, C be a genus 2 curve, let $\alpha' : C \to T'$ be its Albanese map, let $\beta : C \to \mathbb{P}_3(\mathbb{C})$ be an embedding, and let $\gamma : T' \to T$ be a degree d isogeny. Also, let $\alpha := (\gamma \circ d)$, let $f : (\alpha \times \beta) : C \to T \times \mathbb{P}_3(\mathbb{C}) := Z$, let a' be any point of C, and let $a := f(a')$. Then $f_* \cdot \pi_1(C)$ has index d in $\pi_1(Z)$, although Z is easily seen to be $(\{\alpha\}, S)$-connected if S is the irreducible component of $C(Z)_{\{\alpha\}}$ containing the point of $C(Z)$ corresponding to $f(C)$.

3. Rationally connected manifolds

3.1 Definition. Let Z be a normal irreducible compact analytic space. Then Z is said to be *rationally connected*, or R.C. for short (resp. *smoothly rationally connected*, or S.R.C. for short), if there exists (A, S) as in Notation 2.0 such that:

(1) Z is (A, S)-connected,

(2) $A = \{\alpha\}$ is a single point of Z,

(3) Y_s is a rational curve (resp. a smooth rational curve) for s generic in S.

3.2 Remarks. (1) It follows from [9, Theorem 3, p. 206, and Remark, p. 208] that Z is Moishezon if Z is rationally connected.

(2) If $f : Z \to Z'$ is surjective (resp. an unramified covering) and Z is R.C. (resp. Z' is S.R.C), then Z' is R.C. (resp. Z is S.R.C.). In

particular, taking $Z = \mathbb{P}_n(\mathbb{C})$, we see that unirational varieties are R.C., and even S.R.C., if smooth.

(3) Z is R.C. iff $Z_1 := Z \times \mathbb{P}_1(\mathbb{C})$ is S.R.C., as one sees by considering the graph of the composite map $\mathbb{P}_1(\mathbb{C}) \to Z$ of the normalization of Y_s, for s generic in S, and of the inclusion of Y_s in Z.

(4) Let Z be smooth and in $\mathscr{C}$. From [17] it follows that Z is S.R.C. (resp. R.C.) iff it contains a smooth rational curve C (resp. a rational curve C) such that NZ_C (resp. $TZ_{|C}$) is ample, where NZ_C (resp. $TZ_{|C}$) is the normal bundle to C in Z (resp. the restriction to C of the tangent bundle of Z).

3.3 Question. Let Z be an R. C. manifold. Is it unirational? Probably not, in general. Observe that the answer is obviously negative if Z is not smooth (take the cone over an elliptic curve).

3.4 Proposition. *Let Z be an R. C. manifold. Then $h^r(Z, \mathscr{O}_z) = 0$ for $r > 0$ where h^r is the dimension of the r th-cohomology group $H^r(Z, \mathscr{O}_Z)$. In particular, the Euler-Poincaré characteristic $\chi(Z, \mathscr{O}_Z) = 1$.*

Proof. Since Z is Moishezon, it is sufficient by Hodge symmetry to show that $h^0(Z, \Omega^r_Z) = 0$ for $r > 0$. Let $p' : G' \to S$ and $q' : G' \to Z$ be as in the proof of Theorem 2.2. Let (s, z) be a smooth point of $G_{S'}$, with s (resp. z) smooth in S (resp. Z), and with $G'_s := q'^{-1}(s)$ smooth and q of maximal rank of (s, z). Let $\omega \in H^0(Z, \Omega^r_z)$, let Δ be any $(r-1)$-dimensional polydisk of S' centered at s, and let u be any nowhere vanishing section of (Ω^{r-1}_Δ). The holomorphic form $[\omega_\Delta/(p')^*u]$ on G'_s thus vanishes identically, since G'_s is a rational curve, for any such choice, where $\omega_\Delta := (q')^*(\omega)_{|(p')^{-1}(\Delta)}$. For some neighborhood U of s in S, there thus exists a section v of (Ω^r_u) such that $(q')^* \cdot \omega = (p')^* \cdot \nu$. Since $d^{-1}(U \times \{a\})$ is mapped to a by q', v and thus ω vanish.

3.5 Theorem. *Let Z be rationally connected. Then $\pi_1(Z) = 0$.*

Proof. We can assume that Z is S.R.C; possibly we replace it by $Z \times \mathbb{P}_1(\mathbb{C})$. Since $\pi_1(Z)$ is finite by 2.2, the universal cover $u : \widetilde{Z} \to Z$ of Z is S.R.C., so $\tilde{\chi} = \chi(\widetilde{Z}, \mathscr{O}_{\widetilde{Z}}) = 1$. On the other hand, $\tilde{\chi}$ is also the degree of the map u by Riemann-Roch.

4. Moishezon twistor spaces

4.1 Notation. Let $M = (M^{2n}, g, +)$ be a compact connected oriented $2n$-dimensional $(n \geq 2)$ Riemannian manifold. Let $\tau : Z(M) \to M$ be its twistor space as constructed in [4] for arbitrary n, and in [1], [5, §14],

[11], [20], [22] for $n = 2$. The almost complex structure of $Z(M)$ is integrable precisely when g is self-dual, if $n = 2$, and g is conformally flat, if $n \geq 3$. The fibers of τ, called twistor fibers of $Z(M)$, are then rational homogeneous manifolds.

4.2 Proposition. *Let $Z_p := \tau^{-1}(p)$ be the reduced twistor fiber of $Z(M)$ above $p \in M^{2n}$. Let $\{Z_p\}$ be the corresponding point of $C(Z(M))$. Then $C(Z(M))$ is smooth and of dimension $2n$ at $\{Z_p\}$.*

Proof. If $n = 2$, this follows from [17], since $Z_p \simeq \mathbb{P}_1(\mathbb{C})$ has a normal bundle in $Z(M)$ isomorphic to $\mathcal{O}(1)^{\oplus 2}$ [1].

If $n \geq 3$, this follows from [24], since $h^0(Z_p, N) = 2n$, where N is the normal bundle of Z_p in $Z(M)$, and since Z_p has a neighborhood in $Z(M)$ analytically isomorphic to a neighborhood of the zero section in N, because M is then conformally flat.

4.3 Definition. Using Proposition 4.2, there exists a unique irreducible component ZM of $C(Z(M))$ containing all $\{Z_p\}$ for p in M. The map $t : M^{2n} \to ZM$ such that $t(p) = \{Z_p\}$ is then a differentiable totally real embedding of M^{2n} in the smooth locus of ZM. We call ZM the *complexification* of M; it has (complex) dimension $2n$, but it is not compact in general (see Theorem 4.5 below).

4.4 Proposition. *Let $p \in M^{2n}$, let $A := Z_p$ for $n \geq 3$, and let $A = \{a\}$ with $a \in Z_p$ for $n = 2$. Let S be the irreducible component of $(ZM)_A := (ZM \cap C(Z(M))_A)$ containing $\{Z_p\}$. Then $Z(M)$ is (A, S)-connected iff S is compact.*

Proof. By the definition of (A, S)-connectedness, we have only to show the "if" part, and so that S is $Z(M)$-covering.

If $n = 2$, this follows immediately from [17].

Assume that $n \geq 3$. It is sufficient to show the assertion when $M^{2n} = S^{2n}$, since M is then conformally flat. We can thus [24] differentiably identify N with $Z_p \times T_pM$, where T_pM is the tangent space to M^{2n} at p, in such a way that for any holomorphic section S of N over Z_p, there exists $(u, v) \in (T_pM)^2$ such that $s(\tau) = u + \tau \cdot v$, where Z_p is identified with the set of complex structures τ on T_pM compatible with both g and $(+)$. Thus s vanishes at τ_0 if $v = \tau_0 u$, and s vanishes somewhere iff $u^2 = g(u, u) = g(v, v) = v^2$ and $u \cdot v = g(u, v) = 0$. From this we get that $s(\tau) = w$ iff there exists h which is g-orthogonal to w and τw, and such that $u = w/2 + h$ and $v = w/2 - h$. The conditions $u + \tau v = w$, $u^2 = v^2$, and $u \cdot v = 0$ are thus always compatible. Hence the assertion.

4.5 Theorem. *Let* $M = (M^{2n}, g, +)$ *be as in Notation* 4.1 *and such that the complex structure of* $Z(M)$ *is integrable. Then the following conditions are equivalent*:

(1) (ZM) *is compact.*

(2) $Z(M)$ *is in Fujiki's class* $\mathscr{C}$ *(i.e., bimeromorphic to some compact Kähler manifold).*

(3) $Z(M)$ *is Moishezon.*

Moreover, in each case, $\pi_1(M) = 0$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are generally true (the last one follows basically from [6]; see [14] or [19].)

We show that (1) implies $\pi_1(M) = 0$. We use the notation of Proposition 4.4. Since $Z(M)$ is (A, S)-connected, and (ZM) is compact, $\pi_1(A) = 0$, $\pi_1(Y_s) = 0$ for s generic in S, and $\pi_1(Z(M)p) = 0$ for all p in M^{2n}, it follows from Theorem 2.2 that $\pi_1(M) = \pi_1(Z(M))$ is finite.

If $n = 2$, $Z(M)$ is then rationally connected, thus Moishezon and with $\pi_1(Z(M)) = 0$. If $n \geq 3$, $\pi_1(M)$ is thus finite.

Let M' be the (Riemannian) universal covering of M; it is conformally equivalent to S^{2n} [18]. Then $Z(M)$ is covered by $Z(M')$ which is rational homogeneous [24], hence rationally connected. Thus $\pi_1(Z(M)) = 0$, and M is conformally equivalent to S^{2n}.

We have thus shown:

4.6 Corollary. *Let* M *be conformally flat. Then the following are equivalent*:

(1) (ZM) *is compact.*

(2) $Z(M)$ *is Moishezon.*

(3) $Z(M)$ *is rational homogeneous (hence projective).*

(4) M *is conformally equivalent to* S^{2n}.

From this we get a purely Riemannian characterization of S^4, relaxing condition $\pi_1(M^4) = 0$ in Kuiper's theorem:

4.7 Corollary. *Let* $M = (M^4, g, +)$ *be conformally flat with* $b_1(M^4) = 0$ *and* g *having positive scalar curvature where* b_1 *denotes the first Betti number. Then* M *is conformally equivalent to* S^4.

Proof. From [7] it follows that $b_2(M^4) = 0$ where b_2 denotes the second Betti number. Since $b_1(M^4) = 0$, we get $\chi(M^4) = 2$ and $\tau(M^4) = 0$. Using [16], $c_1^3(Z(M)) = 16(2\chi(M^4) - 3\tau(M)) > 0$, where c_1 is the first chern class of the tangent bundle, and c_1^3 its third self-intersection. But Corollary 3.8 of [15] and Serre duality show that $h^2(Z(M), K_{Z(M)}^{-m}) = 0$

for $m > 0$. Riemann-Roch now shows that the Kodaira dimension of $K_{Z(M)}^{-1}$ is 3. Hence $Z(M)$ is Moishezon. The result now follows from Corollary 4.6.

4.8 Remark. Easy examples show that the above conditions do not characterize S^m for $m \geq 5$, and that the condition on scalar curvature cannot be removed.

4.9 Corollary. *Assume that* $M = (M^4, g, +)$ *is self-dual and that* $Z(M)$ *is Moishezon. Then either* $M^4 = S^4$ *or* M^4 *is homeomorphic to the connected sum of* $\tau(M) > 0$ *copies of* $\mathbb{P}_2(\mathbb{C})$.

Proof. It is sufficient to show that $b_2^-(M) = 0$ [12], [10] since $\pi_1(M) = 0$. From [16], where $c_i = c_i(Z(M))$, $\chi := \chi(M)$, and $\tau := \tau(M)$, we have $c_1 \cdot c_2 = 12(\chi - \tau)$. By Riemann-Roch we have $c_1 \cdot c_2 = 24 \cdot \chi(Z(M), \mathcal{O}_{Z(M)}) = 24$, since $Z(M)$ is then rationally connected. Hence $\chi = \tau + 2$. On the other hand $b_1(M) = 0$, so we have $\chi = b_2 + 2$. Hence $b_2^-(M) = 0$, as desired.

4.10 Added in proof. Recently, C. Lebrun and then H. Kurke have constructed examples of Moishezon twistor spaces with M^4 a connected sum of an arbitrary number of copies of $\mathbb{P}_2(\mathbb{C})$. As far as the topology of M^4 is concerned, 4.9 is thus optimal. Question: Does 4.9 remain true with "homeomorphic" replaced by "diffeomorphic"?

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