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RIGIDITY AND THE DISTANCE BETWEEN BOUNDARY POINTS

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Introduction

In this paper we consider some rigidity problems in Riemannian geometry. In particular, we prove

Theorem A. *Any complete Riemannian metric without conjugate points on* $\mathbf{R}^n$ *which is isometric to the Euclidean metric outside a compact set must be isometric to the Euclidean metric.*

This was proved in the case $n = 2$ by Green-Gulliver [8] using E. Hopf's theorem [12] that any metric without conjugate points on a 2-torus must be flat. The corresponding question about n-tori is called the E. Hopf conjecture and is still open.

A corresponding theorem is true for hemispheres of the round Euclidean sphere:

Theorem B. *Any Riemannian metric on the open* n*-ball which has no conjugate points, and for which the complement of a compact set is isometric to the complement of a compact set in an open Euclidean hemisphere, must be isometric to the Euclidean hemisphere.*

The proofs of both these theorems rely on results that come from the following "boundary rigidity" problem: Let (M, H, g) and (M_1, H, g_1) be compact Riemannian manifolds with the same (i.e., diffeomorphic) smooth boundary H. The metric g on M induces a distance function d from $H \times H$ to $\mathbf{R}$, i.e., $d(h_1, h_2)$ is the distance in M between h_1 and h_2. For what (M, H, g) is it true that any (M_1, H, g_1) with $d = d_1$ must have g isometric to g_1? Such an (M, H, g) will be called boundary rigid.

This problem was considered previously by R. Michel [14] and M. Gromov [9]. They have shown that any compact subdomain of $\mathbf{R}^n$, any compact subdomain of an open n-dimensional hemisphere, and any compact subdomain of the hyperbolic plane are boundary rigid (see [9, §5.5B]). In

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fact they show that M needs only admit an isometric immersion into the same dimensional Euclidean space.

In this paper a geodesic segment of M will refer to a segment of a geodesic which lies in the interior of M except possibly for the endpoints (i.e., we do not include grazing geodesics). A segment is said to minimize if its length is the distance between the endpoints, and to strongly minimize if it is the unique such path. By a subdomain we will mean an open set with a smooth boundary.

One cannot expect that all compact manifolds with boundary will be boundary rigid. In §2 we will see examples on surfaces of revolution which will lead us to consider only those (M, H, g) for which a certain condition SGM (Strong Geodesic Minimizing) on the boundary distance function, d, holds. The precise definition of SGM is given in §2 but loosely speaking it means that all geodesic segments are strongly minimizing. Since SGM is a condition on d, if (M, H, g) is SGM and (M_1, H, g_1) has $d_1 = d$ then (M_1, H, g_1) is also SGM. Examples of such M are given by compact subdomains of an open ball B in a Riemannian manifold where all geodesics segments in B are assumed to minimize.

This problem is closely related to other natural problems. One is the uniqueness of (geodesic) lenses. In this problem (M, H, g) is said to be equivalent to (M_1, H, g_1) as lenses if for each geodesic γ entering M the corresponding geodesic γ_1 of M_1 (it enters at the same point of H and makes corresponding angle) exits at the same point of H as γ, making a corresponding angle, after the same amount of time (i.e., γ and γ_1 have the same length). The examples of §2 also lead one to consider this problem in the SGM case. We will see in §1 that if (M, H, g) is SGM, then lens rigidity is equivalent to boundary rigidity.

Another related problem is the special case where M and M_1 are assumed to be diffeomorphic and the metrics pointwise conformal, i.e., $g_1 = f^2 \circ g$, where f is a positive function on M_1. This problem is sometimes referred to as the uniqueness part of the inverse kinematic problem of seismology. That problem is to determine the density (f above) of an object (say the earth) if one knows for each p and q on the boundary the time it takes for a wave started at p to be felt at q ($d(p, q)$ above). In this paper we prove uniqueness:

Theorem C. *Let (M, H, g) be a compact Riemannian manifold with boundary which is SGM. Then if $g_1 = f^2 \cdot g$ is a metric such that $d_1 = d$, then $f(x) = 1$ for all $x \in M$, and hence $g_1 = g$.*

Earlier work on this problem in two dimensions can be seen in [19], [16]. In n dimensions, G. Beylkin [4] has recently proved such a theorem under the stronger assumption that (M, H, g) is convex. He was also able to prove a stability result in that setting.

The paper will be organized as follows: In §1 we give a precise definition of SGM and show how it relates to the loose definition. In §2 on surfaces of revolution we consider examples which show that the assumptions of Theorem C are indeed necessary and that SGM is the right condition to consider for this problem. In §3 we prove Theorem C. In §4 we prove Theorems A and B. In §5 we take up the general question of boundary rigidity reviewing the known facts and developing basic properties. In §6 we will give a new proof that subdomains of $\mathbf{R}^n$ are boundary rigid, and we consider the problem of showing that compact subdomains of a hyperbolic n-space are boundary rigid. Although we cannot yet solve this problem, we will show that such subdomains are rigid if we only consider metrics of negative definite curvature operator (in particular among all nearby metrics). More precisely, we prove

Theorem D. *Let (M, H, g) be a compact subdomain of a hyperbolic n-space, and (M_1, H, g_1) be a metric having negative definite curvature operator (no curvature assumption if $n = 2$). If $d = d_1$, then the spaces are isometric.*

§7 is an appendix where we deal with some differentiability questions left unresolved in earlier sections and papers.

Some recent results on related problems can be found in [18].

Since writing this paper, significant progress was made in the two-dimensional nonpositive curvature case (see [5], [17]). Earlier work on this can be found in [7].

1. Preliminaries

We first give precise definitions of BGM (boundary geodesics minimize) and SGM and then will show how they correspond to our loose definitions.

1.1. Definitions. The class that we will call BGM was considered in [9] as the class of manifolds (M, H, g) satisfying the following: For every $q \in H$ and every tangent vector $v \in T_qH$ with $\|v\| < 1$ there exists a unique point $q_1 \in H$ satisfying the following two conditions:

(i) $\nabla_q d(q_1, \cdot) = v$, where $d(q_1, \cdot): H \to \mathbf{R}$ and ∇_q is the gradient at q,

(ii) there is no point $q_2 \in H$ other than q and q_1 such that $d(q, q_2) + d(q_2, q_1) = d(q, q_1)$.

The condition SGM will be BGM plus the condition that there is no complete, possibly grazing, geodesic in M. (By a grazing geodesic we mean a smooth curve with 0 geodesic curvature which may intersect the boundary in points or segments.) We must state this in terms of d alone. A differentiable curve γ in H from a point p to a point q will be called a straight segment if the length of γ is $d(p, q)$ and, further, there is a sequence $v_i \in T_pH$ converging to the unit vector tangent to γ with $\|v_i\| < 1$ such that the q_i determined from the definition of BGM converge to a point z with $d(z, q) + d(q, p) = d(z, p)$. (This means that γ is a geodesic segment of M as well as a geodesic segment of H.) We say that p and q in H are straight connected if $\{r \in H: d(p, r)+d(r, q) = d(p, q)\}$ is a countable union of straight segments and points (this means that a length minimizing path in M from p to q is a possibly grazing geodesic).

(M, H, g) will be called SGM if it is BGM and, further, there do not exist points p_i, $i \in \mathbf{Z}$, such that p_i is straight connected to p_{i+1}, $\|\nabla_{p_i} d(p_{i-1}, \cdot)\| = 1$, and $\nabla_{p_i} d(p_{i-1}, \cdot) = -\nabla_{p_i} d(p_{i+1}, \cdot)$ and $\sum_0^\infty d(p_i, p_{i+1})$ and $\sum_0^{-\infty} d(p_i, p_{i+1})$ are infinite.

We will now show how these correspond to our loose definition. If γ is a minimizing geodesic from p to q, where p and q are such that $\nabla_q d(p, \cdot)$ exists, then $\gamma'(q)$ must be the unique unit vector at q whose projection onto T_qH is $\nabla_q d(p, \cdot)$. This is true since $\gamma'(q)$ is the gradient of $d(p, \cdot)$ as a function on M (if the gradient exists). Let $U^+H \to H$ represent the bundle of unit vectors u tangent to M at a boundary point, say q, such that $\langle u, N_q \rangle > 0$, where N_q is the inward normal. For given $u \in U^+H$ let v be the orthogonal projection onto T_qH. This gives a one-to-one correspondence between U^+H and the $v \in TH$ with $\|v\| < 1$. If M satisfies BGM, for given u let q_1 be the point corresponding to v. Then by (ii) the minimizing path γ between q and q_1 must be a geodesic segment. Property (i) means that γ is γ_u (the geodesic determined by u) and hence that γ_u is minimizing. By continuity, γ_u will be minimizing even if u is tangent to H. If such a geodesic is not tangent to H at either endpoint, then condition (i) (used from both end points) implies that it is strongly minimizing. On the other hand if all geodesics from the boundary strongly minimize, then for given v let q_1 be the first point on H hit by γ_u.

In the next section on surfaces of revolution we will see examples that satisfy BGM and are not boundary rigid even in their conformal class. These examples however have geodesic segments that are not minimizing, because there are complete geodesics that never hit the boundary. By taking limits of such geodesics we see that this can only happen if there is a complete "grazing" geodesic, i.e., a geodesic that grazes the boundary infinitely often and that minimizes between points of contact. The extra condition in SGM guarantees that this does not happen, so, in particular, all geodesic segments strongly minimize. It is not hard to construct a metric where all geodesics strongly minimize, but which has such a complete grazing geodesic, so is not SGM.

For given $u \in U^+H$ the above arguments show that in the SGM case (or even BGM) d determines the first point (i.e., q_1 above) that the geodesic γ_u hits on H. If (M_1, H, g_1) is such that $d = d_1$, then the corresponding geodesic γ_{1u} also must exit at q_1. Since there is a unique geodesic between q and q_1 in both spaces, reversing the roles of $u = \gamma'(q)$ and $-\gamma'(q_1)$ we see:

1.2. Lemma. *If M is SGM, then the corresponding geodesics γ_u and γ_{1u} have the same length l, and exit H at the same point and angle, i.e., $-\gamma_u'(l) \in U^+H$ corresponds to $-\gamma_{1u}'(1) \in U_1^+H$. In other words, M and M_1 are equivalent as geodesic lenses.*

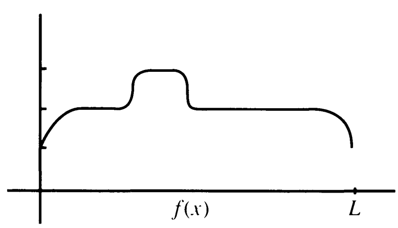
2. Examples on surfaces of revolution

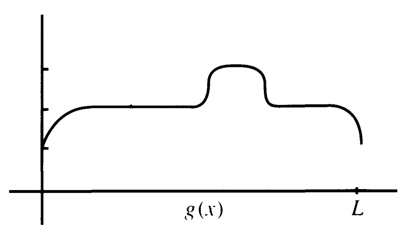
It is easy to see that (M, H, g) is not boundary rigid if for some $x \in M$ there is no minimizing geodesic segment between boundary points that passes through x. For then one could change the metric or even the topology near x without changing d. In this section we will consider some examples that show even the existence of an open set worth of such geodesics through each x is not enough to give boundary rigidity even in the conformal case. The examples show that a reasonable class of manifolds to consider for boundary rigidity as well as Theorem C is the SGM class.

2.1. Examples. We will consider surfaces of revolution obtained by revolving a generating curve $y = f(x)$ around the x-axis. We will refer to the obvious coordinates x and θ. We will assume that $f(0) = f(L) = 1$, $f(x) > 1$ for $x \in (0, L)$, and $f'(0) \neq 0$ and $f'(L) \neq 0$. We note that for any choice of f every point will have an open set of geodesics passing through it, going from one boundary point to another and achieving the distance between them. We now consider a geodesic $\gamma(s) = (x(s), \theta(s))$ with $x(0) = \theta(0) = 0$, which makes an angle φ with the boundary. Our assumptions on f guarantee that the x-component will be monotonic and that when $x = L$ the angle with the boundary will again be φ. By integrating Clairaut's relation (see [6]) we see that when $x = L$, θ will then be

$$\cos(\varphi) \cdot \int_0^L \frac{1}{f(x)} \sqrt{\frac{f'(x)^2 + 1}{f'(x)^2 - \cos^2(\varphi)}}\, dx.$$

Hence if f and g are functions such that the above integrals are the same for all φ, then corresponding geodesics will exit at corresponding points with the same angles and hence by the first variation formula will have the same length up to a constant which will be 0 if the meridian geodesics have the same length. A particular choice of such f and g is illustrated by the figures. In fact we see that there is a one-parameter family of such functions giving rise to a one-parameter family of nonisometric metrics with the same boundary distance functions. Further, it is not hard to see that these metrics are conformal to each other via a diffeomorphism that leaves the boundary fixed. Hence, in particular, Theorem C is not true if we only assume that for each $x \in M$ an open set of geodesics passing through x are geodesics that minimize from one boundary point to another. Note that by choosing f and g symmetric about $x = L/2$ we can make such examples on the Möbius band, hence with only one boundary component.





Surfaces of revolution as above never satisfy condition SGM since the curve $x = c$ is a closed geodesic, where c is the maximum point of f. On the other hand it is easy to construct such surfaces that satisfy BGM. In fact take any such surface and then by adding a sufficiently large constant to f one can make the new metric satisfy BGM (by scaling this metric we could again make the minimum value of $f = 1$, of course changing the value of L). Thus not only are the conditions SGM and BGM different but there are conformal nonisometric metrics satisfying BGM and having the same boundary distance function.

Finally we note that if (M, H, g) represents the standard metric on the hemisphere, then all geodesic segments minimize but not strongly. But in this case any metric $g_1 = f^2 \cdot g$, where f is 1 near the boundary and

≥ 1 in the interior, will have $d_1 = d$. In particular the above example shows that the condition GM, "all geodesics minimize", is not a condition on d as is the condition SGM.

2.2. Question. If (M, H, g) is SGM is it boundary rigid?

In §5 we will return to this general question.

3. Proof of Theorem C

In this section we prove Theorem C. In order to do this we will introduce some notation which will also be useful in later sections. Let UM represent the unit sphere bundle over the interior of M endowed with its usual (local product measure) du. Let $U^+H \to H$ represent the bundle of unit vectors v tangent to M at a boundary point, say q, such that $\langle v, N_q\rangle \geq 0$, where N_q is the inward normal. Endow U^+H with the standard measure dv (again a local product measure where the measure on the fibre is the measure of a hemisphere). For $u \in U^+H$ let $l(v)$ be the first value of $t > 0$ such that $\gamma_v(t) \in H$, where γ_v is the unit speed geodesic with initial tangent v. We will consider the subspace Q of $U^+H \times \mathbf{R}$ given by $\{(v, t)|0 < t < l(v)\}$. There is a natural map $Z: Q \to UM$ given by $(v, t) \to \gamma_v'(t)$ (the geodesic flow). Santaló's formula (see [20, pp. 336–338]) states that Z is measure preserving when Q is given the measure $\langle v, N_q\rangle\, dv\, dt$.

Proof of Theorem C. For u a unit vector with respect to g, its norm $\|u\|_1$ with respect to g_1 is $f(x)$,w here x is the base point of u. Integrating this over all $u \in UM$ and applying Santaló's formula, we obtain

$$\alpha(n-1)\int_M f(x)\,dx = \int_{UM} \|u\|_1\,du = \int_{U^+H}\int_0^{l(v)} \|\gamma_v(t)\|_1\,dt\langle v, N_q\rangle\,dv,$$

where $\alpha(n-1)$ represents the measure of the standard unit $(n-1)$-sphere and n is the dimension of M. Now γ_v is a minimizing geodesic in the metric g from a boundary point $q_0 = \gamma_v(0)$ to a boundary point $q_1 = \gamma_v(l(v))$. In the metric g_1 it is still a curve from q_0 to q_1 and hence, since $d = d_1$, has length $\geq l(v)$. Hence

$$\begin{aligned}\alpha(n-1)\int_M f(x)\,dx \geq \int_{U^+H} l(v)\langle v, N_q\rangle\,dv &= \int_{UM} 1\,du \\ &= \alpha(n-1)\mathrm{Vol}(M, g).\end{aligned}$$

Now by a Hölder inequality we have

$$\begin{aligned}&\mathrm{Vol}(M, g_1)^{1/n}\cdot \mathrm{Vol}(M, g)^{(n-1)/n}\\ &\quad = \left\{\int_M f(x)^n\,dx\right\}^{1/n}\left\{\int_M dx\right\}^{(n-1)/n} \geq \int_M f(x)\,dx\end{aligned}$$

with equality holding if and only if $f(x)$ is constant. Hence we conclude that $\mathrm{Vol}(M, g_1) \geq \mathrm{Vol}(M, g)$ with equality holding if and only if $f(x) = 1$ for all x. Since SGM is defined in terms of d alone, g_1 is also SGM. Hence the theorem follows by reversing the roles of g and g_1.

We will see in §5 that, in the general boundary rigidity case, $\mathrm{Vol}(M_1) = \mathrm{Vol}(M)$ so we could have used this instead of reversing the roles above.

4. Proofs of Theorems A and B

In this section we prove Theorems A and B.

4.1. Proof of Theorem A. By assumption there is a compact set K in $(\mathbf{R}^n, g)$ such that $(\mathbf{R}^n - K, g)$ is isometric to $(\mathbf{R}^n - B(r), g_0)$ for some $r > 0$, where g_0 is the standard Euclidean metric and $B(r)$ is the metric ball of radius r. We will prove that K is isometric to $B(r)$ by showing that $d = d_0$, where $d\colon \partial K \times \partial K \to \mathbf{R}$ is the boundary distance function of the metric on K, and d_0 is the corresponding one for $B(r)$. The theorem will then follow since $B(r)$ is boundary rigid (see [9] or §4).

To prove $d = d_0$ choose p and q in $\partial K = \partial B(r)$. We first show that $d_0(p, q) \leq d(p, q)$. Let $l(t)$ be a line in $\mathbf{R}^n$ parametrized by arclength which does not pass through $B(r)$ but is parallel to and has the same orientation as the line from p to q. $l(t)$ is also a geodesic in g and hence, since g has no conjugate points and is simply connected, l minimizes between any two points on it. In particular for all large t,

$$\begin{aligned}2t &= d_0(l(-t), l(t)) = d(l(-t), l(t))\\ &\leq d(l(-t), p) + d(p, q) + d(q, l(t)).\end{aligned}$$

But for large t the line segment from p to $l(-t)$ (resp. q to $l(t)$) does not intersect $B(r)$ and hence $d(l(-t), p) = d_0(1(-t), p)$ (resp. $d(q, l(t)) = d_0(q, l(t))$). Thus we see $2t - d_0(l(-t), p) - d_0(l(t), q) \leq d(p, q)$. Taking the limit as t goes to ∞ we get $d_0(p, q) \leq d(p, q)$.

To prove the opposite inequality let $\gamma(t)$ be the unit speed geodesic from p to q, i.e., $\gamma(0) = p$ and $\gamma(l) = q$. By our assumptions $\gamma((-\infty, 0))$ and $\gamma((l, \infty))$ are straight lines in $\mathbf{R}^n - B(r)$. It is easy to see that they are parallel (although not necessarily segments of the same straight line) for if not, for large values of t, the line between $\gamma(-t)$ and $\gamma(t + l)$ would not intersect $B(r)$ and hence yield a different geodesic in g between $\gamma(-t)$ and $\gamma(t+l)$ which is not possible since there are not conjugate points. Choose any point z in $\mathbf{R}^n - B(r)$. We thus have

$$2t + l = d(\gamma(-t), \gamma(t+l)) \leq d(\gamma(-t), z) + d(\gamma(t+l), z).$$

Again for large values of t and appropriate choice of z the line segments from z to $\gamma(-t)$ and $\gamma(t+l)$ do not intersect $B(r)$. Hence we get $d(p, q) = l \leq d_0(z, \gamma(-t)) + d_0(z, \gamma(t)) - 2t$. Now letting t to ∞ (and doing a little Euclidean geometry) we get $d(p, q) \leq d_0(p, q)$ which yields the theorem.

4.2. Proof of Theorem B. The proof of Theorem B is similar to that of Theorem A. We will let $(B(\pi/2), g_0)$ represent the hemisphere with the standard metric g_0 (since it is the ball of radius $\pi/2$ about the north pole). Our metric without conjugate points is a metric $(B(\pi/2), g)$ such that $g = g_0$ on $B(\pi/2) - B(r)$ for some $0 < r < \pi/2$, and such that any p and q in the interior are not conjugate to each other.

We must first prove that all geodesics in the metric g are minimizing. Since near the boundary the metric is the standard metric it is easy to see that a minimum length path between any two points must in fact be a geodesic. Hence it is sufficient to prove that for any pair of points p, q in the interior there is a unique geodesic between them. We may assume that p and q both lie in a ball $B(R)$ for $r < R < \pi/2$, and that σ and τ are geodesics from p to q (parametrized on $[0, 1]$). Since any geodesic leaving $B(R)$ hits $\partial B(\pi/2)$ and hence never returns, σ and τ both lie in $B(R)$. Since $B(R)$ is simply connected, σ is homotopic to τ through curves in $B(R)$. Since there are no conjugate points and any geodesic between points in $B(R)$ lies in $B(R)$, there is no obstruction to uniquely lifting this homotopy to $T_pB(R)$. This yields a homotopy from the line segment between 0 and $\sigma(0)$ to the line segment between 0 and $\tau'(0)$ which must leave the point $\sigma'(0)$ fixed. This is a contradiction unless $\sigma = \tau$. Hence all geodesics in the metric g are minimizing.

Let p and q be on the boundary $\partial B(r)$ and let $c: [0, \pi] \to B(\pi/2)$ be the great circle such that $c(t_1) = p$ and $c(t_2) = q$ with $t_1 < t_2$. Choose $\bar{c}$ to be another great circle such that $\bar{c}(0) = c(0)$, $\bar{c}(\pi) = c(\pi)$, and $\bar{c}[0, \pi]$ lies in $B(\pi/2) - B(r)$. Then, by our assumptions, $\bar{c}[0, \pi)$, $c[0, t_1]$, and $c[t_2, \pi]$ are minimizing geodesics in g. Hence we have

$$\begin{aligned}\pi = d(\bar{c}(0), \bar{c}(\pi)) &\leq d(c(0), c(t_1)) + d(p, q) + d(c(t_2), c(\pi)) \\ &= \pi + (t_1 - t_2) + d(p, q),\end{aligned}$$

which implies that $d_0(p, q) \leq d(p, q)$.

Now let $\gamma\colon [0, l] \to B(\pi/2)$ be the unit speed geodesic in g such that $\gamma(0)$ and $\gamma(l)$ are in the boundary $\partial B(\pi/2)$, $\gamma(t_1) = p$, and $\gamma(t_2) = q$ where $0 < t_1 < t_2 < l$. Since $\gamma[0, t_1]$ and $\gamma[t_2, l]$ are geodesic segments in g_0, we get

$$\begin{aligned} t_1 + d(p, q) + (l - t_2) &= d(\gamma(0), \gamma(l)) \\ &= d_0(\gamma(0), \gamma(l)) \le t_1 + d_0(p, q) + (l - t_2), \end{aligned}$$

i.e., $d(p, q) \le d_0(p, q)$. Hence $d(p, q) = d_0(p, q)$.

The theorem follows from the fact that subdomains of hemispheres are boundary rigid as was shown in [14], [9].

4.3. Remark. The proof of boundary rigidity for a domain M in a hemisphere goes as follows (see [14] or [9]]): cut M out of the hemisphere and glue in the manifold M_1 with $d_1 = d$. Doing a corresponding gluing on the opposite hemisphere yields an antipodally symmetric metric on a sphere. Using Lemma 1.2 it is not hard to see that this is a Blaschke sphere. The isometry follows from the proof of the Blaschke conjecture for spheres (see [3, Appendices D and E], or [2]). This argument works without problems in our case above since our gluing is assumed to be smooth. In [14] it was shown that if M is convex then the metric will be C^2, and again there is no problem. In the general case one must show that the proof of the Blaschke conjecture works for these more general metrics. We will discuss this in Remarks 5.3 and 7.3.

In §6 we will give a proof of boundary rigidity of subdomains of $\mathbf{R}^n$ (different from the one in [9]) which is sufficient to prove rigidity in Theorem A. In the general case we also must worry somewhat about differentiability questions, which we do in the appendix.

4.4. Remark. The above arguments apply to a class of metrics on $\mathbf{R}^n$, which includes the universal covering spaces of tori without conjugate points, satisfying strong conditions on the Busemann functions. They yield the property that if g and g_1 are two such metrics that agree in the complement of a compact set K, then the distance functions d and d_1 agree on ∂K. Hence if K (which is SGM) is boundary rigid then $g = g_1$.

5. General boundary rigidity

In this section we will discuss the general boundary rigidity question 2.2. Most of the ideas in this section were known to Gromov and Michel (see [9, §5.5] [14]) in different settings. However some modifications are needed and so we will present them in our setting.

Let (M, H, g) satisfy SGM. If (M_1, H, g_1) is another manifold such that $d_1 = d$, then it is clear that the induced Riemannian metrics $g_{|H}$ and $g_{1|H}$ are the same since they are the length space metrics induced from d and d_1 respectively. The obvious identification of the bundles U^+H and U_1^+H preserves the standard measures. Since all geodesics hit the boundary, we can label them by their initial vector $v \in U^+H$. We can thus define a map Γ from the unit sphere bundle UM of M to the unit sphere bundle UM_1 of M_1 which preserves the geodesic flow as follows. For $u \in UM$ there is a unique $v \in U^+H$ such that $u = \gamma_v'(t)$ for some t and $\gamma_v(s) \notin H$ for all $0 < s < t$. We let $\Gamma(u)$ be $\gamma_{1v}'(t)$. The fact that γ_v has the same length as γ_{1v} means that $\gamma_{1v}'(t)$ is defined. The map Γ is the composition of the inverse Z^{-1} of the map $Z: Q \to UM$ (as defined in §3) and the map $Z_1: Q_1 \to UM_1$. These maps are measure preserving by Santaló's formula if we take the measure on $U^+H \times \mathbf{R}$ to be $\langle v, N_q \rangle\, du dt$ and hence Γ is measure preserving. The map Γ is continuous even though Z^{-1} may not be. Z^{-1} may be discontinuous at u when v is tangent to H. So the possible discontinuities occur on grazing geodesics. But it is not hard to see that grazing geodesics in M correspond naturally to ones in M_1, and hence that Γ is continuous. In conclusion we have

5.1. Lemma. *If M satisfies SGM, then any M_1 with $d = d_1$ will have the same volume as M and will have a homeomorphic unit tangent bundle.*

5.2. Remark. It is still unknown in general if M_1 itself must be homeomorphic to M. However, this will be the case if M can be seen as a subdomain of an open metric ball $B(x, r)$ in a manifold N with center x which is not in M and of radius r less than the injectivity radius of N. In particular this is true for subdomains of complete simply connected manifolds without conjugate points. To see this note that M is diffeomorphic via the exponential map to a subdomain D in T_xN. If we let B_1 be the manifold obtained by cutting M out of B and replacing it with M_1, then D will be diffeomorphic to M via the exponential map on M_1. The only question is whether two geodesics emanating from x might intersect in M_1 (or whether there might be a conjugate point), but then they cannot both be minimizing when they exit from M_1, which they must be since $d_1 = d$.

5.3. Remark. The construction above of cutting M out of a manifold N and replacing it with M_1 clearly yields a manifold $\overline{N}$ with a metric $\overline{g}$ which is smooth except possibly for points on H. In [14] Michel shows that $\overline{g}$ will be C^2 at points of H which are locally convex towards M. In particular $\overline{g}$ will be C^2 whenever M is convex. We however are interested in the case where M is SGM. Of course a positive answer to Question 2.2 would imply that $\overline{g}$ is smooth in this case. It is clear that $\overline{g}$ is C^0 and that geodesics are C^1 curves. Further, geodesic segments in g between points in $N - M$ will be minimizing if and only if the corresponding geodesic in g is. We will call these metrics "almost C^2" since many global rigidity theorems (especially those proved using integral geometry) will hold for these somewhat more general metrics (since almost all geodesics intersect H transversely). In particular, Hopf's theorem [12] on 2-tori without conjugate points, the Blaschke conjecture for spheres [2], Berger's isoembolic inequality [1], Katok's rigidity theorem [13], and Guillemin and Kazhdan's deformation rigidity theorems [10], [11] as well as Min-Oo's generalization [15] should be true for these more general metrics. In [14] boundary rigidity was stated and proved for convex subdomains of the round sphere, while in [9] it was claimed for all subdomains of a convex subset of a round sphere. The proof uses the above construction and then applies Berger's isoembolic inequality (or the Blaschke conjecture). In the appendix we show why these theorems still hold in the almost C^2 case. The proof in [9] for subdomains of the hyperbolic plane uses Katok's theorem along with the above construction. In [14] a deformation rigidity theorem is proved for convex M with pinched negative curvature by using the above construction along with Guillemin and Kazhdan's theorems. This theorem is also true in the SGM case. In the appendix we will discuss this result in the almost C^2 setting.

5.4. Remark. If the map Γ covers a map D on the base, then D must in fact be an isometry. To see this let $x, y \in M$ be such that there is a minimizing geodesic between them. Let γ_v be the geodesic with $\gamma(t_1) = x$, $\gamma(t_2) = y$, and $v \in U^+H$. Then since Γ covers D, we see that $D(x) = \gamma_{1v}(t_1)$ and $D(y) = \gamma_{1v}(t_2)$, and hence the distance in M between x and y is the same as the distance between $D(x)$ and $D(y)$ in M_1, and thus D is an isometry. Hence to answer the question one needs only show that Γ covers a map D.

The above remark leads one to consider the following special case of Question 2.2.

5.5. Question. Let (B, H, g) be a Riemannian metric ball of radius r which is SGM. Must any (B_1, H, g_1) with $d_1 = d$ also be a Riemannian metric ball of radius r?

If the answer to 5.5 is yes (when B_1 has an almost C^2 metric as in 5.3 above), then so is the answer to 2.2 in many cases. In particular, if M is a compact subdomain of a complete simply connected manifold without conjugate points and $x \in M$, choose r so large that M is contained in the ball B about x of radius r. Let B_1 be the manifold obtained by cutting M out of B and replacing it with M_1 yielding that $d_1 = d$ and hence that B_1 is a metric ball with center $x_1 \in M_1$. It is not hard to see that Γ takes vectors at x to vectors at x_1. Hence Γ covers a map on the base, and M_1 is isometric to M.

6. The $\mathbf{R}^n$ and H^n cases

In this section we consider subdomains of $\mathbf{R}^n$ and H^n (hyperbolic n-space). We give a new proof of boundary rigidity for subdomains of Euclidean space having the flavor of the proof of the Blaschke conjecture ([3] or [2]), and we prove Theorem D using the deformation rigidity theorem of Michel [14].

We first consider some general results in the case where (M, H, g) is a subdomain of a complete simply connected manifold without conjugate points N, and (M_1, H, g_1) is a smooth Riemannian manifold such that $d = d_1$. Remark 5.2 states that M is diffeomorphic to M_1, and 5.3 states that if M is replaced by M_1 in N, then the resulting metric $\overline{g}$ is almost C^2 on N without conjugate points since all geodesics still minimize. In this section we will prove our results under the assumption that $\overline{g}$ is C^2. This is good enough for applications such as Theorem A or for convex M (see [14]). In the appendix we will discuss how to modify the proofs in the general case.

We now discuss Jacobi fields of $\overline{g}$ along a geodesic $\overline{\gamma}$. Let $\gamma(t)$ be the corresponding geodesic of g. For t where $\gamma(t) \notin M$, we have $\overline{\gamma}(t) \notin M$ and $\gamma(t) = \overline{\gamma}(t)$. As pointed out in [14], Lemma 1.2 states that if $\overline{J}(t)$ is a Jacobi field of $\overline{g}$ along $\overline{\gamma}(t)$, then there is a Jacobi field $J(t)$ of g along $\gamma(t)$ such that $\overline{J}(t) = J(t)$ when $\gamma(t) \notin M$. Assume that $\gamma(0) = \overline{\gamma}(0) \notin M$, and let $X_1, X_2, \cdots, X_{n-1}$ along γ and $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_{n-1}$ along $\overline{\gamma}$ be parallel orthonormal vector fields perpendicular to γ and $\overline{\gamma}$ respectively and $X_i(0) = \overline{X}_i(0)$. At t such that $\gamma(t) \notin M$ there is an orthogonal transformation taking $X_i(t)$ to $\overline{X}_i(t)$. Let $O(t)$ be the matrix such that $\overline{J}(t) = O(t)J(t)$ when $\overline{J}(t)$ is written as a column vector with respect to the $\overline{X}_i$, and $J(t)$ with respect to the X_i. We will consider matrix solutions $A(t)$ to the resolvent Jacobi equation $Z''(t) + R(t) \cdot Z(t) = 0$, where $R_{ij}(t) = \langle R(\gamma'(t), X_i(t))\gamma'(t), X_j(t)\rangle$, and the corresponding solutions $\overline{A}(t)$ along $\overline{\gamma}$. The above discussion about Jacobi fields states that if $A(0) = \overline{A}(0)$ and $A'(0) = \overline{A}'(0)$ then, when

$\gamma(t) \notin M$, $\overline{A}(t) = O(t)A(t)$. In particular let $A(t)$ be a solution such that $A'(t)A^{-1}(t)$ is symmetric for some $t = t_0$ and hence at all t since R is symmetric. Then

$$(6.1) \qquad B^*(t) = A(0) \cdot \left\{ \int_0^t A^{-1}(x) A^{*-1}(x)\, dx \right\} \cdot A^*(t)$$

defines the solution with $B(0) = 0$ and $B'(0) = I$ as long as $A(x)$ is nonsingular for $x \in [0, t]$ (see, for example, [3, Appendix D]). A similar formula relates the corresponding solutions $\overline{A}$ and $\overline{B}$ and hence for all t such that $\gamma(t) \notin M$:

$$(6.2) \qquad \int_0^t A^{-1}(x) A^{-1*}(x)\, dx = \int_0^t \overline{A}^{-1}(x) \overline{A}^{-1*}(x)\, dx.$$

Using the convexity of $X \to (\mathrm{Det}(X))^{-1/2}$ on the space of positive definite symmetric matrices as in [3, Appendix D], we have

$$(6.3) \quad \mathrm{Det}\left\{ \int_0^t A^{-1}(x) A^{-1*}(x)\, dx \right\} \geq \left\{ \int_0^1 \mathrm{Det}(A(x))^{-2/(n-1)}\, dx \right\}^{n-1}$$

with equality holding if $A(x)$ is diagonal for each $x \in [0, 1]$.

In [9, §§6.1, 2.1, and 4.1] Gromov proves that compact subdomains of a Euclidean space are boundary rigid by showing that the "filling volume" of the boundary is greater than or equal to the volume of the domain. The part that yields the rigidity is that equality in the filing volume estimate implies flatness. In this section we give a more analytic proof having the flavor of the proof of the S^n case i.e., the proof of the Blaschke conjecture for spheres.

6.4. Theorem. *Compact subdomains of the Euclidean metric on* $\mathbf{R}^n$ *are boundary rigid.*

Proof. Let (D, H, g) be a compact subdomain of a Euclidean n-space, and let (M_1, H, g_1) be a Riemannian manifold such that $d_1 = d$ on H. Replacing D by M_1 we get a metric $\overline{g}$ on $\mathbf{R}^n$ which is Euclidean outside a compact set K and has no conjugate points (we assume for now that $\overline{g}$ is C^2). We may assume that K is contained in the interior of the cube $C = \{(x_1, x_2, \cdots, x_n) | |x_i| \leq R\}$ for some R. By 5.1 the volume of C in $\overline{g}$ is $(2R)^n$. Let F be the face of C given by $x_n = -R$ and for each $p \in F$ let $\overline{\gamma}_p$ be the geodesic on $\overline{g}$ from p perpendicular to F. By Lemma 1.2, $\overline{\gamma}_p$ has length $2R$. Let $\overline{A}_p(t)$ be the solution to the Jacobi equation such that $\overline{A}_p(0) = I$ and $\overline{A}'_p(0) = 0$. Then the volume of C is given by

$$\int_F \int_0^{2R} \operatorname{Det}(\overline{A}_p(x))\, dx\, dp.$$

For any positive function $f(x)$ the Hölder inequality implies:

$$\int_a^b f(x)\, dx \geq (b-a)^{(n+1)/2} \left\{ \int_a^b f(x)^{-2/(n-1)}\, dx \right\}^{-(n-1)/2}$$

with equality holding if and only if $f(x)$ is a constant function. Applying this to $\operatorname{Det}(\overline{A}_p(x))$ yields

$$\operatorname{Vol}(C) \geq \int_F (2R)^{(n+1)/2} \left\{ \int_0^{2R} \operatorname{Det}(\overline{A}_p(x))^{-2/(n-1)}\, dx \right\}^{-(n-1)/2} dp$$

with equality holding if and only if $\operatorname{Det}(\overline{A}_p(x))$ is constant hence equal to 1. $\overline{A}(x)$ is a nonsingular matrix for all x since for each x the geodesics $\overline{\gamma}_p$ minimize the distance to the submanifold F and hence there are no focal points to F along $\overline{\gamma}_p$. Since the corresponding matrix $A(x)$ in the Euclidean metric is the identity, an application of (6.2) and (6.3) yields $\operatorname{Vol}(C) \geq (2R)^n$ with equality holding if and only if $\overline{A}_p(x)$ is the identity matrix for all p and x. But we know that equality holds, hence $\overline{A}_p(x)$ is the identity, and the theorem is proved. (See §7.2 for the general almost C^2 case.)

6.5. Remark. The theorem in [9, §5.5B] is more general than the one stated above. However it is not hard to modify the above proof to prove boundary rigidity for M which admits an isometric immersion into the same dimensional Euclidean space.

We now consider the case where N is a hyperbolic n-space. The main tool which we will use is that SGM manifolds with negative definite curvature operators are "deformation rigid". A metric g is deformation rigid if for every 1-parameter family g_t of metrics with $g_0 = g$ and $d_t = d$ we have $g = \Phi_t^*(g_t)$ for diffeomorphisms Φ_t that leave the boundary fixed. Using the rigidity theorem of Guillemin and Kazhdan [11], Michel [14] gave a deformation rigidity theorem for convex manifolds with pinched negative curvature. Min-Oo's extension [15] of [11] extends this deformation rigidity theorem to the case of negative definite curvature operator. The argument can also be extended to the SGM (as opposed to convex) case. In this case rather than using the usual cut and paste method one can apply the [11] or [15] argument directly to M. The global properties of the geodesic flow used in the old argument can be easily replaced by the condition $\frac{d}{dt}(d_t(p, q)) = 0$ for all p and q on the boundary (see §7.4). In the argument below we use the cut and paste construction inside a convex M. We will treat the resulting metric as C^2 and handle the differentiability problems at the boundary in the appendix (§7.5).

6.6. Proof of Theorem D. Since the case $n = 2$ was proved in [9], we need only consider $n \geq 3$. Since M is compact, we can find a large ball $B(x, r)$ in a hyperbolic space such that M lies in an open "Half Ball" B^+ of $B(x, r)$. Let $\overline{g}$ be the (almost C^2) metric gotten by replacing M by M_1 in $B(x, r)$. If we let B_1 be this new Riemannian manifold, it is easy to see that $B_1 = B_1(x, r)$ (i.e., B_1 is a metric ball about x) and hence the orthogonal group $\mathrm{O}(n)$ acts as a group of diffeomorphisms via the exponential map. The function $d_1 = d$ is invariant under this group, and hence $O^*\overline{g}$ has the same boundary distance function as $\overline{g}$ for any $O \in \mathrm{O}(n)$. The deformation rigidity of $\overline{g}$ states that $O^*\overline{g}$ is isometric to $\overline{g}$ via a diffeomorphism Φ that leaves the boundary fixed if O is in $\mathrm{SO}(n)$. Since Φ takes geodesics of $O^*\overline{g}$ to geodesics of $\overline{g}$, we first see that $\Phi(x) = x$ since x is the midpoint of all geodesics between boundary antipodal points in both cases. Now the definition of O implies that the geodesics from x to the boundary are the same in both cases, so we see that $\Phi = \mathrm{Id}$ and hence O is an isometry of $\overline{g}$. Thus $O(M_1)$ is isometric to M_1, and the theorem follows since there is an O in $\mathrm{SO}(n)$ taking the half ball B_1^+ containing M_1 to the opposite half ball.

7. Appendix: Differentiability

In this appendix we fill in the details needed in the proofs of the preceding sections (and in Theorem 5.5B of [9]) when the cutting and pasting yields only an "almost C^2" metric. Our Riemannian manifold N is constructed by replacing a domain M in a smooth manifold by a Riemannian manifold M_1 with the same boundary H. The metric will be smooth everywhere except possibly at H where it will be C^0. For $x \in H$ and $V \in T_xN$ there will be two natural choices for ∇_v both being one sided derivatives; they will be denoted ∇_v^+ and ∇_v^-. Similarly there are two second fundamental forms II^+ and II^-.

Let $x \in H$, and $V \in T_xN$ be transverse to H and written as $aN + W$, where N is the unit normal to H and W is tangent to H. It will be important to consider the linear transformation T_V from T_xN to itself defined by

$$T_V(X) = a \cdot \{\nabla_Y^+ N - \nabla_Y^- N\} + \{\mathrm{II}^+(Y, W) - \mathrm{II}^-(Y, W)\} \cdot N,$$

where $X = bV + Y$, and Y is tangent to H. To see that T_V is symmetric first consider the case $X, Z \in T_X H$. Then $\langle T_V(X), Z\rangle = a\{\langle \nabla^+_X N, Z\rangle - \langle \nabla^-_X N, Z\rangle\}$ which is symmetric in X and Z by the symmetry of the second fundamental forms. The general case follows since it is easy to see that $T_V(V) = 0$ and that $\langle T_V(X), V\rangle = 0$. T_V is a measure of the difference between the two second fundamental forms of H.

We will only look at geodesics γ that intersect the set H transversely since the others form a set of measure 0 in applications. We will call a vector field J along such a γ a Jacobi field if it comes from a variation of geodesics. Such a J clearly satisfies the Jacobi equation except possibly at points of $\gamma(t_0) \in H$. At $\gamma(t_0)$, J is continuous. However it need not be differentiable. We let $V = \gamma'(t_0)$. J has one-sided derivatives at t_0. We will show

$$\nabla^+_V J(t_0) - \nabla^-_V J(t_0) = T_V(J(t_0)). \tag{7.1}$$

Let $F(s, t) = \gamma_s(t)$ be a variation through geodesics giving rise to J. Since V is transverse to H, there is a function $t(s)$ such that $F(s, t(s) + t_0) = \sigma(s) \in H$. We will consider the related variation $G(s, t) = F(s, t(s) + t)$. The Jacobi field $X(t)$ of the new variation is $J(t) + t'(0) \cdot \gamma'(t)$ and is tangent to H at t_0. To show (7.1) for J we need only show that it holds for X. To see this, we extend V to $G_*(d/dt)$ and the variation field to $X = G_*(d/ds)$. Now $\nabla^+_V X = \nabla^+_X V = X(a) \cdot N + a\nabla^+_X N + (\nabla^+_X W)^{\mathsf{T}} + (\nabla^+_X W)^N$, where for a vector Z, Z^{T} and Z^N represent the tangential and normal components respectively. Subtracting the corresponding formula for ∇^-_X yields (7.1) since $(\nabla^+_X W)^{\mathsf{T}}$ depends only on the intrinsic metric of H and hence is equal to $(\nabla^-_X W)^{\mathsf{T}}$.

7.2. We now show how the above allows the proof of Theorem 6.4 to work in the almost C^2 case. Along a geodesic $\overline{\gamma}$ which is transverse to H we let $\overline{A}(t)$ be the matrix solution (with respect to a parallel orthonormal basis) of the Jacobi equation except at points $\overline{\gamma}(t_0) \in H$ where we require $\overline{A}$ to be continuous and $\overline{A}'^{+}(t_0) - \overline{A}'^{-}(t_0) = T_V \cdot \overline{A}(t_0)$. As in the C^2 case since geodesics agree outside M_1 and $\overline{A}$ is defined via variations of geodesics, we see that $\overline{A}(t) = O(t) \cdot A(t)$. All of the arguments follow exactly as before once (6.1) holds for $\overline{A}$. To see this define B by (6.1). B satisfies the initial conditions and will satisfy the Jacobi equation at points $\gamma(x)$ not on H as long as $A'(x)A^{-1}(x)$ is symmetric. To see this we need only that the symmetry is maintained at t_0 if $\gamma(t_0) \in H$, and $A'^{+}(t_0)A^{-1}(t_0) = A'^{-}(t_0)A^{-1}(t_0) + T_V$, where $V = \gamma'(t_0)$. Since T_V is symmetric, symmetry is maintained across t_0. In order that B is the solution we want, we need only check that (7.1) holds at $\gamma(t_0) \in H$. Since we have

$$B^{*/+}(t_0) - B^{*/-}(t_0) = A(0)\left\{\int_0^{t_0} A^{-1}(x)A^{*-1}(x)\,dx\right\}\{A^{*/+}(t_0) - A^{*/-}(t_0)\}$$
$$= B^*(t_0)\cdot T_V^*,$$

(7.1) and thus (6.1) hold, and the argument works.

7.3. Remarks. (6.1) is the only step in the proof of the Blaschke conjecture which does not clearly extend to the almost C^2 case. The above argument thus extends the proof to the almost C^2 case and fills in a step (which does not appear in the literature) in the theorem in [9, §5.5B].

7.4. We will now show that the arguments of [11], [15] are sufficient to prove deformation rigidity if M is SGM and has negative definite curvature operator. We look at a one-parameter family of metrics g_t such that $d_t = d_0$. We will assume that they have the same inward normal at the boundary (if not a diffeomorphism near the boundary will make it so). The argument looks at $g_t^*(V, V)$ as a one-parameter family of functions on the cotangent bundle. The fact that the boundary distance is preserved implies that if $\gamma(t)$ for $t \in [0, 1]$ is a geodesic in g_0 from boundary point to boundary point, then its energy must be smaller in g_0 than in all the metrics g_t and hence

$$\int_0^1 \tfrac{d}{dt}(g_t(\gamma'(s), \gamma'(s)))|_{t=0}\,ds = 0.$$

By dualizing the above we see that $\frac{d}{dt}(g_t^*)|_{t=0}$ integrates to 0 along the integral curves of the geodesic flow ζ in the unit cotangent bundle U^*M and hence (since all geodesics hit the boundary) it is not hard to see that there is a function f on U^*M which is 0 on all vectors at the boundary such that $\zeta(f)(V) = \frac{d}{dt}(g_t^*(V, V))|_{t=0}$. The argument given in [11], [15] now applies to f to show that f is linear on each fiber and hence comes from a vector field Z_0. Z_0 will be 0 at the boundary since all g_t are the same at the boundary. For small values of t, g_t will also have negative definite curvature operator, and hence we get vector fields Z_t. This gives rise to a one-parameter family Φ_t of diffeomorphisms defined by $Z_{t_0}(\Phi_{t_0}(x)) = \frac{d}{dt}(\Phi_t(x))|_{t=t_0}$. Then $g_0 = \Phi_t^*(g_t)$ and hence the result follows (see [11] for more details).

7.5. We now show how to modify the proof of Theorem D to allow for lack of differentiability at the boundary H. Let $O(t)$ be a one-parameter family of orthogonal transformations, and $g_t = O(t)^*\overline{g}$ as in the proof of the theorem. We wish to proceed as above and prove that $g_t = \overline{g}$. The main problem is that $\frac{d}{dt}(g_t)|_{t=0}$ need not be defined at points of H. It is clear however that for $p \notin M_1$ we have $\frac{d}{dt}(g_t)|_{t=0}$ is 0 for covectors at p. Hence the argument of §7.4 can be applied to give a function f on U^*N which is continuous, 0 outside of M_1, and differentiable off H with $\zeta f = \frac{d}{dt}(g_t^*)|_{t=0}$. Now the same argument (which is l_2 in nature) implies that f is linear and hence comes from a continuous vector field Z_0 which is 0 outside M_1 and differentiable in the interior of M_1. The same construction for small values of t yields vectorfields Z_t which are 0 outside and differentiable inside $\Phi_t(M_1)$. The proof will follow when we show that Z_t is 0 everywhere. To see this let γ be a geodesic ray from the center x to the boundary. By construction γ is a geodesic in all our metrics g_t. For $\gamma(t)$ not on H the Z_t will generate a local one-parameter family of diffeomorphisms Φ_t with $\Phi_t^*(g_t) = g_0$, and hence $\Phi_t^{-1}(\gamma)$ is a variation of geodesics in g_0. Therefore Z_0 is a Jacobi field along γ which vanishes at points of H. Since γ minimizes, Z_0 is 0 along γ and hence everywhere. The same argument applied to Z_t yields the result.

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