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QUATERNIONIC KÄHLER 8-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

Y. S. POON & S. M. SALAMON

1. Introduction

Consider the subgroup of $SO(4n)$ consisting of unit quaternions acting on $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by right multiplication. We denote the normalizer of this subgroup by $Sp(n)Sp(1)$, which is a maximal subgroup of $SO(4n)$ for $n \geq 2$. A quaternionic Kähler manifold M is a manifold of dimension $4n$, $n \geq 2$, with a Riemannian metric whose linear holonomy group is contained in $Sp(n)Sp(1)$. It is well known that any quaternionic Kähler manifold is Einstein, so there is a trichotomy according as the constant scalar curvature t of M is positive, negative, or zero. In the latter case M is hyper-Kähler in the sense that it is rendered Kähler by a family of complex structures parameterized by the 2-sphere S^2. However in all three cases there exists a bundle over M with fiber S^2 whose total space Z is a complex manifold. For $t > 0$, Z has a canonical Kähler structure, though in general M will not itself be a complex Kähler manifold.

Wolf [25] showed that each compact simple centerless Lie group G is the isometry group of a quaternionic Kähler symmetric space, equal to the conjugacy class of a three-dimensional subgroup of G determined by a highest root of its Lie algebra. These "Wolf spaces" constitute the only known complete examples for $t > 0$, and the present work is devoted to a proof that there are no others when $n = 2$:

Theorem 1.1. *A complete connected quaternionic Kähler eight-manifold with $t > 0$ is isometric to the quaternionic projective plane $\mathbb{HP}^2$, the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$, or the exceptional space $G_2/SO(4)$.*

In §2 we summarize known facts regarding a quaternionic Kähler eight-manifold M with positive scalar curvature, and the complex Kähler manifold Z. The latter is known as the twistor space of M, because of similarities with four dimensions. Indeed, our approach parallels Hitchin's classification [10] of Kähler twistor spaces of self-dual Riemannian

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four-manifolds, although the details work out rather differently. A key feature special to our Einstein situation is the identification of Lie algebra $\mathfrak{g}$ of Killing vector fields on M with a real form of space of holomorphic sections of a positive line bundle F that arises from a contact structure on Z (Theorem 2.2).

We explain that the twistor space Z is a Fano five-fold of coindex 3 for which F defines a morphism $\Phi: Z \to \mathbb{CP}^{l-1}$, where $l = \dim_{\mathbb{R}} \mathfrak{g}$. Index calculations determine the dimension of the space of holomorphic sections of any power F^n in terms of l, which was shown to equal at least 6 in [22]. We are able to increase this lower bound to 8, with the help of results of Fujita [5], [6], and topological inconsistencies. Observing that G has rank no more than 3 then enables us to apply a combination of local and global techniques, including some case by case arguments using representations of various Lie algebras. The main result (Theorem 3.5) of §3 asserts that the complete linear system of divisors associated to F has empty base locus, i.e., that Φ is regular. In §4 we go on to show that Φ must be an embedding; this is carried out independently of results on Fano n-folds announced by Mukai [18]. The proof of Theorem 1.1 is then completed by showing that the action of the group G forces the only remaining candidates for Z to be singular.

Some remarks place this paper in the context of other recent work. The total space of the $\mathbb{C}^*$-bundle associated to F is the symplectification of the complex contact manifold Z, and its study extends the more familiar geometry of Z itself. Swann [23] has shown that this symplectification admits a hyper-Kähler metric, thereby relating quaternionic Kähler manifolds with $t > 0$ to hyper-Kähler ones admitting an action of the group of nonzero quaternions. In particular, he has identified the singular models that appear at the end of our paper as arising from complex nilpotent coadjoint orbits, which themselves possess a hyper-Kähler structure exhibited by Kronheimer [15] using Yang-Mills theory. The mapping Φ can then be interpreted as the projectivization of a hyper-Kähler moment mapping, and this point of view is likely to have important consequences for the description of quaternionic Kähler spaces with isometries in higher dimensions.

The moment mapping approach was used by Galicki and Lawson [9] to describe a quaternionic Kähler reduction, whose explicit use may provide short cuts to some of our intermediate results. Their techniques, and also those of LeBrun [16], indicate that there is no analogue of Theorem 1.1 for negative scalar curvature. Alekseevskii [3] found examples with $t < 0$, and dimension 16 or more, that are homogeneous but not symmetric.

2. Quaternionic Kähler eight-folds and Fano five-folds

The underlying hypothesis of this paper is that M denotes *a complete connected eight-dimensional quaternionic Kähler manifold with scalar curvature* $t > 0$. Associated to the action of $Sp(1)$ is the adjoint bundle V with fiber $\mathfrak{sp}(1)_x \cong \operatorname{Im}\mathbb{H}$ over $x \in M$. If I, J, K is an orthonormal basis of $\mathfrak{sp}(1)_x$, then $\{aI + bJ + cK: a^2 + b^2 + c^2 = 1\}$ consists of almost complex structures on T_xM. The union Z of these structures over all $x \in M$ is a complex manifold called the twistor space, described by both the second author [22, Theorem 4.1] and Bérard Bergery [4, Theorem 14.6]. Each fiber $Z_x = \pi^{-1}(x) \cong S^2 \cong \mathbb{CP}^1$ of the sphere bundle $\pi: Z \to M$ is a complex submanifold of Z. Reversing the sign of an almost complex structure induces an antilinear involution σ on Z which preserves the fibers, which we shall refer to as the real lines.

The complex manifold Z possesses a contact structure consisting of a holomorphic 1-form β with values in a holomorphic line bundle F. In fact β corresponds to projection of tangent vectors to the fiber directions, and $\ker\beta$ constitutes the bundle of holomorphic horizontal vectors. Because M is a complete manifold with positive Ricci tensor, Myer's Theorem tells us that it (and therefore Z) is necessarily compact. Further properties that will be exploited in the sequel are gathered in the next three theorems; we refer the reader to [22] for more details.

Theorem 2.1. *Z is a simply connected Kähler manifold, and F is a positive holomorphic line bundle such that*

(i) *F is real in the sense that $\sigma^*F \cong \overline{F}$, and F^3 is isomorphic to the anticanonical bundle $(\Lambda^{5,0}Z)^*$;*

(ii) *the restriction of F to a real line $Z_x \cong \mathbb{CP}^1$ is isomorphic to the square $\mathscr{O}(2)$ of the Hopf bundle;*

(iii) *F has a global square root if and only if Z is biholomorphically equivalent to $\mathbb{CP}^5$ and M is isometric to $\mathbb{HP}^2$.*

The positivity of F is a direct consequence of our assumption that the scalar curvature is positive. Kodaira's Embedding Theorem implies that sufficiently high powers of F give projective embeddings of Z. In modern terminology, F is an ample line bundle, and Z is a Fano manifold [11]. The largest integer r for which there exists an rth root of the anticanonical line bundle is called the index of the Fano manifold Z, and $\dim_{\mathbb{C}} Z + 1 - r$ is the coindex. For example $\mathbb{CP}^5$ has index $r = 6$ and coindex 0. In view of Theorem 2.1(iii), this is the only twistor space of index 6, so from now on *we shall suppose that M is not isometric to* $\mathbb{HP}^2$, or equivalently that $r = 3$ and Z has coindex 3. The cube root F is called the fundamental line bundle, and we denote the associated complete linear system of divisors by $|-\frac{1}{3}K|$.

Let G denote the identity component of all isometry group of M, and $\mathfrak{g}$ its Lie algebra consisting of Killing vector fields. If $x \in M$, the isotropy subgroup G_x corresponds to the subalgebra $\mathfrak{g}_x$ of those Killing fields that vanish at x. Any such field is completely determined by the value of the covariant derivative $(\nabla A)|_x$. In fact, for any $A \in \mathfrak{g}$, Kostant's Theorem implies that $(\nabla A)|_x$ belongs to the holonomy algebra (see [14, Theorem 3.3]; [2]). Therefore

$$\mathfrak{g}_x \subseteq \{(\nabla A)_x : A \in \mathfrak{g}\} \subseteq (\mathfrak{sp}(2) + \mathfrak{sp}(1))_x \subset \operatorname{End} T_x M, \tag{2.1}$$

and the isotropy representation is constituted from the two homomorphisms

$$\rho_2 : \mathfrak{g}_x \to \mathfrak{sp}(2)_x, \qquad \rho_1 : \mathfrak{g}_x \to \mathfrak{sp}(1)_x. \tag{2.2}$$

Since Z is defined in terms of the holonomy structure, it follows that any Killing field $A \in \mathfrak{g}$ lifts to a vector field on Z, and an element $g \in G$ acts on Z by sending an almost complex structure I to $\tilde{g}(I) = g_* I g_*^{-1}$. Then $\tilde{g} \circ \sigma = \sigma \circ \tilde{g}$, $\tilde{g}^* \beta = \beta$, and there is a commutative diagram $g \circ \pi = \pi \circ \tilde{g}$. In the sequel, $H^r(Z, F)$ will denote Čech cohomology on Z of the sheaf $\mathscr{O}(F)$ of germs of local holomorphic sections of F, and $h^r(Z, F) = \dim_{\mathbb{C}} H^r(Z, F)$.

Theorem 2.2. (i) *The correspondence $g \leftrightarrow \tilde{g}$ realizes G as a real form of the identity component $G_{\mathbb{C}}$ of holomorphic automorphisms of the contact structure on Z.*

(ii) *The space $H^0(Z, F)$ of holomorphic sections of F is isomorphic to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, and has dimension $I = h^0(Z, F)$ equal to $\frac{1}{2}(c_1(F))^5[Z] + 5 \geq 6$.*

(iii) $h^0(Z, F^n) = \frac{1}{120}(n+1)(n+2)(2n+3)[n(n+3)(I-5)+20]$.

Part (i) has been proved by Nitta and Takeuchi [20]. Part (ii) includes the corresponding infinitesimal statement, which may be established independently as follows. If $A \in \mathfrak{g}$, the component of $(\nabla A)_x$ in the subspace $\mathfrak{sp}(1)_x = \operatorname{span}\{I, J, K\}$ (see (2.1)) determines a section s_A of the adjoint bundle V. It is well known that s_A satisfies a certain first order differential equation, and completely determines A [22, Lemma 6.4] (this fact is the basis of the quaternionic Kähler reduction of [9]). On the other hand, there is a natural isomorphism of $\mathfrak{sp}(1, \mathbb{C})_x$ with the space $H^0(Z_x, \mathscr{O}(2))$ of homogeneous quadratic polynomials. Moreover with the aid of Theorem 2.1(ii), a smooth section s of V gives rise to a holomorphic section of F if and only if $s = s_A$ for some $A \in \mathfrak{g}$ [22, Lemma 6.5]. The required isomorphism then associates this section with A.

Corollary 2.3. *The restriction* $\iota_x^*\colon H^0(Z, F) \to H^0(Z_x, \mathcal{O}(2))$ *can be identified with the linear mapping* $A \mapsto s_A(x)$, $A \in \mathfrak{g}$, *and* $\rho_1(\mathfrak{g}_x) \subseteq \operatorname{Im}(\iota_x^*)$.

Equality (iii) of Theorem 2.2 results from verification of the following facts. By the Riemann-Roch-Hirzebruch Theorem, the Euler characteristic $\chi(n) = \chi(Z, F^n)$ is a polynomial in n of degree no more than 5. By Kodaira vanishing and Serre duality we have $\chi(n) = h^0(Z, F^n)$ for all $n \geq 0$, and $\chi(-1) = 0 = \chi(-2)$. Then $\chi(n)$ is completely determined by the Todd genus $\chi(0) = 1 = -\chi(-3)$ and the definition $\chi(1) = I = -\chi(-4)$. An explicit computation of these indices yields the equality $h^0(Z, F) = \frac{1}{2}(c_1(F))^5[Z] + 5$.

$I = \dim \mathfrak{g}$	6	7	8	9	10	11
$h^0(Z, F^2)$	21	28	35	42	49	56
$[\dim(S^2\mathfrak{g})]$	[21]	[28]	[36]	[45]	[55]	[66]
$h^0(Z, F^3)$	57	84	111	138	165	192
$[\dim(S^3\mathfrak{g})]$	[56]	[84]	[120]	[165]	[220]	[286]
$h^0(Z, F^4)$	132	209	286	363	440	517
$[\dim(S^4\mathfrak{g})]$	[126]	[210]	[330]	[495]	[715]	[1001]

TABLE 1

Some hint of the possibilities for Z is already apparent in Table 1, which compares values of $h^0(Z, F^n)$ with the dimensions of the symmetric powers $S^n H^0(Z, F)$. Further vanishing theorems on Z were used in [22] to prove

Theorem 2.4. *M has Betti numbers $b_1 = 0 = b_3$, $b_4 = b_2 + 1$. Moreover the Kähler manifold Z has $h^{p,q} = 0$ whenever $p \neq q$, and its Euler characteristic is $\chi(Z) = 2\chi(M) = 6(b_2 + 1)$.*

The vanishing of $h^{0,1}$ and $h^{0,2}$ (valid on any Fano manifold, by Serre duality) implies that a holomorphic line bundle L on Z is uniquely determined by its first Chern class in $H^2(Z, \mathbb{Z})$, which is torsion-free. By the Leray-Hirsch Theorem, $c_1(L) = nc_1(F) + \pi^*\omega$, where $n \in \mathbb{R}$ and $\omega \in H^2(M, \mathbb{R})$. If $\sigma^* c_1(L) = -c_1(L)$, we deduce that $\omega = 0$ and n is an integer. Thus,

Corollary 2.5. *Any real holomorphic line bundle L on Z is isomorphic to F^n for some $n \in \mathbb{Z}$.*

The departure of the quaternionic Kähler manifold M from being a symmetric space is measured by the covariant derivative ∇R of its curvature tensor. To describe this, it is convenient to express the complexified cotangent space T^* as the tensor product $E \otimes H$ of the basic $Sp(2)$-module $E \cong \mathbb{C}^4$ with the basic $Sp(1)$-module $H \cong \mathbb{C}^2$, as in [22]. For example, an isotropy subalgebra $\mathfrak{g}_x$ acts on E and H via the representations ρ_2 and ρ_1 of (2.2). Alekseevskii [1] was the first to show that the curvature tensor of M has the form $R = tR_0 + R_1$, where R_0 is a covariant constant tensor representing the curvature of $\mathbb{HP}^2$, and R_1 belongs to the irreducible $Sp(2)$-submodule of $\Lambda^2 T^* \otimes \Lambda^2 T^*$ isomorphic to the fourth symmetric power $S^4 E$. This fact is generalized in the proof of the next result.

Theorem 2.6. *If M has a point x for which $\rho_2(\mathfrak{g}_x)$ has real dimension at least 6, then M is isometric to $\mathbb{HP}^2$.*

Proof. Because of the holonomy reduction, the k-fold covariant derivative $\nabla^{(k)} R_1$ takes values at each point in the space $(\bigotimes^k T^*) \otimes S^4 E$; let C_k be its component in the irreducible $Sp(2)Sp(1)$-submodule $S^{k+4} E \otimes S^k H$ of highest weight. We shall first prove by induction that

$$(2.3_k) \qquad \nabla^{(k)} R_1 = C_k + f_k(R, \nabla R, \nabla\nabla R, \cdots, \nabla^{(k-2)} R), \qquad k \geq 0,$$

where f_k denotes some universal polynomial followed by a contraction. The second Bianchi identity implies that $\nabla R = \nabla R_1$ lies at each point in the kernel of the appropriate mapping

$$b \colon T^* \otimes S^4 E \to \Lambda^3 T^* \otimes \Lambda^2 T^*.$$

Using the fact that

$$E \otimes S^k E \cong S^{k+1} E \oplus V_k \oplus S^{k-1} E$$

is the direct sum of three irreducible $\mathfrak{sp}(2)$-modules, it is easy to see that $\ker b$ is isomorphic to $S^5 E \otimes H$. Thus $C_1 = \nabla R$, $f_1 = 0$, and (2.3_0), (2.3_1) are validated.

More generally, assume (2.3_{k-1}) and (2.3_k). Since C_k is one of the components of ∇C_{k-1}, the irreducible constituents of $\nabla\nabla C_{k-1}$ include all those of ∇C_k, which belongs to the space

$$T^* \otimes S^{k+4}E \otimes S^k H \cong (S^{k+5}E \oplus V_{k+4} \oplus S^{k+3}E) \otimes (S^{k+1}H \oplus S^{k-1}H).$$

This decomposition may be used to verify that the submodule $S^{k+5}E \otimes S^{k+1}H$ is the kernel of the composition

$$\begin{aligned} T^* \otimes S^{k+4}E \otimes S^k H &\hookrightarrow T^* \otimes T^* \otimes S^{k+3}E \otimes S^{k-1}H \\ &\to \Lambda^2 T^* \otimes S^{k+3}E \otimes S^{k-1}H, \end{aligned}$$

sitting ∇C_k inside $\widehat{\nabla\nabla} C_{k-1}$, and then mapping to the skew-symmetrization $\widehat{\nabla\nabla} C_{l-1}$, which by the Ricci identity is some contraction of $R \otimes C_{k-1}$. The component of ∇C_k in this kernel is exactly C_{k+1}. The remaining components of ∇C_k can then be expressed in terms of $\widehat{\nabla\nabla} C_{k-1}$ and the covariant derivatives of components of ∇C_{k-1} orthogonal to C_k. It follows that all the components of $\nabla^{(k+1)}R$ other than C_{k+1} can be expressed in terms of $R \otimes C_{k-1}$ and $\nabla(f_k(R, \nabla R, \cdots, \nabla^{(k-2)}R))$, and (2.3_{k+1}) is established.

The values $(\nabla^{(k)}R)|_x$ and $(C_k)|_x$ have to be invariant by $\mathfrak{g}_x$ for all $k \geq 0$. By hypothesis, $\mathfrak{g}_x$ contains a subalgebra isomorphic to $\mathfrak{sp}(1) + \mathfrak{sp}(1)$, the restriction of ρ_2 to which is injective. If V and W denote the basic 2-dimensional modules of these two $\mathfrak{sp}(1)$'s, then $E \cong V \oplus W$, and the isotropy subgroup has no invariants in $S^{k+4}E \otimes S^k H$, $k \geq 0$, and so $C_k = 0$. Every term of $f_k(R, \nabla R, \cdots, \nabla^{(k-2)}R)$ involves R_1 or one of its covariant derivatives as a factor, so $(\nabla^{(k)}R)|_x$ must vanish for all $k \geq 1$. Because M is Einstein, its metric is real analytic in suitable coordinates, whence $R = tR_0$ at all points of M (cf. [4, Theorem 5.26]), and the result follows. q.e.d.

Consider any point $x \in M$. If the orbit G/G_x has real dimension 8, then $M = G/G_x$ is homogeneous, and must also be symmetric by a theorem of Alekseevskii [2]. We may therefore assume that $d = \dim_{\mathbb{R}}(G/G_x) \leq 7$. An inspection of possible isotropy representations then reveals that ρ_2 can only have rank less than 6 if $d + \dim_{\mathbb{R}} G_x \leq 11$.

Corollary 2.7. *If $I \geq 12$, then M is symmetric.*

3. The fundamental system and its base locus

For M not isometric to $\mathbb{HP}^2$, the fundamental line bundle F of the Fano five-fold Z is the cube root of the anticanonical bundle. We shall study Z by means of the meromorphic mapping

$$\Phi\colon Z \to \mathbb{P}(H^0(Z, F)^*) \cong \mathbb{P}(\mathfrak{g}_{\mathbb{C}}^*) \cong \mathbb{CP}^{I-1\geq 5},$$

associated to the complete linear system $|-\frac{1}{3}K|$. The poles of Φ constitute the base locus B, the set of common zeros of all holomorphic sections of F; the aim of this section is to prove that B is empty. Given a polarized variety, i.e., a pair (Z, F) consisting of an algebraic variety Z and an ample line bundle F, Fujita [5], [7] considers

(i) the genus $g(Z, F) = \frac{1}{2}(c_1(F))^m[Z] + 1$,
(ii) the total deficiency $\Delta(Z, F) = m + (c_1(F))^m[Z] - h^0(Z, F)$,

where $m = \dim_{\mathbb{C}} Z$, and proves the inequality

$$\dim_{\mathbb{C}} B < \Delta(Z, F).$$

In our case, Theorem 2.2(ii) yields

$$g(Z, F) = I - 4, \qquad \Delta(Z, F) = I - 5. \tag{3.1}$$

Lemma 3.1. *The base locus B of $|-\frac{1}{3}K|$ has complex dimension $\dim_{\mathbb{C}} B \leq \min\{3, I-6\}$.*

Proof. It remains to show that B has no fixed component. The normal bundle of any real line Z_x in the complex manifold Z is isomorphic to $\mathcal{O}(1) \otimes \mathbb{C}^4$ (cf. Theorem 2.1(ii)). The argument of [21, Lemma 2.1] generalizes to show that if D is an effective divisor in Z, then $D \cdot Z_x > 0$ for any x. Since $|-\frac{1}{3}K|$ is real, its base locus B is a real subvariety of Z. If B were to contain an effective divisor of Z, it would contain a real effective divisor D, giving $D \cdot Z_x \geq 2 = -\frac{1}{3}K \cdot Z_x$. This would force the variable part of any element of $|-\frac{1}{3}K|$ to have zero intersection with every real line, which is impossible. q.e.d.

The restriction on $\dim_{\mathbb{C}} B$ enables one to handle situations when Z has small total deficiency.

Proposition 3.2. *The case $I = 6$ does not arise.*

Proof. When $I = 6$, the polarized variety Z has total deficiency $\Delta = 1$, and Fujita [6, Theorem 2.5] proves that Z is a double-covering of $\mathbb{CP}^5$ branched along a smooth divisor Y of degree $2g+2$ in $\mathbb{CP}^5$, i.e., a sextic. It follows that

$$\chi(Z) = 2\chi(\mathbb{CP}^5) - \chi(Y) = 12 - 2610 = -2598,$$

contradicting the positivity of $\chi(Z)$ in Theorem 2.4.

Proposition 3.3. *The case $I = 7$ does not arise.*

Proof. Given that $I = 7$, it follows from [7, Theorem 4.1] that a generic element D in the fundamental system is nonsingular irreducible. Applying the same theorem on D, one can find a nonsingular irreducible real V as the intersection of two generic elements in the fundamental system $|-\frac{1}{3}K|$. Then V is a Fano three-fold, with anticanonical bundle $F|_V$ and $g = 3$ (see forward to Lemma 4.2).

The base locus B on Z lies in the base locus of $F|_V$, which can be nonempty only if V is the blow-up of a Fano three-fold W of index 2 along an elliptic curve C, the complete intersection of two elements of the fundamental system of W. The exceptional divisor A is a product $\mathbb{CP}^1 \times C$, and the base locus B equals $\mathbb{CP}^1 \times \{c\}$ for some point $c \in C$ [12, Chapter I, Theorem 6.3], [17, §2]. As V is a Fano manifold, an anticanonical divisor $-K_V$ satisfies $-K_V \cdot B > 0$. From Corollary 2.5, the conjugate divisor $\sigma(A)$ to A satisfies

$$A + \sigma(A) = -nK_V \tag{3.2}$$

for some positive integer n. Since B is real, $A \cdot B = \sigma(A) \cdot B = \frac{1}{2}(-nK_V \cdot B) > 0$. But A is the divisor of a line bundle, whose restriction to the blow-up B of c is the tautological bundle, so $A \cdot B = -1$, which is a contradiction.

With $B = \varnothing$, Z is either a nonsingular quartic hypersurface in $\mathbb{CP}^6$, or a double cover of a smooth hyperquadric in $\mathbb{CP}^6$ with branch the intersection of this quadric and a smooth hyperquartic [8, §0.6]. Both models have negative Euler characteristic, both equal in fact to -540. q.e.d.

We may now assume that $8 \leq I \leq 11$. These remaining cases share the property of being distinguished by a quadratic condition in Table 1.

Proposition 3.4. *The isometry group G has rank less than or equal to 3, so that $\mathfrak{g}$ is isomorphic to one of* (i) $\mathfrak{su}(3)$, (ii) $\mathfrak{sp}(2)$, (iii) $\mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{su}(2)$, (iv) $\mathfrak{su}(3) + \mathfrak{su}(2)$, (v) $\mathfrak{su}(3) + \mathfrak{u}(1)$, *or* (vi) $\mathfrak{sp}(2) + \mathfrak{u}(1)$.

Proof. Recall that G is the real form of a group $G_{\mathbb{C}}$ of holomorphic transformations of Z. Since Z is a Fano manifold, there is an embedding $Z \to \mathbb{P}(H^0(Z, F^n)^*) = \mathbb{CP}^N$ for some n and N. This map realizes Z as an algebraic subvariety of $\mathbb{CP}^N$ invariant by a subgroup G of projective transformations. In these circumstances any maximal torus T of G has a fixed point z on Z (cf. [13, Chapter III, §9]). If $x = \pi(z)$, T is contained in the isotropy subgroup G_x, which by (2.1) is itself a subgroup of $Sp(2)Sp(1)$. The list follows from the classification of Lie algebras.

Theorem 3.5. *The base locus B of the fundamental system $|-\frac{1}{3}K|$ is empty.*

Proof. Let $z \in B$ with $x = \pi(z)$, so that $\pi(B)$ contains the orbit G/G_x, whose tangent space may be identified with $\mathfrak{g}/\mathfrak{g}_x$. We first show that B cannot contain the real line Z_x. If it did, Corollary 2.3 would imply that $\mathfrak{g}_x \subseteq \mathfrak{sp}(2)$ and

$$\dim_{\mathbb{R}} \mathfrak{g} - \dim_{\mathbb{R}} \mathfrak{g}_x \leq \dim_{\mathbb{R}} B - 2.$$

Lemma 3.1 and Proposition 3.4 then imply that $\dim_{\mathbb{R}} \mathfrak{g}_x \geq 6$, which is impossible in view of Theorem 2.6.

We may now assume that the real line Z_x intersects B in two points z and $\sigma(z)$, and that the restriction ι_x^* of Corollary 2.3 has a one-dimensional image. It follows that $\Phi(Z_x) = [\lambda]$, where $\lambda \neq 0$ is a vector in the real subspace $\mathfrak{g}^* \subset \mathfrak{g}_{\mathbb{C}}^*$. Consider the G-equivariant map

$$s_2 \colon S^2\mathfrak{g}_{\mathbb{C}} \cong S^2 H^0(Z, F) \to H^0(Z, F^2); \tag{3.4}$$

if we regard $v \in \mathfrak{g}_{\mathbb{C}}$ as a holomorphic section of F, $s_2(v \otimes v) = v^2$. By definition of Φ, $\lambda \otimes \lambda$ belongs to the annihilator of $\ker s_2$. To prove the theorem, it will suffice to exhibit an irreducible G-submodule $V \subset \ker s_2 \subset S^2\mathfrak{g}_{\mathbb{C}}$ for which $(\lambda \otimes \lambda)|_V \neq 0$. We follow the cases of Proposition 3.4.

(i), (ii): $\mathfrak{g}$ *simple.* There is a well-known decomposition

$$S^2\mathfrak{sl}(3, \mathbb{C}) \cong V^{[27]} \oplus V^{[8]} \oplus \mathbb{C}, \tag{3.5}$$

where $V^{[k]}$ denotes an irreducible k-dimensional $SL(3, \mathbb{C})$-module. The highest weight summand $V^{[27]}$ contains some simple products $v \otimes v$ for $v \in \mathfrak{sl}(3, \mathbb{C})$; clearly $v \neq 0$ implies $s_2(v \otimes v) \neq 0$, so by Schur's Lemma s_2 restricts to an isomorphism on $V^{[27]}$. On the other hand, Table 1 implies that at least one of $V^{[8]}$ or $\mathbb{C}$ lies in $\ker s_2$, and the one-dimensional summand is spanned by the Killing form, which certainly has nonzero contraction with $\lambda \otimes \lambda$. But it is also true that

$$(\lambda \otimes \lambda)|_{V^{[8]}} \neq 0; \tag{3.6}$$

this follows from a computation in which the real element λ belongs to the dual of a fixed Cartan subalgebra of $\mathfrak{su}(3)$.

Case (ii) proceeds in the same way from the decomposition

$$S^2\mathfrak{sp}(2, \mathbb{C}) \cong S^2\mathfrak{so}(5, \mathbb{C}) \cong W^{[35]} \oplus W^{[14]} \oplus W^{[5]} \oplus \mathbb{C}, \tag{3.7}$$

familiar from the study of the Riemann curvature tensor ($W^{[35]}$ is the complexified space of Weyl tensors on a five-manifold, and coincides with the space S^4E of tensors R_1 of Theorem 2.6). This time $\ker s_2$ must contain at least one of the irreducible components $W^{[14]}$ or $\mathbb{C}$, and (3.6) is valid with each of these in place of $V^{[8]}$.

(iii), (iv): $\mathfrak{g}$ *semisimple, but not simple.* In addition to (3.5), there is the decomposition

$$S^2\mathfrak{sl}(2, \mathbb{C}) \cong X^{[5]} \oplus \mathbb{C}. \tag{3.8}$$

For (iv), the irreducible submodules $V^{[27]}$, $\mathfrak{sl}(3, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$, and $X^{[5]}$ of $S^2(\mathfrak{sl}(3, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C}))$ all contain simple tensor products, so s_2 injects then into the 56-dimensional space $H^0(Z, F^2)$. Hence s_2 is surjective, and $\ker s_2$ contains the nondegenerate Killing form, which has nonzero contraction with $\lambda\otimes\lambda$. Case (iii), in which $h^0(Z, F^2) = 42$, is analogous.

(v), (vi): $\mathfrak{g}$ *not semisimple.* The summands $V^{[27]}$, $\mathfrak{sl}(3, \mathbb{C}) \otimes \mathfrak{gl}(1, \mathbb{C})$ and $S^2\mathfrak{gl}(1, \mathbb{C}) = \mathbb{C}$ of $S^2(\mathfrak{sl}(3, \mathbb{C}) + \mathfrak{gl}(1, \mathbb{C}))$ all contain simple products, so s_2 injects them into the 42-dimensional space $H^0(Z, F^2)$. Hence the submodule $V^{[8]}$ of $S^2\mathfrak{sl}(3, \mathbb{C})$ lies in $\ker s_2$. If λ does not annihilate $\mathfrak{sl}(3, \mathbb{C})$, then (3.6) holds and the proof is complete. Otherwise $\pi^{-1}(x)$ lies in the enlarged base locus B' of the real subsystem of $|-\frac{1}{3}K|$ generated by $\mathfrak{sl}(3, \mathbb{C})$, and the arguments that led to Lemma 3.1 and (3.3) then imply that $\dim_{\mathbb{R}} B' \leq 6$ and $\mathfrak{g}/\mathfrak{g}_x \cong \mathfrak{su}(3)/\mathfrak{u}(2)$. The latter (with $\mathfrak{u}(2)$ acting via the adjoint representation) is tangent to $\pi(B')$, and must be a submodule of T_xM (with $\mathfrak{u}(2)$ acting via ρ_2), but these descriptions are incompatible. Case (vi) is similar.

4. Projective embedding of the twistor space

As the fundamental system $|-\frac{1}{3}K|$ on the Fano five-fold Z has no base points, the associated map $\Phi\colon Z \to \mathbb{P}(H^0(Z, F)) = \mathbb{CP}^{I-1}$, $8 \leq I \leq 11$, is holomorphic. We wish to apply Bertini's Theorems [24]:

(1) The generic member of a complete linear system with no fixed component on a projective manifold Y with $\dim\Phi(Y) \geq 2$ is an irreducible subvariety of multiplicity one.

(2) A generic element of a complete linear system on Y cannot have singular points that are not base points of the system.

If Z_x is a real line, the image of the restriction ι_x^* in Corollary 2.3 has to have dimension at least 2, for otherwise $|-\frac{1}{3}K|$ has a base point on Z_x. It follows that $\Phi(Z_x)$ is either a line or a plane conic. Now if $\dim\Phi(Z) \leq 1$, then $\Phi(Z) = \Phi(Z_x)$ is contained in a plane. But by definition, $\Phi(Z)$ is full in $\mathbb{CP}^{I-1\geq 7}$, so $\dim\Phi(Z) \geq 2$. Bertini's Theorems now imply that a generic element of $|-\frac{1}{3}K|$ is irreducible and nonsingular. Pick such a $D \in |-\frac{1}{3}K|$ that is also real; its canonical divisor is determined by the adjunction formula:

$$K_D = K|_D - \tfrac{1}{3}K|_D \quad \text{or} \quad -\tfrac{1}{3}K|_D = -\tfrac{1}{2}K_D \tag{4.1}$$

The exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(F) \to \mathcal{O}_D(F) \to 0$$

and the vanishing of $h^{0,1}$ (Theorem 2.4) show that every section of F over D extends to a section of F over Z. Consequently, the base locus of $|-\frac{1}{2}K_D|$ on D coincides with the base locus of $|-\frac{1}{3}K|$ on Z, and is therefore empty.

Lemma 4.1. *D is a Fano four-fold of index 2 with fundamental line bundle $F|_D$. Moreover* $\mathrm{Pic}(D) = \mathrm{Pic}(Z)$, $b_2(D) = b_2(Z)$, *and the polarized variety $(D, F|_D)$ has the same invariants* (3.1) *as* (Z, F).

Proof. Since D is a divisor of the positive line bundle F on Z, Lefschetz's Hyperplane Theorem implies that the second integral cohomology groups of D and Z are isomorphic. In fact the isomorphism is equivariant with respect to the Hodge decomposition, so the Picard groups of D and Z are isomorphic. Then F restricts to a fundamental line bundle on D, which is therefore a Fano four-fold of index 2. The computation of $g(D, F|_D)$ and $\Delta(D, F|_D)$ is straightforward. q.e.d.

Now the real nonsingular irreducible element D of $|-\frac{1}{3}K|$ must contain at least one real line Z_x. If not, the equality $D \cdot Z_x = -\frac{1}{3}K \cdot Z_x = 2$ would imply that every real line intersects D transversely at a conjugate pair of points. Then the twistor fibration would exhibit D as a double-covering of M, which contradicts the fact that M is simply connected [22, Theorem 6.6].

The associated map of $|-\frac{1}{2}K_D|$ is the restriction of Φ to D, and hence $\dim \Phi(D) \geq 2$. As $|-\frac{1}{2}K_D|$ is base-point free, Bertini's Theorems imply that a generic element V is irreducible nonsingular. The adjunction formula gives

$$K_V = K_D|_V - \tfrac{1}{2}K_D|_V \quad \text{or} \quad -\tfrac{1}{2}K_{D|V} = -K_V\,,$$

and we may repeat our previous argument and Lemma 4.1 to conclude.

Lemma 4.2. *V is a Fano three-fold of index 1 with fundamental line bundle $F|_V$. Moreover* $\mathrm{Pic}(V) = \mathrm{Pic}(Z)$, $b_2(V) = b_2(Z)$, *and $(V, F|_V)$ has the same invariants* (3.1) *as* (Z, F).

Proposition 4.3. *The restriction of Φ to the Fano three-fold V is an embedding.*

Proof. Our assumptions imply that $|-K_V|$ has no base locus, and $g(V, F|_V) \geq 4$. In the terminology of Iskovskikh, Φ can fail to embed

V only if V is hyperelliptic with $b_2(V) \geq 2$ (see [11, Theorem 7.2(iii)], [12, Chapter II, Theorem 2.2], [17, p. 105]), and

(i) Φ exhibits V as a double covering of a Veronese three-fold $\mathbb{CP}^1 \times \mathbb{CP}^2$ in $\mathbb{CP}^5$, branched along a divisor of bidegree $(2, 4)$, or

(ii) V is the blow-up of a Fano three-fold W of index 2 along an elliptic curve C, equal to the complete intersection of two elements of its fundamental system, or

(iii) $V \cong \mathbb{CP}^1 \times S$, where S is a del Pezzo surface obtained by blowing up $\mathbb{CP}^2$ at seven points in general position.

The genus of V in these examples equals 4, 5, and 7 respectively; we explain that they are all incompatible with the real structure of Z. In (i), if $L^{p,q}$ denotes the pullback to V of the line bundle on $\mathbb{CP}^1 \times \mathbb{CP}^2$ of bidegree (p, q), then $F|_V \cong L^{1,1}$, and the ramification locus is a divisor of $L^{1,2}$. The latter is real, since if $d\Phi$ degenerates at z, it also degenerates at $\sigma(z)$. These conclusions contradict Corollary 2.5.

In case (ii), $(-\frac{1}{2}K_W)^3 = 2$ which implies that the degree of the normal bundle of C in W equals 4. Some general theory (as used in [17, §4]) then shows that the exceptional divisor A satisfies

$$(4.2) \quad A^3 = -4, \qquad A^2 \cdot (-K_V) = 0, \qquad A \cdot (-K_V)^2 = (-K_W) \cdot C = 4.$$

If $\sigma(A)$ denotes the divisor conjugate to A, (3.2) gives $2A \cdot (-K_V)^2 = n(-K_V)^3 = 8n$, with $n = 1$. Therefore

$$A^3 = (\sigma(A))^3 = (-K_V - A)^3,$$

which is incompatible with (4.2).

In (iii), pick a divisor A on V which is the product of the factor $\mathbb{CP}^1$ and an exceptional divisor of the blow-up on S. The line bundle L associated to A has bidegree $(0, -1)$ on A, and it follows from (3.2) that $\sigma^*\overline{L}$ is nontrivial on A. Therefore A and $\sigma(A)$ intersect in a real curve C, with positive bidegree on A, and

$$2A \cdot C = (A + \sigma(A)) \cdot C = -nK_V \cdot C > 0,$$

which is a contradiction.

Proposition 4.4. *The natural maps*

$$s_n \colon S^n H^0(Z, F) \to H^0(Z, F^n),$$
$$s_n' \colon S^n H^0(D, F) \to H^0(D, F^n),$$
$$s_n'' \colon S^n H^0(V, F) \to H^0(V, F^n)$$

are all surjective for $n \geq 1$.

Proof. The surjectivity of s_n'' follows, e.g., from [19, §1.16]. Suppose inductively that s_{n-1}' is surjective ($n = 2$ is trivial). If t is a section of $F|_D$ defining the generic V, then

$$o \to \mathcal{O}_D(F^{n-1}) \xrightarrow{\otimes t} \mathcal{O}_D(F^n) \to \mathcal{O}_V(F^n) \to 0$$

induces the exact top row of the commutative diagram:

$$\begin{array}{ccccc} H^0(D, F^{n-1}) & \xrightarrow{t} & H^0(D, F^n) & \xrightarrow{r^n} & H^0(V, F^n) \\ \uparrow{\scriptstyle s_{n-1}'} & & \uparrow{\scriptstyle s_n'} & & \uparrow{\scriptstyle s_n''} \\ S^{n-1}H^0(D, F) & \xrightarrow{t} & S^nH^0(D, F) & \xrightarrow{r_n} & S^nH^0(V, F) \end{array}$$

From (4.1), the positivity of $F|_D$, and Kodaira's Vanishing Theorem, $H^1(D, F^{n-1}) \cong H^3(D, F^{-n-1})^* = 0$ for $n \geq 0$. This implies that r^n and r_n are surjective, and to prove that s_n' is surjective is now just a matter of diagram chasing. The same argument may be repeated replacing D by Z and V by D. q.e.d.

Because F is positive, the associated map of F^n is embedding when n is sufficiently large. This embedding factors through Φ since by the previous lemma, sections of F^n are generated by those of F. Thus Φ is itself an embedding (cf. [18, Proposition 1]), and we shall finish the proof of Theorem 1.1 by rejecting all its possible images. Now Φ is G-equivariant relative to the G-action on Z and the coadjoint action in $\mathfrak{g}_{\mathbb{C}}^*$. The latter must be faithful because Φ is an embedding, so the compact Lie algebra $\mathfrak{g}$ is centerless and a direct sum of simple Lie algebras. Only the cases (i), (ii), (iii), (iv) of Proposition 3.4 remain.

(i) $I = 8$, $\mathfrak{g} \cong \mathfrak{su}(3)$. From Table 1 and Proposition 4.4, the kernel of s_2 (3.4) has dimension 1, and must be spanned by the Killing form, which is the sole invariant in $S^2(\mathfrak{su}(3))$. Similarly, the kernel of $s_3\colon S^3H^0(Z, F) \to H^0(Z, F^3)$ has dimension 9, and must contain the unique invariant in $S^3(\mathfrak{su}(3))$. Hence $\Phi(Z)$ lies in the intersection of the corresponding invariant quadric Q and cubic C on $\mathbb{CP}^7$, and it would follow that $\Phi(Z) = Q \cap C$. However the intersection has singular locus consisting of nonprincipal nilpotent elements.

(ii) $I = 10$, $\mathfrak{g} \cong \mathfrak{sp}(2)$. The kernel of s_2 is spanned by the Killing form together with the submodule $W^{[5]}$ of (3.7). Under the identification $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C}) \cong \Lambda^2\mathbb{C}^5$, we have $W^{[5]} \cong \Lambda^4\mathbb{C}^5$. Thus the locus of points $v \in \Lambda^2\mathbb{C}^5$ for which $s_2(v \otimes v)$ has no component in $W^{[5]}$ defines the Plücker embedding of the Grassmannian $\mathbb{G}r_2(\mathbb{C}^5)$ in $\mathbb{CP}^9$. Therefore $\Phi(Z)$ lies in the intersection of $\mathbb{G}r_2(\mathbb{C}^5)$ with a particular hyperquadric Q in $\mathbb{CP}^9$, and has singular locus

$$\{[v] \in \mathbb{CP}^9 : v \otimes v \in W^{[35]}\} \cong \mathbb{CP}^3,$$

consisting of totally isotropic 2-planes in $\mathbb{C}^5$.

(iii) $I = 9$, $\mathfrak{g} \cong \mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{su}(2)$. Table 1 shows that $\Phi(Z)$ is contained in the singular intersection

$$\{[\lambda, \mu, \nu] \in \mathfrak{g}_{\mathbb{C}} = \mathbb{CP}^8 : \lambda \cdot \lambda = \mu \cdot \mu = \nu \cdot \nu = 0\}$$

of three quadrics, each defined by the Killing form of the respective $\mathfrak{su}(2)$.

(iv) $I = 11$, $\mathfrak{g} \cong \mathfrak{su}(3) + \mathfrak{su}(2)$. The kernel of s_2 has dimension 10, and is spanned by the submodule $V^{[8]}$ of (3.5) and the Killing forms of $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$. The locus of points $v \in \mathfrak{sl}(3, \mathbb{C})$ for which $s_2(v \otimes v) \in V^{[27]}$ defines a three-dimensional flag manifold F in $\mathbb{CP}^7$, corresponding to the fundamental embedding of the twistor space of $\mathbb{CP}^2$. The locus of points $v \in \mathfrak{sl}(2, \mathbb{C})$ for which $s_2(v \otimes v) \in X^{[5]}$ (cf. (3.8)) defines a quadric Q in $\mathbb{CP}^2$. Thus $\Phi(Z)$ is contained in

$$\{[\lambda, \mu] \in \mathbb{P}(\mathfrak{sl}(3, \mathbb{C})^* \oplus \mathfrak{sl}(2, \mathbb{C})^*) = \mathbb{CP}^{10} : [\lambda, 0] \in F, [0, \mu] \in Q\},$$

but if $R \cong \mathbb{CP}^9$ is a generic real hyperplane in $\mathbb{CP}^{10}$, then $\Phi(Z) \cap R$ is singular, which contradicts Lemma 4.1.

A description of the singular quaternionic Kähler eight-folds whose twistor spaces are the above models is implicit in [23]. Cases (i), (ii) and (iii) correspond to quotients of $G_2/SO(4)$, $\mathbb{G}r_2(\mathbb{C}^4)$, and $\mathbb{HP}^2$ by the finite groups $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively. On the other hand, (iv) defines the twistor space of what is locally a quaternionic cone over the self-dual Einstein four-manifold $\mathbb{CP}^2$, and the corresponding quaternionic Kähler metric is not locally symmetric. We expect these models to possess deformations that are twistor spaces of quaternionic non-Kähler manifolds, analogous to the self-dual conformal structures on the connected sum $\mathbb{CP}^3 \# \mathbb{CP}^2$ discovered by the first author [21].

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Rice University
University of Oxford