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ON EMBEDDED COMPLETE MINIMAL SURFACES OF GENUS ZERO

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From the point of view of the classical differential geometry, embedded complete minimal surfaces of finite total curvature in $\mathbf{R}^3$ are interesting objects. However until the last few years no relevant results have been obtained on this field and the only known examples were the plane and the Catenoid.

Some basic properties of these surfaces were described by Jorge and Meeks in [5]. Schoen [10] characterized the Catenoid among the above surfaces as the unique one which has only two ends. The first serious effort to find nontrivial examples was made by Costa [1]. Although he was able to construct the simplest surface of this type, he gave only partial evidence of its embeddedness. The proof of this fact was obtained by Hoffman and Meeks [3] (see [2] for a complete story of this discovery) who also constructed more general examples and gave a nice characterization of those in [4].

Topologically the above surfaces are three times punctured compact surfaces of genus γ, for any $\gamma \geq 1$. No new examples of genus zero have appeared, and it is expected that such a surface does not exist. In this paper we give a proof of this fact. More precisely we will prove that:

> *The plane and the Catenoid are the only embedded complete minimal surfaces of finite total curvature and genus zero in* $\mathbf{R}^3$.

A key step in our reasoning is the proof that for any surface satisfying the hypothesis of the above result we have a one-parameter family of surfaces with the same property. This deformation is also useful in the study of the index of complete minimal surfaces of finite total curvature in $\mathbf{R}^3$ [8].

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1. Preliminaries

In this section we expose some basic results about complete minimal surfaces of finite total curvature in the Euclidean space $\mathbf{R}^3$. More details can be found in [9, Chapter 9],[4, §1].

Let $\psi: M \to \mathbf{R}^3$ be an orientable nonflat complete minimal surface of finite total curvature. We denote by g and ω the meromorphic map and the holomorphic 1-form on M determined by the Weierstrass representation of ψ [9]. Modulo natural identifications, g is the Gauss map of M. We have that $g\omega$ and $g^2\omega$ are holomorphic 1-forms on M and

$$\psi = \text{Real} \int \left((1 - g^2)\frac{\omega}{2}, (1 + g^2)\frac{i\omega}{2}, g\omega \right). \tag{1}$$

It was proved by Osserman [9] that M is conformally equivalent to $\overline{M} - \{p_1, \cdots, p_r\}$, where $\overline{M}$ is a compact Riemann surface. The points p_j correspond to the ends of M. Moreover g and ω extend, in a meromorphic way, to $\overline{M}$. So the normal vector at the ends p_j is defined.

We will assume that the ends of M are all parallel and that each end is embedded (these conditions are necessary for embeddedness of ψ). Then we have the following formula (see [5]):

$$\text{degree}(g) = \text{genus}(\overline{M}) + r - 1. \tag{2}$$

After a rotation, we can suppose that $g(p_j) = 0$ or ∞, $j = 1, \cdots, r$.

If $g(p_j) = 0$, then ω has a pole of order two at p_j, and so $g^2\omega$ is holomorphic at this end. Moreover, as the expression in (1) is well defined, we conclude that ω has no residue at p_j.

Symmetrically, if $g(p_j) = \infty$, then $g^2\omega$ has a pole of order two without residue at p_j, and ω is holomorphic at this end.

Concerning the 1-form $g\omega$, it has a simple pole with real residue at p_j if p_j is not a branch point of g, and it is holomorphic otherwise. In the first case, ψ is asymptotic to a Catenoid at p_j, and the end is called a *Catenoid end.* In the second case, ψ is asymptotic to a plane, and the end is called a *planar end.*

In this paper we also will assume that M is of genus zero, i.e., $\overline{M} = \overline{\mathbf{C}}$. So ω and $g^2\omega$ are exacts. If we put $F = \int \omega/2$, $G = \int g^2\omega/2$, $\phi = \overline{F} - G$, and $h = \text{Real} \int g\omega$, then the immersion ψ is given by

$$\psi(x) = (\phi(x), h(x)) \in \mathbf{C} \times \mathbf{R} = \mathbf{R}^3$$

for any $x \in M$. Moreover F (resp. G) has simple pole at the ends p_j with $g(p_j) = 0$ (resp. $g(p_j) = \infty$), and it is holomorphic at the other points of $\overline{\mathbf{C}}$. The coordinate h is bounded unless precisely at the Catenoid ends of M.

If λ is a positive real number, then we can see easily that the meromorphic map $g_\lambda = \lambda g$ and the holomorphic 1-form $\omega_\lambda = \frac{1}{\lambda}\omega$ determine, via the Weierstrass representation, a complete minimal immersion ψ_λ of M in $\mathbf{R}^3$ with finite total curvature. If we put $\phi_\lambda = \frac{1}{\lambda}\overline{F} - \lambda G$, then $\psi_\lambda \colon M \to \mathbf{C} \times \mathbf{R}$ is given by

$$\psi_\lambda = (\phi_\lambda, h).$$

Note that $g_\lambda(p_j) = 0$ or ∞, $j = 1, \cdots, r$, that each end of ψ_λ is embedded, and that the Catenoid or planar type of an end is independent of λ.

Finally we will need the following result (see Langevin and Rosenberg [6] or Meeks and Rosenberg [7]).

Maximum principle at infinity. *Let M_1 and M_2 be two embedded complete minimal surfaces of finite total curvature and compact boundary in $\mathbf{R}^3$. If* distance$(M_1, M_2) = 0$, *then $M_1 \cap M_2$ is nonempty.*

2. Deformations of embedded minimal surfaces

Let $\psi \colon M \to \mathbf{R}^3$ be an embedded complete nonflat minimal surface of genus zero and finite total curvature, and let $\psi_\lambda \colon M \to \mathbf{R}^3$, $\lambda > 0$, be the deformation described above. In this section we will prove that ψ_λ is an embedding for any λ.

Lemma 1. *Given $x_0 \in \overline{\mathbf{C}}$ and $\lambda_0 \in (0, \infty)$, there are a neighborhood U of x_0 in $\overline{\mathbf{C}}$ and $\epsilon > 0$ such that if $|\lambda - \lambda_0| < \epsilon$, then $\psi_\lambda|_{U \cap M}$ is one-to-one.*

Proof. If $x_0 \in M$ the result is clear.

Suppose now that $x_0 \in \overline{\mathbf{C}} - M$ is an end of M. We can assume that $x_0 = 0$ and that G has a simple pole at the origin. So in some neighborhoods D of x_0 and I of λ_0, we have that

$$G(x) = \frac{a}{x} + G_1(x),$$

with $a \in \mathbf{C} - \{0\}$, G_1 and F being holomorphic in D, and $\phi_\lambda(x) \neq 0$ for all $x \in D - \{0\}$ and $\lambda \in I$.

Then the function $f\colon (D-\{0\})\times I \to \mathbf{C}$ defined by

$$f(x,\lambda)=\frac{1}{\phi_\lambda(x)}=\frac{1}{\frac{1}{\lambda}\overline{F}(x)-\lambda G_1(x)-\lambda a/x}$$
$$=\frac{x}{\frac{x}{\lambda}\overline{F}(x)-\lambda x G_1(x)-\lambda a}$$

extends, in a differentiable way, to $D\times I$. Moreover

$$df_{(0,\lambda_0)}(v,0)=-\frac{v}{\lambda_0 a}\quad \text{for any } v\in\mathbf{C}.$$

We consider the application $\Psi\colon D\times I\to \mathbf{C}\times\mathbf{R}$ given by

$$\Phi(x,\lambda)=(f(x,\lambda),\lambda).$$

As $d\Phi_{(0,\lambda_0)}$ is regular, we conclude that Φ is injective near $(0,\lambda_0)$. This means that $\phi_\lambda|_{U\cap M}$ is one-to-one if $|\lambda-\lambda_0|<\epsilon$, and so we have the same conclusion for the map $\Psi_\lambda|_{U\cap M}$.

Lemma 2. *Let $p_j\in\overline{\mathbf{C}}-M$ be an end of M, λ_0 a positive real number, and $\{\lambda_n\}_{n\in\mathbf{N}}\subset(0,\infty)$, $\{x_n\}_{n\in\mathbf{N}}\subset M$ sequences such that $\lambda_n\to\lambda_0$ and $x_n\to p_j$. Then there exists a sequence $\{x_n'\}_{n\in\mathbf{N}}\subset M$ satisfying that*

$$x_n'\to p_j \text{ and } |\psi_{\lambda_n}(x_n)-\psi_{\lambda_0}(x_n')|\to 0.$$

Proof. We can assume that $p_j=0$ and that G has a simple pole at this point. Choose a neighborhood D of the origin such that

$$G(x)=\frac{a}{x}+G_1(x),\qquad h(x)=b\log|x|+h_1(x)$$

in $D-\{0\}$, with $a\in\mathbf{C}-\{0\}$, $b\in\mathbf{R}$, G_1 and F holomorphic, and h_1 harmonic in D.

As G is invertible near the origin we can consider the sequence $x_n'=G^{-1}(\lambda_n G(x_n)/\lambda_0)$. Observe that $x_n'\to 0$ and $\lambda_0 G(x_n')-\lambda_n G(x_n)=0$, so that

$$\phi_{\lambda_n}(x_n)-\phi_{\lambda_0}(x_n')=\frac{1}{\lambda_n}\overline{F}(x_n)-\frac{1}{\lambda_0}\overline{F}(x_n')\to 0.$$

Moreover for each n we have

$$\frac{a}{x_n'}+G_1(x_n')=\frac{\lambda_n}{\lambda_0}\frac{a}{x_n}+\frac{\lambda_n}{\lambda_0}G_1(x_n),$$

and so

$$\frac{x_n}{x_n'}+\frac{x_n}{a}G_1(x_n')=\frac{\lambda_n}{\lambda_0}+\frac{\lambda_n}{\lambda_0}G_1(x_n)\frac{x_n}{a}.$$

Taking limits we conclude that $x_n/x'_n \to 1$ so that

$$h(x_n) - h(x'_n) = b\log\left|\frac{x_n}{x'_n}\right| + h_1(x_n) - h_1(x'_n) \to 0.$$

Therefore we have proved that

$$\psi_{\lambda_n}(x_n) - \psi_{\lambda_0}(x'_n) \to 0.$$

Proposition 3. *Let $\psi: M \to \mathbf{R}^3$ be an embedded complete minimal surface of finite total curvature and genus zero. Then ψ_λ is an embedding for any positive λ.*

Proof. We reason by contradiction. Suppose that for some λ (we can assume $\lambda > 1$), ψ_λ is not injective. Let

$$\lambda_0 = \text{infimum}\,\{\lambda \geq 1 | \psi_\lambda \text{ if not injective}\}.$$

We discuss the following cases:

(i) ψ_{λ_0} *is injective.* By definition of λ_0 there exist sequences $\{\lambda_n\}_{n\in\mathbf{N}} \subset (0, \infty)$, $\{x_n\}_{n\in\mathbf{N}}$, and $\{y_n\}_{n\in\mathbf{N}} \subset M$, such that $\lambda_n > \lambda_0$, $\lambda_n \to \lambda_0$, $x_n \neq y_n$ and $\psi_{\lambda_n}(x_n) = \psi_{\lambda_n}(y_n)$ for each $n \in \mathbf{N}$.

Without loss of generality, suppose that $\{x_n\}$ and $\{y_n\}$ have limits in $\overline{\mathbf{C}}$, $x_n \to x$ and $y_n \to y$.

If $x = y$, then Lemma 1 gives us a contradiction.

If $x \neq y$, $x, y \in M$, we contradict the injectivity of ψ_{λ_0}.

If $x \in M$ and $y \in \overline{C} - M$, we have that $|\psi_{\lambda_n}(x_n)| \to |\psi_{\lambda_0}(x)|$ and $|\psi_{\lambda_n}(y_n)| \to \infty$, a contradiction.

Finally, if x and y are distinct ends of M, then using Lemma 2 we construct sequences $\{x'_n\}_{n\in\mathbf{N}}$ and $\{y'_n\}_{n\in\mathbf{N}} \subset M$, with $x'_n \to x$ and $y'_n \to y$ such that

$$|\psi_{\lambda_n}(x_n) - \psi_{\lambda_0}(x'_n)| \to 0 \quad \text{and} \quad |\psi_{\lambda_n}(y_n) - \psi_{\lambda_0}(y'_n)| \to 0.$$

Therefore, $|\psi_{\lambda_0}(x'_n) - \psi_{\lambda_0}(y'_n)| \to 0$, which contradicts the maximum principle at infinity.

(ii) ψ_{λ_0} *is not injective.* First note that in this case $\lambda_0 > 1$, and ψ_λ is an embedding for $\lambda \in [1, \lambda_0)$. Given two different points x and y of M, such that $\psi_{\lambda_0}(x) = \psi_{\lambda_0}(y)$, it follows from the maximum principle (see, for example, [10]) that there are neighborhoods of those points with the same image under ψ_{λ_0}. Then we conclude that $N = \psi_{\lambda_0}(M)$ is an orientable embedded complete minimal surface of finite total curvature in $\mathbf{R}^3$, conformally equivalent to $\overline{N} - \{q_1, \cdots, q_s\}$, $\overline{N}$ being a compact Riemann surface. Moreover $\psi_{\lambda_0}: M \to N$ is a finite Riemannian covering, and therefore it has a holomorphic extension to the compact surfaces $\overline{\psi}_{\lambda_0}: \overline{\mathbf{C}} \to \overline{N}$. As $\psi_{\lambda_0}: M \to \mathbf{R}^3$ has embedded ends, we have that $\overline{\psi}_{\lambda_0}$ is also unbranched at each $p_j \in \overline{\mathbf{C}} - M$, and so it has degree one. This contradicts our hypothesis in (ii).

3. The theorem

Note that, using the notation fixed before, given $x, y \in M$ such that $F(x) \neq F(y)$ and $\lambda \in (o, \infty)$ we have that

$$(3) \quad \phi_\lambda(x) = \phi_\lambda(y) \Longleftrightarrow \frac{|F(x) - F(y)|^2}{\lambda^2} = (F(x) - F(y))(G(x) - G(y)).$$

We first state a partial version of our main result.

Lemma 4. *Let $\psi: M \to \mathbf{R}^3$ be an embedded complete minimal surface of finite total curvature and genus zero, and with two Catenoid ends (but, possibly, with other planar ends). Then it must be the Catenoid.*

Proof. We will suppose that $0, \infty \in \overline{\mathbf{C}} - M$ are the Catenoid ends of M. By elementary complex analysis we obtain

$$(4) \qquad h(x) = b \log|x| \quad \text{for some } b \in \mathbf{R} - \{0\}.$$

Define $S_\theta(x)$ for each $\theta \in \mathbf{C}$ with $|\theta| = 1$ a meromorphic function $S_\theta: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$:

$$S_\theta(x) = (F(x) - F(\theta x))(G(x) - G(\theta x)).$$

If S_θ is not a constant function, then it must be onto, and so we can choose $x_0 \in M$ such that $\theta x_0 \in M$ and $S_\theta(x_0) = r^2$ for some positive real number r. Taking $\lambda = \frac{1}{r}|F(x_0) - F(\theta x_0)|$ we conclude from (3) and (4) that $\psi_\lambda(x_0) = \psi_\lambda(\theta x_0)$. As $x_0 \neq \theta x_0$ using Proposition 3 we obtain a contradiction. Therefore S_θ is a finite constant function for every $\theta \in \mathbf{C}$ with $|\theta| = 1$.

If M has a planar end, namely $p_j \in \mathbf{C}$, we can take $\theta \in \mathbf{C}$ with $|\theta| = 1$ such that S_θ must have a simple pole at p_j, which is impossible by the above argument. So M has precisely two ends, and from (2) we see that its Gauss map must be of degree 1. Hence $\psi: M \to \mathbf{R}^3$ is the Catenoid (see [9]).

Theorem. *Let $\psi: M \to \mathbf{R}^3$ be an embedded complete nonflat minimal surface of finite total curvature and genus zero. Then $\psi: M \to \mathbf{R}^3$ is the Catenoid.*

Proof. Let $\overline{\Sigma} \subset \mathbf{CP}^2$ be the projective curve determined by

$$\left\{(x, y) \in \mathbf{C}^2 | [G(x) - G(y)][F(x) - F(y)] = 1\right\},$$

and Σ an irreducible component of $\overline{\Sigma}$. Then the Σ is a compact Riemann surface, and $x, y, F(x), F(y), G(x), G(y): \Sigma \to \overline{\mathbf{C}}$ are meromorphic functions. Let $B \subset \Sigma$ be the finite set formed by the poles of the above six functions, the branch points of x, and the multiple points of Σ in $\mathbf{CP}^2$. Let $M' = \overline{\mathbf{C}} - x(B) \subset M$ and $\Sigma' = x^{-1}(M')$. Then $x: \Sigma' \to M'$ is a finite sheeted topological covering, and Σ' and M' are finitely punctured compact Riemann surfaces. Moreover $x, y, F(x), F(y), G(x), G(y), h(x)$, and $h(y)$ are finite valued functions on Σ', $x(\Sigma')$ and $y(\Sigma')$ are contained in M, $x(p) \neq y(p)$, and $F(x(p)) \neq F(y(p))$ for every point p in Σ'.

If $h(y(p)) = h(x(p))$ for some $p \in \Sigma'$, then taking $\lambda = |F(x(p)) - F(y(p))|$ we have from (3) that $\phi_\lambda(x(p)) = \phi_\lambda(y(p))$, so that $\psi_\lambda(x(p)) = \psi_\lambda(y(p))$ and Proposition 3 give us a contradiction. Therefore the harmonic function $h(x) - h(y)$ has no zeros on Σ'. As Σ' is parabolic, we conclude that $h(y) = h(x) + c$ for some $c \in \mathbf{R} - \{0\}$. In particular for p, $q \in \Sigma'$

$$(5) \qquad x(p) = x(q) \quad \text{implies} \quad h(y(p)) = h(y(q)).$$

We consider now the function $f: \Sigma' \to \mathbf{R}$ defined by

$$f = |F(x) - F(y)|.$$

If there exist two points p and q in Σ' with $x(p) = x(q)$ and $f(p) = f(q)$, then taking $\lambda = f(p)$ it follows from (3) that $\phi_\lambda(x(p)) = \phi_\lambda(y(p))$ and $\phi_\lambda(x(q)) = \phi_\lambda(y(q))$, so that $\phi_\lambda(y(q)) = \phi_\lambda(y(p))$, and using (5) and Proposition 3 we have that $y(p) = y(q)$ and, therefore, $p = q$. Thus f is a continuous function which separates the points in the fibers of the finite covering $x: \Sigma' \to M'$. Therefore x is a conformal diffeomorphism and, so, the same holds for $x: \Sigma \to \overline{\mathbf{C}}$.

By changing the role of x and y we obtain also that the meromorphic function $y: \Sigma \to \overline{\mathbf{C}}$ is of degree one.

In conclusion, the Möbius transformation $H = y \circ x^{-1} : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ satisfies

$$(G(x) - G \circ H(x))\,(F(x) - F \circ H(x)) = 1\,,$$
$$h \circ H(x) = h(x) + c \quad \text{for each } x \in \overline{\mathbf{C}}\,,$$

where $c \neq 0$. From the last identity it follows that H preserves the set of Catenoid ends of M, and that H^n cannot be the identity function on $\overline{\mathbf{C}}$ for any $n \in \mathbf{N}$. So the minimal surface has at most two Catenoid ends. Trivially it must have at least two ends of this type, and using Lemma 4 we conclude the proof of the theorem.

References

[1] C. Costa, *Example of a complete minimal immersion in* $\mathbf{R}^3$ *of genus one and three embedded ends*, Bol. Soc. Brasil. Mat. **15** (1984) 47–54.

[2] D. Hoffman, *The computer aided discovery of new embedded minimal surfaces*, Math. Intelligencer **9** (1987) 8–21.

[3] D. Hoffman & W. H. Meeks III, *A complete embedded minimal surface with genus one, three ends and finite total curvature*, J. Differential Geometry **21** (1985) 109–127.

[4] ——, *Embedded minimal surfaces of finite topology*, Preprint.

[5] L. Jorge & W. H. Meeks III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983) 203–221.

[6] R. Langevin & H. Rosenberg, *A maximum principle at infinity for minimal surfaces and applications*, Duke Math. J. **57** (1988) 819–828.

[7] W. H. Meeks III & H. Rosenberg, *The maximum principle at infinity for minimal surfaces in flat three manifolds*, Preprint.

[8] S. Montiel & A. Ros, *Index of complete minimal surfaces. Schrödinger operator associated with a holomorphic map*, to appear.

[9] R. Osserman, *A survey of minimal surfaces*, 2nd ed., Dover, New York, 1986.

[10] R. Schoen, *Uniqueness, symmetry and embeddedness of minimal surfaces*, J. Differential Geometry **18** (1983) 791–809.

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