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POSITIVE RICCI CURVATURE ON THE CONNECTED SUMS OF $S^n \times S^m$

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0. Introduction and the main results

The topological implications of positive Ricci curvature turned out to be much weaker than what one has expected. For example, it has been shown in [18] that there is no upper bound on the total Betti number for complete Riemannian manifolds with $\mathrm{Ric} > 0$ in a fixed dimension, and the manifold can be of infinite topological type if it is noncompact (compare [1], [12]). In this paper, we prove some existence theorems concerning positive Ricci curvature. It also fills out a gap in [18] in dimensions 4, 5, and 6. Throughout this paper, both n and m will be integers ≥ 2 and we will work in the smooth category. The main results are stated in the following theorems.

Theorem 1. *The connected sum* $\#_{i=1}^k S^n \times S^m$ *of k-copies of* $S^n \times S^m$ *carries a metric with* $\mathrm{Ric} > 0$ *for all* $k = 1, 2, 3, \cdots$, *where* S^p *is the standard p-dimensional sphere.*

Let M^{m+1} be an $(m+1)$-dimensional complete Riemannian manifold with $\mathrm{Ric} > 0$. Set

$$(1) \qquad M_k^{n,m} \equiv S^{n-1} \times \left(M^{m+1} \setminus \coprod_{i=0}^{k} D_i^{m+1}\right) \cup_{\mathrm{Id}} D^n \times \coprod_{i=0}^{k} S_i^m,$$

where D^n, D_i^{m+1} and S^{n-1}, S_i^m, $i = 0, 1, \cdots, k$, are balls and spheres of appropriate dimensions indicated by their superscripts, respectively. $M_k^{n,m}$ is the smooth $(n+1)$-dimensional manifold obtained by removing $(k+1)$-disjoint geodesic balls D_i^{m+1}, $i = 0, 1, 2, \ldots k$, in M^{m+1} and then gluing $S^{n-1} \times (M^{m+1} \setminus \coprod_{i=0}^k D_i^{m+1})$ with $D^n \times \coprod_{i=0}^k S_i^m$ together by

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the identity maps along the corresponding boundaries. Notice that

$$(2)\quad \mathop{\#}_{i=1}^{k} S^n \times S^m \cong S_k^{n,m} \equiv S^{n-1} \times \left(S^{m+1} \setminus \coprod_{i=0}^{k} D_i^{m+1}\right) \cup_{\mathrm{Id}} D^n \times \coprod_{i=0}^{k} S_i^m ,$$

$$(3)\qquad M_k^{n,m} \cong M_0^{n,m} \sharp \left(\mathop{\#}_{i=1}^{k} S^n \times S^m \right),$$

where $\cong$ denotes diffeomorphism. Theorem 1 is a special case of a more general result.

Theorem 2. $M_k^{n,m}$ *carries a complete metric with* Ric > 0 *for all* $k = 0, 1, 2, \ldots$.

In particular, if $M^{m+1} = \mathbf{R}^{m+1}$, set

$$(4)\qquad \mathbf{R}_k^{n,m} \cong \mathbf{R}_0^{n,m} \sharp \left(\mathop{\#}_{i=1}^{k} S^n \times S^m \right).$$

Theorem 3. *The manifold* $\mathbf{R}_k^{n,m}$ *carries a complete metric with* Ric > 0 *for all* $k = \infty, 0, 1, 2, \ldots$.

Remark 1. $\mathbf{R}_\infty^{n,m}$ is of infinite homotopy type. The examples constructed in [18] are of dimension ≥ 7. The theorems here fill out the gap in dimensions 4, 5, and 6. Note also that the group of isometries of $M_k^{n,m}$ contains $O(n)$—the group of orthogonal transformations of dimension n. A generalization of Theorems 2 and 3 is available. See [18] and Remark 5 in §2.

J. Cheeger [7] has constructed metrics with Ric > 0 on the connected sum of two copies of symmetric spaces of rank one, which is actually of nonnegative sectional curvature. In particular, $\mathbf{C}P^2 \sharp (\pm \mathbf{C}P^2)$ carry metrics with Ric > 0 and $K \geq 0$. G. Tian and S. T. Yau [20] have recently proved that $\mathbf{C}P^2 \sharp k(-\mathbf{C}P^2)$ carries Kähler Einstein metric with positive scalar curvature for $3 \leq k \leq 8$. It is an interesting question whether $(k\mathbf{C}P^2) \sharp l(-\mathbf{C}P^2)$ carries a metric with Ric > 0 for all k, $l = 0, 1, 2, 3, \ldots$. The topological classification for smooth closed 1-connected 4-manifolds by S. Donaldson and M. Freedman shows, up to homeomorphism, that $(k\mathbf{C}P^2) \sharp k(-\mathbf{C}P^2)$, $k = 1, 2, 3, \cdots$, are exactly the non-spin smooth closed 1-connected 4-manifolds with zero signature and S^4, $\#_{i=1}^k S^2 \times S^2$, $k = 1, 2, 3, \cdots$, are exactly the spin ones (see [14]). We have

Theorem 4. *Every smoothable closed* 1*-connected* 4*-manifold with zero signature carries a smooth structure and a compatible smooth metric with* Ric > 0.

Remark 2. A closed spin 4-manifold with nonzero signature does not even carry any metric with positive scalar curvature by A. Lichnerowicz [15]. It follows that a closed 1-connected spin 4-manifold carries a metric with $\mathrm{Ric} > 0$ if and only if its signature is zero.

Remark 3. The spin part of Theorem 4 has also been obtained recently by M. Anderson [2] using techniques from gravitational instanton.

Remark 4. Exotic differential structure does exist in dimension 4 in both compact and noncompact cases (cf. [14], [9]).

Smooth closed 1-connected 5-manifolds are classified up to diffeomorphism (cf. [19], [4]). If $H_2(M^5, \mathbf{Z})$ is torsion free, then M^5 is classified up to diffeomorphism by its second Betti number B_2 and second Whitney class ω_2. Assume that $k = B_2(M^5)$. Then M^5 is diffeomorphic to either S^5 $(k = 0)$ or $\#_{i=1}^k S^2 \times S^3$ $(k > 0)$ if $\omega_2 = 0$; or to $S^1 \times (\mathbf{C}P^2 \setminus \coprod_{i=1}^k D_i^4) \cup_{\mathrm{id}} D^2 \times \coprod_{i=1}^k S_i^3$ if $\omega_2 \neq 0$. The following result is therefore a corollary of Theorems 1 and 2.

Theorem 5. *Let M^5 be a smooth closed 1-connected 5-manifold. Assume that $H_2(M^5, \mathbf{Z})$ is torsion free. Then M^5 carries a metric with* $\mathrm{Ric} > 0$.

Since every smooth closed 2-connected 6-manifold is diffeomorphic to S^6 or $\#_{i=1}^k S^3 \times S^3$ (cf [19]), another corollary of Theorem 1 is

Theorem 6. *Every smooth closed 2-connected 6-manifold carries a metric with* $\mathrm{Ric} > 0$.

Combining with the results of J. Nash [17] and Bérard Bergery [5] on the existence of metrics with $\mathrm{Ric} > 0$ on certain fiber bundles, e.g., sphere bundles, over a manifold which carries a metric with $\mathrm{Ric} > 0$, the above theorems show that a substantial number of smooth closed 1-connected 6-manifolds carry metrics with $\mathrm{Ric} > 0$. In particular, an appropriate application of Theorem 2 shows that

$$(5) \qquad \left(\mathop{\#}_{i=1}^{k} S^2 \times S^4\right) \sharp \left(\mathop{\#}_{j=1}^{l} S^3 \times S^3\right), \qquad k, l = 0, 1, 2, \dots,$$

also carry metrics with $\mathrm{Ric} > 0$.

Our results can all be formulated as existence theorems of metrics with $\mathrm{Ric} > 0$ on certain connected sums of simply connected manifolds which carry metrics with $\mathrm{Ric} > 0$. Recall that the connected sum of two manifolds with positive scalar curvature carries a metric with positive scalar curvature. One is naturally tempted to ask whether this is also true for positive Ricci curvature if both of the two manifolds are closed and simply connected. Does a closed simply connected manifold with positive scalar curvature also carry a metric with $\mathrm{Ric} > 0$? While many experts expressed their doubts to a positive answer, counterexamples are not known.

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1. The main lemma

Let ds_m^2 be the Riemannian metric with constant sectional curvature $K \equiv 1$ on the m-dimensional sphere S^m. Let $D^n(N) = \{x \in \mathbf{R}^n | x = (x_1, x_2, \cdots, x_n), \sum_{i=1}^n x_i^2 \le N^2\}$ be the ball in $\mathbf{R}^n$ of radius N. For convenience, we use polar coordinate $(r, y) \in [0, N] \times S^{n-1}/\sim = D^n(N)$ and write D^n for $D^n(N)$.

Lemma 1. *Suppose that* $n, m \ge 2$ *and* $\delta \in (0, \frac{\pi}{2})$. *There exists a constant* $R(n, m, \delta)$ *such that for any* $R \ge R(n, m, \delta)$, $N = \delta^{-1} R$, *there is a smooth warped product metric of the form*

$$g_R = dr^2 + h_R^2(r)\, ds_{n-1}^2 + f_R^2(r)\, ds_m^2 \tag{6}$$

on $D^n(\frac{\pi}{n} N) \times S^m$ *with* $\mathrm{Ric} \ge 0$. *Moreover, there are numbers* $0 < R_0 < R_1 < R$ *such that the Ricci curvature tensor is positive definite on* $(R_0, R_1) \times S^{n-1} \times S^m$ *and*

$$g_R = dr^2 + ds_{n-1}^2 + N^2 \sin^2 \frac{r}{N} \cdot ds_m^2 \tag{7}$$

on $[R, \frac{\pi}{2} N] \times S^{n-1} \times S^m \subset D^n(\frac{\pi}{2} N) \times S^m$.

Consider a warped product metric of the form

$$g = dr^2 + h^2(r)\, ds_{n-1}^2 + f^2(r)\, ds_m^2 \tag{8}$$

on $D^n \times S^m$. Let v and w be unit integral vectors of (S^{n-1}, ds_{n-1}^2) and (S^m, ds_m^2), respectively. Then $U = \partial/\partial r$, $V = h^{-1} \cdot v$, and $W = f^{-1} \cdot w$ are orthonormal tangent vectors of $(D^n \times S^m, g)$. A straightforward calculation gives the Ricci curvature tensor of g.

$$\mathrm{Ric}(U, V) = \mathrm{Ric}(V, W) = \mathrm{Ric}(W, U) = 0, \tag{9}$$

$$\mathrm{Ric}(U, U) = -(n-1)h^{-1}h' - mf^{-1}f'', \tag{10}$$

$$\mathrm{Ric}(V, V) = -h^{-1}h'' + (n-2)h^{-2}[1 - (h')^2] - mh^{-1}h'f^{-1}f', \tag{11}$$

$$\begin{aligned} \mathrm{Ric}(W, W) = & -f^{-1}f'' - (n-1)h^{-1}h'f^{-1}f' \\ & + (m-1)f^{-2}[1 - (f')^2]. \end{aligned} \tag{12}$$

Observe that the warped product metric (8) is smooth if

(a) both f and h are smooth functions on $[0, \frac{\pi}{2}N]$ and positive on $(0, \frac{\pi}{2}N]$,

(b) f is an even function at $r = 0$ and h is odd at $r = 0$, $f(0) = h'(0) = 1$.

We will define f and h essentially by the solution of a second order ordinary differential equation (see [6], [16]). Conditions (a) and (b) will be satisfied automatically.

To begin with, let $\psi \colon \mathbf{R} \to [0, 1]$ be a nonincreasing smooth function such that

$$\psi(r) = \begin{cases} 1 & \text{for } r \le 0, \\ 0 & \text{for } r \ge 1, \end{cases} \tag{13}$$

and

$$\psi'(r) < 0 \quad \text{for } 0 < r < 1. \tag{14}$$

Set

$$\alpha = 2(m-1)/n. \tag{15}$$

Let $t > 0$ be the constant defined by

$$\cos^2 \frac{\delta}{2} = \alpha \int_1^\infty y^{-\alpha-1} \psi(y-t)\, dy. \tag{16}$$

It is obvious that t increases without bound as δ approaches zero.

Let f be the unique solution of the second order initial value problem

$$\begin{aligned} y'' &= \tfrac{\alpha}{2} y^{-\alpha-1} \cdot \psi(y-t), \\ y(0) &= 1, \qquad y'(0) = 0. \end{aligned} \tag{17}$$

Set

$$h(r) = \frac{2}{\alpha} f'(r). \tag{18}$$

A simple calculation shows that the Ricci curvature equations (10), (11), and (12) are nonnegative and positive on $[r_0, r_1] \times S^{n-1} \times S^m$, where $r_0 < r_1$ are determined by the equations

$$f(r_0) = t \quad \text{and} \quad f(r_1) = t + 1. \tag{19}$$

Multiplying

$$f''(r) = \frac{\alpha}{2} f^{-\alpha-1}(r) \cdot \psi(f(r) - t) \tag{20}$$

by $f'(r)$ and integrating, we have

$$f'(r) = \left[\cos^2\frac{\delta}{2} - \alpha\int_{f(r)}^{\infty} y^{-\alpha-1}\psi(y-t)\,dt\right]^{1/2}. \tag{21}$$

We now restrict our attention to $r \geq r_1$. It follows from (19) and (21) that $f(r) \geq t+1$, $f'(r) = \cos\frac{\delta}{2}$, $h(r) = \frac{2}{\alpha}\cos\frac{\delta}{2}$, and $h'(r) = 0$.

To adapt the metric g to the boundary condition (7), we keep h unchanged and modify the function f for $r \geq r_1$ so that the Ricci curvature remains nonnegative. Since h is a constant function for $r \geq r_1$, the Ricci curvature equations (10), (11), and (12) are reduced to

$$R(U, U) = -mf^{-1}f'', \tag{22}$$

$$R(V, V) = \frac{n-2}{4}\alpha^2\sec^2\frac{\delta}{2} \geq 0, \tag{23}$$

$$R(W, W) = - = f^{-1}f'' + (m-1)f^{-2}[1-(f')^2]. \tag{24}$$

So the Ricci curvature will be nonnegative if

$$f''(r) \leq 0 \leq f'(r) \quad \text{for } r \geq r_1. \tag{25}$$

Consider the function

$$k(r) = \frac{r\delta^{-1}\sin\delta - f(r_1)}{r - r_1}. \tag{26}$$

Since

$$\lim_{r\to\infty} k(r) = \delta^{-1}\sin\delta \tag{27}$$

and

$$\cos\delta < \delta^{-1}\sin\delta < \cos\tfrac{\delta}{2}, \tag{28}$$

there exists $r_2 > r_1$, depending only on δ and r_1 and therefore only on n, m, and δ, such that for all $r \geq r_2$,

$$\cos\delta < k(r) < \cos\tfrac{\delta}{2}. \tag{29}$$

Now for any $r_3 \geq r_2$, let $N_3 = \delta^{-1}r_3$. It is obvious that one may smoothly bend the function f for $r > r_1$ such that

$$f''(r) \leq 0 \quad \text{for } r \in [r_1, r_3], \tag{30}$$

$$f(r) = N_3\sin(r/N_3) \quad \text{for } r \in [r_3, \tfrac{\pi}{2}N_3], \tag{31}$$

where

$$k(r_3) = \frac{r_3\delta^{-1}\sin\delta - f(r_1)}{r_3 - r_1} \tag{32}$$

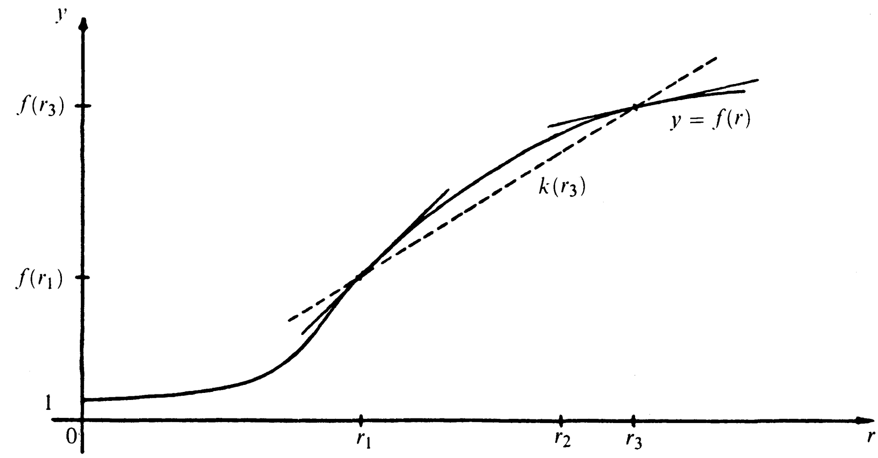

FIGURE 1. THE CONSTRUCTION OF THE FUNCTION $f(r)$

is the slope of the dotted secant line through the two points $(r_1, f(r_1))$ and $(r_3, r_3\delta^{-1}\sin\delta)$ (see Figure 1).

We remark that r_2 is essentially the constant $R(n, m, \delta)$ in the lemma.

It follows that the metric (8) thus defined is a smooth metric on $D^n(\frac{\pi}{2}N_3) \times S^m$ with $\mathrm{Ric} \geq 0$ and $\mathrm{Ric} > 0$ on $(r_0, r_1) \times S^{n-1} \times S^m$, and

$$g = dr^2 + \frac{4}{\alpha^2}\cos^2\frac{\delta}{2} \cdot ds_{n-1}^2 + N_3^2 \sin^2\frac{r}{N_3} \cdot ds_m^2 \tag{33}$$

is a smooth metric on $[r_3, \frac{\pi}{2}N_3] \times S^{n-1} \times S^m$. The lemma is therefore obtained by scaling the metric g by the constant $\frac{\alpha^2}{4}\sec^2\frac{\delta}{2}$.

2. The construction of metrics with $\mathrm{Ric} > 0$

As noticed in (2) of §0, $\#_{i=0}^k S^n \times S^m$ is diffeomorphic to

$$S_k^{n,m} \equiv S^{n-1} \times \left(S^{m+1} \backslash \coprod_{i=0}^{k} D_i^{m+1}\right) \cup_{\mathrm{Id}} D^n \times \coprod_{i=0}^{k} S_i^m.$$

For any positive integer k, choose δ such that $0 < \delta < \pi/(k+1)$. Let $R = R(n, m, \delta)$ and $N = \delta^{-1}R(n, m, g)$ be the constants as in the lemma. Let $D^n \times S_i^m$, $i = 0, 1, 2, \cdots, k$, be $(k+1)$-copies of $D^n(R) \times S^m \subset D^n(\frac{\pi}{2}N) \times S^m$ and g_i the restriction of g_R onto $D^n \times S_i^m$. The scaled round sphere $(S^{m+1}, N^2 ds^2_{m+1})$ contains $(k+1)$-disjoint geodesic balls D_i^{m+1}, $i = 0, 1, 2, \cdots, k$, of radius R. It is obvious that the restriction of the product metric $ds^2_{n-1} + N^2 ds^2_{M+1}$ onto $S^{n-1} \times (S^{m+1} \setminus \coprod_{i=0}^k D_i^{m+1})$ is extended to a smooth metric g with $\mathrm{Ric} \geq 0$ on $S_k^{n,m}$ by setting $g = g_i$ on $D^n \times S_i^m$, $i = 0, 1, 2, \cdots, k$. Since $\mathrm{Ric} > 0$ somewhere, one may deform g to a metric with $\mathrm{Ric} > 0$ everywhere (see [3], also [10]). This proves Theorem 1.

For a proof of the second theorem, one first notices a metric deformation result of [11], which states that if M carries a metric with $\mathrm{Ric} > 0$, then for any point $p \in M$, one can deform the metric in a small neighborhood U of p to a metric with constant sectional curvature in a smaller neighborhood $V \subset U$ of p while keeping Ricci curvature positive on M.

Now let (M^{m+1}, g) be a complete Riemannian manifold with $\mathrm{Ric} > 0$. One may therefore assume that M contains a geodesic ball $D^{m+1}(\varepsilon)$ of radius $\varepsilon > 0$ with constant sectional curvature 1 on $D^{n+1}(\varepsilon)$. Choose $\delta \in (0, \varepsilon/(k+1))$, and let $R = R(n, m, \delta)$ and $N = \delta^{-1}R$. Then $(M^{m+1}, N^2 g)$ is of positive Ricci curvature, and $D^{m+1}(\varepsilon)$ contains $(k+1)$-disjoint geodesic balls D_i^{m+1}, $i = 0, 1, 2, \cdots, k$, with radius R and constant sectional curvature $K = N^{-2}$. Proceeding as in the proof of Theorem 1, one obtains a metric with $\mathrm{Ric} > 0$ on $M_k^{n,m}$.

For the construction of a metric with $\mathrm{Ric} > 0$ on $\mathbf{R}_\infty^{n,m}$, we refer to [18].

Remark 5. One can replace D^n in the lemma in §1 by a disc bundle of rank ≥ 2 over a closed Riemannian manifold with $\mathrm{Ric} > 0$ and thus obtain a generalization of Theorems 2 and 3. The examples in [18] are actually of this kind.

The spin part of Theorem 4 and Theorems 5 and 6 are corollaries of the relevant classification theorems and Theorems 1 and 2. We devote the next section to a proof of the nonspin part of Theorem 4.

3. Deform the metric on $\mathbf{C}P^2 \sharp (-\mathbf{C}P^2)$

In [7], J. Cheeger constructed metrics with $K \geq 0$ and $\mathrm{Ric} > 0$ on the connected sum of two copies of simply connected Riemannian symmetrical spaces of rank one, in particular, on $\mathbf{C}P^2 \sharp (-\mathbf{C}P^2)$. Topologically, $\mathbf{C}P^2 \sharp (-\mathbf{C}P^2)$ is the nontrivial S^2 bundle over S^2 which can be described in the following way.

Let $S^1 \to S^3 \to S^2$ be the Hopf fibration. Consider the associated fibration

$$S^1 \times [0, \pi] \to S^3 \times [0, \pi] \to S^2 \tag{34}$$

over S^2 with fiber $S^1 \times [0, \pi]$. $\mathbf{C}P^2 \sharp(-\mathbf{C}P^2) = S^3 \times [0, \pi]/\sim$ is the resulting manifold obtained by identifying each component of the boundary of each fiber with a point.

Use the Euclidean coordinate for $S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbf{R}^4 ;\ x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}$. Let

$$X_1 = -y_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_2}, \tag{35}$$

$$X_2 = -x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} + x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}, \tag{36}$$

$$X_3 = -y_2\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial y_2}. \tag{37}$$

X_1, X_2, X_3 form a global orthonormal frame for S^3. One may assume that X_1 is the unit tangent vector field of the Hopf fibration. Let ω_1, ω_2, ω_3 be the dual of X_1, X_2, X_3. Let θ be the parameter for $[0, \pi]$. Then the smooth metric

$$d\theta^2 + \sin^2\theta \cdot \omega_1^2 + \omega_2^2 + \omega_3^2 \tag{38}$$

on $\mathbf{C}P^2 \sharp(-\mathbf{C}P^2)$ is of nonnegative sectional curvature and positive Ricci curvature.

It is convenient to use the normal polar coordinate of a geodesic circle for S^3. Set

$$x_1 = \cos r \cdot \cos\theta_1, \qquad y_1 = \sin r \cdot \sin\theta_1, \tag{39}$$

$$x_2 = \sin r \cdot \cos\theta_2, \qquad y_2 = \sin r \cdot \sin\theta_2, \tag{40}$$

where $r \in [0, \frac{\pi}{2}]$, θ_1, $\theta_2 \in [0, 2\pi]$. The metric (38) in the new coordinate is given by

$$\begin{aligned} d\theta^2 + \tfrac{1}{4}\sin^2\theta[d\theta_1 + d\theta_2 + \cos 2r(d\theta_1 - d\theta_2)]^2 \\ + dr^2 + \sin^2 r \cdot \cos^2 r \cdot (d\theta_1 - d\theta_2)^2. \end{aligned} \tag{41}$$

Our idea is to deform the above metric on $\mathbf{C}P^2 \sharp(-\mathbf{C}P^2)$ to a metric g_0 with $\mathrm{Ric} \geq 0$ such that $D^3_{\pi N/4} \times S^1 \subset (S^3 \times S^1, N^2 ds_3^2 + dt^2)$ is isometrically embedded in $(\mathbf{C}P^2 \sharp(-\mathbf{C}P^2), g_0)$, where $D^3_{\pi N/4}$ is a geodesic ball in $(S^3, N^2 ds_3^2)$ with radius $\pi N/4$.

Thus we consider a metric of the form

$$f^2(r)\{d\theta^2 + \tfrac{1}{4}\sin^2\theta[d\theta_1 + d\theta_2 + h'(r)(d\theta_1 - d\theta_2)]^2\} + dr^2 + h^2(r)(d\theta_1 - d\theta_2)^2, \tag{42}$$

where $r \in [-R, R]$, $\theta \in [0, \pi]$, $\theta_1, \theta_2 \in [0, 2\pi]$, f is a positive smooth function on $[-R, R]$ which is even at the two ends, and h is smooth, positive on $(-R, R)$, odd at the two ends, and $h'(-R) = -h'(R) = 1$.

Using homogeneous coordinates, it is a direct computation to check that the metric (42) defines a smooth metric on $\mathbf{C}P^2\sharp(-\mathbf{C}P^2)$.

To write down the Ricci curvature tensor, let Y_0, Y_1, Y_2, Y_3 be an orthonormal frame of the metric (42):

$$Y_0 = f^{-1}\frac{\partial}{\partial\theta}, \tag{43}$$

$$Y_1 = \csc\theta \cdot f^{-1}\cdot\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right), \tag{44}$$

$$Y_2 = \frac{1}{2}h^{-1}\left\{\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2} - h'\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right)\right\}, \tag{45}$$

$$Y_3 = \frac{\partial}{\partial r}. \tag{46}$$

Then

$$\mathrm{Ric}(Y_i, Y_j) = 0 \quad \text{for } i < j,\ (i, j) \neq (1, 2), \tag{47}$$

$$\mathrm{Ric}(Y_1, Y_2) = -\tfrac{1}{4}\sin\theta\{f\cdot\tfrac{\partial}{\partial r}(h^{-1}h'') + 4f'\cdot h^{-1}h''\}, \tag{48}$$

$$\mathrm{Ric}(Y_0, Y_0) = f^{-2} - (f^{-1}f')^2 - f^{-1}f'\cdot h^{-1}h' - f^{-1}f'', \tag{49}$$

$$\mathrm{Ric}(Y_1, Y_1) = \mathrm{Ric}(Y_0, Y_0) + \tfrac{1}{8}[fh^{-1}h''\cdot\sin\theta]^2, \tag{50}$$

$$\mathrm{Ric}(Y_2, Y_2) = -h^{-1}h'' - 2f^{-1}f'h^{-1}h' - \tfrac{1}{8}[fh^{-1}h''\sin\theta]^2, \tag{51}$$

$$\mathrm{Ric}(Y_3, Y_3) = -h^{-1}h'' - 2f^{-1}f'' - \tfrac{1}{8}[fh^{-1}h''\sin\theta]^2. \tag{52}$$

A slight modification of the construction in §1 for $n = m = 2$ and $\alpha = 7 - 4\sqrt{2}$ gives the following lemma.

Lemma 2. *For any $N_0 > 0$, there exists a positive constant $N \geq N_0$ and a smooth metric g_0 of the form* (42) *on $\mathbf{C}P^2\sharp(-\mathbf{C}P^2)$ with* $\mathrm{Ric} \geq 0$ *such that for $r \in [-\frac{\pi}{4}N, \frac{\pi}{4}N]$*

$$g_0 = N^2\cos^2\frac{r}{N}\left\{d\theta^2 + \frac{1}{4}\sin^2\theta(d\theta_1 + d\theta_2)^2\right\} + dr^2 + \frac{1}{4}(d\theta_1 - d\theta_2)^2. \tag{53}$$

Corollary. $D^3_{\pi N/4} \times S^1$ *with the product metric is isometrically embedded in* $(\mathbf{C}P^2\sharp(-\mathbf{C}P^2),\ g_0)$

Notice that any nonspin smooth closed simply connected 4-manifold M with zero signature is homeomorphic to $\mathbf{C}P^2\sharp(-\mathbf{C}P^2)\sharp(\#^k_{i=1} S^2 \times S^2)$ for some nonnegative integer k by S. Donaldson and M. Freedman. Now by choosing N_0 sufficiently large and applying Lemma 1 k-times on $(\mathbf{C}P^2\sharp(-\mathbf{C}P^2),\ g_0)$, we obtain a smooth metric with $\mathrm{Ric} > 0$ on M. This completes the proof of Theorem 4.

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