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UNIQUENESS OF THE COMPLEX STRUCTURE ON KÄHLER MANIFOLDS OF CERTAIN HOMOTOPY TYPES

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1. Introduction

In this note we show that the homotopy types of certain complex projective spaces and quadrics support a unique complex structure of Kähler type. Structures on complex projective space have attracted much attention. Hirzebruch and Kodaira [14], [11, p. 231] showed that a Kähler manifold V with the homotopy type and Pontryagin classes of CP_n is analytically equivalent to CP_n; their additional assumption that $c_1(V) \neq -(n-1)x$ for even n was later removed by Yau's work [31]. Here x denotes the generator of $H^2(V; Z)$ which is positive in the sense that it is the fundamental class of some Kähler metric on V [13, §18.1]. On the other hand it is known that for every $n > 2$ the homotopy type of CP_n supports infinitely many inequivalent differentiable structures distinguished by their Pontryagin classes (see Montgomery and Yang [25] or Wall [30] for $n = 3$ and Hsiang [15] for $n > 3$). Moreover for $n = 3$ or 4 each of these smooth structures can be shown to support almost complex structures. In §7 we prove this for the case $n = 4$ by applying results of Brumfiel and Heaps. The main result of this paper is that for $n \leq 6$ these other smoothings of a homotopy CP_n do not support a Kähler structure.

Theorem 1. *A Kähler manifold homotopy equivalent to CP_n for $n \leq 6$ is analytically equivalent to CP_n.*

It follows from the Kodaira embedding theorem that any homotopy complex projective space with a Kähler structure is projective algebraic, i.e., is analytically equivalent to a nonsingular subvariety of a higher dimensional projective space [13, §18.1]. In [31], Yau applied a criterion of Kodaira to show that a complex manifold homotopy equivalent to CP_2 is algebraic (hence Kähler) and showed moreover that it is analytically equivalent to CP_2. It is still an open question whether a complex manifold

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homotopy equivalent to CP_3 is algebraic. If this were so it would settle another open question by showing that S^6 has no complex structure (cf. [11, p. 223]).

Fujita [6], [7, Appendix 2] proved Theorem 1 for $n \leq 5$ under the additional assumption that $c_1(V)$ is positive; this paper extends his techniques. In [20] Lanteri and Struppa studied algebraic manifolds with the homology of CP_3 and proved Theorem 1 for $n = 3$. The three-dimensional case uses Iskovskih's results on the classification of Fano 3-folds; we return to this in §3.

The result of Hirzebruch and Kodaira on CP_n has been extended to the quadric $X_n(2)$ by Brieskorn [2] and Morrow [26] who showed that for $n > 2$ a Kähler manifold V with the homotopy type and Pontryagin classes of $X_n(2)$ is analytically equivalent to $X_n(2)$ provided $c_1(V) \neq -nx$ when n is even. For dimension two, however, Hirzebruch [10] exhibited infinitely many distinct complex algebraic structures on $S^2 \times S^2$, the smooth manifold underlying the quadric $X_2(2)$. For the quadric we have the following result.

Theorem 2. *If V_n is a Kähler manifold homotopy equivalent to $X_n(2)$, then*

(a) *for $n = 3$, V is analytically equivalent to $X_3(2)$,*
(b) *for $n = 4$, V is analytically equivalent to $X_4(2)$ provided $c_1(V) \neq -4x$.*

For hypersurfaces of degree greater than two the space of deformations of the complex structure has positive dimension, so there can be no analytic uniqueness [18, §14]. The same fact for complete intersections of degree greater than two can be deduced from the formula for the moduli dimension in [23]. One can ask for conditions on a complex (or Kähler or algebraic) manifold, which imply that any such manifold homotopy equivalent to it lie in the same component of the moduli space of complex structures or at least have a diffeomorphic underlying smooth manifold. Complete intersections provide a class of algebraic spaces whose homotopy type is nearly as simple as projective space and the quadric [21]. But they include homotopy equivalent 3-folds which are not diffeomorphic [22, Example 9.2] and diffeomorphic 3-folds not related by deformation [22, Example 9.3]. For algebraic surfaces the study of Donaldson's invariants by Friedman, Moishezon, and Morgan [5] shows that for complete intersections of even geometric genus p_g, the divisibility of c_1 is a diffeomorphism invariant. Using the search procedure described in [22, §9] one can find pairs of homeomorphic 2-dimensional complete intersections which are not diffeomorphic by this criterion. The pair with the smallest values of c_1 has multidegrees $(9, 5, 3, 3, 3, 3, 3, 2, 2)$ with $c_1 = -21$ and $(10, 7, 7, 6, 6, 3, 3)$ with $c_1 = -27$.

To prove Theorems 1 and 2 we use results of Kobayashi and Ochiai who showed [17] that if V is an n-dimensional Kähler manifold with $c_1(V) \geq (n+1)x$, where x is positive, then V is analytically equivalent to CP_n, and if $c_1(V) = nx$, then V is analytically equivalent to $X_n(2)$. For a Kähler manifold, V, homotopy equivalent to CP_n or $X_n(2)$ the Hodge numbers and hence the χ_y-genus of V are uniquely determined. The Riemann-Roch theorem then yields equations satisfied by the Chern numbers. The most striking of these is a formula, in Proposition 2.3 below, for $c_{n-1}c_1$ which generalizes to higher dimensions Todd's formula for c_2c_1 [29, p. 215] and Hirzebruch's formula for c_3c_1 [12, p. 124], [13, §0.6(8)], and which implies the following.

Theorem 3. *For a compact complex manifold V, the Chern number $c_{n-1}c_1[V]$ is determined by $\chi_y(V)$ and hence by the Hodge numbers.*

This is proved in §2. Additional restrictions are imposed by various integrality conditions such as the mod 2 invariance of the Chern classes or the mod 24 invariance of Pontryagin classes and by Yau's inequality. To complete the proof of Theorems 1 and 2 it remains to check that the first Chern class is uniquely determined by these conditions. These ad hoc methods work through complex dimension six. The authors hope more general methods may give a more general result. §§3–6 treat dimensions 3–6 respectively. Almost complex structures are studied in §7. We thank A. O. L. Atkin for suggesting the algorithm described at the end of §2 and for recommending the use of quadratic residues in §6.

2. The generalized Todd genus

Let $Q(x) = 1 + q_1x + q_2x^2 + \cdots$ be the characteristic power series for the multiplicative sequence K_n, where $K_n(c_1, \cdots, c_n)$ is a homogeneous polynomial of weight n in $c_1, \cdots, c_n$ where c_i has weight i. The identity

$$(1 + c_1 + c_2 + \cdots)(1 + \gamma) = 1 + (\gamma + c_1) + (\gamma c_1 + c_2) + \cdots$$

implies [13, §1.2]

$$\begin{aligned}(1 + K_1(c_1) + \cdots + K_n(c_1, \cdots, c_n) + \cdots)Q(\gamma) \\ = 1 + K_1(\gamma + c_1) + K_2(\gamma + c_1, \gamma c_1 + c_2) + \cdots \\ + K_n(\gamma + c_1, \cdots, \gamma c_{n-1} + c_n) + \cdots .\end{aligned}$$

Hence $K_1(c_1) + q_1\gamma = K_1(c_1 + \gamma)$. Since K_1 has weight one, $K_1(c_1)$ is a multiple of c_1, so we must have $K_1(c_1) = q_1c_1$. If $K_1, \cdots, K_{n-1}$ are known, the identity

$$(2.1)\qquad \begin{aligned} &K_n(\gamma + c_1, \gamma c_1 + c_2, \cdots, \gamma c_{n-1} + c_n) - K_n(c_1, \cdots, c_n) \\ &\quad = q_n\gamma^n + \cdots + q_1\gamma K_{n-1}(c_1, \cdots, c_{n-1}) \end{aligned}$$

uniquely determines $K_n(c_1, \cdots, c_n)$, for example the coefficient of c_1^n must be q_n. This uniqueness is a restatement of [13, Lemma 1.2.1]. At the end of this section we describe a recursive procedure for computing $K_n(c_1, \cdots, c_n)$.

The χ_y-genus of an n-dimensional compact complex manifold V_n is defined in terms of the Hodge numbers of V by

$$\chi^p(V) = \sum_{q=0}^{n}(-1)^q h^{p,q}, \qquad \chi_y(V) = \sum_{p=0}^{n}\chi^p(V)y^p.$$

By the Riemann-Roch theorem χ_y is given in terms of the Chern numbers by the multiplicative sequence $T_n(y; c_1, \cdots, c_n) = \sum_{p=0}^{n} T_n^p y^p$. Because of the symmetry relations [13, 1.8(13)] the χ_y-genus imposes $\lfloor (n+2)/2 \rfloor$ independent conditions on the Chern numbers of V:

$$\chi^p(V_n) = T_n^p(V) \quad \text{for } 0 \le p \le \lfloor n/2 \rfloor.$$

That these conditions are independent can be seen by noting that the polynomials

$$\chi_y(CP_j \times CP_{n-j}) = \frac{(1 + (-1)^j y^{j+1})(1 + (-1)^{n-j} y^{n-j+1})}{(1+y)^2}$$

are independent for $0 \le j \le \lfloor n/2 \rfloor$.

For our purpose it is easier to use the variable $z = y + 1$; we set

$$t_n(z; c_1, \cdots, c_n) = T_n(z - 1; c_1, \cdots, c_n).$$

Then the characteristic power series for t_n is

$$q(z; x) = 1 + \left(1 - \frac{1}{2}z\right)x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}(xz)^{2k}.$$

Lemma 2.2. *$t_n(z; c_1, \cdots, c_n)$ is a polynomial in z of degree n with initial terms*

$$c_n - \frac{1}{2}nc_n z + \frac{1}{12}\left\{\frac{1}{2}n(3n-5)c_n + c_{n-1}c_1\right\}z^2 + \cdots.$$

Proof. The sequence $t_n(z; c_1, \cdots, c_n)$ is determined by (2.1). In particular $t_1(z; c_1) = (1 - \frac{1}{2}z)c_1$. Assuming the formula is correct for $n < k$, it is straightforward to check that the formula for $t_k(z; c_1, \cdots, c_k)$ satisfies (2.1) through terms of the second degree in z, i.e., to check that

$$\begin{aligned} t_k(z; \gamma + c_1, \cdots, \gamma c_{k-1} + c_k) - t_k(z; c_1, \cdots, c_k) \\ = \left(1 - \frac{1}{2}z\right) \gamma t_{k-1}(z; c_1, \cdots, c_{k-1}) \\ + \frac{1}{12}(\gamma z)^2 t_{k-2}(z; c_1, \cdots, c_{k-2}) + \cdots \end{aligned}$$

holds through terms of degree two in z.

A precise version of Theorem 3 is the following.

Proposition 2.3. *For a compact complex manifold V,*

$$\sum_{p=2}^{n}(-1)^p \binom{p}{2} \chi^p(V_n) = \frac{1}{12}\left\{\frac{1}{2}n(3n-5)c_n + c_{n-1}c_1\right\}[V_n].$$

Proof. Using the Riemann-Roch formula we have

$$\begin{aligned} t_n(z)[V] = T_n(z-1)[V] = \sum_{p=0}^{n} \chi^p(V)(z-1)^p \\ = \sum_{p=0}^{n}(-1)^p \chi^p(V) - z\sum_{p=1}^{n}(-1)^p \binom{p}{1}\chi^p(V) \\ + z^2 \sum_{p=2}^{n}(-1)^p \binom{p}{2}\chi^p(V) - + \cdots . \end{aligned}$$

By the lemma, equating the coefficients of z^2 gives the proposition. The identity for the Euler characteristic [13, Theorem 15.8.1],

$$c_n[V] = \sum_{p=0}^{n}(-1)^p \chi^p(V),$$

is given by equating the constant terms.

Remark 2.4. It follows from the proposition and Milnor's result that algebraic varieties generate the cobordism ring of almost complex manifolds [12, §6 and §7(1)], that the Chern numbers of any almost complex $2n$-manifold satisfy the congruence

$$\tfrac{1}{2}n(3n-5)c_n + c_{n-1}c_1 \equiv 0 \pmod{12}.$$

Corollary 2.5. *If V is a Kähler manifold with the same Betti numbers as CP_n, then*

$$c_{n-1}c_1[V] = \tfrac{1}{2}n(n+1)^2.$$

If the Betti numbers are the same as $X_n(2)$ *for* n *even, then* $c_{n-1}c_1[V] = \frac{1}{2}n^2(n+2)$.

Proof. In each case the relation between the Hodge numbers and the Betti numbers of a Kähler manifold [13, §5.7], the relation $h^{r,s} = h^{s,r}$, and the inequality $h^{r,r} \geq 1$ for $0 \leq r \leq n$ imply that the Hodge numbers of V are the same as those of CP_n or $X_n(2)$ respectively. It follows from the proposition that $c_{n-1}c_1[V]$ is also the same as the corresponding Chern numbers of CP_n or $X_n(2)$.

Remarks. Lemma 2.2 can be generalized in a qualitative way as follows. Set $t_n(z; c_1, \cdots, c_n) = \sum_{p=1}^{n} t_n^p(c_1, \cdots{}_n)z^p$. The polynomials t_n^p are homogeneous of weight n in the Chern classes. The initial terms, t_n^0 and t_n^1, involve only c_n. For $k \geq 1$ and $p \leq 2k+1$ each term of t_n^p is divisible by some c_j with $j \geq n - 2k + 1$.

The $\lfloor (n+2)/2 \rfloor$ independent conditions on the Chern numbers are given by the coefficients of even powers of z. For p odd, t_n^p is a linear combination of $t_n^1, \cdots, t_n^{p-1}$.

The coefficient of z^n is $t_n^n = T_n^n = (-1)^n T_n^0$; the Todd polynomials, T_n^0, are listed for $1 \leq n \leq 6$ in Todd's paper [29] and in [13, p. 14].

A. O. L. Atkin has suggested a recursive technique for computing the terms in a multiplicative sequence. As above let $Q(t) = 1 + \sum_{n=1}^{\infty} q_n t^n$ be the characteristic power series for the multiplicative sequence K_n and let

$$\frac{d}{dt}\log Q(t) = \sum_{n=0}^{\infty} r_{n+1}t^n.$$

Then

$$\frac{d}{dt}\log \prod_{j=1}^{m} Q(x_j t) = \sum_{n=0}^{\infty} r_{n+1}s_{n+1}t^n,$$

where $s_k = \sum_{j=1}^{m} x_j^k$. Recall [13, p. 10] that

$$\prod_{j=1}^{m} Q(x_j t) = \sum_{j=0}^{m} K_j(c_1, \cdots, c_j)t^j + \text{ higher order terms.}$$

Then

$$\frac{d}{dt}\log \prod_{j=1}^{m} Q(x_j t) = \left\{\sum_{j=0}^{m} jK_j t^{j-1} + \cdots\right\} \Big/ \left\{\sum_{j=0}^{m} K_j t^j + \cdots\right\}.$$

Hence

$$mK_m = \sum_{i=1}^{m} r_i s_i K_{m-i}.$$

The special case $Q(t) = 1 + t$ gives the Newton formulas [13, p. 92] which express the s_i in terms of the elementary symmetric polynomials c_i:

$$s_m = -(-1)^m m c_m - \sum_{i=1}^{m-1} (-1)^i c_i s_{m-i}.$$

The coefficients r_i are expressed in terms of the coefficients of Q by

$$r_m = m q_m - \sum_{i=1}^{m-1} q_i r_{m-i}.$$

Given the q_i and using these relations it is easy to compute K_m. For the sequence $t_n(z; c_1, \cdots, c_n)$ we obtain

$$\begin{aligned}
t_1 &= c_1 + \tfrac{1}{2}(-c_1)z, \\
t_2 &= c_2 - c_2 z + \frac{1}{12}(c_2 + c_1^2)z^2, \\
t_3 &= c_3 + \frac{1}{2}(-3c_3)z + \frac{1}{12}(6c_3 + c_2c_1)z^2 + \frac{1}{24}(-c_2c_1)z^3, \\
t_4 &= c_4 - 2c_4 z + \frac{1}{12}(14c_4 + c_3c_1)z^2 + \frac{1}{12}(-2c_4 - c_3c_1)z^3 \\
&\quad + \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4)z^4, \\
t_5 &= c_5 + \frac{1}{2}(-5c_5)z + \frac{1}{12}(25c_5 + c_4c_1)z^2 + \frac{1}{8}(-5c_5 - c_4c_1)z^3 \\
&\quad + \frac{1}{720}(30c_5 + 29c_4c_1 + c_3c_1^2 + 3c_2^2c_1 - c_2c_1^3)z^4 \\
&\quad + \frac{1}{1440}(c_4c_1 - c_3c_1^2 - 3c_2^2c_1 + c_2c_1^3)z^5, \\
t_6 &= c_6 - 3c_6 z + \frac{1}{12}(39c_6 + c_5c_1)z^2 + \frac{1}{6}(-9c_6 - c_5c_1)z^3 \\
&\quad + \frac{1}{720}(186c_6 + 69c_5c_1 + 3c_4c_2 + c_4c_1^2 - 3c_3^2 + 3c_3c_2c_1 - c_3c_1^3)z^4 \\
&\quad + \frac{1}{720}(-6c_6 - 9c_5c_1 - 3c_4c_2 - c_4c_1^2 + 3c_3^2 - 3c_3c_2c_1 + c_3c_1^3)z^5 \\
&\quad + \frac{1}{60480}(2c_6 - 2c_5c_1 - 9c_4c_2 - 5c_4c_1^2 - c_3^2 + 11c_3c_2c_1 + 5c_3c_1^3 \\
&\qquad\qquad + 10c_2^3 + 11c_2^2c_1^2 - 12c_2c_1^4 + 2c_1^6)z^6.
\end{aligned}$$

3. 3-folds

If V is a Kähler manifold which has the same Betti numbers as CP_3, then $c_2c_1[V] = 24$ and $c_1(V) > 0$. The inequality is a consequence of Yau's result [31]. First, since $H^{1,1}(V;Z) = H^2(V;Z) = Z$, the positive integral classes are positive multiples of a generator x (cf. [13, §18.1]). If $-c_1(V)$ were positive, Yau's result would give the inequality $8c_2c_1[V] \le 3c_1^3[V]$. But $c_1(V) < 0$ implies $c_1^3[V] < 0$ (since $x^3[V] > 0$ by the Wirtinger theorem [8, p. 31]) which gives a contradiction.

By the result of Kobayashi and Ochiai, if $c_1(V) \ge 4x$, then V is analytically equivalent to CP_3 and, if $c_1(V) = 3x$, V is analytically equivalent to $X_3(2)$. Since c_1 mod 2 is a homotopy invariant, it remains to show that $c_1(V) = 2x$ is impossible if V is homotopy equivalent to CP_3 and that $c_1(V) = 1$ is impossible if V is homotopy equivalent to $X_3(2)$. We remark that there are smooth almost complex manifolds V which do satisfy these conditions (see [22, §9], [30, §7]).

The condition $c_1(V) > 0$ means that V is a Fano 3-fold. A proof of the uniqueness result for CP_3 has been given by Fujita [7] (assuming c_1 positive) and by Lanteri and Struppa [20, 2.1] using Iskovskih's work [16], [27] on the classification of Fano 3-folds. This classification is complete for 3-folds V with $c_1(V)$ equivalent to two or more times an indivisible class, the case of index greater than or equal to 2. The only Fano 3-fold with $c_1(V) = 2x$ and $x^3[V] = 1$ has $h^{1,2} = 21$ (cf. [16, I.1 and IV.3.5]). Hence V homotopy equivalent to CP_3 implies $c_1(V) = 4x$, so V is analytically equivalent to CP_3.

For the three-dimensional case of Theorem 2, from the viewpoint of [20, Theorem 1.5] it remains to show that V cannot be a so-called pathological Fano 3-fold, a problem caused by the incompleteness of the classification of Fano 3-folds of index 1. The following lemma is a consequence of Iskovskih's results.

Lemma. *If V is a Fano 3-fold with $c_1(V) = x$ and $x^3[V] = 2$, then V is a double cover of CP_3 branched over a smooth hypersurface of degree 6.*

For this V, $h^{1,2} = 52$ [16, IV.3.5], and hence V is not homotopy equivalent to $X_3(2)$.

Proof of Lemma. The hypotheses imply V is Fano of index $r = 1$ and degree $d = 2$. Let H be the anticanonical line bundle, $c_1(H) = x$, and consider the system of divisors $H^0(V, \mathcal{O}_V(H))$ of dimension $h^0(\mathcal{O}(H)) = 4$. (The dimension is computed in [16, I.4.2ii] using Riemann-Roch and the Kodaira vanishing theorem.) It follows from [16, I.6.1b] that this system is base point free since $m = 1 + \frac{1}{2}x^3[V] < 3$. Hence the corresponding map, the anticanonical map $\varphi: V \to CP_3$, has degree 2 and by [16, II.2.2] φ is branched over a smooth hypersurface of degree 6. A direct proof of the Lemma has been given by Shepherd-Barron (unpublished).

4.

If V is homotopy equivalent to CP_4, then $c_4 = 5x^4$ and, by (2.5), $c_3c_1 = 50x^4$ where x is the positive generator of $H(V; Z)$. Also c_1 is odd since $w_2 \neq 0$, so the possible values of c_1 are $\pm x$, $\pm 5x$, and $\pm 25x$. One more equation is given by the Riemann-Roch formula for the arithmetic genus, $\chi^0(V) = 1$, hence [13, p. 14]

$$3c_2^2 + 4c_2c_1^2 - c_1^4 = 675x^4. \tag{4.1}$$

Solving for c_2, the discriminant is

$$4(7c_1^4 + 2025x^4)$$

which, for the possible values of c_1, is a square only for $c_1 = \pm 5x$. Thus the only integer solutions of (4.1) have $c_1 = \pm 5x$ and $c_2 = 10x^2$. But if $c_1 = -5x$, Yau's result [31] implies that V is covered by the ball, contradicting $\pi_1(V) = 0$.

If V is homotopy equivalent to $X_4(2)$, the cohomology ring of V is generated by $x \in H^2(V; Z)$ and $y \in H^4(V; Z)$ with $x^3 = 2xy$, $x^4[V] = 2$, $x^2y[V] = 1$, and $y^2[V] = 1$ (cf. [19, p. 253]). Since the total Chern class $c(X_4(2)) = 1 + 4x + 7x^2 + 6x^3 + 3x^4$, and since $X_4(2)$ and V have the same Hodge numbers, $c_4(V) = 3x^4$ and $c_3c_1(V) = 24x^4$ (cf. (2.5)). From the homotopy invariance of the Stiefel-Whitney classes we have $c_1 \equiv 0 \pmod 2$, $c_2 \equiv x^2 \pmod 2$, and $c_3 \equiv 0 \pmod 2$.

Since $\chi^0 = 1$,

$$3c_2^2 + 4c_2c_1^2 - c_1^4 = 339x^4.$$

Let $c_1 = ax$ and $c_2 = ux^2 + vy$. Then

$$6u^2 + 6uv + 3v^2 + 8a^2u + 4a^2v - 2a^4 = 678. \tag{4.2}$$

Also a is even, a divides 24, u is odd, and v is even.

Lemma. $v \equiv 0 \pmod 3$ *and* $a \not\equiv 0 \pmod 3$.

Proof. The Pontryagin class p_1 mod 3 is an invariant of homotopy type, hence

$$c_1^2 - 2c_2 \equiv 2x^2 \qquad \pmod 3.$$

This implies both $a^2 - 2u \equiv 2 \pmod 3$ and $-2v \equiv 0 \pmod 3$. Now if $a \equiv 0 \pmod 3$, then (4.2) implies $6u^2 \equiv 3 \pmod 9$ hence $2u^2 \equiv 1 \pmod 3$ which has no solution, hence $a \not\equiv 0 \pmod 3$.

Equation (4.2) is quadratic in u with discriminant $4(28a^4-9v^2+4068)$. For the possible values of a; ±2, ±4, or ±8; this discriminant is a square giving integer solutions only in the following cases:

$$\begin{aligned}
&a=\pm 2, \quad u=-5, \quad && v=18\\
&a=\pm 2, \quad u=13, \quad && v=-18\\
&a=\pm 4, \quad u=7, \quad && v=0\\
&a=\pm 4, \quad u=-35, \quad && v=30\\
&a=\pm 4, \quad u=-5, \quad && v=-30.
\end{aligned}$$

Any prime $p\equiv 3 \pmod 4$ appears to an even power in the prime decomposition of a sum of two squares. Taking $p=11$ shows that for $a=\pm8$ there are no solutions since $28a^4+4068$ is not a sum of two squares. The other cases are done by looking through the list of possible values of v.

There is a line bundle L over V with $c_1(L)=x$, hence

$$\begin{aligned}
\chi(V, L) &= \{e^x T\}[V]\\
&= \chi^0(V)+\tfrac{1}{24}\{xc_2c_1+x^2(c_2+c_1^2)+2x^3c_1+x^4\}[V]
\end{aligned}$$

is an integer. This implies

$$2u(a+1)+v(a+1)+2(a+1)^2\equiv 0 \pmod{24}$$

so, since a is an even integer,

$$2u+v+2(a+1)\equiv 0 \pmod 8.$$

The only solution above which satisfies this condition is $c_1=\pm4x$, $c_2=7x^2$.

Unfortunately, the inequality in Yau's theorem reduces to $35>32$ and so does not rule out the case $c_1=-4x$.

5.

Assume V is homotopy equivalent to CP_5. Then $c_5(V)=6x^5$ and $c_4c_1=90x^5$. Since $\chi^0(V)=1$, we have

$$c_3c_1^2+3c_2^2c_1-c_2c_1^3=1530x^5. \tag{5.1}$$

Lemma. $c_1=6x$ *or* $-2x$.

Proof. Since c_2 is odd and c_1 and c_3 are even, reducing (5.1) modulo 8 yields $3c_1\equiv 2x \pmod 8$, so $c_1\equiv 6x \pmod 8$. Also $c_1\not\equiv 0 \pmod 9$ since otherwise (5.1) mod 27 yields $0\equiv 18 \pmod{27}$. The remaining possible values for c_1 are 30, 6, -2, and -10.

If $c_1 \equiv 0 \pmod 5$, then (5.1) implies $3c_2^2c_1 \equiv 5 \pmod{25}$. If $c_1 = 30$, then $15c_2^2 \equiv 5 \pmod{25}$ or $c_2^2 \equiv 2 \pmod 5$ which is impossible. If $c_1 = -10$ we find $c_2^2 \equiv 4 \pmod 5$. Now recall that the characteristic class $p_1^2 - 2p_2 \bmod 5$ is an invariant homotopy type [24, p. 229] and hence, under the assumption $c_1 \equiv 0 \pmod 5$, we have $2c_2^2 - 4c_4 \equiv 1 \pmod 5$. But $c_4 = -9$ so $c_2^2 \equiv 0 \pmod 5$, a contradiction. This establishes the lemma.

It remains to show that $c_1 = -2x$ leads to a contradiction. First, if $c_1 = -2x$, then (5.1) implies

$$2c_3x - 3c_2^2 + 4c_2x^2 = 765x^4,$$

so, since c_2 is odd, $2c_3+4c_2x \equiv 0 \pmod 8$ and hence $c_3 \equiv 2x^3 \pmod 4$.

Second, since the homology of V is torsion free, $p_1 \bmod 24$ is a homotopy invariant [1, p. 207]. With $c_1 = -2x$ this implies $c_2 \equiv -x^2 \pmod 4$.

Finally we apply an integrality result for continuous vector bundles over CP_n [13, Theorem 22.4.1] to the bundles $f^*\tau V \otimes H^r$ where $f\colon CP_5 \to V$ is a homotopy equivalence. By composing with complex conjugation if necessary we may assume $f^*x = h$, the positive generator of $H^2(CP_5; Z)$. If the total Chern class $c(V)$ factors formally as

$$(1 + x\delta_1)\cdots(1 + x\delta_5),$$

then $T(CP_5, f^*\tau V \otimes H^r)$ is given by the symmetric function

$$\sum_{i=1}^{5} \binom{5 + r + \delta_i}{5},$$

which therefore is integer valued. We can write

$$T(CP_5, f^*\tau V \otimes H) - T(CP_5, f^*\tau V) = \sum_{i=1}^{5} \binom{5 + \delta_i}{4}$$

in terms of the Chern classes of V (the elementary symmetric functions of $\delta_1, \cdots, \delta_5$) as

$$\tfrac{1}{24}\{-4c_4 + 4c_3c_1 + 2c_2^2 - 4c_2c_1^2 + c_1^4 + 14x(3c_3 - 3c_2c_1 + c_1^3) \\ + 71x^2(-2c_2 + c_1^2) + 154x^3c_1 + 120x^4\}.$$

Then $c_1 = -2x$ and $c_4 = -45x$ imply

$$c_2^2 + xc_3 - x^2c_2 \equiv 2x^4 \pmod 4.$$

This contradicts the results for c_2 and c_3 above. Hence $c_1(V) = 6x$ and V is analytically equivalent to CP_5.

6.

The case of V homotopy equivalent to CP_6 involves rather more computation than the previous cases. Writing the χ_y-genus of V in terms of $z = y + 1$ we have

$$\chi_y(V) = 7 - 21z + 35z^2 - 35z^3 + 21z^4 - 7z^5 + z^6 = t_6(V),$$

where t_6 is the polynomial in z introduced in §2. Equating coefficients of z gives equations satisfied by the Chern numbers of V; the coefficients of even powers of z give a maximal independent set of equations. The constant term gives $c_6 = 7x^6$ and, using this, the quadratic term gives $c_5c_1 = 147x^6$ (cf. (2.5)). We pass from equations in $H^*(V; Z) = Z[x]/(x^7 = 0)$ to equations over the integers by replacing c_i by c_ix^i and then equating coefficients of x^6. Setting

$$\begin{aligned} e_4 = & -3c_4c_2 - c_4c_1^2 + 3c_3^2 - 3c_3c_2c_1 + c_3c_1^3 + 3675, \\ e_6 = & -9c_4c_2 - 5c_4c_1^2 - c_3^2 + 11c_3c_2c_1 + 5c_3c_1^3 + 10c_2^3 \\ & + 11c_2^2c_1^2 - 12c_2c_1^4 + 2c_1^6 - 60760, \end{aligned}$$

the remaining two equations can be written as

$$e_4 = 0, \qquad e_6 = 0.$$

By the remark following the proof of (2.2), the equation $e_6 = 0$ is equivalent to the formula for the arithmetic genus of V in terms of the Todd polynomial T_6. In the presence of the other equations, $e_4 = 0$ is equivalent to the condition imposed by the signature formula:

$$L_3(V)[V] = 1.$$

(The polynomials L_3 and T_6 are given in [13, pp. 12 and 14].)

Lemma. *For the only integer solutions to these equations we have* $c_1 = \pm 7$ *and* $c_2 = 21$.

But if $c_1 = -7$ and $c_2 = 21$, it follows from Yau's result that V is covered by the unit ball which contradicts the assumption that V is simply connected. Therefore the lemma implies V is analytically equivalent to CP_6.

Proof of Lemma. Write

$$e_4 = k_1c_4 + k_0, \qquad e_6 = l_1c_4 + l_0$$

and set

$$e_1 = k_1 l_0 - k_0 l_1 = a_2 c_3^2 + a_1 c_3 + a_0$$

to eliminate c_4. We compute

$$a_2 = 2(15c_2 + 8c_1^2),$$

$$a_1 = -4c_2 c_1 (15c_2 + 8c_1^2),$$

$$a_0 = -30c_2^4 - 43c_2^3 c_1^2 + 25c_2^2 c_1^4 + 6c_2 c_1^6 + 215355c_2 - 2c_1^8 + 79135c_1^2.$$

Given an integer solution to $e_1 = 0$, it follows that $15c_2 + 8c_1^2$ divides a_0. By division we may write

$$1125a_0 = (15c_2 + 8c_1^2)(-2250c_2^3 - 2025c_2^2 c_1^2 + 2955c_2 c_1^4 - 1126c_1^6 + 16151625) + c_1^2(6758c_1^6 - 40186125).$$

Therefore $15c_2 + 8c_1^2$ divides

$$R(c_1) = c_1^2(6758c_1^6 - 40186125).$$

Now since c_1 divides $147 = 3 \cdot 7^2$, there are six possible values of $|c_1|$. For each we compute and factor $R(c_1)$. Then for each divisor of $R(c_1)$, positive or negative, we check whether the resulting c_2 is an integer and, if it is, whether the discriminant of $e_1 = 0$ is a perfect square. This discriminant is

$$\begin{aligned} D &= a_1^2 - 4a_2 a_0 \\ &= 8(15c_2 + 8c_1^2)(30c_2^4 + 73c_2^3 c_1^2 - 9c_2^2 c_1^4 - 6c_2 c_1^6 - 215355c_2 \\ &\qquad + 2c_1^8 - 79135c_1^2). \end{aligned}$$

The only cases giving rise to a square discriminant have $c_1 = \pm 7$ and $c_2 = 21$.

The computations indicated above were done with the aid of a computer. The algebraic programming system REDUCE was used for the manipulation of polynomials and to compute and factor the integers $R(c_1)$. A program written in PL/I was used to complete the test. Potential discriminants were computed modulo $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ and modulo $19 \cdot 23 \cdot 29 \cdot 31$ and then reduced modulo each of these primes and checked against a computed table of quadratic residues. Two cases which pass these tests are ruled out modulo 37 or modulo 47. We thank A. O. L. Atkin for recommending this method to us.

7. Almost complex structures

Theorem 7.1. *Each smooth manifold M homotopy equivalent to CP_4 supports a (nonzero) finite number of almost complex structures. The Pontryagin class $p_1(M) = (5 + 24m)x^2$ for some $m \equiv 0$ or $6 \pmod{14}$. Almost complex structures on M correspond to integers a dividing $25 + \frac{3}{7}(24^2m^2 + 10 \cdot 24m)$ under the correspondence $c_1(M) = ax$.*

For the standard smooth CP_4, $m = 0$ and almost complex structures correspond to integers a dividing 25, a result of Thomas [28, Theorem 3.2]. For a fixed divisor a, the two complex structures on the tangent bundle of M with c_1 equal to $+ax$ and $-ax$ are conjugate bundles (cf. [24, p. 167]).

Proof. By surgery theory and work of Brumfiel [4, I.4], [3, 8.2] we have $\tau M = \tau CP_4 + \xi$ in $KO^0(CP_4)$, where $\xi = m\xi_1 + n\xi_2$ is a linear combination of the generators $\xi_1 = 24\omega + 98\omega^2$ and $\xi_2 = 240\omega^2$ of

$$\operatorname{im}\{[CP_4, G/O] \to [CP_4, BSO]\}.$$

Here $\omega = r(H-1)$ generates $KO^0(CP_4)$ as a ring. Computing the surgery obstruction, index $M-$ index CP_4, in terms of Pontryagin classes yields the relation [3, p. 58]

$$14n = 2m^2 - 5m.$$

Hence $m \equiv 0$ or $6 \pmod{14}$. Brumfiel shows that for each such m there corresponds four distinct smoothings of CP_4. The Pontryagin classes of M are

$$\begin{aligned} p_1(M) &= (5 + 24m)x^2, \\ p_2(M) &= \{10 + \tfrac{1}{7}(24^2m^2 + 10 \cdot 24m)\}x^4. \end{aligned} \tag{7.2}$$

Almost complex structures on 8-manifolds have been studied by Thomas [28] and Heaps [9]. If M has the homotopy type of CP_4, then [9, Theorem 1] implies there is an almost complex structure on M with

$$c_1 = ax \quad \text{and} \quad c_3 = bx^3$$

if and only if the following two conditions hold:

$$a \text{ is odd} \quad \text{and} \quad b \equiv 2 \pmod 4). \tag{7.3}$$

(Hence $2\chi(M)+qb \equiv 0 \pmod 4$. The necessity of this condition follows from (2.4).)

$$(7.4) \qquad 40x^4 = 4p_2 + 8abx^4 - a^4x^4 + 2a^2x^2p_1 - p_1^2$$

(i.e., $\chi(M) = c_4[M]$, where c_4 is determined by p_2, p_1, c_3, and c_1).
Substituting (7.2) in (7.4) yields

$$(7.5) \qquad 3 \cdot 24^2m^2 + 30 \cdot 24m + 7 \cdot 25 = 7a(8b - a^3 + 10a + 2a \cdot 24m).$$

It follows that a is a divisor of the left-hand side of (7.5) which is nonzero. Also a determines the Chern classes which determine the complex bundle over M. Thus the number of almost complex structures on a given smooth M is finite.

Moreover, since $m \equiv 0$ or $6 \pmod{14}$, the left-hand side is congruent to 7 modulo 14. Choose any integer a dividing

$$25 + \tfrac{3}{7}(24^2m^2 + 10 \cdot 24m)$$

and then solve (7.5) for b. It follows that a is odd so $a^2 \equiv 1 \pmod 8$. Since m is even, (7.5) implies

$$24 \equiv 8ab - a^4 + 10a^2 \qquad \pmod{32}$$

and therefore

$$8ab \equiv (a^2 - 5)^2 \equiv 16 \qquad \pmod{32}.$$

Hence $ab \equiv 2 \pmod 4$ so condition (7.3) is satisfied. Thus divisors do yield almost complex structures on M.

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