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LOCAL DIFFERENTIAL GEOMETRY AND GENERIC PROJECTIONS OF THREEFOLDS

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The purpose of this note is to prove a result concerning the 4-secant lines of a nondegenerate irreducible, say smooth, threefold

$$X \subset \mathbf{P}^r, \qquad r \geq 9;$$

namely we prove essentially that all these lines together fill up at most a fourfold (see Theorem 1 below); equivalently, the generic projection of X to $\mathbf{P}^4$ has no fourfold points that come from collinear quadruples of points on X.

The (very classical) subject of generic projections of n folds to $\mathbf{P}^{n+1}$ and the multiple points of such projections has recently come into focus in connection with work of Pinkham [4], Lazarsfeld [2], and Peskine [3], which has shown how certain properties (both known and conjectural) of such projections can be used to establish various cohomological properties of the n folds in question, in particular Castelnuovo regularity. Indeed, Lazarsfeld's paper [2] shows, among other things, that the above statement concerning fourfold points of projections to $\mathbf{P}^4$ is exactly what is needed to establish a sharp Castelnuovo regularity bound for smooth nondegenerate threefolds in $\mathbf{P}^r$, $r \geq 9$ (see Corollary 3 below).

We now proceed with a precise statement.

Theorem 1. *Let X be an irreducible nondegenerate three-dimensional subvariety of $\mathbf{P}^r$, $r \geq 9$, whose tangent variety is six-dimensional, and let $\{L_y : y \in Y\}$ be a family of lines in $\mathbf{P}^r$ with the property that for any general $y \in Y$, the part of the scheme-theoretic intersection $L_y \cap X$ supported at smooth points of X has length at least 4. Then we have*

$$\dim\left(\bigcup_{y \in Y} L_y\right) \leq 4.$$

Remarks 2.1. Any *smooth* threefold has six-dimensional tangent variety (cf. [1]). The hypothesis that X has six-dimensional tangent variety

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is presumably unnecessary, especially in view of the fact that the threefolds with tangent variety of dimension < 6 have been classified in [1]; this hypothesis enters in the proof only to help handle certain 'degenerate' cases.

2.2. It seems likely that the theorem is true for $r = 7, 8$ as well, but the proof does not yield this.

2.3. It is reasonable to expect that the analogue of the theorem is true for (nondegenerate) n folds X in $\mathbf{P}^r$, $r \geq 2n+1$: namely that the $(n+1)$-secant lines of X fill up at most an $(n+1)$ fold. The proof below 'almost' shows this for $r \geq 2n+3$, but breaks down at some degenerate cases. In any event, Corollary 3 below would not follow from the analogue of Theorem 1 for $n \geq 4$. For surfaces, on the other hand, the proof does work for all $r \geq 6$, and this result is apparently new (notwithstanding some assertions to the contrary in the literature). Actually, the analogue of Theorem 1 is in fact true for $r = 5$ as well, but the proof of that case is considerably more difficult.

2.4. For any $n \geq 2$, $r \geq 2n+1$, and $k \geq n+1$, it is easy to construct examples of smooth nondegenerate n folds X in $\mathbf{P}^r$ whose k-secant lines fill up an $(n+1)$ fold: e.g., unions of ∞^1 plane curves of degree k. Thus Theorem 1 is essentially sharp.

Corollary 3. *Let X be a smooth nondegenerate irreducible threefold of degree d in $\mathbf{P}^r$, $r \geq 9$. Then X is $(d-r+4)$-regular, i.e., the ideal sheaf $I = I_{X/\mathbf{P}^r}$ satisfies $H^i(\mathbf{P}^r, I(d-r+4-i)) = 0$ for $i > 0$.*

Proof. Given Theorem 1, this essentially follows from Lazarsfeld's paper [2]. Namely Lazarsfeld shows, at least implicitly, that X is $(d-r+4)$-regular provided the following statement is true:

If $Z \subset X$ is any fibre of a generic projection

$$\pi: X \to \overline{X} \subset \mathbf{P}^4,$$

(*) *then Z imposes independent conditions on quadrics, i.e., the restriction map*

$$H^0(\mathcal{O}_{\mathbf{P}^4}(2)) \to H^0(\mathcal{O}_Z(2))$$

is surjective.

Now in our case, it follows from [5] that no fibres Z can exist having length ≥ 5; on the other hand, it is trivial that any scheme of length ≤ 3 imposes independent conditions on quadrics. As for fibres of length 4, Theorem 1 implies that Z cannot be contained in a line, and if Z were to span a $\mathbf{P}^3$, it would impose independent conditions on linear forms, hence *a fortiori* on quadrics. It remains to consider the case where Z is a length-4 subscheme of a plane. If Z failed to impose independent conditions on quadrics, there would exist three independent (possibly singular) conics C_1, C_2, C_3 through Z. By Noether's $Af + Bg$ theorem, it follows that the C_i must have a common component, which clearly must be a line M. But then $C_1 \cap C_2 \cap C_3 = M$ scheme-theoretically, so that $Z \subset M$, which cannot be. This completes the proof of statement $(*)$, hence that of Corollary 3.

Remark 4. It seems likely that the foregoing argument extends to the case $n = 4$ as well; the case $n \geq 5$ however seems more difficult, inasmuch as it would eventually involve dealing with fibres Z of length 6 contained in a plane, for which one would have to show Z is not on any conic, a property which at the moment seems too subtle to handle.

We now turn to the proof of Theorem 1. Let $\{L_y : y \in Y\}$ be a family of four-secant lines of X as in the statement of the theorem. Without loss of generality, we may assume Y is an irreducible four-dimensional subvariety of the Grassmannian $G = G(1, \mathbf{P}^r)$ such that $\bigcup_{y \in Y} L_y$ is a fivefold. We fix a general member $L = L_y$ of the family and work locally in an analytic neighborhood of y on Y. We will assume, to begin with, that $L \cap X$ contains four distinct points $p_1, \cdots, p_4$ smooth on X. By [5] it follows that $p_1, \cdots, p_4$ are general on X, that L meets X transversely at p_i, $i = 1, \cdots, 4$, and moreover that there are no further smooth points of X on L. Put $T = T_y Y$, $L = \mathbf{P}(A)$, $\mathbf{P}^r = \mathbf{P}(B)$, and $N = B/A$. Then we have

$$T \subseteq T_y G = \mathrm{Hom}(A, N),$$

whence a map $A \to \mathrm{Hom}(T, N)$, which must be injective, hence induces

$$\delta : L = \mathbf{P}(A) \to \mathbf{P}(\mathrm{Hom}(T, N)) =: \mathbf{P}.$$

Let $D \subset \mathbf{P}$ denote the determinantal variety of singular (i.e., noninjective) homomorphisms. As in [5], we see that $\delta(p_i) \in D$, $i = 1, \cdots, 4$, and moreover that the $\delta(p_i)$ must have rank exactly 3. Let $u_i \in T$ be a basis for $\mathrm{Ker}(\delta(p_i))$, $i = 1, \cdots, 4$.

Lemma 5. (i) *$u_1, \cdots, u_4$ are linearly independent.*

(ii) *There is a four-dimensional subspace $N_0 \subset N$, and none smaller, such that d factors through $\mathbf{P}(\mathrm{Hom}(T, N_0))$.*

Proof. (i) If $u_1, \cdots, u_4$ were to span a subspace $T_1 \subset T$ of dimension $k < 4$, let N_1 be a generic k-dimensional quotient of N and

$$\delta_1 : L \to \mathbf{P}(\mathrm{Hom}(T_1, N_1)) =: \mathbf{P}_1$$

the induced map. Then $\delta(L)$ must be entirely contained in the analogous determinantal variety $D_1 \subset \mathbf{P}_1$ (because $\delta(p_i) \in D_i$, $i = 1, \cdots, 4$), and because N_1 was generic this implies that $\delta(L) \subset D$ also, which then implies that the lines L_y only fill up a fourfold, which is a contradiction.

(ii) Let N_0 be the span of $\operatorname{im}(u_i)$, $i = 1, \cdots, 4$ (considering the u_i as rank-1 homomorphisms $A \to N$). Then clearly we have $\dim N_0 \leq 4$ and δ factors as indicated; on the other hand, if δ were to factor through a subspace of dimension ≤ 3, it would follow as above that $\delta(L) \subset D$, which is not the case.

To formulate the conclusion of part (ii) of the lemma in a slightly more intuitive way, there is a five-dimensional linear subspace $R = R_y \subset \mathbf{P}^r$, containing L, such that the first order deformations of L in Y stay within, and in fact span R.

Now consider the embedded tangent spaces

$$T_i := \widetilde{T}_{P_i} X, \qquad i = 1, \cdots, 4.$$

As p_i was general on X, any first order deformation of p_i in X lifts to a deformation of L in Y, hence we have

$$T_i \subset R, \qquad i = 1, \cdots, 4.$$

Moreover, for any $i \neq j$, T_i and T_j together must span R: indeed, a deformation of a line is determined by that of any two distinct points on it, so if T_i and T_j span $R' \subseteq R$, then the first order deformations of L must stay within R', so that $R' = R$. Now set

$$\overline{M}_{ij} = T_i \cap T_j \subset R, \tag{1}$$

which is therefore a $\mathbf{P}^1$. Moreover $p_i, p_j \notin \overline{M}_{ij}$, because L was transverse to X, hence $\overline{M}_{ij}$ corresponds to a two-dimensional subspace $M_{ij} \subset T_{P_i} X$.

Now let $K(p_j)$ be the two-dimensional cone obtained by varying L within Y while keeping p_j fixed, and let S_j be the embedded tangent plane to $K(p_j)$ at a general point $q \in L$ (this is independent of q). Thus S_j is the $\mathbf{P}^2$ containing L corresponding to the one-dimensional subspace $\operatorname{im}(u_j) \subset N$ encountered above. Note that S_j meets T_i in a line through p_i for all $i \neq j$ and let $v_{ij} = v_{ij,y} \in T_{P_i} X$ be the corresponding direction (defined up to scalar multiple). Note that

$$v_{ij} \in M_{ik} \tag{2}$$

whenever i, j, k are all distinct. By Lemma 5, the v_{ij} for any fixed i are independent.

The idea now will be to differentiate the identity (1) in the various directions u_k, thus obtaining various identities involving the *second fundamental form* of X, for whose definition and basic properties we refer to [1]; we will just set up some notation. We denote the second fundamental form of X at a point p by II_p, and view it as a symmetric bilinear form on the tangent space $T_p(X)$, whose values are vectors in the vector space B corresponding to $\mathbf{P}^r$, well defined modulo $\widetilde{T}_pX$ (more precisely, modulo the corresponding linear subspace of B, but we will allow ourselves the luxury of such abuses of terminology).

Now differentiating (1) in the direction u_j, we obtain

$$\mathrm{II}_{p_i}(v_{ij}, M_{ij}) \equiv 0 \mod R, \qquad i \neq j. \tag{3}$$

On the other hand, differentiating (1) in the direction u_k, $k \neq i, j$, we obtain

$$\mathrm{II}_{p_i}(v_{ik}, v_{ik}) \equiv \mathrm{II}_{p_j}(v_{jk}, v_{jk}) \mod R, \qquad i, j, k \text{ all distinct.} \tag{4}$$

Now set

$$U_i = U_{i,y} = \mathrm{Span}(v_{ij} \cdot M_{ij},\ j \neq i \subset \mathrm{Sym}^2(T_{p_i}X)),$$

a three-dimensional subspace. Then (3) yields

$$\mathrm{II}_{p_i}(U_i) \subset R, \qquad i = 1, \cdots, 4. \tag{5}$$

Assume for now that equality holds in (5) for some i; it follows in particular that

$$R \subset T^2_{p_i}, \tag{6}$$

where T^2_p denotes the second-order tangent space to X at p, considered as a subspace of $\mathbf{P}^r$ (i.e., this is just the image of II_p; cf. [1]). Now (6) clearly yields $\mathrm{II}_{p_j}(v_{ji}, v_{ji}) \subseteq T^2_{p_i}$, and since moreover the $T^2_{p_i}$ all have the same dimension, it follows by (4) that we have

$$T^2_{p_1} = \cdots = T^2_{p_4}. \tag{7}$$

Now note that $\mathrm{Sym}^2(T_{p_i}X)$ is spanned by U_i plus the v_{ij}, $j \neq i$, hence $T^2_{p_i}X$ is at most a $\mathbf{P}^8$. Thus (7) implies that as we vary our initial y to a nearby $y' \in Y$ while fixing any of the p_i, the lines $L_{y'}$ remain in a fixed linear subspace of $\mathbf{P}^r$ of dimension ≤ 8. The following elementary observation now yields a contradiction to our hypothesis that X was nondegenerate in $\mathbf{P}^r$, $r \geq 9$, and $p_1, \cdots, p_4$ were general on X.

Lemma 6 (*The Goose-Step principle*). *For a general $y \in Y$, let $\mathcal{F}(y)$ be the set of $y' \in Y$ connectable to y by a finite chain of irreducible curves $C_1 \cup \cdots \cup C_k \subset Y$ such that for j, as y'' varies within C_j, one of the points of $L_{y''} \cap X$, which is a deformation of one of the points of $L_y \cap X$, stays fixed. Then $\mathcal{F}(y)$ is dense in y.*

Proof. If this were false, then the closures of the $\mathcal{F}(y)$ would form a nontrivial foliation of (some open subspace of) Y. As $y \in Y$ is general, there is a leaf of this foliation through y, and the vectors $u_1, \cdots, u_4$ must be tangent to it, contradicting Lemma 5(i).

Next, we consider the case where the inclusion (5) is strict for all $i = 1, \cdots, 4$. Suppose first that for some i we have

$$\dim(T^2_{p_i} \cap R) = 4. \tag{8}$$

In particular, it follows that for all j, $T^2_{p_i}$ is at most seven-dimensional and meets T_j at least in a $\mathbf{P}^2$, hence a p_i is kept fixed, p_j varies at most in a fixed $(\dim^2_{p_i} + 1)$-dimensional linear space, which must coincide with $T^2_{p_i} + R$, and as above we may conclude that

$$T^2_{p_i} + R = T^2_{p_j} + R \quad \text{for all } j \neq i.$$

Moreover by (3) and (4) the latter space, which is at most eight-dimensional, stays infinitesimally fixed, hence fixed, as L varies fixing any of the p_i, so the Goose-Step Principle yields a contradiction as above.

Suppose next that we have

$$\dim(T^2_{p_i} \cap R) < 4, \qquad i = 1, \cdots, 4. \tag{9}$$

In other words, we have $\mathrm{II}_{p_i}(U_i) = 0$. Since the kernel of II_{p_i} is at most three-dimensional anyway, it follows that this kernel must coincide with U_i, and in particular $U_i = U_{i,y}$ stays fixed as L varies fixing p_i; as U_i determines the $v_{ij} = v_{ij,y}$ these stay similarly fixed, up to scalar multiple.

We may now conclude that, locally at each p_i, X possesses three mutually transverse one-dimensional foliations, tangent to the v_{ij} and compatible with the foliations of Y tangent to the u_i. Let C_{ij} be a local integral arc of the v_{ij}-foliation. Then we may conclude, e.g., that an arbitrary chord joining C_{12} and C_{21} is in our family $\{L_y\}$, hence meets X elsewhere. By the trisecant lemma for analytic arcs, either C_{12} and C_{21} are both in some $\mathbf{P}^2$, or they are in a $\mathbf{P}^3$ that meets X in a surface. The first alternative clearly implies that our lines L_y fill up only a fourfold; the second alternative implies that X contains a two-parameter family of surfaces S_α each contained in a $\mathbf{P}^3$. Since two generic points of X will lie on some S_α, the embedded tangent spaces to X at these points must meet in a generally-positioned line, and this contradicts our hypothesis that the tangent variety of X is six-dimensional. This completes the discussion of the case where L is transverse to X.

It remains to consider the case where L is tangent to X at some smooth point. First, if L is a simple bitangent, tangent at two points $p_1 \neq p_2$, then, using notation introduced above, we have $S_1 \subset T_2$. On the other hand, obviously $K(p_1) \subset T_1$ so $S_1 \subset T_1$, hence T_1 and T_2 meet in a $\mathbf{P}^2$ which as we have seen cannot be. Next, if L is a flex tangent at p_1, say, then by [5] there is a two-dimensional subspace $V \subset T_{p_1}X$ such that $\mathrm{II}_{p_1}(T_{p_1}L, V) = 0$, which implies that all first order infinitesimal deformations of L in Y span only a $\mathbf{P}^4$, which again is impossible.

Finally, consider the case where L is simply tangent at p_1 and transverse at $p_2 \neq p_3$. As we have $R \subseteq T^2_{p_1}$, it follows as above that

$$T^2_{p_1} = T^2_{p_2} = T^2_{p_3},$$

and again we may apply the Goose-Step Principle to contradict the nondegeneracy of X (the point is, goose-stepping through p_1, p_2, and p_3 is sufficient to fill up a dense subset of X).

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