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ON A SET OF POLARIZED KÄHLER METRICS ON ALGEBRAIC MANIFOLDS

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0. Introduction and statement of main theorems

A projective algebraic manifold M is a complex manifold in certain projective space CP^N, $N \geq \dim_C M = n$. The hyperplane line bundle of CP^N restricts to an ample line bundle L on M. This bundle L is a polarization on M. For the Kähler metric g on M, we can associate a positive, d-closed $(1, 1)$-form ω_g. In any local coordinate system $(z_1, \cdots, z_n)$ of M, the metric g is expressed by a tensor $(g_{i\bar{j}})_{1\leq i, j\leq n}$, and ω_g is defined to be $\frac{\sqrt{-1}}{2\pi}\sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. We call this ω_g the Kähler form associated to the metric g. By a polarized Kähler metric with respect to L, we mean a Kähler metric with its associated Kähler form representing the Chern class $C_1(L)$ of L in $H^2(M, Z)$. We denote by $\mathrm{Ka}(M)$ the set of all polarized Kähler metrics on M with respect to L. Given a Kähler metric g in $\mathrm{Ka}(M)$, one can find a hermitian metric h on L with its Ricci curvature form equal to ω_g (cf. [7], or Lemma 1.1 in §1). For each positive integer $m > 0$, the hermitian metric h induces a hermitian metric h^m on L^m. Choose an orthonormal basis $\{S_0^m, \cdots, S_{N_m}^n\}$ of the space $H^0(M, L^m)$ of all holomorphic global sections of L^m. Here the inner product on $H^0(M, L^m)$ is the natural one induced by the Kähler metric g and the hermitian metric h^m on L^m, i.e., $\langle S_\alpha^m, S_\beta^m\rangle = \int_m h^m(S_\alpha^m, S_\beta^m)\, dV_g$. Such a basis $S_0^m(x), \cdots, S_{N_m}^n(x)$ induces a holomorphic embedding φ_m of M into CP^{Nm} by assigning the point x of M to $[S_0^m(x), \cdots, S_{N_m}^m(x)]$ in CP^{N_m}. Let g_{FS} be the standard Fubini-Study metric on CP^{N_m}, i.e., $\omega_{g_{\mathrm{FS}}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\sum_{i=0}^{N_m}|w_i|^2)$ for a homogeneous coordinate system $[w_0, \cdots, w_{N_m}]$ of CP^{N_m}. The $\frac{1}{m}$-multiple of g_{FS} on CP^{N_m} restricts to a Kähler metric $\frac{1}{m}\varphi_m^* g_{\mathrm{FS}}$ on M. This metric is obviously in $\mathrm{Ka}(M)$, i.e., polarized by L, and it is called

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the Bergmann metric with respect to L. One of our main theorems here is the following.

Theorem A. *Let M be an algebraic manifold with a polarization L and let g be a polarized Kähler metric in* $\mathrm{Ka}(M)$. *Define the Bergmann metric $g_m = \frac{1}{m}\varphi_m^* g_{\mathrm{FS}}$ as above. Then*

$$\max_M \left\{ \|g_m - g\|, |Dg_m - Dg|\,\|, \left\|D^2 g_m - D^2 g\right\|, \|R(g_m) - R(g)\| \right\} = O\left(\frac{1}{\sqrt{m}}\right),$$

where D is the covariant derivative with respect to the metric g, $R(g_m)$ and $R(g)$ are the curvature tensors of g_m and g, respectively, and $O(1/\sqrt{m})$ means a constant bounded by $C/\sqrt{m}$ with C depending only on the metric g; the norm $\|\cdot\|$ is taken with respect to the metric g.

In particular, the theorem implies that the g_m converge to g in the C^2-topology on the space S^2M of all symmetric covariant 2-tensors. It is likely that the g_m actually converge to g in the C^∞-topology on S^2M. When M is naturally polarized by its canonical line bundle, the theorem solves a problem of S. T. Yau [15]; the problem asks whether or not the Kähler-Einstein metric on M can be the limit of a sequence of Bergmann metrics induced by pluricanonical line bundles K_M^m. This problem of Yau is one of the motivations in proving the above theorem. As we have noticed, the Bergmann metric $\frac{1}{m}\varphi^* g_{\mathrm{FS}}$ depends on the choice of the basis $\{S_0^m, \cdots, S_{N_m}^m\}$ of $H^0(M, L^m)$, so it depends on the metric g. But the set of Bergmann metrics $P_m = \{\frac{1}{m}\varphi_m^* g_{fs} | \sigma \in \mathrm{Aut}(CP^{N_m})\}$ is independent of the metric g. A corollary of Theorem A is the following density theorem.

Theorem B. *Let M be an algebraic manifold with a polarization L. Then the union $\bigcup_{m=1}^\infty P_m = P$ is dense in* $\mathrm{Ka}(M)$ *in the C^2-topology induced by the one on S^2M, where* $\mathrm{Ka}(M)$ *and P_m are defined as above.*

One can regard the Kähler metric in P as the metric defined by polynomials on M. Hence, both Theorems A and B are analogues of the famous Stone's approximation theorem on the space of continuous functions in the case of Kähler geometry. In particular, Theorem A and Theorem B imply that any function ψ can be approximated by the logarithm of polynomials whenever $\omega + \partial\bar\partial\psi \geq 0$ for an L-polarized Kähler form ω. Such a function is sometimes said to be almost pluriharmonic. These theorems also throw light on making use of the variational method in finding extremal Kähler metrics [2], such as Kähler-Einstein metrics. For instance, one can first study whether or not there is a Kähler metric g_m in P_m which reaches the minimum of the functional of an L^2-norm of scalar curvature [2] on P_m. Then one considers the convergence of those g_m as m goes to infinity. Note that the group $PGL(N_m+1) = \mathrm{Aut}(CP^{N_m})$ acts naturally on the set P_m; the latter is dominated by the affine space C^{N_m}. If this action is proper for the functional of an L^2-norm of scalar curvature, then it admits a minimum on P_m. The properness of the group action might be related to the stability of M in the Chow variety as defined in [9]. It is another motivation of proving these theorems for studying the connection between the existence of Kähler-Einstein metrics on M and Mumford's stability.

The proof is based on Hörmander's L^2-estimate of the $\bar\partial$-operator [6]; it is local in nature. Therefore, we can generalize Theorem A to complete Kähler manifolds with some conditions on Ricci curvature (see §4, Theorem 4.1). As applications of the generalization of Theorem A in case of quasiprojective manifolds, we can prove the following.

Theorem C. *Let X be a quasiprojective manifold, and let g be a complete Kähler metric on X with its Ricci curvature bounded from above by $-\lambda g$ for some $\lambda > 0$. Define the Ricci form* $\mathrm{Ric}(g)$ *as*

$$\mathrm{Ric}(g) = \frac{\sqrt{-1}}{2\pi} R_{i\bar{j}}\, dz_i \wedge d\bar{z}_j,$$

where $\{R_{i\bar{j}}\}_{1\le i,j\le n}$ is the Ricci tensor of g in local coordinates $(z_1, \cdots, z_n)$. Suppose that $\overline{X}$ is a smooth projective compactification of X. Then the positive $(1,1)$-form $-\mathrm{Ric}(g)$ can be naturally extended across the infinity of X in $\overline{X}$, and

(1) $0 < \int_X (-\mathrm{Ric}(g))^n \le C$, *where* $n = \dim_C X$,

(2) $0 < \int_X (-\mathrm{Ric}(g) \wedge \omega^{n-1}) \le C$, *where ω is a Kähler form on $\overline{X}$, C depends only on λ and $\overline{X}$, and $(-\mathrm{Ric}(g))^n$ is the n-exterior product of $-\mathrm{Ric}(g)$, similar to ω^{n-1}.*

In the case that X is a quasiprojective surface, we can say more about the extension of $\mathrm{Ric}(g)$ in the above theorem. This is stated in Theorem 5.1 in §5.

The organization of this paper is as follows. In §1, we construct L^2-holomorphic global peak sections of L^m with the L^2-norm almost concentrated at one point, i.e., the peak point. The tool used for this is Hörmander's L^2-estimate for $\bar\partial$-operator [6]. In §2, we calculate the Taylor expansion of the peak section constructed in §1 at the peak point. In §3, we use those peak sections of L^m constructed in §1 to prove Theorem A. Theorem B easily follows from Theorem A. In §4, we consider the generalization of Theorem A to noncompact, complete Kähler manifolds. Then in §5 we will use the generalization of Theorem A to prove Theorem C. We end the section with an improvement of Theorem C in the case of complex surfaces (Theorem 5.1). In §6, we discuss briefly the application of Theorem A or B to evaluating the holomorphic invariant introduced in [11].

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1. Construction of peak global sections of some line bundles

Let M be an n-dimensional algebraic manifold with a polarization L, and let g be a polarized Kähler metric with respect to L. In local coordinates $(z_1, \cdots, z_n)$, g is represented by a positive hermitian matrix $(g_{\alpha\overline{\beta}})$ and the associated Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta=1}^{n} g_{\alpha\overline{\beta}}\, dz_\alpha \wedge d\overline{z}_\beta.$$

ω_g is in $C_1(L)$.

Lemma 1.1 (See [7] for the proof). *There exists a hermitian metric h on L such that the curvature form* $\mathrm{Ric}(h)$ *of h is just ω_g.*

In the local holomorphic frame e_L of the line bundle L, the hermitian metric h is represented by a positive function $a(z)$, $z \in M$, i.e., for a local section $s = fe_L$ of L, $\|S\|_h^2 = a|f|^2$, where f is a local holomorphic function. Then the curvature $\mathrm{Ric}(h) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log a$.

Proposition 1.1. *Suppose that (M, g) is a complete Kähler manifold of complex dimension n, L is a line bundle on M with the hermitian metric h, and ψ is a function on M, which can be approximated by a decreasing sequence of smooth functions $\{\psi_l\}_{1\le l<+\infty}$. If*

$$(1.1)\qquad \left\langle \partial\overline{\partial}\psi_l + \frac{2\pi}{\sqrt{-1}}(\mathrm{Ric}(h) + \mathrm{Ric}(g)),\, v \wedge \overline{v} \right\rangle_g \ge C\,\|v\|_g^2$$

for any tangent vector v of type $(1, 0)$ at any point of M and for each l, where $C > 0$ is a constant independent of l, and $\langle\ ,\ \rangle_g$ is the inner product induced by g, then for any C^∞ L-valued $(0, 1)$-form w on M

with $\bar\partial w = 0$ *and* $\int_M \|w\|^2 e^{-\psi} dV_g$ *finite, there exists a* C^∞ *L-valued function* u *on* M *such that* $\bar\partial u = w$ *and*

$$\int_M \|u\|^2 e^{-\psi} dV_g \leq \frac{1}{C} \int_M \|w\|^2 e^{-\psi} dV_g, \tag{1.2}$$

where dV_g *is the volume form of* g *and the norms* $\|\cdot\|$ *induced by* h *and* g; *for instance, in the local coordinates* $(z_1, \cdots, z_n)$ *and the local frame* e_L *as above,* $g = (g_{\alpha\bar\beta})$, $w = w_\alpha d\bar z_\alpha$, *and* $\|w\|^2 = a(z) g^{\alpha\bar\beta} \overline{w}_\alpha w_\beta$.

This proposition can be proved easily by modifying the proof of [6, Theorem 4.4.1, p. 92] with the use of the Bochner-Kodaira Laplacian formula (see e.g. [7]).

Now let M and L be given as at the beginning of this section, and we construct peak sections of L^m for m large. Fix a point x_0 in M. Choose a local normal coordinate $(z_1, \cdots, z_n)$ at x_0, such that $x_0 = (0, \cdots, 0)$ and the hermitian matrix $(g_{\alpha\bar\beta})$ satisfies

$$\begin{aligned} &g_{\alpha\bar\beta}(x_0) = \delta_{\alpha,\beta}, \qquad dg_{\alpha\bar\beta}(x_0) = 0, \\ &\frac{\partial^2 g_{\alpha\bar\beta}}{\partial z_\gamma \partial z_\delta}(x_0) = 0 \quad \text{for } \gamma, \delta, = 1, 2, \cdots, n, \\ &\frac{\partial^3 g_{\alpha\bar\beta}}{\partial z_\gamma \partial z_\delta \partial z_\lambda}(x_0) = 0 \quad \text{for } \lambda, \gamma, \delta = 1, 2, \cdots, n, \\ &\frac{\partial^4 g_{\alpha\bar\beta}}{\partial z_\gamma \partial z_\delta \partial z_\lambda \partial z_\mu}(x_0) = 0 \quad \text{for } \lambda, \mu, \gamma, \delta = 1, 2, \cdots, n. \end{aligned} \tag{1.3}$$

Next we choose a local holomorphic frame e_L of L at x_0 such that the local representation function a of the hermitian metric h has the properties

$$\begin{aligned} &a(x_0) = 1, \qquad da(x_0) = 0, \qquad \partial\left(\frac{\partial a}{\partial z_i}\right)(x_0) = 0, \\ &\partial\left(\frac{\partial^2 a}{\partial z_i \partial z_j}\right)(x_0) = 0, \qquad \partial\left(\frac{\partial^3 a}{\partial z_i \partial z_j \partial z_k}\right)(x_0) = 0, \end{aligned} \tag{1.4}$$

where $i, j, k = 1, 2, \cdots, n$.

Suppose that this local coordinate $(z_1, \cdots, z_n)$ is defined on the open neighborhood U of x_0 in M. Define a function ρ on U, $\rho(z) = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for $z \in U$, where $|\cdot|$ is the euclidean norm.

Lemma 1.2. *For an n-tuple of integers* $(p_1, \cdots, p_n) \in Z^n_+$ *and an integer* $p' > p = p_1 + \cdots + p_n$, *there exists an* $m_0 > 0$ *such that, for* $m >$

m_0, there is a holomorphic global section S in $H^0(M, L^m)$, satisfying

$$(1.5)\quad \int_M \|S\|_{h^m}^2\, dV_g = 1, \qquad \int_{M\setminus\left\{\rho(z)\le \frac{\log m}{\sqrt{m}}\right\}} \|S\|_{h^m}^2\, dV_g = O\left(\frac{1}{m^{2_{p'}}}\right),$$

and locally at x_0,

(1.6)

$$S(z) = \lambda_{(p_1,\cdots,p_n)}\left(z_1^{p_1}\cdots z_n^{p_n} + O\left(|z|^{2p'}\right)\right)e_L^m\left(1+O\left(\frac{1}{m^{2p'}}\right)\right),$$

where $\|\cdot\|_{h^m}$ is the norm on L^m given by h^m, and $O(1/m^2p')$ denotes a quantity dominated by $C/m^{2p'}$ with the constant C depending only on p' and the geometry of M, moreover

$$(1.7)\qquad \lambda_{(p_1,\cdots,p_n)}^{-2} = \int_{\rho(z)\le\log m/\sqrt{m}} \left|z_1^{p_1}\cdots z_n^{p_n}\right|^2 a^m\, dV_g,$$

where $dV_g = \det(g_{i\bar{j}})(\sqrt{-1}/(2\pi))^n dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$ is the volume form.

Proof. We apply Proposition 1.1. Take a cut-off function η from a positive half real line R_+^1 to R_+^1 such that $\eta(t)\equiv 1$ for $t\le\frac{1}{2}$, $\eta(t)\equiv 0$ for $t\ge 1$, $0\le -\eta'(t)\le 4$, and $|\eta''(t)|\le 8$. Define the weight function

$$\psi(z) = \left(n+2p'\right)\eta\frac{m\rho^2(z)}{(\log m)^2}\log\left(\frac{m\rho^2(z)}{(\log m)^2}\right).$$

Then

$$\begin{aligned}
&\partial\bar\partial\psi(z)\\
&= \left(n+2p'\right)\left\{\left[\eta''\frac{m\rho^2(z)}{(\log m)^2}\frac{m^2}{(\log m)^4}\partial\rho^2\wedge\bar\partial\rho^2\right.\right.\\
&\qquad\left.+\eta'\frac{m\rho^2(z)}{(\log m)^2}\frac{m}{(\log m)^2}\partial\bar\partial\rho^2\right]\log\left(\frac{m\rho^2(z)}{(\log m)^2}\right)\\
&\qquad+2\,\mathrm{Re}\left[\eta'\frac{m\rho^2(z)}{(\log m)^2}\frac{m}{(\log m)^2}\partial\rho^2(z)\wedge\bar\partial\log\rho^2(z)\right]\\
&\qquad\left.+\eta\frac{m\rho^2(z)}{(\log m)^2}\partial\bar\partial\log\rho^2(z)\right)\Bigg\}.
\end{aligned}$$

Either $\eta''m\rho^2(z)/(\log m)^2<0$ or $\eta'm\rho^2(z)/(\log m)^2<0$ only if $\frac{1}{2}\le m\rho^2(z)/(\log m)^2\le 1$, i.e., $(\log m)^2/(2m)\le\rho^2(z)\le(\log m)^2/m$. For m

sufficiently large, $\partial\bar\partial\rho^2(z) \geq 0$ for $\rho^2(z) \leq (\log m)^2/m$. We will always use C to denote a constant independent of m. Hence

$$\partial\bar\partial\psi \geq \frac{-mC\left(n+2p'\right)}{(\log m)^2}\frac{2\pi}{\sqrt{-1}}\omega_g \quad \text{for } \rho(z)>0,$$

which implies that there is a decreasing sequence $\{\psi_l\}$ with $\lim_{l\to\infty}\psi_l = \psi$, and for any unit vector v of type $(1,0)$ at any point of M we have

$$\begin{aligned}(1.8)\qquad &\left\langle \partial\bar\partial\psi_l + \frac{2\pi}{\sqrt{-1}}\left(\mathrm{Ric}\left(h^m\right)+\mathrm{Ric}(g)\right),\, v\wedge\bar v\right\rangle_g \\ &\geq m\left(1-\frac{C\left(n+2p'\right)}{(\log m)^2}\right)\|v\|_g^2.\end{aligned}$$

Put $w=(1/4)\bar\partial(\eta(m\rho^2(z)/(\log m)^2)z_1^{p_1}\cdots z_n^{p_n}e_L^m)$, and by applying Proposition 1.1 we obtain an L-valued section u, solving $\bar\partial u = w$, and

$$(1.9)\qquad \int_M \frac{\|u\|_{h^m}^2}{e^\psi}\,dV_g \leq \frac{1}{m\left[1-\frac{C(n+2p')}{(\log m)^2}\right]}\int_M\frac{\|w\|_{h^m}^2}{e^\psi}\,dV_g,$$

where the norms $\|\ \|_{h^m}$ are induced by h^m and g. For m large, $C(n+2p')/(\log m)^2 \leq \frac{1}{2}$, so

$$\begin{aligned}&\int_M \|u\|_{h^m}^2 e^{-\psi}\,dV_g \\ &\leq \frac{1}{8m}\int_M\left|\eta'\frac{m\rho^2(z)}{4(\log m)^2}\right|^2 g^{i\bar j}\rho^2(z)_i\,\rho^2(z)_{\bar j}\left(\frac{m^2}{(\log m)^4}\right) \\ &\quad\cdot\left|z_1^{p_1}\cdots z_n^{p_n}\right|^2 a^m e^{-\psi}\,dV_g \\ (1.10)\qquad &\leq \frac{C}{(\log m)^2}\int_{\frac{2(\log m)^2}{m}\leq\rho^2(Z)\leq\frac{4(\log m)^2}{m}}\left|z_1^{p_1}\cdots z_n^{p_n}\right|^2 a^m e^{-\psi}\,dV_g.\end{aligned}$$

From the definition it follows that $\psi(z)=0$ for $\rho^2(z)\geq(\log m)^2/m$, and by (1.4) and Taylor expansion we have

$$\begin{aligned}(1.11)\qquad a(z) &= 1+\frac{\partial^2 a}{\partial z_i\bar\partial z_j}(x_0)\,z_i\bar z_j + O\left(|z|^3\right) \\ &= 1-|z|^2+O\left(|z|^3\right) \geq \left(1-\frac{1}{2}|z|^2\right) \quad \text{for } |z| \text{ small},\end{aligned}$$

where we have used the fact that $g_{i\bar{j}}(x_0) = -(\partial^2/\partial z_i \bar{\partial} z_j)\log(a(z))|_{z=x_0} = \delta_{ij}$. Hence, by (1.10) and (1.11) and the nonpositivity of ψ we obtain, for m large enough,

$$\begin{aligned}
&\int_M \|u\|_{h^m}^2 \, dV_g \\
&\quad \le \frac{C}{(\log m)^2} \int_{2\frac{(\log m)^2}{m} \le \rho^2(z) \le \frac{4(\log m)^2}{m}} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 \left(1 - \frac{1}{2}|z|^2\right)^m dV_g \\
&\quad \le \frac{C}{(\log m)^2} \left(\frac{(\log m)^2}{m}\right)^{n+p} \left(1 - \frac{(\log m)^2}{m}\right)^m \\
&\quad = O\left(\frac{1}{m^{8p'+2n}}\right).
\end{aligned} \tag{1.12}$$

Thus $(1-(\log m)^2/m)m = e^{m\log(1-(\log m)^2/m)} \le e^{-(\log m)^2/2} = m^{-(\log m)/2}$ for m large.

Put $\widetilde{S}(z) = \eta(m\rho^2(z)/(\log m)^2) z_1^{p_1} \cdots z_n^{p_n} e_L^m - u(z)$. By (1.10) and the definition of ψ, we have $u(z) = O(|z|^{2p'})$ at x_0, so at x_0

$$\widetilde{S}(z) = z_1^{p_1} \cdots z_n^{p_n} + O\left(|z|^{2p'}\right).$$

Now using the same argument as in (1.12) yields

$$\begin{aligned}
\left\|\widetilde{S}\right\|_{h^m}^2 &= \int_M \left| \eta\left(\frac{m\rho^2(z)}{(\log m)^2}\right)\right|^2 \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 a^m \, dV_g \\
&\quad + 2\,\mathrm{Re}\left(\int_M \left\langle \eta\left(\frac{m\rho^2(z)}{(\log m)^2}\right) z_1^{p_1} \cdots z_n^{p_n} e_L^m , u\right\rangle_{h^m} dV_g\right) + \|u\|_{h^m}^2 \\
&= \int_{\rho^2(z) \le (\log m)^2/m} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 a^m \, dV_g + O\left(\frac{1}{m^{4p'+n}}\right).
\end{aligned}$$

Define $S(z) = \widehat{S}(z)/\|\widehat{S}\|_{h_m}$. Then S is the holomorphic section needed.

Those sections constructed by Lemma 1.2 are called peak sections of the line bundle L^m.

2. Taylor expansions of the peak sections

In this section, we will evaluate $\lambda_{(p_1, \cdots, p_n)}$ to obtain the Taylor expansions of the peak sections constructed in the last section.

Lemma 2.1. *Let* $(p_1, \cdots, p_n)$ *be an n-tuple of integers, and* $p = p_1 + \cdots + p_n$. *Then*

$$\left(\frac{\sqrt{-1}}{2}\right)^n \int_{\rho(z)\leq \log m/\sqrt{m}} \left|z_1^{p_1}, \cdots, z_n^{p_n}\right|^2 \left(1-|z|^2\right)^m$$

$$\cdot dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$$

$$= \frac{\pi^n p_1! \cdots p_n! m!}{(p+n+m)!} + O\left(\frac{1}{m^{2p'}}\right) \quad \textit{for m large.}$$

Proof. This follows from a straightforward computation and the fact that $[1-(\log m)^2/m]^m = O(1/m^{2p'})$.

Denote by $\{R_{i\overline{j}k\overline{l}}\}_{1\leq i,j,k,l\leq n}$ the bisectional curvature tensor of the Kähler metric g in local coordinates $(z_1, \cdots, z_n)$. Then

$$\mathrm{Ric}_{i\overline{j}}(z) = \sum_{k=1}^n R_{i\overline{j}k\overline{k}}(z), \qquad r(g)(z) = \sum_{k=1}^n \mathrm{Ric}_{k\overline{k}}(z),$$

where $r(g)$ denotes the scalar curvature of g.

Lemma 2.2. *Let* μ *be a positive function on* R^1. *Then*

$$\left(\frac{\sqrt{-1}}{2}\right) \int_{\rho^2(z)\leq (\log m)^2/m} z_1^{p_1} \cdots z_n^{p_n} \overline{z}_1^{q_1} \cdots \overline{z}_n^{q_n} \mu\left(\rho^2(z)\right)$$

$$\cdot dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n = 0$$

for $(p_1, \cdots, p_n) \neq (q_1, \cdots, q_n)$.

Proof. Using polar coordinates, the above integral is equal to

$$\int_{S^{2n-1}} \left(\frac{z_1}{|z|}\right)^{p_1} \cdots \left(\frac{z_n}{|z|}\right)^{p_n} \left(\frac{\overline{z}_1}{|z|}\right)^{q_1} \cdots \left(\frac{\overline{z}_n}{|z|}\right)^{q_n} d\sigma$$

$$\cdot \int_0^{\log m/\sqrt{m}} t^{2n-1+p+q} \mu\left(t^2\right) dt,$$

where $q = q_1 + \cdots + q_n$, and $d\sigma$ is the standard measure on S^{2n-1}. But $(z_1/|z|)^{p_1} \cdots (z_n/|z|)^{p_n}$ and $(\overline{z}_1/|z|)^{q_1} \cdots (\overline{z}^n/|z|)^{q_n}$ are the eigenfunctions of the Laplacian on S^{2n-1} and are induced by harmonic functions on C^n; moreover, they are orthogonal to each other when $(p_1, \cdots, p_n) \neq (q_1, \cdots, q_n)$. Hence the lemma follows.

Lemma 2.3. *We have the following expressions*:

$$\lambda_{(2,0,\cdots,0)}^{-2} = \frac{m!}{(m+n+2)!}\left[R_{1\bar{1}1\bar{1}}(x_0) - r(g)(x_0) + n^2 + m + 6n + 10\right] + O\left(\frac{1}{m^{n+4}}\right),$$

$$\lambda_{(1,0,\cdots,0)}^{-2} = \frac{m!}{(m+n+1)!}\left[1 + \frac{1}{2(m+n+2)}\left(-r(g)(x_0) + n^2 + 3n + 2\right)\right] + O\left(\frac{1}{m^{n+3}}\right),$$

$$\lambda_{(0,0,\cdots,0)}^{-2} = \frac{m!}{(m+n)!}\left[1 + \frac{1}{2(m+n+2)}\left(-r(g)(x_0) + n^2 + n\right)\right] + O\left(\frac{1}{m^{n+2}}\right).$$

Proof. By (1.3),

$$\det\left(g_{\alpha\bar{\beta}}\right) = 1 - \mathrm{Ric}(g)_{i\bar{j}}\, z_i \bar{z}_j + b(z, \bar{z}) + O\left(|z|^4\right), \tag{2.1}$$

where $b(z, \bar{z})$ is a homogeneous polynomial in z_i and $\bar{z}_j$, and $\deg b(z, \bar{z}) = 3$. Since $-\partial\bar{\partial}\log a = 2\pi\omega_g/\sqrt{-1} = \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, dz_\alpha \wedge d\bar{z}_\beta$, using (1.3) and (1.4), we have the Taylor expansion

$$a(z) = 1 - |z|^2 + \frac{1}{4}\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(x_0)\, z_i \bar{z}_j z_k \bar{z}_l + \frac{1}{2}|z|^4 + d(z, \bar{z}) + O\left(|z|^6\right), \tag{2.2}$$

where $d(z, \bar{z})$ is a homogeneous polynomial of degree 5 in z_i and $\bar{z}_j$. Set

$$\lambda_{(2,0,\cdots,0)}^{-2} = \int_{\rho(z)\le \log m/\sqrt{m}} |z_1|^4 a^m \det\left(g_{\alpha\bar{\beta}}\right)\left(\frac{\sqrt{-1}}{2\pi}\right)^n \cdot dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

$$
\begin{aligned}
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\rho(z)\le \log m/\sqrt{m}} |z_1|^4 \\
&\qquad \times \left[1 - |z|^2 + \frac{1}{4}\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}(x_0)\, z_i \bar{z}_j z_k \bar{z}_l + \frac{1}{2}|z|^4 + d(z,\bar{z}) + O\left(|z|^6\right)\right]^m \\
&\qquad \cdot \left(1 - \mathrm{Ric}(g)_{i\bar{j}}(x_0)\, z_i \bar{z}_j + b(z,\bar{z}) + O\left(|z|^4\right)\right) \\
&\qquad \cdot dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{|z|\le \log m/\sqrt{m}} |z_1|^4 \\
&\quad \cdot \Bigg[\left(1 - |z|^2\right)^m + m\left(1 - |z|^2\right)^{m-1} \\
&\qquad \cdot \left(\frac{1}{4}\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}(x_0)\, z_i \bar{z}_j z_k \bar{z}_l + \frac{1}{2}|z|^4 + d(z,\bar{z})\right) \\
&\qquad + \left(1 - \frac{1}{2}|z|^2\right)^{m-2} O\left(m|z|^6 + m^2|z|^8\right)\Bigg] \\
&\qquad \cdot \left(1 - \mathrm{Ric}(g)_{i\bar{j}}(x_0)\, z_i \bar{z}_j + b(z,\bar{z}) + O\left(|z|^4\right)\right) \\
&\qquad \cdot dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{|z|\le \log m/\sqrt{m}} |z_1|^4 \\
&\qquad \cdot \Bigg[\left(1 - |z|^2\right)^m \left(1 - \mathrm{Ric}(g)_{i\bar{j}}(x_0)\, z_i \bar{z}_j + b(z,\bar{z})\right) \\
&\qquad + m\left(1 - |z|^2\right)^{m-1} \\
&\qquad \cdot \left(\frac{1}{4}\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}(x_0)\, z_i \bar{z}_j z_k \bar{z}_l + \frac{1}{2}|z|^4 + d(z,\bar{z})\right) \\
&\qquad + \left(1 - \frac{1}{2}|z|^2\right)^m O\left(|z|^4 + m|z|^6 + m^2|z|^8\right)\Bigg] \\
&\qquad \cdot dz_1 \wedge d\bar{z}_1 \wedge \cdots dz_n \wedge d\bar{z}_n.
\end{aligned}
$$

By applying either Lemma 2.1 or Lemma 2.2 to the integral of each term, obtain

$$\begin{aligned}
\lambda^{-2}_{(2,0,\cdots,0)} &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \\
&\cdot \int_{|z|\leq \log m/\sqrt{m}} \Bigg[|z_1|^4 \left(1-|z|^2\right)^m \\
&\qquad - \sum_{i=1}^n \mathrm{Ric}(g)_{i\bar{i}}(x_0)\, |z_1|^4 |z_i|^2 \left(1-|z|^2\right)^m \\
&\qquad + m\, |z_1|^4 \left(1-|z|^2\right)^{m-1} \\
&\quad \times \left(\frac{1}{4}\sum_{i=1}^n R_{i\bar{i}i\bar{i}}(x_0)\, |z_i|^4 \right. \\
&\qquad \left. + \sum_{i<j} R_{i\bar{i}j\bar{j}}(x_0)\, |z_i|^2 \left|z_j\right|^2 + \frac{1}{2}|z|^4 \right) \Bigg] \\
&\cdot dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n + O\left(\frac{1}{m^{n+4}}\right) \\
&= \frac{2m!}{(m+n+2)!}\left(1 + \frac{1}{2(m+n+3)} \right. \\
&\qquad \left. \cdot \left(R_{1\bar{1}1\bar{1}}(x_0) - r(g)(x_0) + n^2 + 5n + 6\right)\right) \\
&\quad + O\left(\frac{1}{m^{n+4}}\right).
\end{aligned}$$

Similarly, one can compute $\lambda^{-2}_{(1,0,\cdots,0)}$ and $\lambda^{-2}_{(0,\cdots,0)}$.

We end this section by summarizing the above into the following.

Proposition 2.1. *We have for a point x_0 in M, a local coordinate $(z_1, \cdots, z_n)$ with the properties stated in (1.3), and a local frame e_L of L with properties stated in (1.4). Let $\mu = (\mu_1, \cdots, \mu_n)$ be a unit vector in C^n. Then for m large enough, there are holomorphic peak global sections S, S_μ, and S_{μ^2} of L^m such that at x_0,*

$$(2.3)\quad S(z)=\sqrt{\frac{(m+n)!}{m!}}\left[1+\frac{1}{2(m+n+1)}\left(r(g)(x_0)-n^2-n\right)+O\left(\frac{1}{m^2}\right)\right]\cdot\left[1+O\left(|z|^8\right)\right],$$

(2.4)

$$S_\mu(z)=\sqrt{\frac{(m+n+1)!}{m!}}\left[1+\frac{1}{2(m+n+2)}\left(r(g)(x_0)-n^2-3n-2\right)+O\left(\frac{1}{m^2}\right)\right]\cdot\left[\sum_{i=1}^{n}\mu_i z_i+O\left(|z|^9\right)\right],$$

(2.5)

$$S_{\mu^2}(z)=\sqrt{\frac{(m+n+2)!}{2m!}}\left[1+\frac{1}{4(m+n+3)}\left(r(g)(x_0)-R_{\mu\bar\mu\mu\bar\mu}(x_0)-n^2-5n-6\right)+O\left(\frac{1}{m^2}\right)\right]\cdot\left[\left(\sum_{i=1}^{n}\mu_i z_i\right)^2+O\left(|z|^{10}\right)\right],$$

where $R_{\mu\bar\mu\mu\bar\mu}(x_0)=\sum_{i,j,k,l}R_{i\bar j k\bar l}\mu_i\bar\mu_j\mu_k\bar\mu_l$. *Moreover, those sections of* L^m *satisfy two equations in* (1.5) *for* $p'=4$.

Proof. Without loss of generality, we may assume that $\mu=(1,0,\cdots,0)$. Then the proposition follows from Lemmas 1.2 and 2.3 for proper p'.

3. The proof of Theorem A

We adopt the notation of the last section. The m-multiple of the polarization L induces an embedding $\varphi_m\colon M\to CP^{Nm}$, where $N_m+1=\dim_C H^0(M,L^m)$. Such an embedding is not canonical and is up to $\mathrm{Aut}(CP^{N_m})$. We define φ_m by choosing an orthonormal basis $\{S_0^m,\cdots,S_{N_m}^m\}$ of $H^0(M,L^m)$, i.e., $\varphi_m(x)=[S_0^m(x),\cdots,S_{N_m}^m(x)]$ in CP^{N_m} for any point x of M. Although this φ_m is not uniquely defined and is up to $U(N_m+1)$, we observe that the pull-back metric $g_m=\frac{1}{m}\varphi_m^* g_{FS}$ is uniquely defined. This observation is important and will often be used in the following proof of Theorem A. Let ω_m be the associated Kähler form of g_m. Then

$$\begin{aligned}\omega_m &= \frac{\sqrt{-1}}{2\pi m}\partial\overline{\partial}\log\left(\left|S_0^m\right|^2+\cdots+\left|S_{N_m}^m\right|^2\right)\\ &= \frac{1}{m}\operatorname{Ric}\left(h^m\right)+\frac{\sqrt{-1}}{2m\pi}\partial\overline{\partial}\log\left(\left\|S_0^m\right\|_{h^m}^2+\cdots+\left\|S_{N_m}^m\right\|_{h^m}^2\right)\\ &= \omega_g+\frac{\sqrt{-1}}{2m\pi}\partial\overline{\partial}\log\left(\left\|S_0^m\right\|_{h^m}^2+\cdots+\left\|S_{N_m}^m\right\|_{h^m}^2\right).\end{aligned} \tag{3.1}$$

We should clarify the meaning of $\partial\overline{\partial}\log(|S_0^m|^2+\cdots+|S_{N_m}^m|^2)$, since S_i^m $(0\le i\le N_m)$ is no longer a function and its euclidean norm is meaningless. Here in the local frame e_L^m of L^m, $S_i^m = f_i^m e_L^m$ for a holomorphic function f_i^m as a form, and $\partial\overline{\partial}\log(|S_0^m|^2+\cdots+|S_{N_m}^m|^2) = \partial\overline{\partial}\log(|f_0^m|^2+\cdots+|f_{N_m}^m|^2)$ is independent of the choice of the local frame e_L^m of L^m.

The proof of Theorem A is local in nature. It suffices to estimate the differences of g_m and g, and their derivatives, in a pointwise manner. Fix a point x_0 at M. Choose the local coordinate $(z_1, \cdots, z_n)$ at x_0 and a local frame e_L of L at x_0 such that (1.3) and (1.4) are satisfied. For each $\mu=(\mu_1,\cdots,\mu_n)\in C^n$, $|\mu|^2=1$, define $y_\mu=\sum_{i=1}^n \mu_i z_i$ and $Y_\mu=\partial/\partial y_\mu$, where Y_μ is a local holomorphic vector field of M at x_0. It is well known that the sectional curvature tensor is dominated by the holomorphic sectional curvature. Thus in order to prove Theorem A, it is sufficient to prove that

$$\begin{aligned}\max\Big\{&\left|g_m\left(Y_\mu, Y_\mu\right)-g\left(Y_\mu, Y_\mu\right)\right|(x_0),\\ &\left|D_{Y_\mu}g_m\left(Y_\mu, Y_\mu\right)-D_{Y_\mu}g\left(Y_\mu, Y_\mu\right)\right|(x_0),\\ &\left|D_{Y_\mu}^2g_m\left(Y_\mu, Y_\mu\right)-D_{Y_\mu}^2g\left(Y_\mu, Y_\mu\right)\right|(x_0),\\ &\left|R_{\mu\overline{\mu}\mu\overline{\mu}}(g_m)-R_{\mu\overline{\mu}\mu\overline{\mu}}(g)\right|(x_0)\Big\}\\ &= O\left(1/\sqrt{m}\right),\end{aligned} \tag{3.2}$$

where $O(1/\sqrt{m})$ denotes a quantity dominated by $C/\sqrt{m}$ with C independent of x_0 and μ, and the covariant derivative D is taken with respect to the metric g.

By (1.3) and the definition of g_m, one can easily see that (3.2) is equivalent to

$$\max\left\{\left|\frac{1}{m}\frac{\partial^2}{\partial y_\mu \partial \bar{y}_\mu}\log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^n\right|^2\right)-1\right|(x_0),\right.$$

$$\left|\frac{1}{m}\frac{\partial^3\log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial y_\mu^2\partial\bar{y}_\mu}\right|(x_0),$$

$$\left|\frac{1}{m}\frac{\partial^4\log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial y_\mu^3\partial\bar{y}_\mu}\right|(x_0),\tag{3.3}$$

$$\left.\left|\frac{-1}{m}\frac{\partial^4\log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial y_\mu^2\partial\bar{y}_\mu^2}-R_{\mu\bar{\mu}\mu\bar{\mu}}(g)(x_0)\right|\right\}$$

$$=O\left(\frac{1}{\sqrt{m}}\right),$$

where the $\{f_i^m\}$ are local representations of $\{S_i^m\}$ in the frame e_L^m of L^m. It suffices to prove (3.3). First we investigate the orthogonal discrepancy among the sections in the chosen basis $\{S_i^m\}$ and those sections constructed in either Lemma 1.2 or Proposition 2.1.

Lemma 3.1. *Let S be a holomorphic global section constructed in either Lemma 1.1 or Proposition 2.1. Then it is known that*

$$S(z)=\sqrt{\frac{(p+n+m)\,1}{p_1!\cdots p_n!\,m!}}\left(1+O\left(\frac{1}{m^{p'}}\right)\right)\left(z_1^{p_1}\cdots z_n^{p_n}+O\left(|z|^{2p'}\right)\right)e_L^m$$

at x_0, where $p'\gg p$. Let T be another section of L^m with $\|T\|_{h^m}=1$, which contains no term $z_1^{p_1}\cdots z_n^{p_n}$ in its Taylor expansion at x_0. Then

$$\langle S, T\rangle_{h_m}=O\left(\tfrac{1}{m}\right),\tag{3.4}$$

where $\langle\ ,\ \rangle_{h^m}$ is the inner product on the linear space $H^0(M, L^m)$ induced by the metric h^m. Furthermore, if T contains no term $z_1^{q_1}\cdots z_n^{q_n}$ with

$q_1 + \cdots + q_n = p$, *then*

$$(3.5) \qquad \langle S, T \rangle_{h^m} = O\left(1/m^{3/2}\right).$$

Proof. We prove only (3.5), since the proof for (3.4) is almost the same. By (1.5) and Schwartz inequality we obtain

$$\langle S, T \rangle_{h^m} = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{|z| \le \log m/\sqrt{m}} f_S \cdot \overline{f}_T a^m \det\left(g_{\alpha\overline{\beta}}\right) dz \wedge d\overline{z} = O\left(\frac{1}{m^{p'}}\right),$$

where $dz \wedge d\overline{z} = dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$, $S(z) = f_S(z) e_L^m$, and $T(z) = f_T(z) e_L^m$ in a neighborhood of x_0. From (1.6) it follows that

$$f_S(z) = \frac{\sqrt{(p+n+m)!}}{m! p_1! \cdots p_n!} \left[z_1^{p_1} \cdots z_n^{p_n} + O\left(\frac{1}{m^{(p'-1)/2}}\right)\right]$$

for $|z| \le \log m/\sqrt{m}$. By taking p' large enough, we only need to prove

$$(3.6) \qquad \begin{aligned} I = & \int_{|z| \le \log m/m} z_1^{p_1} \cdots z_n^{p_n} \cdot \overline{f}_T a^m \det\left(g_{\alpha\overline{\beta}}\right) dz \wedge d\overline{z} \\ & = O\left(\frac{1}{m^{(n+p+3)/2}}\right). \end{aligned}$$

Substituting (2.2) and (2.1) in the integrand on the left-hand side of (3.6), we obtain

$$\begin{aligned} I = \int_{|z| \le \log m/\sqrt{m}} & z_1^{p_1} \cdots z_n^{p_n} \cdot \overline{f}_T(z) \left[1 - |z|^2 + \frac{1}{4} \sum_{i,j,k,l} R_{i\overline{j}k\overline{l}} z_i z\Big|_j z_k z\Big|_l \right. \\ & \left. + \frac{1}{2}|z|^4 + O\left(|z|^5\right)\right]^m \cdot \left(1 - \sum_{i,j} \mathrm{Ric}(g)_{i\overline{j}} z_i \overline{z}_j + O\left(|z|^3\right)\right) dz \wedge d\overline{z} \end{aligned}$$

$$= \int_{|z| \leq \log m/\sqrt{m}} z_1^{p_1} \cdots z_n^{p_n} \cdot \overline{f}_T(z)$$

$$\cdot \left[\left(1 - \sum_{i,j} \mathrm{Ric}(g)_{i\bar{j}} z_i \overline{z}_j \right) \left(1 - |z|^2\right)^m \right.$$

$$\left. + \left(\frac{m}{2} |z|^4 + \frac{m}{4} \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}(x_0) z_i \overline{z}_j z_k \overline{z}_l \right) \left(1 - |z|^2\right)^m - 1 \right] dz \wedge d\overline{z}$$

$$+ \int_{|z| \leq \log m/\sqrt{m}} z_1^{p_1} \cdots z_n^{p_n} \cdot \overline{f}_T(z) \left(O\left(m|z|^5 + |z|^3 \right) \right) a^m \, dz \wedge d\overline{z}.$$

By Lemma 2.2 and the assumption that the Taylor expansion of $f_T(z)$ at x_0 has no term $z_1^{q_1} \cdots z_n^{q_n}$ with $q_1 + \cdots + q_n = p$, one easily sees that the first integral above is zero. Thus by Schwartz inequality we have

$$|I| \leq C \left[\int_{|z| \leq \log m/\sqrt{m}} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 \left(m^2 |z|^{10} + |z|^6 \right) a^m \, dV_g \right]^{1/2} \|T\|_{h^m}$$

$$= O\left(\frac{1}{m^{(p+n+3)/2}} \right).$$

Without loss of generality, we may assume that $\mu = (1, 0, \ldots, 0)$. Then $y_\mu = z_1$. Since the metric g_m is independent of the choice of the orthonormal basis $\{S_0^m, \ldots, S_{N_m}^n\}$ of $H^0(M, L^m)$, by an orthogonal transformation we may further assume that

$$f_i^m(0) = 0 \quad \text{for } i \geq 1,$$

$$\frac{\partial f_i^m}{\partial z_j}(0) = 0 \quad \text{for } i \geq j+1, \; j = 1, 2, \ldots, n, \tag{3.7}$$

$$\frac{\partial^2 f_i^m}{\partial z_1^2}(0) = 0 \quad \text{for } i \geq n+2.$$

Lemma 3.2. *Under the above assumption on the orthonormal basis* $\{S_0^m, \ldots, S_{N^m}^m\}$, *we have the following estimates:*

$$\left|f_0^m(x_0)\right| = \sqrt{\frac{(n+m)!}{m!}}\left[1 + \frac{1}{2(m+n+1)}\left(r(g)(x_0) - n^2 - n\right) + O\left(\frac{1}{m^2}\right)\right]$$

(i)

$$\frac{\partial f_0^m}{\partial z_i}(x_0) = O\left(m^{n/2-1}\right),$$

$$\frac{\partial^2 f_0^m}{\partial z_1^2}(x_0) = O\left(m^{(n-1)/2}\right), \quad \frac{\partial^3 f_0^m}{\partial z_1^3}(x_0) = O\left(m^{n/2}\right),$$

(ii)

$$\left|\frac{\partial f_1^m}{\partial z_1}(x_0)\right| = \sqrt{\frac{(n+m+1)!}{m!}} \cdot \left[1 + \frac{1}{2(m+n+1)}\left(r(g)(x_0) - n^2 - 3n - 2\right) + O\left(\frac{1}{m^2}\right)\right],$$

$$\frac{\partial^2 f_1^m}{\partial z_1^2}(x_0) = O\left(m^{(n-1)/2}\right), \quad \frac{\partial^3 f_1^m}{\partial z_1^3}(x_0) = O\left(m^{n/2}\right),$$

(iii)

$$\frac{\partial^2 f_i^m}{\partial z_1^2}(x_0) = O\left(m^{(n-1)/2}\right) \quad \textit{for } i = 1, 2, 3, \dots, n;$$

(iv)

$$\left|\frac{\partial^2 f_n + 1^m}{\partial z_1^2}(x_0)\right| = \sqrt{\frac{2(m+n+2)!}{m!}}\left[1 + \frac{1}{4(m+n+3)}\left(r(g)(x_0) - R_{1\bar{1}1\bar{1}}(x_0) - n^2 - 5n - 6\right) + O\left(\frac{1}{m^2}\right)\right].$$

Proof. The proof of (i) is almost the same as that of (ii), so we omit it. Let us first prove (iii) and (iv). Use Lemma 1.1 to construct holomorphic sections $T_1, \dots, T_n$ of L^m satisfying (1.5) and (1.6) for $(p_1, \dots, p_n) = (1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)$, respectively. There are constants β_{ij} $(1 \leq i \leq n,\ 0 \leq j \leq N_m)$ such that

(3.8)
$$\sum_{j=0}^{N_m} \left|\beta_{ij}\right|^2 = 1,$$

(3.9)
$$T_i = \sum_{j=0}^{N_m} \beta_{ij}, S_j, \qquad i = 1, 2, \dots, n.$$

By (3.7), $\beta_{ij} = 0$ for $j < i$. Applying Lemma 3.1 to T_i, S_j, and $\sum_{j=n+1}^{N_m} \beta_{ij} S_j$, we conclude from (3.7) that for large p' in Lemma 1.1,

(3.10)
$$\beta_{ij} = O(1/m) \quad \text{for } i = 1, 2, \cdots, n;\ i < j < n+1,$$
$$\sqrt{|\beta_{in+1}|^2 + \cdots + \left|\beta_{iN_m}\right|^2} = O\left(1/m^{3/2}\right) \quad \text{for } i = 1, 2, \cdots, n.$$

Take derivatives on both sides of (3.9), and use the fact that $(\partial^2 f_{T_i}/\partial z_1^2)(x_0) = 0$ (if we take p' large in the construction of T_i, by Lemma 3.1), where $T_i = f_{T_i} e_L^m$ at x_0. One then obtains

$$0 = \sum_{j=1}^{N_m} \beta_{ij} \frac{\partial^2 f_i^m}{\partial z_1^2}(x_0), \qquad i = 1, 2, \cdots, n. \tag{3.11}$$

It follows from (3.8) and (3.10) that $|\beta_{ii}| = 1 + O(1/m^2)$. Inductively, one solves (3.11) for $(\partial^2 f_i^m/\partial z_1^2)(x_0)$, $i = 1, 2, \cdots, n$, and $(\partial^2 f_i^m/\partial z_1^2) \cdot (x_0) = (\partial^2 f_{n+1}^m/\partial z_1^2)(x_0) \cdot O(1/m^{3/2})$.

By using Proposition 2.1 to construct S_{μ^2} and proceeding as above, we have

$$\begin{aligned}\frac{\partial^2 f_{n+1}^m}{\partial z_1^2}(x_0) &= \frac{\partial^2 f_{\mu^2}}{\partial z_1^2}(x_0)\left[1 + O\left(\frac{1}{m^2}\right)\right] \qquad \left(S_{\mu^2} = f_{\mu^2} e_L^m\right) \\ &= \sqrt{\frac{2(m+n+2)!}{m!}}\left[1 + \frac{1}{4(m+n+3)}\left(r(g)(x_0)\right.\right. \\ &\qquad \left.\left. - R_{1\bar{1}1\bar{1}}(x_0) - n^2 - 5n - 6\right) + O\left(\frac{1}{m^2}\right)\right].\end{aligned}$$

Hence (iii) and (iv) follow.

By similar arguments, one can prove that $(\partial^3 f_1^m/\partial z_1^3)(x_0) = O(m^{n/2})$. To finish the proof of (ii), it remains to evaluate $(\partial f_1^m/\partial z_1)(x_0)$. For this purpose, we simply take the section S_μ constructed in Proposition 2.1 for $\mu = (1, 0, \cdots, 0)$, and express it in terms of $\{S_i^m\}$ as follows:

$$S_\mu = \sum_{j=0}^{N_m} \beta_j S_j^m, \qquad \sum_{j=0}^{N_m} \left|\beta_j\right|^2 = 1.$$

One sees immediately that $\beta_0 = 0$ and $\beta_j = O(\frac{1}{m})$ for $j \geq 2$ by Lemma 3.1, so we obtain the evaluation of $(\partial f_1^m/\partial z_1)(x_0)$ required in the statement of (ii).

Now we are ready to prove the estimate (3.3). We split the proof into the following four lemmas, and note that we may assume $y_\mu = z_1$.

Lemma 3.3. *With the above notation and assumptions, we have*

$$\frac{1}{m}\frac{\partial^2 \log\left(|f_0^m|^2 + \cdots + \left|f_{N_m}^m\right|^2\right)}{\partial z_1 \bar{\partial} z_1}(x_0) - 1 = O\left(\frac{1}{m}\right).$$

Proof. By (3.7) and Lemma 3.2 (i), (ii), we have

$$\begin{aligned}
&\frac{1}{m}\frac{\partial^2 \log\left(|f_0^m|^2 + \cdots + \left|f_{N_m}^m\right|^2\right)}{\partial z_1 \bar{\partial} z_1}(x_0) \\
&= \frac{|\partial f_1^m/\partial z_1|^2}{m\,|f_0^m|^2}(x_0) \\
&= \frac{1}{m}(m+n+1)\left[\frac{1 + \frac{1}{2(m+n+2)}\left(r(g)(x_0) - n^2 - 3n - 2\right) + O\left(1/m^2\right)}{1 + \frac{1}{2(m+n+1)}\left(r(g)(x_0) - n^2 - n\right) + O\left(1/m^2\right)}\right]^2 \\
&= 1 + O\left(\frac{1}{m}\right).
\end{aligned}$$

Lemma 3.4. *With the above notation and assumptions, we have*

$$\frac{1}{m}\frac{\partial^3 \log\left(|f_0^m|^2 + \cdots + \left|f_{N_m}^m\right|^2\right)}{\partial z_1^2 \bar{\partial} z_1}(x_0) = O\left(\frac{1}{m}\right).$$

Proof. By (3.7) and a direct computation, we obtain

$$\begin{aligned}
&\frac{1}{m}\frac{\partial^3 \log\left(|f_0^m|^2 + \cdots + \left|f_{N_m}^m\right|^2\right)}{\partial z_1^2 \partial \bar{z}_1}(x_0) \\
&\quad = \frac{\left(\partial^2 f_1^m/\partial z_1^2\right)\left(\overline{\partial f_1^m/\partial z_1}\right)}{m\,|f_0^m|^2}(x_0) \\
&\quad\quad - \frac{2\,|\partial f_1^m/\partial z_1|^2\,\mathrm{Re}\left((\partial f_0^m/\partial z_1)\, f|_0^m\right)}{m\,|f_0^m|^4}(x_0).
\end{aligned}$$

Then the lemma follows from Lemma 3.2 (i), (ii).

Lemma 3.5. *With the notation and the assumptions as in Lemma* 3.3 *and* 3.4, *we have*

$$\frac{1}{m}\frac{\partial^4 \log\left(|f_0^m|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial z_1^3\partial \overline{z}_1}(x_0)=O\left(\frac{1}{\sqrt{m}}\right).$$

Proof. By (3.7) and a complicated, but straightforward, computation, we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\partial^4 \log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial z_1^3\partial \overline{z}_1}(x_0)\\
&\quad=-2\frac{\partial^3 \log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial z_1^2\partial \overline{z}_1}(x_0)\cdot\frac{\partial f_0^m/\partial z_1}{m f_0^m}(x_0)\\
&\quad\quad-2\frac{\partial^2 \log\left(\left|f_0^m\right|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial z_1\partial z|_1}(x_0)\\
&\quad\quad\cdot\frac{\overline{f_0^m}\left|f^m)0\right|^2\left(\partial^2 f_0^m/\partial z_1^2\right)(x_0)-\left(\overline{f_0^m}\right)^2\left(\partial f_0^m/\partial z_1\right)^2}{m\left|f_0^m\right|^4}\\
&\quad\quad+\frac{\left(\partial^3 f_1^m/\partial z_1^3\right)\left(\partial f_1^m/\partial z_1\right)}{m\left|f_0^m\right|^2}(x_0)\\
&\quad\quad-\frac{\left(\partial^2 f_1^m/\partial z_1^2\right)\left(\overline{\partial f_1^m/\partial z_1}\right)\left(\partial f_0^m/\partial z_1\right)\overline{f_0^m}+f_0^m\left(\partial^2 f_0^m/\partial z_1^2\right)\left|\partial f_1^m/\partial z_1\right|^2}{m\left|f_0^m\right|^4}(x_0).
\end{aligned}$$

Then this lemma follows from Lemmas 3.3 and 3.4, and (i), (ii) of Lemma 3.2.

Lemma 3.6. *With the notation and assumptions as in the above lemmas, we have*

$$\left|\frac{-1}{m}\frac{\partial^4 \log\left(|f_0^m|^2+\cdots+\left|f_{N_m}^m\right|^2\right)}{\partial z_1^2\partial \overline{z}_1^2}-R_{1\overline{1}1\overline{1}}(x_0)\right|=O\left(\frac{1}{m}\right).$$

Proof. By a straightforward computation and (3.7), we obtain

$$
\begin{aligned}
II &= \frac{1}{m}\frac{\partial^4 \log\left(|f_0^m|^2 + \cdots + |f_{N_m}|^2\right)}{\partial z_1^2 \partial \bar{z}_1^2} \\
&= -4\,\mathrm{Re}\left(\frac{\left(\partial^2 f_1^m/\partial z_1^2\right)\left(\overline{\partial f_1/\partial z_1}\right) f_0^m \left(\overline{\partial f_0^m/\partial z_1}\right)}{m\,|f_0^m|^4}\right) \\
&\quad + 4\frac{|\partial f_1^m/\partial z_1|^2\,|\partial f_0^m/\partial z_1|^2}{m\,|f_0^m|^4} \\
&\quad + \frac{1}{m}\left(\frac{\sum_{i=1}^{n+1}\left|\partial^2 f_i^m/\partial z_1^2\right|^2}{|f_0^m|^2} - \frac{2\,|\partial f_1^m/\partial z_1|^4}{|f_0^m|^4}\right).
\end{aligned}
$$

By Lemma 3.2(i), (ii), the first two terms on the right-hand side are both of order $O(1/m^2)$. By Lemma 3.2(iii), we have $\sum_{i=1}^{n} |\partial f_i^m/\partial z_1^2|^2 = O(m^{n-1})$, and therefore

$$
\begin{aligned}
II &= \frac{1}{m}\left(\frac{\left|\partial^2 f_{n+1}^m/\partial z_1^2\right|^2}{|f_0^m|^2} - \frac{2\,|\partial f_i^m/\partial z_1|^4}{|f_0^m|^4}\right) + O\left(\frac{1}{m^2}\right) \\
&\qquad\qquad \text{(by Lemma 3.2 again)} \\
&= \frac{1}{m} 2(m+n+2)(m+n+1) \\
&\quad \cdot \left[1 + \frac{1}{2(m+n+3)}\left(r(g)(x_0) \cdot -R_{1\bar{1}1\bar{1}}(x_0) - n^2 - 5n - 6\right) + O\left(\frac{1}{m^2}\right)\right] \\
&\quad \cdot \left[1 + \frac{1}{2(m+n+1)}\left(r(g)(x_0) - n^2 - n\right) + O\left(\frac{1}{m^2}\right)\right]^{-1} \\
&\quad - 2(m+n+1)^2\left(1 + \frac{1}{2(m+n+2)}\left(r(g)(x_0) - n^2 - 3n - 2\right) + O\left(\frac{1}{m^2}\right)\right)^2
\end{aligned}
$$

$$\cdot\left(1+\frac{1}{2(m+n+1)}\left(r(g)(x_0)-n^2-n\right)+O\left(\frac{1}{m^2}\right)\right)^{-2}$$

$$+O\left(\frac{1}{m^2}\right)$$

$$=-R_{1\bar{1}1\bar{1}}(x_0)+O\left(\frac{1}{m}\right).$$

The estimate (3.3) follows from Lemmas 3.3–3.6. Hence Theorem A is proved. Obviously, Theorem B follows from Theorem A.

Remark. It is likely that one can estimate the difference between g_m and g up to higher derivatives, but one should have a neater method. For Riemann surfaces we can show that g_m converges to g in C^3-topology by computations analogous to the above ones.

4. Generalizations to noncompact manifolds

It is easy to see that the previous proof for Theorem A is local in nature. This makes us believe that some generalizations of it should be possible for noncompact Kähler manifolds. Let X be a complete Kähler manifold with Kähler metric g and let L be a line bundle on X with a hermitian metric h and Ricci curvature $\mathrm{Ric}(h)$ greater than $\varepsilon\max\{\omega_g, -\mathrm{Ric}(g)\}$ for some $\varepsilon>0$. Denote by $H^0_{(2)}(X, L^m)$ the space of all L^2-integrable holomorphic global sections of the line bundle L^m. This space is a Hilbert space with a natural inner product induced by the hermitian metric h^m. Choose an orthonormal basis $\{S_i^m\}_{i\geq 0}$ of $H^0_{(2)}(X, L^m)$.

Lemma 4.1. *For each local frame e_L of L at any point x in X, write $S_i^m=f_i^m e_L^m$. Then $\sum_{i=0}^\infty |f_i^m|^2$ is a smooth function near x.*

Proof. Each f_i^m is holomorphic near x. Then by the Cauchy integral formula (cf. [5]), it suffices to prove that $\sum_{i=0}^\infty |f_i^m|^2$ is locally uniformly bounded at x. First we observe that for a holomorphic section S in $H^0_{(2)}(X, L^m)$ with $\|S\|_{h^m}=1$, there is a neighborhood U of x and a constant C which is independent of S, such that $S=fe_L^m$, $|f|^2$ is bounded by C in U. Now for any $N>0$, take an orthogonal transformation σ from $U(N+1)$ such that if $\sigma=(\sigma_{ij})_{0\leq i,j\leq N}$ and $T_i=\sum_{j=0}^N \sigma_{ij}S_j^m$, then

$T_i(x) = 0$ for $i \geq 1$. Since σ is orthogonal,

$$\sum_{i=1}^{n} \left|f_i^m\right|^2 (x) = \frac{1}{\left\|e_L^m\right\|_{h^m}^2 (x)} \sum_{i=0}^{n} \left\|S_i^m\right\|_{h^m}^2$$
$$= \frac{1}{\left\|e_L^m\right\|_{h^m}^2 (x)} \sum_{i=0}^{n} \left\|T_i\right\|_{h^m}^2 = \left|\frac{T_0}{e_L^m}\right|^2 (x) \leq C + \infty.$$

Thus the lemma is proved.

By this lemma, one can define a positive, d-closed, $(1, 1)$-current ω_m on X in the following way: At any point x in X, define

$$\omega_m (x) = \frac{\sqrt{-1}}{2m\pi} \partial\overline{\partial} \log \left(\sum_{i=0}^{\infty} \left|f_i^m\right|^2\right) (x), \tag{4.1}$$

where $S_i^m = f_i^m e_L^m$ in a neighborhood of x ($\{S_i^m\}$ is the basis as in Lemma 4.1.) Formally, we can write

$$\omega_m = \frac{\sqrt{-1}}{2m\pi} \partial\overline{\partial} \log \left(\sum_{i=0}^{\infty} \left|S_i^m\right|^2\right).$$

Since $\mathrm{Ric}(h) \geq \varepsilon \max\{\omega_g, -\mathrm{Ric}(g)\}$ for some $\varepsilon > 0$, it follows from the standard L^2-estimate of the $\overline{\partial}$-operator (Proposition 1.1) that for any compact set $K \subset X$, ω_m will be regular $(1, 1)$-form in K for m large enough. By the same arguments as in the proof of Theorem A, one can prove the following.

Theorem 4.1. *Set X, L, g, and h as above. Then for any compact set K of X, there are constants m_0 and C, depending only on the geometry of K in M, such that for $m > m_0$, ω_m is regular in K and so induces a Kähler metric g_m on K, and*

$$\max_M \{\|g_m - \mathrm{Ric}(h)\|, \|Dg_m - D\,\mathrm{Ric}(h)\|\} \leq C_K/\sqrt{m}, \tag{4.2}$$

where the covariant derivative D is taken with respect to g, and $\|\cdot\|$ is a norm on tensors induced by g. In the case that (X, g) is a complete Kähler-Einstein manifold with $\mathrm{Ric}(g) = -\omega_g$, and $L = K_X$ is the canonical line bundle, we have in the compact set K

$$\max_K \left\{\|g_m - g\|, \|Dg_m\|, \left\|D^2 g_m\right\|, \|R(g_m) - R(g)\|\right\}$$
$$\leq C_K/\sqrt{m} \quad \textit{for } m > m_0, \tag{4.3}$$

where $R(g_m)$ and $R(g)$ are curvature tensors of g_m and g, respectively. Furthermore, if the metric g has bounded curvature tensor, and the injectivity radius at each point is bounded from below by a uniform constant

$c > 0$, *then* C_K *in* (4.2) *or* (4.3) *can be taken independent of the compact set* K.

Remark. As a corollary of Theorem 4.1, one can prove that for any strictly pluriharmonic function φ on the unit ball $B_1(0)$ of C^n, the function φ can be approximated up to its third derivatives in the ball $B_{1/2}(0)$ by functions of the form $\lambda \log(\sum_{i=1}^N |f_i|^2)$, where the $\{f_i\}$ are holomorphic functions on $B_1(0)$.

5. The proof of Theorem C

In this section, we will use Theorem 4.1 to prove Theorem C. We start with the following lemma.

Lemma 5.1. *Let* X *be a quasiprojective manifold with a complete Kähler metric* g, *where* $\mathrm{Ric}(g) \leq -\lambda\omega_g$ *for some* $\lambda > 0$. *Let* $\overline{X}$ *be a smooth compactification of* X. *Then for each* m, *the space* $H^0_{(2)}(X, K_X^m)$ *is of finite dimension, and any holomorphic section* S *in* $H^0_{(2)}(X, K_X^m)$ *can be extended to be a meromorphic section of* $(K_{\overline{X}})^m$ *on* $\overline{X}$ *with poles along* D *and the order less than* m.

Proof. It suffices to prove the second statement. Fix a Kähler metric $\overline{g}$ on $\overline{X}$ induced by the Kähler form ω given in the statement of Theorem C. Put $\overline{h} = (\det(\overline{g}_{\alpha\overline{\beta}}))^{-1}$; then $\overline{h}$ is a hermitian metric on $K_{\overline{X}}$. Let h be the hermitian metric on K_X induced by g on X, and put $e^{\varphi} = h/\overline{h}$. Then by $K_{\overline{X}}|_X = K_X$ we have

$$\int_X \|S\|^2_{\overline{h}^m} e^{(m-1)\varphi}\, dV_{\overline{g}} = \int_X \|S\|^2_{h^m}\, dV_g < +\infty. \tag{5.1}$$

Put $D = \overline{X} - X$; then D is a divisor in $\overline{X}$. Let $\mathrm{Sing}(D)$ be the singular points of D, and $\mathrm{codin}_{\overline{X}}(\mathrm{Sing}(D)) \geq 2$. By Hartog's theorem (cf. [5]), we only need to extend S across the regular part of D, i.e., across $D - \mathrm{Sing}(D)$. Take any point x in $D - \mathrm{Sing}(D)$. Then there is a neighborhood U_x of x in $\overline{X}$ such that $U_x \cap X \approx \Delta^* \times \Delta^{n-1}$, where $\Delta = \{z \in C^1 \mid |z| < 1\}$, and $\Delta^* = \Delta \setminus \{0\}$. Thus there is a complete hyperbolic metric g_x on $U_x \cap X$. This metric g_x is actually the product metric of those Poincaré metrics on Δ or Δ^*. Moreover, if z_1 is the local defining function of $D \cap U_x$, then for a smaller neighborhood V_x of x in U_x,

$$\overline{h}\, |z_1|^2 (-\log|z_1|)^2 \leq C h_x \quad \text{in } V_x \cap X, \tag{5.2}$$

where C is a positive constant, and h_x is the hermitian metric on $K_{\overline{X}}|_{U_x \cap X}$ induced by g_x.

On the other hand, by our assumption on $\mathrm{Ric}(g)$ and Yau's generalized Schwartz lemma [14], the volume form of g uniformly dominates that of g_x on $U_x \cap X$, i.e.,

$$e^{\varphi}\overline{h} = H \geq C(\lambda)\, h_x \quad \text{on } U_x \cap X, \tag{5.3}$$

where $C(\lambda)$ is a positive constant depending only on λ and n.

Now by (5.1), (5.2), and (5.3), one easily sees that S can be extended to be a meromorphic section of $K_{\overline{X}}$ possibly with poles along D and the order less than m.

Let D be as above and write $D = \sum_{i=1}^{l} D_i$, D_i smooth $(i = 1, 2, \cdots, l)$. Let T_i be the defining section of D_i. Then by Lemma 5.1, there are integers p_i^m, $0 \leq p_i^m \leq m-1$, such that for each S in $H^0_{(2)}(X, K_X^m)$, $T_1^{p_1^m} \cdots T_l^{p_l^m} S$ can be extended to be a holomorphic section of $K_{\overline{X}}^m + \sum_{i=1}^{l} p_i^m D_i$. Define

$$\overline{\omega}_m = \frac{\sqrt{-1}}{2m\pi} \log\left(\sum_{j=0}^{N_m} \left| S_j \prod_{i=1}^{l} T_i^{p_i^m} \right|^2\right),$$

where $N_m + 1 = \dim_C H^0_{(2)}(X, K_X^m)$. Then $\overline{\omega}_m$ is a positive, d-closed, $(1, 1)$-current on $\overline{X}$, extending ω_m on X. Moreover,

$$\begin{aligned}\int_X \overline{\omega}_m \wedge \omega^{n-1} &= \frac{1}{m}\left(K_{\overline{X}}^m + \sum_{j=0}^{l} p_j^m D_j\right) \cdot [\omega]^{n-1} \\ &\leq (K_{\overline{X}} + D) \cdot [\omega]^{n-1} < +\infty,\end{aligned}$$

where $[\omega]$ is the cohomological class represented by ω.

It follows from Theorem 4.1 that

$$\int_X (-\mathrm{Ric}(g)) \wedge \omega^{n-1} \leq (K_{\overline{X}} + D) \cdot [\omega]^{n-1} < +\infty,$$

so that part (2) of Theorem C is proved.

Note that we also proved that $\overline{\omega}_m$ converges weakly to a positive, d-closed, $(1, 1)$-current $\overline{\omega}$ extending $-\mathrm{Ric}(g) = \mathrm{Ric}(K_X)$.

Choose a hermitian metric h_i for the line bundle $[D_i]$ $(i = 1, 2, \cdots, l)$ to obtain

$$\overline{\omega}_m = -\operatorname{Ric}(\overline{g}) + \sum_{i=1}^{l} \frac{p_i^m}{m} \operatorname{Ric}(h_i) + \frac{\sqrt{-1}}{2m\pi}\partial\overline{\partial}\log l\left[\sum_{j=0}^{N_m}\left(\|S\|_{h^m}^2 \prod_{i=1}^{l} \|T_i\|_{h_i}^{2p_i^m}\right)\right].$$

Define $\overline{\omega}_{m,\varepsilon}$ by

$$\overline{\omega}_{m,\varepsilon} = -\operatorname{Ric}(\overline{g}) + \sum_{i=1}^{l} \frac{p_i^m}{m} \operatorname{Ric}(h_i) + \frac{\sqrt{-1}}{2m\pi}\partial\overline{\partial}\log\left[\varepsilon + \sum_{j=0}^{N_m}\left(\|S\|_{h^m}^2 \prod_{i=1}^{l} \|T_i\|_{h_i}^{2p_i^m}\right)\right].$$

By Theorem 4.1, one can choose a sequence $\{\varepsilon_m\}$, $\lim_{m\to\infty}\varepsilon_m = 0$, such that on each compact set K of X, $\overline{\omega}_{m,\varepsilon_m}$ converges uniformly to $-\operatorname{Ric}(g)$. On the other hand, by a straightforward computation, $\overline{\omega}_{m,\varepsilon_m}$ is bounded from below by $\operatorname{Min}\{0, -\operatorname{Ric}(\overline{g}) + \sum_{i=1}^{l}(p_i^m/m)\operatorname{Ric}(h_i)\}$. Since $0 \le p_i^m \le m-1$, by Fatou's lemma we obtain

$$\begin{aligned} 0 < \int_X (-\operatorname{Ric}(g))^n &\le \overline{\lim_{m\to\infty}} \int_X \omega_{m,\varepsilon_m}^n \\ &= \overline{\lim_{m\to\infty}}\left(C_1(K_{\overline{X}}) + \sum_{i=1}^{l} \frac{p_i^m}{m} C_1(D_i)\right)^n \\ &= \left(C_1(K_{\overline{X}}) + \sum_{i=1}^{l} p_i C_1(D_i)\right)^n < \infty, \end{aligned}$$

where $p_i = \overline{\lim}_{m\to\infty} p_i^m/m$, $0 \le p_i \le 1$. Hence Theorem C is proved.

Before we end this section, we give an improved version of Theorem C in the case $n = 2$ for later use. For example, by using this improved version of Theorem C and some tricks for solving complex Monge-Ampere equations, one can prove that the volume of a complete Kähler-Einstein metric on a quasiprojective surface is always a rational number. In fact, one should be able to prove that the volume is an integer.

Theorem 5.1. *Let X be a quasiprojective surface with a complete Kähler metric g and $-\operatorname{Ric}(g) \ge \varepsilon\omega_g$, and let $\overline{X}$ be a compactification of X. Then the positive, d-closed, $(1,1)$-current $-\operatorname{Ric}(g)$ can be extended to* *a positive, d-closed,* $(1, 1)$*-current* $\overline{\omega}_g$ *on* $\overline{X}$ *such that it represents a cohomological class* $C_1(K_{\overline{X}})+\sum_{i=1}^l p_i C_1(D_i)$ *in* $H^{1,1}(\overline{X}, C)\cap H^2(\overline{X}, Z)$*, where* $p_i \leq 1$ *and* $\overline{X} - X = \bigcup_{i=1}^l D_i$*,* D_i *irreducible. Moreover*

(1) $0 < \int_X (\mathrm{Ric}(g))^2 \leq (C_1(K_{\overline{X}}) + \sum_{il=1}^l p_i C_i(D_i))^2$.

(2) *For any irreducible effective curve* E *in* $\overline{X}$*,*

$$(C_1(K_{\overline{X}}) + \sum_{i=1}^l p_i C_1(D_i))[E] \geq 0$$

and > 0 *if* E *is not contained in* $\bigcup_{i=1}^l D_i$*, where* $[E]$ *denotes the cohomology class of* E *in* $H^2(\overline{X}, Z)$*. In particular,* $C_1(K_{\overline{X}}) + \sum_{i=1}^l p_1 C_1(D_i)$ *is numerically effective.*

Remark. According to the terminology of algebraic geometry, such a divisor $K_{\overline{X}} + D$ is numerically positive.

Proof. For each $m > 0$, choose an orthonormal basis $\{S_i^m\}_{0\leq i\leq N_m}$ of $H^0_{(2)}(X, K_X^m)$. Define $p_i^m = \max_{0\leq j\leq N_m}\{$order of the pole of S_j^m along $D_i\}$, $i = 1, 2, \cdots, l$, where we regard the zero of S_j^m as the pole with negative order. Lemma 5.1 can be applied here to conclude that $p_i^m \leq m - 1$ for $i = 1, 2, \cdots, l$. Let T_i be the defining section of the divisor D_i. Then the holomorphic section $\{(\prod_{i=1}^l T_i^{p^m)i})S_j^m\}_{0\leq j\leq N_m}$ of $K_{\overline{X}}^m + \sum_{i=1}^l p_i^m D_i$ has no divisor as common zero along D. Define a positive, d-closed, $(1, 1)$-current $\overline{\omega}_m$ on $\overline{X}$ as before by

$$\overline{\omega}_m = \frac{\sqrt{-1}}{2m\pi}\partial\overline{\partial}\log\left(\prod_{i=1}^l \left|T_i^{2p_i^m}\right| \cdot \sum_{j=0}^{N_m}\left|S_j^m\right|^2\right). \tag{5.4}$$

By $0 < \int_{\overline{X}} \overline{\omega}_m \wedge \omega$, where ω is a positive Kähler form on $\overline{X}$, we have a uniform lower bound for all $p_i^m/m \leq 1$ $(i = 1, 2, \cdots, l)$, in fact,

$$-\int_{\overline{X}} C_1\left(K_{\overline{X}}\right) \wedge \omega \leq \mathrm{Ric}\, p_i^m m \int_{\overline{X}} C_1\left(D_i\right) \wedge \omega. \tag{5.5}$$

By taking a sequence of $\{m\}_{m\geq 0}$, we may assume that $p_i = \lim_{m\to\infty} p_i^m/m$ exists for each i. Since $\int_{\overline{X}} \overline{\omega}_m \wedge \omega \leq (C_1(K_{\overline{X}}) + \sum(p_i^m/m)C_1(D_i)) \cdot \omega$ is bounded uniformly from above, $\overline{\omega}_m$ converges weakly to a positive, d-closed, $(1, 1)$-current $\overline{\omega}_g$, which extends to $-\mathrm{Ric}(g)$ by Theorem 4.1. As in the proof of Theorem C, one sees

$$0 < \int_X (\mathrm{Ric}\,(g))^2 = \int_X \overline{\omega}_g^2 \leq \left(C_1\left(K_{\overline{X}}\right) + \sum_{i=1}^l p_i C_1\left(D_i\right)\right)^2. \tag{5.6}$$

Note that $\overline{\omega}_g$ represents the cohomological class $C_1(K_{\overline{X}})+\sum_{i=1}^{l} p_i C_1(D_i)$. For each irreducible curve E, since $\{\prod_{i=1}^{l}(T_i^{p_i^m})S_j^m\}_{0\leq j\leq N_m}$ has no common zero divisor along D for m large enough, the restriction of the $(1, 1)$-current $\overline{\omega}_m$ to E as current is well defined and

$$0 \leq \int_E \overline{\omega}_m = \left(C_1(K_{\overline{X}}) + \sum_{i=1}^{l} \frac{p_i^m}{m} C_1(D_i)\right) \cdot [E]. \tag{5.7}$$

By taking the limit, one sees $(C_1(K_{\overline{X}})+\sum_{i=1}^{l} p_i C_1(D_i))\cdot[E] \geq 0$. It is also easy to see from (5.7) that $(C_1(K_{\overline{X}})+\sum_{i=1}^{l} p_i C_1(D_i))\cdot[E] > 0$ if E is not contained in D.

Remark. In the joint work [12] by S. T. Yau and the author, we prove that the quasiprojective manifold X admits an "almost" complete Kähler-Einstein metric with negative Ricci curvature if $X = \overline{X}-D$ with D normal crossing and $K_{\overline{X}}+D$ positive in X and numerically positive in $\overline{X}$. Note that such a metric which we constructed on X should be complete. This makes us believe that all p_i in Theorem 5.1 are actually equal to one.

6. A final remark

Let (M, g) be a compact Kähler manifold with positive first Chern class and the metric g naturally polarized by the ample anticanonical line bundle K_M^{-1} on M, i.e., ω_g represents $C_1(M)$. There is an unsolved problem of E. Calabi whether or not M admits a Kähler-Einstein metric with positive scalar curvature. In general, there are obstructions to the existence of such a Kähler-Einstein metric. They were found by Matsushima [8], Futaki [4], Donaldson [3], and Uhlenbeck and Yau [13]. However, in [11] a numerical criterion is given for the existence of Kähler-Einstein metrics on M. Such a criterion is expressed in terms of a holomorphic invariant $\alpha(M)$. We will recall the definition of this invariant later. This $\alpha(M)$ is an analogue of the conformal invariant in the study of Yamabe's equation [1], [10].

Let G be a maximal compact group in the automorphism group $\mathrm{Aut}(M)$ of M, and let the metric g be G-invariant. Define

$$P_G(M, g) = \left\{\varphi \in C^2\left(M, R^1\right) \middle| \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0, \right.$$
$$\left. \sup_M \varphi = 0, \ \varphi \text{ is } G\text{-invariant}\right\}.$$

Then we define
(6.1)
$$\alpha(M) = \sup\left\{\alpha \,\middle|\, \exists C_\alpha > 0 \text{ s.t. } \forall\varphi \in P_G(M, g),\ \int_M e^{-\alpha\varphi}\, dV_g \le C_\alpha \right\}.$$

Although the definition of $\alpha(M)$ depends on the choice of the metric g and the maximal compact group G, the number $\alpha(M)$ does not. In [11], the author proved that M admits a Kähler-Einstein metric with positive scalar curvature whenever $\alpha(M) \ge qn/(n+1)$, where n is the complex dimension of M. So a natural question arises: How do we estimate $\alpha(M)$ from below? Theorem B throws light on it.

Define

$$\begin{aligned} P_m(M, g) = \{\varphi \in C^\infty(M, R^1) \sup_M \varphi = 0,\ &\varphi \text{ is } G\text{-invariant and} \\ &\exists \text{ a basis } \{S_i^m\}_{0\le i\le N_m} \text{ of } H^0(M, K_M^{-m}) \text{ such that} \\ &\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi = \frac{\sqrt{-1}}{2m\pi}\log(\sum_{i=0}^{N_m}|S_i^m|^2)\}, \end{aligned}$$

where $N_m + 1 = \dim H^0(M, K_m^{-m})$ and m is large.

A direct corollary of Theorem B is that we can confine ourselves to $\bigcup_{m_0}^\infty P_m(M, g)$ in the definition (4.1) of $\alpha(M)$, where m_0 is large. Precisely, we have

Proposition 6.1. *The holomorphic invariant $\alpha(M)$ is equal to*

$$(6.2)\qquad \sup\left\{\alpha \,|\, \exists C > 0 \ s.t.\ \forall\varphi \ in\ P_m(M, g)\ with\ m \ge m_0,\ \int_M e^{-\alpha\varphi}\, dV_g \le C\right\}.$$

This proposition turns the estimate of $\alpha(M)$ into the evaluation of some rational integrals. But it is still hard to deal with the estimate since we must compute infinitely many integrals. So we propose the following approach. One can define, for m large,

$$\alpha_m(M) = \sup\left\{\alpha \,\middle|\, \exists C > 0 \text{ s.t. } \forall\varphi \subset P_m(M, g),\ \int_M e^{-\alpha\varphi}\, dV_g \le C_1 \right\}.$$

Each $\alpha_m(M)$ is a holomorphic invariant of M, and obviously, $\alpha(M) \le \inf_m \alpha_m(M)$.

Question 1. Does $\alpha_m(M) = \alpha(M)$ for m sufficiently large?

The Kähler metric g induces a hermitian norm $\|\cdot\|$ on K_M^{-m}. Let S_m be the unit sphere in $H^0(M, K_M^{-m})$ with respect to the induced hermitian norm $\|\cdot\|_m$. Then one can check that

$$(6.3) \quad \alpha_m(M) = \sup\left\{\alpha \,\middle|\, \begin{array}{l} C > 0 \text{ s.t. for any global section } S \text{ in} \\ \text{the sphere } S_m,\ \int_M \left(1/\|S\|_m^2\right)^{\alpha/m} dV_g \le C \end{array}\right\}.$$

This interpretation indicates that the invariant $\alpha_m(M)$ reflects to some extent how bad the singularities of the divisors of M are cut by the global sections of K_M^{-m}.

For fixed $\alpha > 0$, the integral $\int_M (1/\|S\|_m^2)^{\alpha/m}\, dV_g$ defines a function on S_m. Note that this function may take the value infinity. The Fatou lemma implies that this function, denoted by F_m, is upper semicontinuous.

Question 2. Is F_m continuous on S_m? Or, more weakly, can the supreme of F_m on S_m be achieved on S_m?

In the case that M is a complex surface with positive first Chern class and m small, we can affirm Question 2 by some direct, but complicated, computations of rational integrals. The affirmation of these equations will make the evaluation of $\alpha(M)$ much more tractable.

Added in proof. Lemmas 1.2 and 2.3 also yield the following result. Let $\|\cdot\|$ be a hermitian metric on an ample line bundle L on M such that its curvature form ω is a Kähler form on M. Then for large m, $\sum_{i=0}^{N_m} \|S_i\|_m^2$ is equal to $[(m+n)!/m!](1 + 0(1/m))$, where $\|\cdot\|_m$ is the induced hermitian metric on L^m by $\|\cdot\|$, and $\{S_i\}_{0\le i\le m}$ is an orthonormal basis of $H^0(M, L^m)$ with respect to the induced inner product by $\|\cdot\|_m$ and ω.

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