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A RIGIDITY THEOREM FOR PROPERLY EMBEDDED MINIMAL SURFACES IN $\mathbf{R}^3$

HYEONG IN CHOI, WILLIAM H. MEEKS, III, & BRIAN WHITE

Abstract

We consider the question of when an intrinsic isometry of a properly embedded minimal surface is induced by an ambient isometry. We prove it always extends when the surface has at least two ends.

There are several interesting theorems and conjectures on the rigidity of complete surfaces in $\mathbf{R}^3$, which satisfy some natural geometric constraint. Perhaps the best known theorem of this type is the Cohn-Vossen theorem which shows that a compact Riemannian surface of positive curvature has a unique immersion into $\mathbf{R}^3$. A similar result is conjectured for complete surfaces of nonpositive or nonnegative curvature with the additional hypothesis that the surface has finite area. It is also conjectured that tight surfaces are rigid.

In this paper we prove a rigidity theorem for properly embedded minimal surfaces in $\mathbf{R}^3$, which have more than one end. This theorem states that the inclusion map of the surface into $\mathbf{R}^3$ represents the unique isometric minimal immersion of such a surface up to a rigid motion of $\mathbf{R}^3$. In particular it follows from our theorem that every intrinsic isometry of this type of surface extends to an isometry of $\mathbf{R}^3$.

In §1 we derive a geometric condition on an immersed minimal surface which guarantees that the surface is minimally rigid. We prove that if the surface intersects a plane transversally along an immersed closed curve, then any other isometric minimal immersion of the surface into $\mathbf{R}^3$ differs from the original immersion by an ambient isometry. In §2 we obtain some asymptotic properties of solutions of the minimal surface equation over annular planar domains. In §3 we prove that if a properly embedded minimal surface M in $\mathbf{R}^3$ has more than one end, then M is transverse

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to a plane P, and also some component α of $P \cap M$ is a closed Jordan curve. It seems likely that P can also be chosen so that $P \cap M$ consists entirely of pairwise disjoint closed Jordan curves but we are unable to prove this result. The existence of the Jordan curve α together with the already mentioned rigidity result in §1 proves our main rigidity theorem.

Recently, examples of properly embedded minimal surfaces with three ends were described by Hoffman and Meeks [6]. Their surfaces have finite total curvature. They prove our rigidity result with the additional assumption that the properly embedded surfaces have finite total curvature. Examples of properly embedded minimal surfaces with an infinite number of ends is described in Riemann's complete works [14] as well as in [1]. It should be noted that our rigidity theorem fails to hold when the surface has one end, as is pfnstrated by the nonrigidity of the helicoid. However, every other known properly embedded minimal surface satisfies the rigidity criterion of §1. Therefore, we would like to conjecture that any properly embedded nonsimply-connected minimal surface is minimally rigid and that an isometry of any properly embedded minimal surface extends to an ambient isometry of $\mathbf{R}^3$.

1. The basic rigidity theorem

One of the interesting questions in the theory of minimal surfaces is to determine whether a given minimal surface can be deformed in a nontrivial way. The first step to this problem is to understand when a given minimal surface can be isometrically deformed to a noncongruent minimal surface. Lawson [9] has shown that two isometric minimal immersions in a three-dimensional space form have the same second fundamental form, except that one may be a rotation of the other by a constant angle everywhere. This generalizes the concept of associated surfaces in the Weierstrass Representation, and, in this sense, two isometric minimal immersions in a three-dimensional space form can still be called associated surfaces. (See [9] for the more general case of constant mean curvature surfaces.)

In this section, we mention a few well-known elementary facts about the local geometry of minimal immersions in a three-dimensional manifold of constant sectional curvature, and give the basic rigidity theorem on which our work is based.

Let $N^3(c)$ be a three-dimensional manifold of constant sectional curvature c, and let $f\colon M^2 \to N^3$ be a minimal immersion. Choose local orthonormal vector fields e_1, e_2, e_3 on N such that e_1, e_2 are tangent to

M and e_3 is normal to M. Let w_1, w_2, w_3 be the dual 1-forms. The first structure equations are

$$dw_i = -\sum_{j=1}^{3} w_{ij} \wedge w_j, \qquad i = 1, 2, 3,$$
$$w_{ij} + w_{ji} = 0, \qquad i, j = 1, 2, 3,$$

where w_{12} is the connection form of the induced metric on M, and w_{31}, w_{32} define the second fundamental form by

$$w_{3i} = \sum_{j=1}^{2} h_{ij} w_j, \qquad i = 1, 2.$$

Differentiating w_{12} gives the Gauss equation

$$K_M = c - \tfrac{1}{2}|A|^2,$$

where K_M is the Gaussian curvature of M, and $|A|^2 = \sum_{i,j=1}^{2} h_{ij}^2$. Differentiation of w_{3i} gives the Codazzi equation which states that h_{ijk} is symmetric in all three indices, where h_{ijk} is the covariant derivative of h_{ij} defined by the formula

$$dh_{ij} = \sum_{k=1}^{2} (h_{ijk} w_k + h_{ik} w_{kj} + h_{kj} w_{ki}), \qquad i, j = 1, 2.$$

The Codazzi equations can be neatly expressed in terms of holomorphicity of Hopf's quadratic differential $Q = (h_{12} + \sqrt{-1}h_{11})(w_1 + \sqrt{-1}w_2)^2$. In other words, if e_1 and e_2 are chosen so that $w_1 + \sqrt{-1}w_2 = \lambda dz$, then $(h_{12} + \sqrt{-1}h_{11})\lambda^2$ is a holomorphic function, where $\lambda^2|dz|^2$ is the metric of M. The theorem of Lawson about the associated surfaces follows directly from this fact: Let $f_1\colon M \to N$ and $f_2\colon M \to N$ be isometric minimal immersions. By the Gaussian equation, $|\tilde{A}| = |A|$ where $|A|$ (resp. $|\tilde{A}|$) is the length of the second fundamental form of f_1 (resp. f_2). Therefore

$$\tilde{h}_{12} + \sqrt{-1}\tilde{h}_{11} = e^{\sqrt{-1}\theta}(h_{12} + \sqrt{-1}h_{11})$$

for some real valued function θ. Multiplying both sides by λ^2, and using the holomorphicity, we obtain that $e^{\sqrt{-1}\theta}$ is a holomorphic function. Then the maximum modulus principle implies that θ is a constant function.

We are now in a position to ask the fundamental question: Given a minimal immersion $f\colon M \to N$, is there another minimal immersion isometric to f? If $N = \mathbf{R}^3$, $\mathbf{H}^3$ or S^3, and M is simply connected, then the answer is yes. The construction is a straightforward consequence of the fundamental theorem of surface theory: the Gauss and the Codazzi equations are a complete set of local invariants, and they are clearly satisfied even if the second fundamental form is rotated by a constant angle. Therefore the procedure is first to rotate the second fundamental form by a constant angle, then to integrate out the Gauss and Codazzi equations. This procedure is a local one, and cannot be carried out in general if $\pi_1(M) \neq 0$. In this case it is not hard to describe the obstruction as a homomorphism $h\colon \pi_1(M) \to \mathrm{Iso}(N)$, where $\mathrm{Iso}(N)$ is the isometry group of N. If N is not simply connected, then the obstruction is a homomorphism $h\colon \pi_1(M) \to \mathrm{Iso}(\tilde{N})$ which commutes with the deck transformations, where $\tilde{N}$ is the universal cover of N. We are now ready to discuss the notion of minimal rigidity.

Definition 1.1. An isometric minimal immersion $f\colon M \to N$ is called *minimally rigid* if any isometric minimal immersion into N differs from f by an ambient isometry.

Since the rigidity phenomenon occurs only when $\pi_1(M) \neq 0$, we first look at what happens to a closed curve. Let γ be a closed curve in M. Choose orthonormal vector fields e_1, e_2, e_3 such that $e_1 = \dot{\gamma}$, e_1, e_2 are tangent to M, and e_3 is normal to M. Then $\overline{D}_{e_1} e_1 = a_{12}e_2 + h_{11}e_3$, $\overline{D}_{e_1} e_2 = h_{12}e_3 - a_{12}e_1$ of N and $a_{12} = w_{21}(e_1)$. In the matrix form,

$$(1.1)\qquad \begin{pmatrix} e_1 \\ e_3 \\ e_2 \end{pmatrix}' = \begin{pmatrix} 0 & h_{11} & a_{12} \\ -h_{11} & 0 & -h_{12} \\ -a_{12} & h_{12} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_3 \\ e_2 \end{pmatrix},$$

which is essentially the "Serret-Frenet Formula" for the curve γ, although h_{11} and h_{12} are allowed to have variable signs and a_{12} may not vanish. Here prime $(')$ denotes the covariant differentiation $\overline{D}_{\dot{\gamma}}$. As we described above, we want to rotate the second fundamental form by a constant angle θ and integrate the Codazzi and Gauss euations to produce a new $\tilde{f}\colon M \to N$. So now let us imagine that $\tilde{f}$ exists. Then

$$\tilde{h}_{12} + \sqrt{-1}\tilde{h}_{11} = e^{\sqrt{-1}\theta}(h_{12} + \sqrt{-1}h_{11}).$$

The curve γ must be integrated out to a closed curve. Therefore the O.D.E.

$$(1.2)\qquad \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_3 \\ \tilde{e}_2 \end{pmatrix}' = \begin{pmatrix} 0 & \tilde{h}_{11} & a_{12} \\ -\tilde{h}_{11} & 0 & -\tilde{h}_{12} \\ -a_{12} & \tilde{h}_{12} & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_3 \\ \tilde{e}_2 \end{pmatrix}$$

must have a solution $(\tilde{e}_1, \tilde{e}_3, \tilde{e}_2)^t$ such that $\tilde{e}_1$ integrates out to a closed curve $\tilde{\gamma}$ (i.e., $\dot{\tilde{\gamma}} = e_1$). For simplicity, we call $\tilde{\gamma}$ a solution curve of (1.2).

Clearly if some closed curve of M does not integrate out to a new closed solution curve of (1.2) for any $\theta \in (0, 2\pi)$, then M is minimally rigid. This O.D.E. can be handled easily if $N = \mathbf{R}^3$. However, if N is S^3 or $\mathbf{H}^3$, we are unable to obtain a simple geometric criterion for the solution curve of (1.2) to be closed, which should give a corresponding "Rigidity Theorem" in S^3 or $\mathbf{H}^3$. We now assume that $N = \mathbf{R}^3$.

Let γ_θ be a solution curve of O.D.E. (1.2) in $\mathbf{R}^3$. It is easy to check that $(\tilde{e}_1, \tilde{e}_3, \tilde{e}_2)^t = (e_2, e_3, -e_1)^t$ is a solution of (1.2) for $\theta = \pi/2$. Therefore

$$\gamma_{\pi/2}(t) = \int_0^t \tilde{e}_1(s)\, ds = \int_0^t e_2(s)\, ds.$$

Hence $\gamma_{\pi/2}$ is a closed curve if and only if $\int_0^l e_2(s) = 0$, where l is the length of γ_0. Once γ_0 and $\gamma_{\pi/2}$ are known it is easy to obtain γ_θ. In fact γ_θ is given by

$$\gamma_\theta(s) = \cos\theta\gamma_0(s) + \sin\theta\gamma_{\pi/2}(s).$$

Therefore γ_θ is closed for all θ if and only if γ_0 and $\gamma_{\pi/2}$ are closed. To summarize, we have the basic rigidity formula

Theorem 1.2. *Let $f: M \to \mathbf{R}^3$ be an isometric minimal immersion. Let γ be a closed curve in M such that*

$$\int_0^l B(s)\, ds \neq 0,$$

where l is the length of γ and $B: [0, l] \to \mathbf{R}^3$ is a unit vector field along γ such that B is tangent to M, but normal to $\dot{\gamma}$. Then M is minimally rigid.

Proof. Suppose $\tilde{f}: M \to \mathbf{R}^3$ is another isometric minimal immersion. Since they are associated surfaces, the second fundamental form of $\tilde{f}$ is that of f rotated by a constant angle, say, θ. Thus γ must be a solution curve of O.D.E. (1.2). On the other hand, since the integral condition implies that $\gamma_{\pi/2}$ is not a closed curve, no γ_θ is a closed curve unless $\theta = 0$. Hence θ must be zero, and the proof follows from the fundamental theorem of surface theory.

Corollary 1.3 (*Basic rigidity theorem*). *Suppose $f: M \to \mathbf{R}^3$ is an isometric minimal immersion. If M contains a compact minimal annulus A whose image boundary curves lie on opposite sides of a plane P, then M is minimally rigid.*

Proof. Since the intersection of P with A is compact and disjoint from ∂A, P can be moved slightly to meet A transversally and remain disjoint from ∂A. Thus $P \cap A$ contains an immersed circle γ and P

is transverse to A along γ. Let v be a unit vector perpendicular to the plane P, and note that $\langle B, v\rangle$ is never zero. Therefore $\int_\gamma B \neq 0$.

Another proof of Theorem 1.2. Suppose $f\colon M \to \mathbf{R}^3$ is an isometric minimal immersion. Then the Weierstrass representation tells us that f can be written in the form

$$f(p) = \operatorname{Re}\left[\int_{p_0}^{p} (w_1, w_2, w_3)^t\right],$$

where w_i, $i = 1, 2, 3$, are holomorphic 1-forms on M. An associated surface is then up to a rigid motion

$$f_\theta(p) = \operatorname{Re}\left[e^{-\sqrt{-1}\theta}\int_{p_0}^{p} (w_1, w_2, w_3)^t\right].$$

Suppose f_θ is defined for some $\theta \neq 0, \pi$. Then the imaginary periods vanish, and $\hat{f}\colon (f, -f_\pi)\colon M \to \mathbf{C}^3 = \mathbf{R}^3 \oplus i\mathbf{R}^3$ defines a holomorphic map from the Riemannian surface M. Let J be the associated almost complex structure on $\mathbf{C}^3$. Let $\eta\colon S^1 \to M$ be a closed curve, and denote $\gamma = f\circ\eta$ and $\hat{\gamma} = \hat{f}\circ\gamma$. Let T and $\hat{T}$ be the associated tangent vector fields and let $\hat{B} = J(T)$. Since $\hat{f}$ is conformal, $\hat{T} = T \oplus i\hat{f}_*(J(T))$. Now let $\gamma_\theta = f_\theta \circ \eta$. Then $\gamma_\theta' = \operatorname{Re}(e^{i\theta}\hat{\gamma}) = \cos\theta T + \sin\theta B$. Since f_θ is well defined, γ_θ is a closed curve, i.e.,

$$\begin{aligned} 0 &= \int_{S^1} \gamma_\theta'(t)\,dt = \cos\theta \int_{S^1} T\,dt + i\sin\theta\int_{S^1} B\,dt \\ &= i\cdot\sin\theta\int_{S^1} B\,dt, \end{aligned}$$

which is a contradiction.

Remark 1.4. Similarly Corollary 1.3 can be directly proved without the use of Theorem 1.2 by the following argument. In the proof of the corollary assume that P is the xy-plane. Since f^*dz is never tangent along the curve γ, the conjugate 1-form $*f^*(dz)$ has the property that $\langle *f^*(dz), \dot{\gamma}\rangle$ is never zero along γ, where $\langle\ ,\ \rangle$ is the pairing of a 1-form with a vector and $*$ is the Hodge star operator. The existence of the nonzero period $P(\gamma) = \int_\gamma *f^*(dz)$ implies the corollary.

2. A maximum principle at infinity

Proposition 2.1. *Let* $f, g\colon \{x \in \mathbf{R}^2 : |x| > R\} \to \mathbf{R}$ *be solutions of the minimal surface equation such that* $g > f$ *and*

$$\lim_{x\to\infty} f(x) = 0.$$

Then $\liminf_{x\to\infty} g(x) > 0$.

Proof. It is well known (for example, by the Weierstrass representation [13]) that

$$|f(x)| = O(|x|^{-1}),$$

from which it readily follows (from the estimates in [5, Corollary 16.7] applied to balls $B_{|x|/2}(x)$) that

$$|D^k f(x)| = O(|x|^{-1-k}).$$

Let Q be the minimal surface operator:

$$Q(v) = (1 + |Dv|^2)\Delta v - D_i v D_j v D_{ij} v.$$

Let $u(x) = |x|^{-1/2}$, so that

$$|Du(x)| = O(|x|^{-3/2}), \quad |D^2 u(x)| = O(|x|^{-5/2}), \quad \Delta u(x) = |x|^{-5/2}/4.$$

Then for $0 < t < 1$:

$$\begin{aligned} Q(f + tu) &= Q(f + tu) - Q(f) \\ &\geq (1 + |D(f + tu)|^2)t\Delta u + (2tDf \cdot Du + t^2|Du|^2)\Delta f \\ &\quad - t \cdot (|Du| + |Df|)^2 \cdot (|D^2 f| + |D^2 u|) \\ &\geq t \cdot [|x|^{-5/2}/4 + o(|x|^{-5/2})]. \end{aligned}$$

Thus $Q(f + tu) > 0$ for $|x|$ sufficiently large, say $|x| > R$. Choose a value $t < 1$ such that

$$0 < t < R^{1/2} \inf\{g(x) - f(x) : |x| = R\}.$$

Thus

$$g(x) \geq f(x) + t|x|^{-1/2} \quad \text{when } |x| = R$$

and

$$\liminf_{x \to \infty} g(x) \geq 0 = \lim_{x \to \infty} f(x) + tu(x).$$

Since $f + tu$ is a subsolution of the minimal surface equation, the maximum principle implies

$$g(x) > f(x) + t|x|^{-1/2}$$

for $|x| \geq R$. But $|f(x)| = O(|x|^{-1})$, so this implies $g(x) > 0$ for sufficiently large x, say $|x| \geq r$. Let $c = \inf\{g(x) : |x| = r\}$, and for each $s > r$, let v_s be the solution of the minimal surface equation in $\{x : r \leq |x| \leq s\}$ with

$$v_s(x) = c \quad \text{if } |x| = r,$$
$$v_s(x) = 0 \quad \text{if } |x| = s.$$

(The graph of v_s is a portion of a catenoid.) By the maximum principle, $g(x) \geq v_s(x)$ for $r \leq |x| \leq s$. But $\lim_{s\to\infty} v_s(x) = c$. (To see this, either write down the equation of the catenoid or note that $v = \lim_{s\to\infty} v_s$ is a positive solution of the minimal surface equation, so the argument above (with g and u replaced by v and 0) shows that $v(x) \geq t|x|^{-1/2}$ for some $t > 0$. But if $\lim_{x\to\infty} v(x) = 0$, then $|v(x)| = O(|x|^{-1})$, a contradiction.) Thus $g \geq c$.

Theorem 2.2. *Let $A_R = \{x \in \mathbf{R}^2 | \|x\| \geq R\}$ be the annular region of $\mathbf{R}^2$, which is the exterior of a disk of radius R. Let $g\colon A_R \to \mathbf{R}$ be a solution of the minimal surface equation such that*

$$\lim_{x\to\infty} g(x) = 0.$$

Let M be a complete properly immersed minimal surface in $\mathbf{R}^3$ with compact, possibly empty, boundary. If M does not intersect the graph G of g, then $\operatorname{dist}(M, g) > 0$.

Proof. Since ∂M is compact, there exists a number $T > R$ such that ∂M is contained in the interior of the ball of radius T centered at the origin. Let C_T be the catenoid whose waist is the circle $\{x \in \mathbf{R}^3 | x_3 = 0, |x| = T\}$. Since G is asymptotic to the x_1x_2-plane, we can choose T large enough so that C_T separates G into two annular components. Let W be the closure of the nonsimply connected component of $\mathbf{R}^3 - C_T$. Note that $\hat{G} = G \cap W$ separates W into two components W_1 and W_2 and $\partial\hat{G}$ is homologically nontrivial in either W_1 or W_2.

Assume now that M is disjoint from G. If M is disjoint from W, then $\operatorname{dist}(M, G)$ is clearly positive. Suppose now that $M \cap W \neq \varnothing$ and suppose, after possibly reindexing, that $M \cap W_1 \neq \varnothing$. Let $\hat{W}$ be the closure of the component of $W_1 - M$ which contains G. The manifold $\hat{W}$ has piecewise smooth boundary. Furthermore the nonsmooth boundary points of $\hat{W}$ arise locally in $\mathbf{R}^3$ from the intersection of a finite number of embedded minimal disks. It follows from this description of $\hat{W}$ that $\partial\hat{W}$ is an appropriate barrier for solving least area problems in W. (See the discussion before Theorem 1 in [11] concerning this boundary condition for $\hat{W}$.) Since $\operatorname{dist}(M, \partial G) > 0$, we can choose a curve $\hat{\gamma} \subset (\partial\hat{W}) \cap C_T$ which is a small perturbation of $\partial \cap G \subset \partial\hat{W}$ and such $\hat{\gamma} \cap (M \cup \hat{G}) = \varnothing$. For integers $n > T$ let Γ_n be the intersection of the cylinder $\Delta_n = \{x \in \mathbf{R}^3 | x_1^2 + x_2^2 = n\}$ of radius n with G. The curves Γ_n and $\hat{\gamma}$ are homologous in $\partial\hat{W}$ (in fact $\Gamma_n \cup \hat{\gamma}$ is the boundary of an annulus in $\partial\hat{W}$), and hence Γ_n and $\hat{\gamma}$ are homologous in $\hat{W}$. Let Σ_n be a smooth surface that minimizes area mod 2 in $\hat{W}$ and has boundary Γ_n and $\hat{\gamma}$.

Note that the intersection of Σ_n with any ball B of radius r inside W has area $\leq 2\pi r^2$ (otherwise replace $\Sigma_n \cap B$ by the smaller of the two regions into which Σ_n divides ∂B). Thus as $n \to \infty$, a subsequence of Γ_n must converge to a surface Σ that is stable and locally area minimizing mod 2.

Since the surface Σ is area minimizing mod 2, it is embedded. The maximum principle implies that Σ intersects $\partial \hat{W}$ only along its boundary curve $\hat{\gamma}$. Recall that $\hat{\gamma}$ is the boundary of a proper annulus $A \subset \partial \hat{W}$ which contains the annulus $\hat{G}$. Since $\Sigma \cup A$ is a properly embedded piecewise smooth surface in $\mathbf{R}^3$, it must be orientable and hence Σ is an orientable stable minimal surface. In this case Theorem 2.1 in [12] implies that Σ has finite total curvature. Since Σ has finite total curvature and it is embedded, then Σ has a finite number of ends each of which converges to a catenoid or a flat plane. Since the ends of Σ are disjoint from the catenoid C_T and G, it is clear that each end can be represented as the graph of a solution to the minimal surface equation defined on the exterior of some disk in $\mathbf{R}^3$. Theorem 2.1 implies that each end of Σ is a positive distance from $\hat{G}$, and hence the distance from Σ to $\hat{G}$ is greater than some $\varepsilon > 0$. However since Σ separates $\hat{W}$ into two components, one of which contains $M \cap \hat{W}$ and one of which contains $\hat{G}$, $\operatorname{dist}(M \cap \hat{W}, \hat{G}) \geq \operatorname{dist}(\Sigma, \hat{G}) > \varepsilon$. A similar argument shows that $M \cap W_2$ is a positive distance from $\hat{G}$. It follows from the triangle inequality that M is a positive distance from G. This completes the proof of Theorem 2.2.

Remark 2.3. Theorems 2.1 and 2.2 are examples of maximum principles at infinity, and recently have had useful generalizations (see [8], [10]).

3. Some special properties of properly embedded minimal surfaces with more than one end

In this section we prove that a properly embedded minimal surface M in $\mathbf{R}^3$ with more than one end always intersects transversally some plane P so that $P \cap M$ contains a component γ which is a simple closed curve. The existence of such a curve γ together with Corollary 1.3 proves that M is minimally rigid. The technique of proof of this fact is one developed by Meeks and Yau in their proof of the topological uniqueness of properly embedded minimal surfaces of finite type in $\mathbf{R}^3$ [12]. Their technique of proof is to derive geometric information on the behavior of the ends of M through the existence and geometry of stable minimal surfaces contained in one of the components of $\mathbf{R}^3 - M$.

Proposition 3.1. *If M is a properly embedded minimal surface in $\mathbf{R}^3$ with more than one end, then there exists a plane P which is transverse to M and such that $P \cap M$ contains a component which is a simple closed curve.*

Proof. Since M has more than one end, there exists a simple closed curve γ on M, which separates M into two noncompact subsurfaces M_1 and M_2. Alexander duality implies that M disconnects $\mathbf{R}^3$ into two components and hence M is orientable. If M_1 and M_2 are both stable, then the work of Fischer-Colbrie [3], as adapted by Meeks-Yau in Theorem 2.1 in [12], proves that M has finite total curvature. In the case where M has finite total curvature, it is relatively easy to prove the lemma and this is actually carried out by Hoffman and Meeks [6]. Hence from now on we assume that the component M_2 does not have finite total curvature. In this case M_2 is unstable by the above mentioned results of Fischer-Colbrie and Meeks-Yau.

Assertion 3.2. *The curve γ is nonhomologous to zero in a component X of $\mathbf{R}^3 - M$.*

Proof of Assertion 3.2. This follows immediately from the Mayer-Vietoris sequence. However, we give more geometric proof. Suppose γ is homologous to zero in the closures X, Y of the two components $\mathbf{R}^3 - M$. In this case γ is the boundary of compact surfaces Σ_1 and Σ_2 embedded in X and Y respectively such that $\Sigma_1 \cap \Sigma_2 = \gamma = (\Sigma_1 \cup \Sigma_2) \cap M$, and therefore $\Sigma = \Sigma_1 \cup \Sigma_2$ is a closed embedded surface in $\mathbf{R}^3$. Let Q be the closure of the bounded component of $\mathbf{R}^3 - \Sigma$. Suppose that M_1 enters Q near γ. Since $M_1 \cap \partial Q = \gamma$, the surface M_1 cannot escape Q. On the other hand, M_1 is properly embedded in Q and therefore is a closed subset of the compact set Q. However, this implies that M_1 is compact contrary to our assumption that M_1 is noncompact. This contradiction proves the assertion.

We now complete the proof of Proposition 3.1. By Assertion 3.2 there exists a smooth simple closed curve γ which separates M into two noncompact unstable components M_1 and M_2, and in the closure X of one of the components of $\mathbf{R}^3 - M$ the curve γ is nonhomologous to zero. Let $N_1 \subset N_2 \subset \cdots \subset N_n \cdots$ be a compact exhaustion of M_1 by smooth compact subsurfaces where $\gamma \subset \partial N_1$. Let Γ_i denote the boundary curves of N_i. Since the boundary of X has nonnegative mean curvature, the collection of curves Γ_i is the boundary of a least-area embedded minimal surfaces $\Sigma_i \subset X$. Standard compactness theorems for area minimizing surfaces show that a subsequence Σ_i converges to a properly embedded smooth least-area surface Σ in X with boundary γ. For the moment assume that Σ is not equal to either M_1 or M_2. The maximum principle theorem implies that $\Sigma \cap \partial X = \gamma$. Since $\Sigma \cup M_1$ is a properly embedded piecewise smooth surface in $\mathbf{R}^3$, Σ must be orientable.

The result of Fischer-Colbrie [3] (see also Theorem 2.1 in [12]) implies that Σ has finite total curvature, so that Σ is conformally diffeomorphic to a compact Riemannian surface with one boundary curve and a finite number of punctures. Each puncture point of Σ corresponds to an annular end of Σ. It is well known that an embedded minimal annular end of finite total curvature converges smoothly to a flat plane or to a catenoid at infinity (see [6] or [15]). Hence, each end of Σ converges to a flat plane or to a catenoid. Suppose for the moment that E is a given end which converges to a catenoid. In this case there exists a plane P_E which intersects E in a simple closed curve γ_E and such that P_E is transverse to M. The curve γ_E is the boundary of a disk $D_E \subset P_E$. If $D_E \cap M \neq \varnothing$, then $D_E \cap M$ contains a simple closed curve component. Suppose for the moment that all of the ends of Σ are of catenoid type. Let $E_1, \cdots, E_k$ be these ends and let $D_{E_1}, \cdots, D_{E_k}$ be the disks defined above. Since γ is not homologous to zero in X and γ is homologous to $\gamma_{E_1} + \gamma_{E_2} + \cdots + \gamma_{E_k}$, at least one of the disks, D_{E_i}, must intersect M transversely and this disk contains a component which is a simple closed curve. Thus we conclude that the proposition can only fail if some end of M converges to a flat plane.

Suppose now that at least one end of Σ converges to a flat plane P. We can choose a representative E of this flat end to be a graph over the complement of a disk D in P. We can also choose E to be disjoint from γ so that $E \cap M = \varnothing$. Theorem 2.1 shows that the distance between E and M is greater than some $\varepsilon > 0$. Let P' be a plane which is transverse to M and is within a distance of $\varepsilon/2$ from P. In this case $P' \cap M$ is compact and consists of a finite number of simple closed curves. The strong half-space theorem [7] guarantees that $P' \cap M \neq \varnothing$. This completes the proof of the proposition in the case $\Sigma \neq M_1$ and $\Sigma \neq M_2$.

If $\Sigma = M_1$, then M_1 has finite total curvature. If M_1 has a catenoid end, then it is clear that one can find a plane P such that $P \cap M$ contains a simple closed curve (just choose a plane near the end orthogonal to the limiting normal vector). If an end E converges to a plane P, then the argument in the previous paragraph shows that a nearby plane intersects M compactly. This completes the proof of the proposition.

The next theorem is an immediate consequence of Corollary 1.3 and Proposition 3.1.

Theorem 3.3. *A properly embedded minimal surface M in $\mathbf{R}^3$ which has more than one end is minimally rigid. In particular any intrinsic isometry of M extends to an isometry of $\mathbf{R}^3$.*

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University of Iowa
University of Massachusetts
Stanford University