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POISSON LIE GROUPS, DRESSING TRANSFORMATIONS, AND BRUHAT DECOMPOSITIONS

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0. Introduction

A Poisson Lie group is a Lie group together with a compatible Poisson structure. The notion of Poisson Lie group was first introduced by Drinfel'd [2] and studied by Semenov-Tian-Shansky [17] to understand the Hamiltonian structure of the group of dressing transformations of a completely integrable system. The purpose of this paper is to investigate some aspects of the geometry and the dressing transformations of a Poisson Lie group by using the notion of double Lie groups, and to present some new examples. At the end of this paper, we prove that every connected compact semisimple Lie group G is a Poisson Lie group and that every coadjoint orbit $\mathcal{O}$ of G has an induced Poisson structure making $\mathcal{O}$ a Poisson-homogeneous G-space. Moreover, the symplectic leaves of $\mathcal{O}$ coincide with the cells of a Bruhat decomposition and also with the orbits of a Poisson action by a Poisson Lie group. We call such a Poisson structure on a coadjoint orbit a Bruhat-Poisson structure. The relation of this example to the recent work on quantum groups ([3], [13], [19], [22], [23], [24]) should be an interesting problem for further study. We have already checked that the semiclassical limit as $\mu \to 1$ of the quantum groups (pseudogroups) $S_\mu U(2)$ in [23], considered as a Poisson Lie group SU(2), is isomorphic to the one we have in this paper (Theorem 4.7). We also have learned recently of reference [19], in which this Poisson structure is studied and it is shown that the primitive ideal space of the C^*-algebra $S_\mu U(2)$, for $\mu < 1$, is in one-to-one correspondence by a highest weight construction to the symplectic leaf space of this limit Poisson structure on SU(2). On the other hand, the C^*-algebras $S_\mu U(2)$ are defined by generators and relations and not (except formally) as noncommutative multiplications on $C(\mathrm{SU}(2))$, so it is not yet clear whether they can be understood as strict

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deformation quantizations of the Poisson manifold SU(2) in the sense of Rieffel [14], [15].

A double Lie group consists of a triple (G, G_+, G_-) of Lie groups where G_+ and G_- are both Lie subgroups of G such that the map $\alpha: G_+ \times G_- \to G$ defined by $(g_+, g_-) \mapsto g_+g_-$ is a diffeomorphism. For such a triple, there are naturally induced actions of the factor groups on each other by "twisted automorphisms" which reflect how the two actions are "twisted together" to build a Lie group structure on the product manifold $G_+ \times G_-$ (Theorem 3.8). This notion of double Lie groups goes back at least to G. Mackey [10], and it is a natural generalization of the semidirect product of two Lie groups, where one of the actions is the trivial one. We also have the counterpart for the corresponding Lie algebras, the so-called double Lie algebras, also called "matched pair of Lie algebras" by Takeuchi [18] and Majid [12], and "twilled extension" by Kosmann-Schwarzbach and Magri [7].

A Poisson Lie group G has a dual group G^* which is also a Poisson Lie group, and G^* acts on G by dressing transformations, whose orbits are the symplectic leaves of G [17]. We will construct a (local, in some cases) double Lie group $(G \bowtie G^*, G, G^*)$ from each Poisson Lie group G (Theorem 3.12). It turns out that the induced actions of G and G^* on each other from this double Lie group give rise to the dressing actions (Theorem 3.13).

In §1, we summarize the basic properties of Poisson Lie groups and dressing transformations. Results in this section can be found in various references [2], [6], [7], [16], [17], [21]. In §2, we discuss the characterizing properties of the left and right dressing actions. An infinitesimal criterion for an action of a Poisson Lie group on a Poisson manifold to be a Poisson action is given, and it is used to prove that the dressing actions are Poisson actions. The construction of a double Lie group from a Poisson Lie group and some explicit formulas for the dressing actions are presented in §3. §4 is devoted to the examples connected with compact semisimple Lie groups.

Reference [12] contains some of our results here, independently obtained. (See the paragraph following Theorem 4.1 of this paper.) We also learned from [12] that the Jimbo-Drinfel'd solution of the Modified Classical Yang-Baxter Equations for a complex semisimple Lie algebra restricts to a solution for the compact real form; the corresponding Lie bialgebra is just that given by the Iwasawa decomposition, which is our starting point to obtain a Poisson Lie group structure on the compact real semisimple Lie group.

1. Poisson Lie groups and Lie bialgebras

Definition 1.1. A Lie group G is called *a Poisson Lie group* if it is also a Poisson manifold such that the multiplication map $m: G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure. In this case, we say that the Poisson structure on G is multiplicative (or grouped).

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and π a Poisson tensor on G. Pulling π back to the identity element e of G by left and right translations, we get two maps $\pi_l: G \to \mathfrak{g}\wedge\mathfrak{g}$ and $\pi_r: G \to \mathfrak{g}\wedge\mathfrak{g}$ defined by $\pi_l(g) = l_{g_*^{-1}}\pi(g)$ and $\pi_r(g) = r_{g_*^{-1}}\pi(g)$ for $g \in G$, where l_{g^*} and r_{g_*} denote the tangent maps of the left and right translations of G by g.

Theorem 1.2. (*Drinfel'd* [2], *Weinstein* [20]). *The following conditions are equivalent:*

(1) *the Poisson structure π is multiplicative, i.e. G is a Poisson Lie group*;

(2) $\pi(gh) = l_{g_*}\pi(h) + r_{h_*}\pi(g)\ \forall g, h \in G$;

(3) $\pi_l(gh) = \pi_l(h) + \mathrm{Ad}_{h^{-1}}\pi_l(g)\ \forall g, h \in G$;

(4) $\pi_r(gh) = \pi_r(g) + \mathrm{Ad}_g\pi_r(h)\ \forall g, h \in G$, *i.e. π_r is a cocycle on G with respect to the coadjoint representation of G on $\mathfrak{g} \wedge \mathfrak{g}$*;

(5) (*assuming that G is connected*) $\pi(e) = 0$ *and $\mathscr{L}_X\pi$ is left invariant whenever X is a left invariant vector field*;

(6) (*assuming that G is connected*) $\pi(e) = 0$ *and $\mathscr{L}_X\pi$ is right invariant whenever X is a right invariant vector field, where $\mathscr{L}_X\pi$ denotes the Lie derivative of π with respect to X.*

Definition 1.3. A multivector field K on a Lie group G is called *multiplicative* if it satisfies

$$K(gh) = l_{g_*}K(h) + r_{h_*}K(g) \quad \forall g, h \in G.$$

According to Theorem 1.2, a Poisson structure π on a Lie group G is multiplicative if and only if the Poisson bivector field π is multiplicative in the sense of the definition above.

Remark 1.4. Theorem 1.2 still holds when π is replaced by any multivector field K.

Let K be any k-vector field on G with $K(e) = 0$. The intrinsic derivative of K at e [4, Chapter II, §6] is defined to be the linear map

$$d_eK: \mathfrak{g} \to \Lambda^k\mathfrak{g}$$

given by $X \mapsto \mathscr{L}_{\overline{X}}K(e)$, where $\overline{X}$ can be any vector field on G with $\overline{X}(e) = X$.

Proposition 1.5. *Let G be connected.*

(1) *A multiplicative multivector field K on G is identically zero if and only if its intrinsic derivative at e is zero.*

(2) *If K and L are multiplicative k- and l-vector fields respectively, then their Schouten bracket $[K, L]$ is a multiplicative $(k+l-1)$-vector field.*

Proof. (1) Assume that K is multiplicative and that $d_e K = 0$. By definition, $\mathscr{L}_X K(e) = 0$ for any left invariant vector field X on G. By Remark 1.4, $\mathscr{L}_X K$ is left invariant, so $\mathscr{L}_X K$ must be identically zero. Since X can be any left invariant vector on G, K must be right invariant. Therefore K is identically zero because $K(e) = 0$.

(2) Let X be a left invariant vector field and Y a right invariant vector field. Then

$$\mathscr{L}_X[K, L] = [\mathscr{L}_X K, L] + [K, \mathscr{L}_X L],$$
$$\mathscr{L}_Y \mathscr{L}_X[K, L] = [\mathscr{L}_Y \mathscr{L}_X K, L] + [\mathscr{L}_X K, \mathscr{L}_Y L] + [\mathscr{L}_Y K, \mathscr{L}_X L] + [K, \mathscr{L}_Y \mathscr{L}_X L].$$

By Remark 1.4, $\mathscr{L}_X K$ and $\mathscr{L}_X L$ are left invariant and $\mathscr{L}_Y K$ and $\mathscr{L}_Y L$ are right invariant. Therefore, $\mathscr{L}_Y \mathscr{L}_X K = 0$, $\mathscr{L}_Y \mathscr{L}_X L = 0$, $[\mathscr{L}_X K, \mathscr{L}_Y L] = 0$, and $[\mathscr{L}_Y K, \mathscr{L}_X L] = 0$. Hence,

$$\mathscr{L}_Y \mathscr{L}_X[K, L] = 0.$$

Now $K(e) = 0$ and $L(e) = 0$ imply that $[K, L](e) = 0$. Again by Remark 1.4, $[K, L]$ is multiplicative. q.e.d.

Let π be an arbitrary bivector field on G with $\pi(e) = 0$. Let $d_e\pi \colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be the intrinsic derivative of π at e. The dual map of $d_e\pi$ is an antisymmetric bilinear map

$$[\ ,\ ]_\pi \colon \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$$

given by $[\alpha, \beta]_\pi = d_e(\pi(\overline{\alpha}, \overline{\beta}))$, where $\alpha, \beta \in \mathfrak{g}^*$, and $\overline{\alpha}$ and $\overline{\beta}$ can be any 1-forms on G with $\overline{\alpha}(e) = \alpha$ and $\overline{\beta}(e) = \beta$. When π is a Poisson tensor, $[\ ,\ ]_\pi$ is exactly the Lie bracket on $\mathfrak{g}^*$ obtained by linearizing the Poisson structure at e (see [20]).

In the next theorem, we give criteria for π to be multiplicative and/or Poisson in terms of the maps $d_e\pi$ and $[\ ,\ ]_\pi$.

Theorem 1.6. *Let π be a bivector field on G with $\pi(e) = 0$.*

(1) *If π is multiplicative, then $d_e\pi \colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. Conversely, if G is connected and simply connected, then for any 1-cocycle $\varepsilon \colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$, there is a unique multiplicative bivector field π such that $\varepsilon = d_e\pi$.*

(2) *If π is a Poisson tensor, then the bracket $[\ ,\ ]_\pi \colon \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ induced by π satisfies the Jacobi identity, i.e. it is a Lie bracket on $\mathfrak{g}^*$.*

Moreover, when G is connected, a multiplicative bivector field π is a Poisson tensor if and only if its derivative at e defines a Lie bracket $[\ ,\ ]_\pi$ on $\mathfrak{g}^*$.

Proof. (1) If π is multiplicative, then $\pi_r: G \to \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle on G relative to the adjoint representation of G on $\mathfrak{g} \wedge \mathfrak{g}$ (Theorem 1.2). Differentiating π_r at the identity $e \in G$, we get $d_e\pi: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ as a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. Conversely, if G is connected and simply connected, any 1-cocycle $\varepsilon: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ on $\mathfrak{g}$ can be integrated to get a 1-cocycle $\varepsilon_\pi: G \to \mathfrak{g} \wedge \mathfrak{g}$ on G ([1], [7]). The bivector field π on G defined by $\pi(g) = r_{g_*}\varepsilon_\pi(g)$ is then a multiplicative bivector field on G with $\pi(e) = 0$ and $d_e\pi = \varepsilon$. Moreover, if π is multiplicative and $d_e\pi = 0$, then by Proposition 1.5 $\pi \equiv 0$, which proves the uniqueness.

(2) The first part of the statement is clear [20]. For the second part, assume that G is connected. By Proposition 1.5, we know that π being multiplicative implies that its Schouten bracket $[\pi, \pi]$ with itself is also multiplicative. The bracket $[\ ,\ ]_\pi$ on $\mathfrak{g}^*$ being a Lie bracket implies that the intrinsic derivative of $[\pi, \pi]$ at e is zero. Again by Proposition 1.5, $[\pi, \pi]$ is identically zero, i.e. π is a Poisson tensor.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. We say that $(\mathfrak{g}, \mathfrak{g}^*)$ form a *Lie bialgebra* if there is given a Lie algebra structure on $\mathfrak{g}^*$ such that the map $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the Lie bracket map $\mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

As a corollary of Theorem 1.6, we have

Theorem 1.8. *If (G, π) is a Poisson Lie group, then there is a Lie algebra structure on $\mathfrak{g}^*$ such that $(\mathfrak{g}, \mathfrak{g}^*)$ form a Lie bialgebra, called the tangent Lie bialgebra to (G, π). Conversely, if G is connected and simply connected, then every Lie bialgebra structure on $(\mathfrak{g}, \mathfrak{g}^*)$ defines a unique multiplicative Poisson structure π on G such that $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra to the Poisson Lie group (G, π).*

A very important family of Poisson Lie groups arises from the so-called classical **r**-matrices [16].

Let $\mathbf{r} \in \mathfrak{g} \wedge \mathfrak{g}$ be arbitrary. Define a bivector π on G by

$$\pi(g) = r_{g_*}\mathbf{r} - l_{g_*}\mathbf{r}, \qquad g \in G.$$

Then π is easily seen to be multiplicative; the induced $\pi_r: G \to \mathfrak{g} \wedge \mathfrak{g}$ is actually a coboundary on G. The intrinsic derivative $d_e\pi: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ of π at e is given by

$$d_e\pi(X) = \tfrac{d}{dt}|_{t=0} r_{\exp(-tX)_*}\pi(\exp tX) = -\mathrm{ad}_X\mathbf{r},$$

which is a coboundary on $\mathfrak{g}$. The dual map of $d_e\pi$ is given by

$$[\ ,\ ]_\pi: \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*: (\alpha, \beta) \mapsto \mathrm{ad}^*_{\mathbf{r}\alpha}\beta - \mathrm{ad}^*_{\mathbf{r}\beta}\alpha,$$

where $\mathbf{r}$ also denotes the linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ induced by $\mathbf{r}$. Now a short calculation shows that such a bracket on $\mathfrak{g}^*$ satisfies the Jacobi identity if and only if the element $[\mathbf{r}, \mathbf{r}] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ defined by

$$\begin{aligned}[\mathbf{r}, \mathbf{r}](\alpha \otimes \beta \otimes \gamma) &= \langle \alpha, \mathbf{r}[\beta, \gamma]_\pi - [\mathbf{r}\beta, \mathbf{r}\gamma] \rangle \\ &= \langle \alpha, [\mathbf{r}\beta, \mathbf{r}\gamma] \rangle + \text{c.p.}(\alpha, \beta, \gamma)\end{aligned}$$

is ad-invariant, where $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and the right-hand side of the last equality means cyclically permuting α, β and γ in the first term and adding them (see also [7]).

Definition 1.9. We say that $\mathbf{r} \in \mathfrak{g} \wedge \mathfrak{g}$ satisfies the Yang-Baxter equation if $[\mathbf{r}, \mathbf{r}] = 0$.

Using Theorem 1.6, we obtain the following theorem of Drinfel'd [2].

Theorem 1.10 (*Drinfel'd*). *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. For $\mathbf{r} \in \mathfrak{g} \wedge \mathfrak{g}$, define a bivector field π on G by*

$$\pi(g) = r_{g_*}\mathbf{r} - l_{g_*}\mathbf{r} \quad \forall g \in G.$$

Then (G, π) is a Poisson Lie group if and only if $[\mathbf{r}, \mathbf{r}] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. In particular, when $\mathbf{r}$ satisfies the Yang-Baxter equation, it defines a Poisson Lie group structure on G.

Since every cocycle on a connected semisimple Lie group or on a compact Lie group is a coboundary, we have the following result.

Theoiem 1.11. *If G is connected and semisimple, or if G is compact, then every multiplicative Poisson structure π on G is of the form*

$$\pi(g) = r_{g_*}\mathbf{r} - l_{g_*}\mathbf{r} \quad \forall g \in G,$$

where $\mathbf{r} \in \mathfrak{g} \wedge \mathfrak{g}$ is such that $[\mathbf{r}, \mathbf{r}] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$.

We conclude this section by proving the following theorem of Yu. I. Manin and recalling the definition of a Manin triple [3].

Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$, and let $\langle\ ,\ \rangle$ denote the nondegenerate symmetric bilinear scalar product on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by

$$\langle X_1 + \alpha_1, X_2 + \alpha_2 \rangle = \alpha_1(X_2) + \alpha_2(X_1), \qquad X_1, X_2 \in \mathfrak{g},\ \alpha_1, \alpha_2 \in \mathfrak{g}^*.$$

Assume that $\mathfrak{g}^*$ also has a given Lie algebra structure. We use $[\ ,\ ]$ to denote both the bracket on $\mathfrak{g}$ and the bracket on $\mathfrak{g}^*$, and use $\mathrm{ad}^*_X \alpha$ and $\mathrm{ad}^*_\alpha X$ to denote the coadjoint representations of $\mathfrak{g}$ on $\mathfrak{g}^*$ and of $\mathfrak{g}^*$ on $\mathfrak{g} = (\mathfrak{g}^*)^*$ respectively.

Theorem 1.12. *Let the notations be as above. Then the only antisymmetric bracket operation, also denoted by $[\ ,\ ]$, on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$*

such that (1) *it restricts to the given brackets on* $\mathfrak{g}$ *and* $\mathfrak{g}^*$ *and* (2) *the scalar product* $\langle\ ,\rangle$ *on* $\mathfrak{g}\oplus\mathfrak{g}^*$ *is invariant is given by*

$$[X_1+\alpha_1, X_2+\alpha_2] = [X_1, X_2] - \mathrm{ad}^*_{\alpha_2}X_1 + \mathrm{ad}^*_{\alpha_1}X_2 + [\alpha_1, \alpha_2] + \mathrm{ad}^*_{X_1}\alpha_2 - \mathrm{ad}^*_{X_2}\alpha_1.$$

Moreover, it is a Lie bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$ *if and only if* $(\mathfrak{g}, \mathfrak{g}^*)$ *forms a Lie bialgebra.*

Proof. The first part is clear. For the second part, we observe that the bracket as defined satisfies the Jacobi identity if and only if

$$[\alpha, [X, Y]] + [X, [Y, \alpha]] + [Y, [\alpha, X]] = 0$$

for $X, Y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. This is then equivalent to

$$\mathrm{ad}^*_\alpha[X, Y] = [\mathrm{ad}^*_\alpha X, Y] + [X, \mathrm{ad}^*_\alpha Y] + \mathrm{ad}^*_{\mathrm{ad}^*_Y\alpha}X - \mathrm{ad}^*_{\mathrm{ad}^*_X\alpha}Y.$$

Let $\rho\colon \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g} \subset \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$ be the map dual to the Lie bracket map on $\mathfrak{g}^*$, i.e.

$$\rho(X)\colon \alpha \mapsto \mathrm{ad}^*_\alpha X.$$

Then the above identity can be rewritten as

$$\begin{aligned}\rho([X, Y]) &= -\mathrm{ad}_Y \circ \rho(X) + \mathrm{ad}_X \circ \rho(Y) + \rho(X)\circ \mathrm{ad}^*_Y - \rho(Y)\circ\mathrm{ad}^*_X \\ &= (\mathrm{ad}_X\circ\rho(Y) - \rho(Y)\circ\mathrm{ad}^*_X) - (\mathrm{ad}_Y\circ\rho(X) - \rho(X)\circ\mathrm{ad}^*_Y),\end{aligned}$$

which says exactly that ρ is a cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}\wedge\mathfrak{g}$. q.e.d.

There is a one-to-one correspondence between Lie bialgebras and the so-called Manin triples.

Definition 1.13. A *Manin triple* consists of a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ and a nondegenerate invariant symmetric scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that

(1) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$;

(2) $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces;

(3) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the scalar product $\langle\ ,\ \rangle$.

The correspondence between Lie bialgebras and Manin triples mentioned above is constructed as follows: given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, by Theorem 1.12, there is a Lie algebra structure on the vector space $\mathfrak{g}\oplus\mathfrak{g}^*$, denoted by $\mathfrak{g}\bowtie\mathfrak{g}^*$, such that $(\mathfrak{g}\bowtie\mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ together with the natural scalar product on $\mathfrak{g}\oplus\mathfrak{g}^*$ form a Manin triple. Conversely, given a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \langle\ ,\ \rangle)$, then $\mathfrak{g}_-$ is naturally isomorphic to $\mathfrak{g}_+^*$ under $\langle\ ,\ \rangle$. Hence $\mathfrak{g} \cong \mathfrak{g}_+^*$ as vector spaces, and $\langle\ ,\ \rangle$ becomes the natural scalar product on $\mathfrak{g}$ relative to the above decomposition. Again by Theorem 1.12, $(\mathfrak{g}_+, \mathfrak{g}_+^*)$ becomes a Lie bialgebra.

2. Dressing actions as Poisson actions

Our approach in this section follows that of [21].

As we have seen in the last section, if (G, π) is a Poisson Lie group, then $\mathfrak{g}^*$ inherits a Lie algebra structure given by linearizing π at the identity element e. The connected and simply connected Lie group G^* with $\mathfrak{g}^*$ as its Lie algebra is called by Drinfel'd [2] the dual Poisson Lie group of (G, π). By Theorems 1.8 and 1.12, G^* is also a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$.

It is a general fact ([8], [11]) that the space $\Omega^1(P)$ of 1-forms on a Poisson manifold (P, π) has a Lie algebra structure with Lie bracket

$$(2.1)\qquad \begin{aligned}[][\omega_1, \omega_2] &= d\pi(\omega_1, \omega_2) - \pi\omega_1 \rfloor d\omega_2 + \pi\omega_2 \rfloor d\omega_1 \\ &= -d\pi(\omega_1, \omega_2) + \mathscr{L}_{\pi\omega_2}\omega_1 - \mathscr{L}_{\pi\omega_1}\omega_2,\end{aligned}$$

where π denotes both the Poisson bivector field and the bundle map $T^*P \to TP$ given by $\pi\omega \rfloor \eta = \pi(\eta, \omega)$. This Lie bracket on $\Omega^1(P)$ together with $-\pi$ make $\Omega^1(P)$ a Lie algebroid; i.e. $-\pi$ is a Lie algebra homomorphism into the space $\chi(G)$ of vector fields with the commutator Lie algebra structure, and the following derivation law holds:

$$[f\omega_1, \omega_2] = f[\omega_1, \omega_2] - (-\pi\omega_2 \cdot f)\omega_1.$$

Theorem 2.1 (*Weinstein* [21]). *The right* (*left*) *invariant* 1*-forms on a Poisson Lie group* (G, π) *form a Lie subalgebra with respect to the bracket* (2.1). *The corresponding Lie algebra structure on* $\mathfrak{g}^*$ *coincides with the one given by linearizing* π *at the identity element* e.

For each $\alpha \in \mathfrak{g}^*$, let α_l and α_r be the *left* and *right* invariant 1-forms on G with $\alpha_l(e) = \alpha$ and $\alpha_r(e) = \alpha$ respectively. Denote

$$\lambda(\alpha) = \pi\alpha_l, \qquad \rho(\alpha) = -\pi\alpha_r.$$

Then we get two linear maps $\lambda, \rho: \mathfrak{g}^* \to \chi(G)$. By Theorem 2.1, λ is a Lie algebra antihomomorphism, while ρ is a Lie algebra homomorphism.

Definition 2.2. We call $\lambda(\alpha) \in \chi(G)$ (resp. $\rho(\alpha)$) the *left* (resp. *right*) *dressing vector field* on G corresponding to α. Integrating λ (resp. ρ) gives rise to a local (and global if the dressing vector fields are complete) left (resp. right) action of G^* on G. We call this action the *left* (resp. *right*) *dressing action* of G^* on G, and we say that this left (resp. right) dressing action consists of *left* (resp. *right*) *dressing transformations*.

We now characterize the dressing actions by (1) their "twisted multiplicativity" and (2) their linearizations at the identity element of G. Recall the following fact.

Lemma 2.3. *If G is a Lie group acting on a manifold M with a fixed point $x_0 \in M$, then linearizing the action of x_0 gives rise to a representation of G on $T_{x_0}M$, and differentiating this action of G gives rise to a representation of* $\mathfrak{g}$ *on $T_{x_0}M$. We call this representation of* $\mathfrak{g}$ *on $T_{x_0}M$ the linearization of the action of G on M at x_0.*

Theorem 2.4. *Let G be a Lie group with tangent Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *and dual Poisson Lie group G^*. Then*

(1) *("twisted multiplicativity" of the dressing actions) for $\alpha \in \mathfrak{g}^*$ and $g, h \in G$, we have*

$$\lambda(\alpha)(gh) = l_{g_*}\lambda(\alpha)(h) + r_{h_*}\lambda(\mathrm{Ad}^*_{h^{-1}}\alpha)(g),$$
$$\rho(\alpha)(gh) = l_{g_*}\rho(\mathrm{Ad}^*_g\alpha)(h) + r_{h_*}\rho(\alpha)(g);$$

(2) *the linearization at $e \in G$ of the left dressing action of G^* on G is the coadjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}$, while the linearization at $e \in G$ of the right dressing action of G^* on G is minus the coadjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}$;*

(3) *properties* (1) *and* (2) *uniquely characterize the dressing actions.*

Proof. (1) Directly follows from the multiplicativity of the Poisson tensor π.

(2) Since each $\lambda(\alpha)$ vanishes at e, it is linearized at e to give a map $\mathfrak{g} \to \mathfrak{g}$ by

$$X \mapsto [\overline{X}, \lambda(\alpha)](e),$$

where $\overline{X}$ can be any vector field on G with $\overline{X}(e) = X$. Choose $\overline{X}$ to be right invariant. Then for any $\beta \in \mathfrak{g}^*$, we have

$$\begin{aligned}[\overline{X}, \lambda(\alpha)](e)(\beta) &= \tfrac{d}{dt}\big|_{t=0} l_{\exp(-tX)_*}\lambda(\alpha)(\exp tX)(\beta) \\ &= \tfrac{dt}{dt}\big|_{t=0}\pi_l(\exp tX)(\beta, \alpha) \\ &= X([\beta, \alpha]) = (\mathrm{ad}^*_\alpha X)(\beta).\end{aligned}$$

Hence, $[\overline{X}, \lambda(\alpha)](e) = \mathrm{ad}^*_\alpha X$. The statement for the right dressing vector fields is proved similarly.

(3) If $\lambda_1\colon \mathfrak{g}^* \to \chi(G)$ is another Lie algebra antihomomorphism satisfying properties (1) and (2), then λ_1 must be equal to λ. Indeed, let $\lambda_0 = \lambda_1 - \lambda$; then λ_0 satisfies (1). Define a bivector field π_0 on G by

$$\pi_0(\alpha_l, \beta_l) = \langle \alpha_l, \lambda_0(\beta) \rangle$$

for $\alpha, \beta \in \mathfrak{g}^*$, where α_l and β_l are the corresponding left invariant 1-forms. Then $\pi(e) = 0$ and the fact that λ_0 satisfies (1) implies that π_0 is multiplicative. Furthermore, since $[\overline{X}, \lambda_0(\alpha)](e) = 0$ for all $\alpha \in \mathfrak{g}^*$ and $\overline{X} \in \chi(G)$, π_0 is linearized to zero at e. By Proposition 1.5, π_0 must be zero. Therefore, $\lambda_0 = 0$ and so $\lambda_1 = \lambda$. q.e.d.

As noticed by Semenov-Tian-Shansky [17], the dressing transformations of a Poisson Lie group G do not in general preserve the Poisson structure on G; rather, the dressing actions of G^* on G are Poisson actions in the following sense.

Definition 2.5. A left action $\sigma: G \times P \to P$ of a Poisson Lie group G on a Poisson manifold P is called a *Poisson action* if σ is a Poisson map, the space $G \times P$ being equipped with the product Poisson structure. Similarly, a right action $\tau: P \times G \to P$ is called a Poisson action if τ is a Poisson map.

When (G, π) is a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, G^* is also a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ (Theorem 1.8). Semenov-Tian-Shansky [17] has proved that the right dressing action of G^* on G is a Poisson action. In this section, we will give another proof of this fact by using the following infinitesimal criterion for an action to be a Poisson action.

Let $\sigma: G \times P \to P$ be an action of a Poisson Lie group (G, π_G) on a Poisson manifold (P, π_P). For $g \in G$ and $x \in P$, denote by $\sigma_g: P \to P$ and by $\sigma_x: G \to P$ the maps

$$\sigma_g: x \mapsto \sigma(g, x) = gx, \qquad \sigma_x: g \mapsto \sigma(g, x) = gx,$$

and denote by σ_{g_*} and σ_{x_*} the derivatives of σ_g and σ_x, respectively, extended from vectors to bivectors. Let $\lambda: \mathfrak{g} \to \chi(P)$ be the Lie algebra antihomomorphism which defines the infinitesimal generators of this action, i.e. $\lambda(X)(x) = \sigma_{x_*}X$ for $X \in \mathfrak{g}$, $x \in P$.

Theorem 2.6. *The following conditions are equivalent*:

(1) *σ is a Poisson action*;

(2) *for all $g \in G$ and $x \in P$*,

$$\pi_P(gx) = \sigma_{g_*}\pi_P(x) + \sigma_{x_*}\pi_G(g);$$

(3) (*assuming that G is connected*) *for each $X \in \mathfrak{g}$*

$$\mathscr{L}_{\lambda(X)}\pi_P = (\lambda \wedge \lambda)(d_e\pi_G)(X),$$

where $d_e\pi_G: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is the intrinsic derivative of π_G at e;

(4) (*assuming that G is connected*) *for any 1-forms ω_1 and ω_2 on P*,

$$(\mathscr{L}_{\lambda(X)}\pi_P)(\omega_1, \omega_2) = \langle [\xi_{\omega_1}, \xi_{\omega_2}], X \rangle,$$

where ξ_ω is the $\mathfrak{g}^*$-valued function on P defined by

$$\langle \xi_\omega, X \rangle = \langle \omega, \lambda(X) \rangle, \qquad X \in G,$$

and $[\xi_{\omega_1}, \xi_{\omega_2}]$ denotes the pointwise bracket in $\mathfrak{g}^*$.

Proof. (1) $\Rightarrow$ (2): By definition, $\sigma: G \times P \to P$ is a Poisson map if and only if for any $\phi, \varphi \in C^\infty(P)$, $g \in G$, and $x \in P$,

$$\begin{aligned} &\{\phi \circ \sigma, \varphi \circ \sigma\}_{G\times P}(g, x) = \{\phi, \varphi\}_P(gx) \\ &\quad \Leftrightarrow \{\phi \circ \sigma_x, \varphi \circ \sigma_x\}_G(g) + \{\phi \circ \sigma_g, \varphi \circ \sigma_g\}_P(x) = \{\phi, \varphi\}_P(gx) \\ &\quad \Leftrightarrow \sigma_{x_*}\pi_G(g) + \sigma_{g_*}\pi_P(x) = \pi_P(gx). \end{aligned}$$

Therefore, (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3): From (2), we get

$$\sigma_{g^{-1}_*}\pi_P(gx) = \pi_P(x) + \sigma_{x_*} l_{g^{-1}_*}\pi_G(g).$$

Substituting $\exp tX$ for g yields

$$\sigma_{\exp(-tX)_*}\pi_P(\exp tX \cdot x) = \pi_P(x) + \sigma_{x_*} l_{\exp(-tX)_*}\pi_G(\exp tX). \tag{2.2}$$

Differentiating (2.2) with respect to t at $t = 0$, we get (3).

(3) $\Rightarrow$ (2): Assuming (3), we first prove that (2.2) holds for all real numbers t. Clearly (2.2) holds when $t = 0$. Differentiating both sides of (2.2), we have:

$$\begin{aligned} \tfrac{d}{dt}\text{lhs} &= \sigma_{\xi p(-tX)_*}(\mathscr{L}_{\lambda(X)}\pi_P)(\exp tX \cdot x) \\ &= \sigma_{\exp(-tX)_*}\sigma_{(\exp tX\cdot x)_*}(\mathscr{L}_X\pi_G)(e) \\ &= \sigma_{x_*}\mathrm{Ad}_{\exp(-tX)}(\mathscr{L}_X\pi_G(e)), \\ \tfrac{d}{dt}\text{rhs} &= \sigma_{x_*}\tfrac{d}{dt} l_{\exp(-tX)_*}\pi_G(\exp tX) \\ &= \sigma_{x_*}\mathrm{Ad}_{\exp(-tX)}(\mathscr{L}_X\pi_G(e)). \end{aligned}$$

Hence, $\frac{d}{dt}$lhs $= \frac{d}{dt}$rhs, and (2.2) holds for all t. Therefore, (2) holds for all $x \in P$ and g in an open neighborhood of e in G. But since G is connected, any open neighborhood of e in G generates G. Using the multiplicativity of π_G, it follows that (2) holds for all $g \in G$ and $x \in P$. This proves that (3) $\Rightarrow$ (2).

The equivalence of (3) and (4) follows by applying $\omega_1 \wedge \omega_2$ to both sides of (3). q.e.d.

We now use the infinitesimal criterion (4) to prove the following main result:

Theorem 2.7. *For a Poisson Lie group (G, π), both the left and right dressing actions of G^* on G are Poisson actions.*

Proof. We will prove the theorem for the left action. (The proof for the right action is similar.) Let $\alpha, \beta, \gamma \in \mathfrak{g}^*$ and $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ be the corresponding left invariant 1-forms on G. By (4) of Theorem 2.6, we need to prove that

$$(\mathscr{L}_{\lambda(\alpha)}\pi)(\overline{\beta}, \overline{\gamma}) = \langle \alpha, [\xi_{\overline{\beta}}, \xi_{\overline{\gamma}}]\rangle,$$

where $\xi_{\overline{\beta}}$ and $\xi_{\overline{\gamma}}$ are the $\mathfrak{g}$-valued functions on G given by

$$\xi_{\overline{\beta}}(g) = -l_{g_*^{-1}}\lambda(\beta)(g) \quad \forall g \in G.$$

We need to prove that

$$(\mathscr{L}_{\lambda(\alpha)}\pi)(\overline{\beta},\overline{\gamma})(g) = \langle \alpha, [l_{g_*^{-1}}\lambda(\beta)(g), l_{g_*^{-1}}\lambda(\gamma)(g)]\rangle. \tag{2.3}$$

Now,

$$\begin{aligned}
\text{lhs} &= \lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}) - \pi(\mathscr{L}_{\lambda(\alpha)}\overline{\beta},\overline{\gamma}) - \pi(\overline{\beta},\mathscr{L}_{\lambda(\alpha)}\overline{\gamma}) \\
&= \lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}) - \langle\mathscr{L}_{\lambda(\alpha)}\overline{\beta},\lambda(\gamma)\rangle + \langle\mathscr{L}_{\lambda(\alpha)}\overline{\gamma},\lambda(\beta)\rangle \\
&= \lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}) - \lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}) + \langle\overline{\beta},[\lambda(\alpha),\lambda(\gamma)]\rangle \\
&\quad + \lambda(\alpha)\cdot\pi(\overline{\gamma},\overline{\beta}) - \langle\overline{\gamma},[\lambda(\alpha),\lambda(\beta)]\rangle \\
&= -\lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}) - \langle\overline{\beta},[\lambda(\gamma),\lambda(\alpha)]\rangle - \langle\overline{\gamma},[\lambda(\alpha),\lambda(\beta)]\rangle,
\end{aligned}$$

all evaluated at $g \in G$. Define three functions $a_{\alpha,\beta,\gamma}$, $b_{\alpha,\beta,\gamma}$ and $c_{\alpha,\beta,\gamma}$ on G by

$$\begin{aligned}
a_{\alpha,\beta,\gamma}(g) &= (\lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}))(g), \\
b_{\alpha,\beta,\gamma}(g) &= \langle\overline{\alpha},[\lambda(\beta),\lambda(\gamma)]\rangle(g), \\
c_{\alpha,\beta,\gamma}(g) &= \langle\alpha,[\xi_{\overline{\beta}},\xi_{\overline{\gamma}}]\rangle(g) \\
&= \langle\alpha,[l_{g_*^{-1}}\lambda(\beta)(g), l_{g_*^{-1}}\lambda(\gamma)(g)]\rangle.
\end{aligned}$$

Then (2.3) is equivalent to

$$a_{\alpha,\beta,\gamma} + b_{\beta,\gamma,\alpha} + b_{\gamma,\alpha,\beta} + c_{\alpha,\beta,\gamma} = 0. \tag{2.4}$$

Lemma 2.8. *The following relations hold*:

(1) $a_{\alpha,\beta,\gamma} = b_{\alpha,\beta,\gamma} + c_{\beta,\gamma,\alpha} + c_{\gamma,\alpha,\beta}$;

(2) $a_{\alpha,\beta,\gamma} + b_{\alpha,\beta,\gamma} + \text{c.p.}(\alpha,\beta,\gamma) = 0$.

Assuming the lemma, we get

$$b_{\alpha,\beta,\gamma} + c_{\alpha,\beta,\gamma} + \text{c.p.}(\alpha,\beta,\gamma) = 0.$$

(2.4) then follows immediately.

Proof of Lemma 2.8. (1) By definition,

$$\begin{aligned}
a_{\alpha,\beta,\gamma}(g) &= (\lambda(\alpha)\cdot\pi(\overline{\beta},\overline{\gamma}))(g) = \langle d_g\pi(\overline{\beta},\overline{\gamma}),\lambda(\alpha)(g)\rangle \\
&= \langle l_g^* d_g\pi(\overline{\beta},\overline{\gamma}), l_{g_*^{-1}}\lambda(\alpha)(g)\rangle \\
&= -\langle d_e(\pi(\overline{\beta},\overline{\gamma})\circ l_g),\xi_{\overline{\alpha}}(g)\rangle,
\end{aligned}$$

but

$$(\pi(\overline{\beta},\overline{\gamma})\circ l_g)(h) = \pi_l(gh)(\beta,\gamma) = (\pi_l(h) + \mathrm{Ad}_{h^{-1}}\pi_l(g))(\beta,\gamma),$$

so

$$\begin{aligned} a_{\alpha,\beta,\gamma}(g) &= -\langle[\beta,\gamma],\xi_{\overline{\alpha}}(g)\rangle + \mathrm{ad}_{\xi_{\overline{\alpha}(g)}}\pi_l(g)(\beta,\gamma) \\ &= -\langle\overline{\alpha},\lambda([\beta,\gamma])\rangle - \pi_l(g)(\mathrm{ad}^*_{\xi_{\overline{\alpha}}(g)}\beta,\gamma) - \pi_l(g)(\beta,\mathrm{ad}^*_{\xi_{\overline{\alpha}(g)}}\gamma) \\ &= \langle\overline{\alpha},[\lambda(\beta),\lambda(\gamma)]\rangle + \langle\mathrm{ad}^*_{\xi_{\overline{\alpha}(g)}}\beta,\xi_{\overline{\gamma}}(g)\rangle - \langle\mathrm{ad}^*_{\xi_{\overline{\alpha}(g)}}\gamma,\xi_{\overline{\beta}}(g)\rangle \\ &= b_{\alpha,\beta,\gamma} + c_{\beta,\gamma,\alpha} + c_{\gamma,\alpha,\beta}. \end{aligned}$$

(2) Starting with $[\pi,\pi]=0$, we get

$$\langle\mathscr{L}_{\lambda(\gamma)}\overline{\beta},\lambda(\alpha)\rangle + \text{c.p.}(\alpha,\beta;\gamma) = 0.$$

But

$$\begin{aligned} \text{lhs} &= \lambda(\gamma)\cdot\pi(\overline{\beta},\overline{\alpha}) - \langle\overline{\beta},[\lambda(\gamma),\lambda(\alpha)]\rangle + \text{c.p.}(\alpha,\beta,\gamma) \\ &= -a_{\gamma,\alpha,\beta} - b_{\beta,\gamma,\alpha} + \text{c.p.}(\alpha,\beta,\gamma). \end{aligned}$$

Therefore (2) holds.

3. Double Lie groups, double Lie algebras, and more on dressing transformations

In this section, we develop the idea of double Lie groups and double Lie algebras (see also [10], [12], [18]). From a Poisson Lie group G and its dual group G^*, we will construct a double Lie group which can be used to describe the dressing transformations of G^* on G.

Definition 3.1. Three Lie groups (G, G_+, G_-) form a *double Lie group* if G_+ and G_- are both closed Lie subgroups of G such that the map $\alpha: G_+ \times G_- \to G$ defined by $(g_+, g_-) \mapsto g_+g_-$ is a diffeomorphism.

Definition 3.2. Three Lie groups (G, G_+, G_-) form a *local double Lie group* if there exist Lie subgroups G'_+ and G'_- of G such that G_i is locally isomorphic to G'_i for $i = +, -$, and such that the map $\alpha: G'_+ \times G'_- \to G$ defined by $(g'_+, g'_-) \mapsto g'_+g'_-$ is a local diffeomorphism near the identities.

Definition 3.3. Three Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ form a *double Lie algebra* if $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces.

Example 3.4. Let G and H be two Lie groups such that G acts on H by automorphisms, and let $H \times_{1/2} G$ be the Lie group semidirect product of G and H relative to this action. Then $(H \times_{1/2} G, H, G)$ form a double Lie group. Likewise, let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras such that $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations, and let $\mathfrak{h} \times_{1/2} \mathfrak{g}$ be the Lie algebra semidirect sum of $\mathfrak{g}$ and $\mathfrak{h}$ relative to this action. Then $(\mathfrak{h} \times_{1/2} \mathfrak{g}, \mathfrak{h}, \mathfrak{g})$ form a double Lie algebra. This example shows that the notions of double Lie groups and double Lie algebras are natural generalizations of that of semidirect products of Lie groups and Lie algebras.

Example 3.5 (*Iwasawa's decomposition* [5]). Let G be a finite-dimensional connected complex semisimple Lie group with complex Lie algebra $\mathfrak{g}$, let G^R be G considered as a real Lie group with real Lie algebra $\mathfrak{g}^R$, and let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$. Then there exists a (real) solvable Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}^R$ such that $\mathfrak{g}^R = \mathfrak{u} \oplus \mathfrak{b}$ as (real) vector spaces. Let U and B be the connected subgroups of G^R corresponding to the Lie subalgebras $\mathfrak{u}$ and $\mathfrak{b}$. Then U is a compact Lie group, and B is a simply connected solvable Lie group. Moreover, the mapping $(u, b) \mapsto ub$ for $u \in U$, $b \in B$ is a diffeomorphism from the manifold $U \times B$ to the manifold G^R. Therefore, (G^R, U, B) form a double Lie group and $(\mathfrak{g}^R, \mathfrak{u}, \mathfrak{b})$ form a double Lie algebra. In Theorem 4.3, this example is carried further to show that every compact semisimple Lie group has a nontrivial Poisson Lie group structure.

As expected, there is a simple correspondence between local double Lie groups and double Lie algebras.

Theorem 3.6. *Let (G, G_+, G_-) be a (local) double Lie group, and let $\mathfrak{g}, \mathfrak{g}_+$ and $\mathfrak{g}_-$ be their Lie algebras. Then $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ form a double Lie algebra. Conversely, let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a double Lie algebra, and let G, G_+, and G_- be any Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{g}_+$ and $\mathfrak{g}_-$ respectively. Then (G, G_+, G_-) form a local double Lie group.*

In some special cases, we can get (global) double Lie groups from double Lie algebras.

Theorem 3.7. *Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a double Lie algebra, and let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Let G_+ and G_- be the connected Lie subgroups of G with Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ respectively. If G_+ is compact and G_- is closed in G, then (G, G_+, G_-) form a (global) double Lie group.*

Proof. Consider the smooth map $\alpha: G_+ \times G_- \to G$ given by

$$\alpha: (g_+, g_-) \mapsto g_+ g_-.$$

Claim 1. α is a submersion, hence a local diffeomorphism (see [5, p. 271]).

Claim 2. α is a proper map.

Let $\{g_{+,n}\} \subset G_+$ and $\{g_{-,n}\} \subset G_-$ such that $g_{+,n} g_{-,n} \to g_0$ in G. Since G_+ is compact, there is a sequence of $\{g_{+,n}\}$, also denoted by $\{g_{+,n}\}$, such that $g_{+,n} \to g_+$ for some g_+ in G_+. Therefore, $g_{+,n} \to g_+$ in G. Hence,

$$g_{-,n} = g_{+,n}^{-1}(g_{+,n} g_{-,n}) \to g_+^{-1} g_0 \quad \text{in } G.$$

But since G_- is closed in G, we have $g_+^{-1}g_0 = g_-$ for some $g_- \in G_-$. Therefore

$$(g_{+,n}, g_{-,n}) \to (g_+, g_-) \quad \text{in } G_+ \times G_-.$$

This proves that α is a proper map.

Now since any proper local diffeomorphism to a connected manifold is a covering map, and since $G_+ \times G_-$ is connected and G is connected and simply connected, α must be a diffeomorphism. Therefore, (G, G_+, G_-) form a (global) double Lie group. q.e.d.

We now show that for any double Lie group there are induced actions of the factor groups on each other.

Let (G, G_+, G_-) be a double Lie group. (The following results hold locally for local double Lie groups.) If $g_+ \in G_+$ and $g_- \in G_-$, then $g_-g_+ \in G$, so there exist unique $g_+^{g_-} \in G_+$ and $g_-^{g_+} \in G_-$ such that

$$g_-g_+ = g_+^{g_-}\, g_-^{g_+}.$$

Fix $g_+ \in G_+$; we get a C^∞ map given by

$$G_- \to G_-: g_- \mapsto g_-^{g_+}.$$

Fix $g_- \in G_-$; we get a C^∞ map given by

$$G_+ \to G_+: g_+ \mapsto g_+^{g_-}.$$

Another way of looking at this is by considering the following sequences of maps:

$$G_- \times G_+ \xrightarrow{m} G \xrightarrow{\alpha^{-1}} G_+ \times G_- \xrightarrow{\Pi_+} G_+,$$

$$G_- \times G_+ \xrightarrow{m} G \xrightarrow{\alpha^{-1}} G_+ \times G_- \xrightarrow{\Pi_-} G_-.$$

Composing the sequences, we get two maps:

$$\Pi_+ \circ \alpha^{-1} \circ m: G_- \times G_+ \to G_+: (g_-, g_+) \mapsto g_+^{g_-},$$

$$\Pi_- \circ \alpha^{-1} \circ m: G_- \times G_+ \to G_-: (g_-, g_+) \mapsto g_-^{g_+}.$$

The first two parts of the following theorem show that these two maps define a left action of G_- on G_+ and a right action of G_+ on G_- respectively. The remaining parts show that these actions are by "twisted automorphisms".

Theorem 3.8. *Let (G, G_+, G_-) be a double Lie group, let $g_+, h_+ \in G_+$, and let $g_-, h_- \in G_-$. Then*

(1) $(g_-^{g_+})^{h_+} = g_-^{g_+h_+}$,

(2) $(g_+^{g_-})^{h_-} = g_+^{h_-g_-}$,

(3) $(g_-h_-)^{g_+} = g_-^{(g_+^{h_-})} h_-^{g_+}$,

(4) $(g_+h_+)^{g_-} = g_+^{g_-} h_+^{(g_-^{g_+})}$.

Conversely, if G_+ and G_- are two Lie groups which act on each other such that properties (1)–(4) *hold, then there is a Lie group structure on the product manifold $G_+ \times G_-$, denoted by $G_+ \bowtie G_-$, such that $(G_+ \bowtie G_-, G_+, G_-)$ form a double Lie group and the induced actions are exactly the given ones.*

Proof. The first part of the proof depends on straightforward calculations. For the second part, we assume that G_+ and G_- act on each other with properties (1)–(4). Define a multiplication on the product manifold $G = G_+ \times G_-$ by

$$(g_+, g_-)(h_+, h_-) = (g_+ h_+^{g_-}, g_-^{h_+} h_-).$$

Then one checks, by fully using properties (1)–(4), that this makes G into a Lie group. q.e.d.

A similar result holds for a double Lie algebra. Here we get induced representations of the Lie algebras on each other by "twisted derivations" (see also [7]).

Theorem 3.9. *Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a double Lie algebra. Then there are induced representations of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ on each other given by the formula*

$$[x_-, x_+] = x_+^{x_-} + x_-^{x_+}, \qquad x_+ \in \mathfrak{g}_+,\ x_- \in \mathfrak{g}_-,\ x_+^{x_-} \in \mathfrak{g}_+,\ x_-^{x_+} \in \mathfrak{g}_-.$$

They have the following properties:

(1) $x_-^{[x_+, y_+]} = (x_-^{x_+})^{y_+} - (x_-^{y_+})^{x_+}$,

(2) $x_+^{[x_-, y_-]} = (x_+^{y_-})^{x_-} - (x_+^{x_-})^{y_+}$,

(3) $[x_-, y_-]^{x_+} = [x_-^{x_+}, y_-] + [x_-, y_-^{x_+}] + x_-^{(x_+^{y_-})} - y_-^{(x_+^{y_-})}$,

(4) $[x_+, y_+]^{x_-} = [x_+^{x_-}, y_+] + [x_+, y_+^{x_-}] - x_+^{(x_-^{y_+})} + y_+^{(x_-^{y_+})}$.

Moreover, if $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are two Lie algebras such that $\mathfrak{g}_+$ acts on $\mathfrak{g}_-$ on the right and $\mathfrak{g}_-$ acts on $\mathfrak{g}_+$ on the left with properties (1)–(4)*, then there is a Lie algebra structure on the vector space $\mathfrak{g}_+ \oplus \mathfrak{g}_-$, denoted by $\mathfrak{g}_+ \bowtie \mathfrak{g}_-$, such that $(\mathfrak{g}_+ \bowtie \mathfrak{g}_-, \mathfrak{g}_+, \mathfrak{g}_-)$ form a double Lie algebra and the induced representations are exactly the given ones.*

Remark 3.10. As found in [7], properties (3) and (4) can be respectively expressed as cocycle conditions on $\mathfrak{g}_+$ and $\mathfrak{g}_-$ with values in the $\mathfrak{g}_+$-module $\mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}_+)$ and the $\mathfrak{g}_-$-module $\mathrm{Hom}(\mathfrak{g}_+, \mathfrak{g}_-)$.

For a double Lie group (G, G_+, G_-), the induced actions of G_+ and G_- on each other fix the identity elements of G_+ and G_- respectively. Therefore linearizing them at these points, we get representations of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ on each other (Lemma 2.3). The first part of the following theorem shows that these representations of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ on each other are precisely those that are induced from the double Lie algebra $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ described in Theorem 3.9. The second part shows that the infinitesimal generators of the group actions have "twisted multiplicativity".

Theorem 3.11. *Let (G, G_+, G_-) be a double Lie group and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ the corresponding double Lie algebra. Then*

(1) *linearization of the induced actions from (G, G_+, G_-) of G_+ and G_- on each other at their identities gives rise to the representations of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ on each other induced from the double Lie algebra $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$;*

(2) *denote by $\rho_+ : \mathfrak{g}_+ \to \chi(G_-)$ and $\lambda_- : \mathfrak{g}_- \to \chi(G_+)$ the Lie algebra homomorphism and antihomomorphism defining the infinitesimal generators of these two group actions. Then they have the following "twisted multiplicativity" for $X_+ \in \mathfrak{g}_+$, $X_- \in \mathfrak{g}_-$, $g_+, h_+ \in G_+$, and $g_-, h_- \in G_-$:*

$$\rho_+(X_+)(g_-h_-) = l_{g_-*}\rho_+(X_+)(h_-) + r_{h_-*}\rho_+(X_+^{h_-})(g_-), \tag{3.1}$$

$$\lambda_-(X_-)(g_+h_+) = l_{g_+*}\lambda_-(X_-^{g_+})(h_+) + r_{h_+*}\lambda_-(X_-)(g_+), \tag{3.2}$$

where $X_+^{h_-} = \frac{d}{dt}|_{t=0}(\exp tX_+)^{h_-}$ and $X_-^{g_+} = \frac{d}{dt}|_{t=0}(\exp tX_-)^{g_+}$ denote the induced actions of G_- on $\mathfrak{g}_+$ and of G_+ on $\mathfrak{g}_-$ respectively.

Proof. The first part is proved by differentiating the following identity with respect to $t = 0$ and $s = 0$ successively:

$$\exp sX_- \cdot \exp tX_+ = (\exp tX_+)^{\exp sX_-} \cdot (\exp sX_-)^{\exp tX_+}.$$

Identity (3.1) is proved by the definitions and property (3) in Theorem 3.8 as follows:

$$\begin{aligned}\rho_+(X_+)(g_-h_-) &= \frac{d}{dt}\Big|_{t=0}(g_-h_-)^{\exp tX_+} = \frac{d}{dt}\Big|_{t=0} g_-^{(\exp tX_+)^{h_-}} h_-^{\exp tX_+} \\ &= l_{g_-*}\rho_+(X_+)(h_-) + r_{h_-*}\rho_+(X_+^{h_-})(g_-).\end{aligned}$$

Similarly, one proves identity (3.2). q.e.d.

Given a Poisson Lie group G, we can now construct a local double Lie group from G.

Theorem 3.12. (1) *For every Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, denote by $\mathfrak{g} \bowtie \mathfrak{g}^*$ the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ together with the Lie bracket given by Theorem 1.12. Then $(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ form a double Lie algebra.*

(2) *Let G be a Poisson Lie group with dual Poisson Lie group G^* and tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Denote by $G \bowtie G^*$ the connected and simply connected Lie group with Lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$. Then $(G \bowtie G^*, G, G^*)$ form a local double Lie group.*

Proof. The proof follows immediately from Theorems 1.12 and 3.6.

Consider now the left action of G^* on G induced from the double Lie group $(G \bowtie G^*, G, G^*)$. Its linearization at $e \in G$ is the coadjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}$, for by the formula of the Lie bracket on $\mathfrak{g} \bowtie \mathfrak{g}^*$ given in Theorem 1.12, we have for $X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$

$$[\alpha, X] = \mathrm{ad}^*_\alpha X - \mathrm{ad}^*_X \alpha.$$

Hence

$$X^\alpha = \mathrm{ad}^*_\alpha X.$$

Theorem 3.13. *Making the left action of G^* on G induced from the double Lie group $(G \bowtie G^*, G, G^*)$ into a right one in the natural way, we get the right dressing action of G^* on G.*

Proof. Denoting this right action by $\sigma: G \times G^* \to G$, we need to prove that σ satisfies the two characterizing properties of the right dressing action stated in Theorem 2.4. We already know that the linearization of σ at $e \in G$ gives rise to minus the coadjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}$, so we only need to check that it has the same "twisted multiplicativity" as does the right dressing action. Denote by $\rho: \mathfrak{g}^* \to \chi(G)$ the Lie algebra homomorphism defining the infinitesimal generators for σ. Then for $\alpha \in \mathfrak{g}^*$ and $g, h \in G$

$$\rho(\alpha)(gh) = l_{g*}\rho(\alpha^g)(h) + r_{h*}\rho(\alpha)(g),$$

where $\alpha^g \stackrel{\mathrm{def}}{=} \frac{d}{dt}|_{t=0}(\exp t\alpha)^g$. One checks that $\alpha^g = \mathrm{Ad}^*_g \alpha$; therefore, we get the following "twisted multiplicativity" for ρ:

$$\rho(\alpha)(gh) = l_{g*}\rho(\mathrm{Ad}^*_g \alpha)(h) + r_{h*}\rho(\alpha)(g).$$

Comparing with Theorem 2.4, we see that σ satisfies the two characterizing properties of the right dressing action, so σ must be the right dressing action of G^* on G.

Theorem 3.14. *Let G be a connected Poisson Lie group with dual group G^* and tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Let $G \bowtie G^*$ be the local double Lie group with Lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$. Denote*

$$\Pi_+: G \bowtie G^* \to G \quad (g, l) \mapsto g,$$
$$\Pi_+: \mathfrak{g} \bowtie \mathfrak{g}^* \to \mathfrak{g} \quad (X, \alpha) \mapsto X,$$
$$\Pi_-: G \bowtie G^* \to G^* (g, l) \mapsto l,$$
$$\Pi_-: \mathfrak{g} \bowtie \mathfrak{g}^* \to \mathfrak{g}^* \quad (X, \alpha) \mapsto \alpha,$$

where we use the same symbol to denote both the Lie group and the Lie algebra projections. Then

(1) *the right dressing action of G^* on G is given by*

$$G \times G^* \to G \quad (g, l) \mapsto \Pi_+(l^{-1}g),$$

where $l^{-1}g$ is the product of l^{-1} and g in $G \bowtie G^*$;

(2) *the right dressing vector fields are given by*

$$\rho(\alpha)(g) = -l_{g_*}\Pi_+(\mathrm{Ad}_{g^{-1}}\alpha) = l_{g_*}(\mathrm{Ad}_g^*\alpha) - r_{g_*}\alpha$$

for $\alpha \in \mathfrak{g}^*$, $g \in G$, and $\mathrm{Ad}_g^*\alpha \in \mathfrak{g}^* \subset \mathfrak{g} \bowtie \mathfrak{g}^*$;

(3) *the multiplicative Poisson structure π on G is given by*

$$\pi_r(g)(\alpha, \beta) = \langle \Pi_-(\mathrm{Ad}_{g^{-1}}\alpha), \Pi_+(\mathrm{Ad}_{g^{-1}}\beta)\rangle,$$

where $\alpha, \beta \in \mathfrak{g}^*$, $\mathrm{Ad}_{g^{-1}}\alpha, \mathrm{Ad}_{g^{-1}}\beta \in \mathfrak{g} \bowtie \mathfrak{g}^*$, and $\pi_r(g) \stackrel{\text{def}}{=} r_{g_*^{-1}}\pi(g)$ $\forall g \in G$.

Proof. (1) The proof follows immediately from Theorem 3.13.

(2) For $\alpha \in \mathfrak{g}^*$ and $g \in G$, we have

$$g^{-1} \cdot \exp t\alpha \cdot g = g^{-1}g^{\exp t\alpha}(\exp t\alpha)^g.$$

Differentiating at $t = 0$ yields

$$\mathrm{Ad}_{g^{-1}}\alpha = -l_{g_*^{-1}}\rho(\alpha)(g) + \alpha^g = -l_{g_*^{-1}}\rho(\alpha)(g) + \mathrm{Ad}_g^*\alpha,$$

which leads to (2).

(3) By the definition of the right dressing vector fields, we have

$$\begin{aligned}\pi_r(g)(\alpha, \beta) &= -\rho(\beta)(r_{g^{-1}}^*\alpha) = -l_{g_*^{-1}}\rho(\beta)(g)(\mathrm{Ad}_g^*\alpha) \\ &= \langle \alpha, \mathrm{Ad}_g\Pi_+(\mathrm{Ad}_{g^{-1}}\beta)\rangle = \langle \mathrm{Ad}_{g^{-1}}\alpha, \Pi_+(\mathrm{Ad}_{g^{-1}}\beta)\rangle \\ &= \langle \Pi_-(\mathrm{Ad}_{g^{-1}}\alpha), \Pi_+(\mathrm{Ad}_{g^{-1}}\beta)\rangle. \quad \text{q.e.d.}\end{aligned}$$

We conclude this section by proving the following fact about the symplectic leaves of a Poisson Lie group G.

Theorem 3.15 (*Semenov-Tian-Shansky* [17]). *The symplectic leaf of a Poisson Lie group G through a point $g \in G$ is exactly the image under the projection $\Pi_+: G \bowtie G^* \to G$ of the left coset G^*g.*

Proof. By the definition of the dressing action, the symplectic leaf through the point g coincides with the orbit of the right (or left) dressing action through g. But by Theorem 3.14, this orbit of the dressing action is exactly as described in the statement.

4. Burhat-Poisson structures

Recall that the category of finite dimensional connected and simply connected Poisson Lie groups is equivalent to the category of finite dimensional Lie bialgebras, and that there is a one-to-one correspondence between Lie bialgebras and Manin triples (Theorems 1.8 and 1.12). Our idea here is to use Iwasawa's decompositions of certain Lie algebras to get examples of Manin triples. We quote the following theorem from [5, Theorem 6.3, p. 275].

Theorem 4.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{g}^R$ the Lie algebra $\mathfrak{g}$ considered as a Lie algebra over $\mathbb{R}$. Let $\mathfrak{u}$ be any compact real form of $\mathfrak{g}$, and let $\mathfrak{a}$ be any maximal abelian subalgebra of $\mathfrak{u}$. Then the algebra $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let Δ^+ be the set of positive roots with respect to some ordering of Δ. If $\mathfrak{n}_+$ denotes the space $\sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$ considered as a real subspace of $\mathfrak{g}^R$, then the following decomposition of real vector spaces is valid:*

$$\mathfrak{g}^R = \mathfrak{u} \oplus i\mathfrak{a} \oplus \mathfrak{n}_+.$$

Let G^R be any connected Lie group with Lie algebra $\mathfrak{g}^R$ and let U, A_*, and N be the connected subgroups of G^R with Lie algebras $\mathfrak{u}$, $i\mathfrak{a}$ and $\mathfrak{n}_+$ respectively. Then the mapping

$$(u, a, n) \mapsto uan, \qquad u \in U,\ a \in A_*,\ n \in N,$$

is a diffeomorphism of $U \times A_* \times N$ onto G^R. The groups A_* and N are simply connected.

Let $\mathfrak{b} = i\mathfrak{a} \oplus \mathfrak{n}_+$. Then $\mathfrak{b}$ is a solvable subalgebra of $\mathfrak{g}^R$, and $\mathfrak{g}^R = \mathfrak{u} \oplus \mathfrak{b}$ as real vector spaces. Let B be the connected Lie subgroup of G^R with Lie algebra $\mathfrak{b}$. Then the map

$$(u, b) \to ub, \qquad u \in U,\ b \in B,$$

is a diffeomorphism from $U \times B$ to G^R. Therefore, $(\mathfrak{g}^R, \mathfrak{u}, \mathfrak{b})$ is a double Lie algebra and (G^R, U, B) is a double Lie group.

To see that $(\mathfrak{g}^R, \mathfrak{u}, \mathfrak{b})$ is actually a Manin triple, we consider the Killing form K of $\mathfrak{g}$. It is a complex-valued nondegenerate invariant symmetric bilinear form on $\mathfrak{g}^R$. Therefore its imaginary part, $\operatorname{Im} K$, is a real-valued nondegenerate invariant symmetric bilinear form on $\mathfrak{g}^R$. We claim that the Lie subalgebras $\mathfrak{u}$ and $\mathfrak{b}$ are both isotropic with respect to $\operatorname{Im} K$. Notice that since $\mathfrak{u}$ is a real form of $\mathfrak{g}$, K takes real values on $\mathfrak{u}$, so $\operatorname{Im} K = 0$ on $\mathfrak{u}$. To see that $\operatorname{Im} K = 0$ on $\mathfrak{b}$, we recall that $K(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ if $\alpha + \beta \neq 0$, so

$$K(\mathfrak{b}, \mathfrak{b}) = K(i\mathfrak{a} \oplus \mathfrak{n}_+, i\mathfrak{a} \oplus \mathfrak{n}_+) = -K(\mathfrak{a}, \mathfrak{a}) \subset \mathbb{R}.$$

Therefore $\operatorname{Im} K(\mathfrak{b}, \mathfrak{b}) = 0$, and both $\mathfrak{u}$ and $\mathfrak{b}$ are isotropic with respect to $\operatorname{Im} K$. Hence $(\mathfrak{g}^R, \mathfrak{u}, \mathfrak{b})$ together with the bilinear form $\operatorname{Im} K$ is a Manin triple. This then induces Poisson Lie group structures on U and B such that they can be identified with the dual groups of each other.

Example 4.2. When $\mathfrak{g} = \mathrm{sl}(n, \mathbb{C})$ and $\mathfrak{u} = \mathrm{su}(n)$, we can take

$$\mathfrak{a} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} : \theta_j \in \mathbb{R},\ j = 1, \cdots, n,\ \theta_1 + \cdots + \theta_n = 0 \right\},$$

$\mathfrak{n}_+ = \{$all $n \times n$ strictly upper triangular complex matrices$\}$.

Then

$$i\mathfrak{a} = \left\{ \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{pmatrix} : \theta_j \in \mathbb{R},\ j = 1, \cdots, n,\ \theta_1 + \cdots + \theta_n = 0 \right\},$$

$$\mathfrak{b} = \mathrm{sb}(n, \mathbb{C}) = \left\{ \begin{array}{l} \text{all } n \times n \text{ traceless upper triangular} \\ \text{complex matrices with real diagonal elements} \end{array} \right\},$$

and the nondegenerate invariant bilinear form on $\mathrm{sl}(n, \mathbb{C})$ can be taken as

$$\langle X, Y \rangle = \mathrm{Im}(\mathrm{trace}(XY)).$$

Let $\mathrm{SB}(n, \mathbb{C})$ be the group of all $n \times n$ upper triangular complex matrices with determinant one and real positive diagonal elements, whose Lie algebra is $\mathfrak{b}$. Then there are Poisson Lie group structures on $\mathrm{SU}(n)$ and on $\mathrm{SB}(n, \mathbb{C})$ such that they can be identified with the dual groups of each other.

Returning to the general case, we denote by Π_+ both the projections

$$\begin{aligned} &\Pi_+ : \mathfrak{g}^R \to \mathfrak{u} \quad X_\mathfrak{u} + X_\mathfrak{b} \mapsto X_\mathfrak{u}, \\ &\Pi_+ : G^R \to U \quad ub \mapsto u, \end{aligned}$$

and denote by Π_- both the projections

$$\begin{aligned} &\Pi_- : \mathfrak{g}^R \to \mathfrak{b} \quad X_\mathfrak{u} + X_\mathfrak{b} \mapsto X_\mathfrak{b}, \\ &\Pi_- : G^R \to B \quad ub \mapsto b. \end{aligned}$$

By our discussion in the last section, we get the following result.

Theorem 4.3. *Let notations be as in Theorem* 4.1.

(1) *There are multiplicative Poisson structures on the groups U and B such that they can be identified with the dual groups of each other.*

(2) *The right dressing action of B on U is given by*

$$U \times B \to U \quad (u, b) \mapsto \Pi_+(b^{-1}u).$$

(3) *The multiplicative Poisson tensor π on U is given by*

$$\pi_r(u)(\alpha, \beta) = \mathrm{Im}\, K(\Pi_-(\mathrm{Ad}_{u^{-1}}\alpha), \Pi_+(\mathrm{Ad}_{u^{-1}}\beta)),$$

where $\pi_r(u) \stackrel{\mathrm{def}}{=} r_{u_*^{-1}}\pi(u)$ *for* $u \in U$, $\alpha, \beta \in \mathfrak{u}^* \cong \mathfrak{b}$, *and* K *is the Killing form of* $\mathfrak{g}$.

Proof. The proof follows immediately from Theorem 3.14.

We can carry on this example further to show that every coadjoint orbit of U has an induced Poisson structure. To this end, we recall the notion of a Poisson Lie subgroup.

Definition 4.4. A Lie subgroup H of a Poisson Lie group G is called a *Poisson Lie subgroup* if it also a Poisson submanifold of G, i.e. if it also has a Poisson structure such that the inclusion map $i: H \to G$ is a Poisson map.

Proposition 4.5 (*Semenov-Tian-Shansky* [17]). *For a given subgroup* H *of* G, *the following are equivalent:*

(1) H *is a Poisson Lie subgroup*;

(2) (*Assuming that* H *is connected*) $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ *is an ideal, where* $\mathfrak{h}$ *is the Lie algebra of* H;

(3) H *is invariant under the right* (*therefore also the left*) *dressing action of* G^* *on* G.

Theorem 4.6. *If* H *is a closed Poisson Lie subgroup of* G, *then there is an induced Poisson structure on the left coset space* G/H *such that*

(1) *the natural projection* $\tau: G \to G/H$ *is a Poisson map*;

(2) *the natural action of* G *on* G/H *by left translations is a Poisson map. Hence the Poisson manifold* G/H *becomes a "Poisson-homogeneous"* G*-space*;

(3) the right dressing action of G^* *on* G *induces a right action of* G^* *on* G/H *which is also a Poisson action; its orbits coincide with the symplectic leaves of* G/H;

(4) *the Poisson structure on* G/H *has zero rank at the point* eH, *and its linearization at this point is isomorphic to the Lie Poisson space* $(\mathfrak{h}^\perp)^*$.

Proof. (1) Simply define a bivector field $\overline{\pi}$ on G/H by

$$\overline{\pi}(gH) \stackrel{\mathrm{def}}{=} \tau_*\pi(g).$$

One checks, by using the fact that π is multiplicative and H is invariant under the dressing action, that $\overline{\pi}$ is a well-defined Poisson tensor on G/H. (2)–(4) are consequences of this definition. q.e.d.

With notations as before, we consider the Lie algebra $\mathfrak{p}_0 \stackrel{\mathrm{def}}{=} \mathfrak{a} \oplus i\mathfrak{a} \oplus \mathfrak{n}_+$ and its Lie group $P_0 = AA_*N$, which is a minimal parabolic subgroup of G^R. Subgroups of G^R containing P_0 or its conjugates are called parabolic subgroups of G^R [9]. If P is a parabolic subgroup containing P_0, then there is a subgroup W_P of the Weyl group W of the pair $(\mathfrak{g}^R, i\mathfrak{a})$ (which is the same as that of $(\mathfrak{u}, \mathfrak{a})$), such that

$$P = P_0 W_P P_0,$$

and we have the following Bruhat decomposition of G^R:

$$G^R = \bigcup_{w \in W/W_P} P_0 w P \quad \text{(disjoint union)}$$

(see [9]).

Now every coadjoint orbit $\mathscr{O}$ of U is of the form $\mathscr{O} \cong G^R/P \cong U/U \cap P$ for some parabolic subgroup P of G^R containing P_0. As a subgroup of the Poisson Lie group U, we can check that $U \cap P$ is a Poisson Lie subgroup of U. To see this, let $u \in U \cap P$ and $b \in B$ be arbitrary. Then we have $b^{-1}u = u_1 b_1$ for some $u_1 \in U$ and $b_1 \in B$. Hence

$$u_1 = \Pi_+(b^{-1}u) = b^{-1}ub_1^{-1} \in U \cap P.$$

By (2) of Theorem 4.3, we see that $U \cap P$ is invariant under the dressing action of B on U. By Proposition 4.5, $U \cap P$ is a Poisson Lie subgroup of U, and by Theorem 4.6, there is a Poisson structure on the coadjoint orbit $\mathscr{O} = U/U \cap P$ and an induced right Poisson action of B on $\mathscr{O}$ such that the symplectic leaves in $\mathscr{O}$ coincide with the B-orbits. An explicit formula for this B-action on $\mathscr{O}$ is given by

$$(gP, b) \to (b^{-1}g)P, \qquad g \in G^R, \; gP \in \mathscr{O} \cong G^R/P.$$

Hence, if we equip $\mathscr{O}$ with the complex manifold structure induced from that of G^R and P, then the B-action on $\mathscr{O}$ is by holomorphic maps. Furthermore, by the Bruhat decomposition of G^R, we have the following decompositions for $\mathscr{O}$:

$$\begin{aligned} \mathscr{O} \cong G^R/P &= \bigcup_{w \in W/W_P} P_0 w P/P \\ &= \bigcup_{w \in W/W_P} P_0 w/P = \bigcup_{w \in W/W_P} BAw/P = \bigcup_{w \in W/W_P} Bw/P. \end{aligned}$$

The decomposition of $\mathscr{O}$ given by the last equality is called a Bruhat decomposition of $\mathscr{O}$. It is exactly the decomposition of $\mathscr{O}$ into B-orbits. We call this Poisson structure on $\mathscr{O}$ a Bruhat-Poisson structure. Summarizing our results, we have the following theorem.

Theorem 4.7. (1) *Every connected compact semisimple Lie group G is a (nontrivial) Poisson Lie group;*

(2) *Each coadjoint orbit $\mathscr{O}$ of a connected compact semisimple Lie group G has a Poisson structure such that it becomes a Poisson-homogeneous G-space with respect to the Poisson Lie group structure on G given by* (1).

Moreover, the symplectic leaves of $\mathscr{O}$ coincide with the Bruhat cells of a Bruhat decomposition and also with the orbits of a Poisson action by the dual Poisson Lie group of G. We call such a Poisson structure a Bruhat-Poisson structure.

Question 4.8. Is Theorem 4.7 the classical limit of a theorem about quantum groups and their representations ([13], [22], [23], [24]]? Generalizing the results in [13], can we "quantize" a general Poisson-homogeneous space?

Example 4.9. When $G = \mathrm{SU}(2)$, we can check that the linearization of the Poisson structure on SU(2) (Theorem 4.3) is isomorphic to the Lie-Poisson space $\mathfrak{b}^*$, where $\mathfrak{b}$ is the three-dimensional "book" algebra (so called because its regular coadjoint orbits resemble the pages of an open book, with the singular orbits as the binding) with brackets

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = 0.$$

Consider now the quantum groups $\mathrm{S}_\mu\mathrm{U}(2)$ in [23]. By letting $\mu \to 1$, we get a Poisson Lie group structure on SU(2). We can calculate the Poisson brackets for some set of coordinate functions near the identity element. The linearization of this Poisson structure turns out to be isomorphic to the "book" algebra. Since a multiplicative Poisson structure on a connected Lie group is uniquely determined by its linearization at the identity, the two Poisson structures on SU(2) must be isomorphic. Therefore the answer to the first part of Question 4.8 is affirmative, at least for the case when $G = \mathrm{SU}(2)$.

There are only two nondiffeomorphic types of coadjoint orbits of SU(2), one consisting of a single point and the other diffeomorphic to the two sphere. Let $\mathscr{O} \cong S^2$ be such a principal orbit. The Bruhat-Poisson structure on $\mathscr{O}$ has rank 0 at one point $\mathbf{n} \in \mathscr{O}$ and rank 2 everywhere else. Using the coordinates (α, β) obtained by the stereographic projection with respect to $\mathbf{n}$, we can calculate the Bruhat-Poisson structure on the 2-cell $S^2\backslash\{\mathbf{n}\}$ to get

$$\pi = -\tfrac{1}{2}(1 + \alpha^2 + \beta^2)\tfrac{\partial}{\partial\alpha} \wedge \tfrac{\partial}{\partial\beta}.$$

Notice that the symplectic manifold $(S^2\backslash\{\mathbf{n}\}, -(2/(1+\alpha^2+\beta^2))\, d\alpha \wedge d\beta)$ has infinite volume and is therefore isomorphic to the plane. If we use the coordinates (s, t) obtained by the sterographic projection with respect to the antipodal point $\mathbf{s}$ of $\mathbf{n}$, then the Bruhat-Poisson structure on $\mathscr{O}$ is given by

$$\pi = \tfrac{1}{2}(s^2 + t^2)(1 + s^2 + t^2)\tfrac{\partial}{\partial s} \wedge \tfrac{\partial}{\partial t}.$$

The point $\mathbf{n}$ has coordinates $(s, t) = (0, 0)$. The linearization of π at $\mathbf{n}$ is abelian, so π is not linearizable at $\mathbf{n}$. However, employing the change of coordinates $x = s/\sqrt{1+s^2+t^2}$ and $y = t/\sqrt{1+s^2+t^2}$, we get

$$\pi = \tfrac{1}{2}(x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Hence π is quadratic in this coordinate system.

Let $\tau \colon \mathrm{SU}(2) \to S^2$ be the Hopf fibration, which is a Poisson map by Theorem 4.6. The global decomposition of SU(2) into symplectic leaves (also found by Vaksman and Soibel'man in [19], who related it to the primitive ideal space of the quantum $\mathrm{S}_\mu\mathrm{U}(2)$ as defined in [23]), consists of a "binding" circle $T = \tau^{-1}(\mathbf{n})$ of singular points and "pages" which are mapped symplectomorphically by τ onto $S^2\backslash\{\mathbf{n}\}$. Each page is obtained by choosing a point of the circle $\tau^{-1}(\mathbf{s})$ (its "page number") and following the horizontal (i.e. perpendicular to the fibers of τ) lifts of all the geodesics in S^2 from $\mathbf{s}$ to $\mathbf{n}$. A dressing transformation on SU(2) fixes points on the "binding" circle T and maps each page into itself by the lift via τ of a fixed conformal transformation of S^2 fixing $\mathbf{n}$. In stereographic coordinates with respect to $\mathbf{n}$, these transformations become the translations and dilations (but not rotations) of the complex plane.

Question 4.10. How is the Poisson structure on the sphere related to the "quantum spheres" in [13]? (Presumably, it is a limit in the same way that the structure on SU(2) is.)

Note added in proof. The most general Poisson structure on S^2 for which the action of our Poisson SU(2) is a Poisson action is found by adding to the Bruhat-Poisson structure a real multiple of the SU(2)-invariant structure. This one-parameter family of Poisson structures contains two of Bruhat-Poisson type, two open intervals of symplectic structures, and an open interval of structures whose symplectic leaves are two open discs and the points of the circle separating them. There is a very close relation between the symplectic leaves of these structures and the irreducible quantum sphere algebras of [13] for all the values of Podles' parameter c, so the Poisson spheres give quite an accurate "picture" of the quantum ones.

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