J. DIFFERENTIAL GEOMETRY
30 (1989) 833-845

ON THE DISTRIBUTION OF CONJUGATE POINTS

WALTER VANNINI

1. Introduction and main theorems

In 1958, L. W. Green published a curvature inequality for compact riemannian manifolds without conjugate points [4]. For M a riemannian manifold as above, he showed that $\int_M \mathrm{Scal} > 0$. He also established that equality occurs precisely when M is flat.

By Ambrose's criterion for conjugate points, and an observation of A. Avez regarding the use of Birkhoff's Ergodic Theorem, Green's inequality can be quickly derived [1], [2]. By the same argument, and with the use of a new criterion for conjugate points, we have a generalization of Green's inequality.

Theorem 1. *Let M be a complete riemannian manifold of dimension n with a finite volume and Ricci curvature bounded above. Then*

$$\int_M \mathrm{Scal} \le \frac{\pi(n-1)^{1/2}n}{\mathrm{vol}(S^{n-1}, can)} \sqrt{\sup(0, \mathrm{Ric})} \int_{SM} \underline{\psi}.$$

In the above theorem, SM is the unit tangent bundle with the induced Liouville measure, Ric: $SM \to R$ is the Ricci curvature function, and $\underline{\psi}\colon SM \to [0, \infty]$ is defined by

$$\underline{\psi}(v) = \liminf_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \text{ where } c_v(t) = \exp(tv) \end{pmatrix}.$$

For M the standard n-sphere of constant sectional curvature 1, $\underline{\psi} = 1/\pi$, $\mathrm{Ric} = n-1$, $\mathrm{Scal} = n(n-1)$, and $\mathrm{vol}(SM) = \mathrm{vol}(M)\mathrm{vol}(S^{n-1}, can)$. The standard n-sphere shows that the above generalization of Green's inequality is sharp.

It would be desirable to also generalize Green's equality statement. It seems plausible that equality occurs in the generalized Green's inequality precisely when M has constant sectional curvature.

The new criterion for conjugate points, mentioned above, is

Received May 16, 1988.

Theorem 2. *Let $c\colon [0, L] \to M$ be a unit speed geodesic on a riemannian manifold of dimension n. If*

$$\int_0^L \operatorname{Ric}(c'(t))\, dt \geq \pi(n-1)^{1/2} \sqrt{\max_{t\in[0,L]}(0, \operatorname{Ric}(c'(t)))},$$

and $\operatorname{Ric}(c')$ *is not identically zero, then $c(0)$ has a conjugate point $c(T)$ along c for some T in $(0, L]$.*

Furthermore, if the smallest such T is L, then $K(\sigma) = \pi^2/L^2$ for all tangent two-planes σ containing a tangent vector to c.

This theorem provides an analogue to Myer's criterion for conjugate points [7]. His criterion is the same as ours, except that the curvature condition is replaced by $\operatorname{Ric}(c') \geq (n-1)\pi^2/L^2$. Our criteria coincide when $\operatorname{Ric}(c') = (n-1)\pi^2/L^2$.

It should be noted that the curvature condition in Theorem 2 cannot be replaced by

$$\int_0^L \operatorname{Ric}(c'(t))\, dt \geq (n-1)\pi^2/L;$$

a sufficiently long geodesic beginning from the vertex of the paraboloid $z = x^2 + y^2$ demonstrates this. However, the curvature condition

$$\int_0^L \operatorname{Ric}(c'(t))(1 - \cos(2\pi t/L))\, dt \geq (n-1)\pi^2/L$$

is a valid replacement, as shown by L. W. Green [5] in 1963. At present, this is the strongest generalization of Myers' criterion.

An immediate consequence of Theorem 2 is the following supplement to a result of Ambrose [1].

Theorem 3. *Let $c\colon [0, \infty) \to M$ be a unit speed geodesic on a riemannian manifold of dimension n, which gives rise to no conjugate points of $c(0)$. Then*

$$\limsup_{T\to\infty} \int_0^T \operatorname{Ric}(c'(t))\, dt \leq \pi(n-1)^{1/2} \sqrt{\sup_{t\in[0,\infty)}(0, \operatorname{Ric}(c'(t)))}.$$

Ambrose showed that with the same hypotheses, $\lim_{T\to\infty} \int_0^T \operatorname{Ric}(c'(t))\, dt$ is not $+\infty$.

Corresponding to Theorem 2, there is a theorem about second order differential equations.

Theorem 4. *Consider the second order differential equation $x'' + Fx = 0$, where F is a continuous function defined on $[0, L]$. Let $z\colon [0, L] \to R$ be* *a solution for which* $z(0) = 0$ *and* $z'(0) \neq 0$. *If*

$$\int_0^L F(t)\,dt \geq \pi\sqrt{\max_{t\in[0,L]}(0, F(t))}$$

and F *is not identically zero, then* $z(T) = 0$ *for some* T *in* $(0, L]$.

Furthermore, if the smallest such T *is* L, *then* $F(t) = \pi^2/L^2$ *for all* t *in* $[0, L]$.

Theorem 1 can be restated in terms of the integral of the Ricci curvature. It is then apparent that it follows from the following stronger result.

Theorem 5. *Let* M *be a finite volume complete riemannian manifold of dimension* n.

(1) *If* Ric *has an integrable positive or negative part, then*

$$\int_Z \mathrm{Ric} \leq 0.$$

(2) *If* Ric *is bounded above, then*

$$\int_{SM-Z} \mathrm{Ric} \leq \pi(n-1)^{1/2}\sqrt{\sup(0, \mathrm{Ric})}\int_{SM} \underline{\psi}.$$

Here Z denotes the subset of SM consisting of unit vectors v for which the geodesic $c_v\colon [0, \infty) \to M$ defined by $c_v(t) = \exp(tv)$ gives rise to no conjugate points of $c_v(0)$.

The author would like to thank Detlef Gromoll for many helpful conversations and suggestions. Thanks are also due to Demir Kupeli, for pointing out that any estimate for conjugate points should be a special case of a more general result for differential equations.

2. Basic facts and technical lemmas

2.1. Conjugate points along geodesics. We refer to [3] as a basic reference.

Let M be a riemannian manifold, and let $c\colon [0, L] \to M$ be a unit speed geodesic on M. $c(\tau)$ is said to be a conjugate point of $c(0)$ along c (where $0 < \tau \leq L$) if $\exp_{c(0)}\colon TM_{c(0)} \to M$ is singular at $\tau' c(0)$. The multiplicity of the conjugate point is defined to be the dimension of the nullspace of the differential of $\exp_{c(0)}$ at $\tau c'(0)$.

Equivalently, $c(\tau)$ is conjugate to $c(0)$ along c when there is a Jacobi field J along c, other than the zero field, that vanishes at 0 and τ. The multiplicity of $c(\tau)$ is the dimension of the vector space generated by all such Jacobi fields.

The index form I_c associated to c is defined by

$$I_c(V,W) = \int_0^L \left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle - \langle R(V, c'(t))c'(t), W\rangle \, dt,$$

where V, W are continuous piecewise smooth vector fields on c that are orthogonal to c, and vanish at 0 and L. Such vector fields will be called admissible (or c-admissible, whenever there might be confusion).

There exists a conjugate point $c(\tau)$ to $c(0)$ along c, $0 < \tau \leq L$, precisely when there exists an admissible vector field V on c, where V is not the zero field, for which $I_c(V,V) \leq 0$. If the only conjugate point to $c(0)$ along c is $c(L)$, then the admissible V for which $I_c(V,V) \leq 0$ are precisely the Jacobi fields vanishing at 0 and L.

We wish to recall the well-known

Morse Index Theorem. *The number of conjugate points to $c(0)$ along $c|_{[0,L)}$, counted according to multiplicity, is equal to the dimension of a maximal subspace of admissible fields for which I_c is negative definite.*

This theorem is used here to count conjugate points. Counting will usually not be according to multiplicity.

Lemma 1. *If $0 < T_1 < T_2 \leq L$ and it is known that $c(0)$ has a conjugate point along $c|_{[0,T_1]}$ and $c(T_1)$ has a conjugate point along $c|_{[T_1,T_2]}$, then it follows that $c(0)$ has a conjugate point $c(T)$ along c, where $T_1 < T \leq T_2$.*

By repeated application of Lemma 1, we have

Lemma 2. *If $0 < T_1 < T_2 < \cdots < T_k \leq L$ and it is known that $c(0)$ has a conjugate point along $c|_{[0,T_1]}$, $c(T_1)$ has a conjugate point along $c|_{[T_1,T_2]}, \cdots, c(T_{k-1})$ has a conjugate point along $c|_{[T_{k-1},T_k]}$, then it follows that $c(0)$ has at least k conjugate points along c.*

In Lemma 3, conjugate points are counted according to multiplicity.

Lemma 3. *If $0 < T_1 < T_2 < \cdots < T_k \leq L$ and it is known that $c(0)$ has α_1 conjugate points along $c|_{[0,T_1]}$, $c(T_1)$ has α_2 conjugate points along $c|_{[T_1,T_2]}, \cdots, c(T_{k-1})$ has α_k conjugate points along $c|_{[T_{k-1},T_k]}$, then it follows that*

$$\sum_{i=1}^{k} \alpha_i - k(n-1) \leq \alpha \leq \sum_{i=1}^{k} \alpha_i + k(n-1),$$

where α is the number of points conjugate to $c(0)$ along $c|_{[0,T_k]}$.

Proof of Lemma 1. Without loss of generality $T_2 < L$. Pick $\varepsilon_2 > 0$ ($\varepsilon_2 < L - T_2$). Then there exists $\varepsilon_1 > 0$ for which $c(T_1 + \varepsilon_1)$ has a conjugate point along $c|_{[T_1+\varepsilon_1, T_2+\varepsilon_2)}$.

Take a maximal subspace of $c|_{[0,T_1+\varepsilon_1]}$-admissible vector fields for which the index form is negative definite, and extend it to a subspace of $c|_{[0,T_2+\varepsilon_2]}$-admissible vector fields by taking the vector fields to be zero outside of $[0, T_1 + \varepsilon_1]$. Call the resulting subspace W_1.

Now take a maximal subspace of $c|_{[T_1+\varepsilon_1,T_2+\varepsilon_2]}$-admissible vector fields for which the index form is negative definite, and extend it in a similar way to give W_2. By the Morse Index Theorem $\dim(W_2) > 0$, so that $\dim(W_1 + W_2) > \dim(W_1)$. Applying the Morse Index Theorem again, we see that $c(0)$ has a conjugate point $c(T)$ along c, where $T_1+\varepsilon_1 \leq T < T_2+\varepsilon_2$.

Now, there exists S satisfying $T_1 < S < T_2$, for which $c(0)$ has no conjugate points $c(\tau)$ satisfying $T_1 < \tau < S$. This tells us that $c(0)$ has a conjugate point $c(T)$ along c, where $S \leq T < T_2 + \varepsilon_2$. This is true for any positive ε_2 and $S > T_1$, so that $c(0)$ has a conjugate point $c(T)$ along c, where $T_1 < T \leq T_2$.

Proof of Lemma 2. By repeated application of Lemma 1, there exist conjugate points $c(\tau_1), c(\tau_2), \cdots, c(\tau_k)$ to $c(0)$ along c, satisfying

$$0 < \tau_1 \leq T_1 < \tau_2 \leq T_2 < \cdots < \tau_k \leq T_k.$$

Proof of Lemma 3. Define T_0 to be 0. Note that conjugate points are counted according to multiplicity in this proof. Take a maximal subspace of $c|_{[T_{i-1},T_i]}$-admissible vector fields for which the index form is negative definite. Call it $\underline{W_i}$. Extend it to a subspace of $c|_{[0,T_k]}$-admissible vector fields by taking the vector fields of $\underline{W_i}$ to be zero outside of $[T_{i-1}, T_i]$. Call this subspace W_i. Let W be the direct sum of the W_i's.

Since $c(T_{i-1})$ has at least $\alpha_i - (n-1)$ conjugate points along $c|_{[T_{i-1},T_i)}$, we have that $\dim(W_i) \geq \alpha_i - (n-1)$, and so

$$\dim(W) \geq \sum_{i=1}^{k} \alpha_i - k(n-1).$$

Since $\alpha \geq \dim(W)$,

$$\sum_{i=1}^{k} \alpha_i - k(n-1) \leq \alpha.$$

Since $c(T_{i-1})$ has at most α_i conjugate points along $c|_{[T_{i-1},T_i)}$, it follows that $\dim(W_i) \leq \alpha_i$, so that

$$\dim(W) \leq \sum_{i=1}^{k} \alpha_i.$$

Now extend W to a maximal subspace of $c|_{[0,T_k]}$-admissible vector fields for which the index form is negative definite. Call this space X, and note that

$$\alpha \leq \dim X + (n-1).$$

Let W' be the orthogonal complement of W in X (with respect to I). By showing that $\dim W' \leq (k-1)(n-1)$, we are done, since then

$$\alpha \leq \dim X + (n-1) = \dim W + \dim W' + (n-1)$$
$$\leq \sum_{i=1}^{k} \alpha_i + (k-1)(n-1) + (n-1).$$

Suppose that $\dim W' > (k-1)(n-1)$. Since the maximum possible dimension of the image of the linear map sending V' in W' to $(V'(T_1), V'(T_2), \cdots, V'(T_{k-1}))$ is $(k-1)(n-1)$, there exists a nontrivial vector field V in W' for which

$$V(T_1) = 0, V(T_2) = 0, \cdots, V(T_{k-1}) = 0.$$

For some j from 1 to k, $V_j = V|_{[T_{j-1}, T_j]}$ is a nontrivial vector field for which $I(V_j, V_j) < 0$, and $I(V_j, W_j) = 0$, since $I(V, W) = 0$. This contradicts the maximal property of W_j.

2.2. The geodesic flow and the ergodic theorem

We refer to [8] as a basic reference.

Let M be a complete riemannian manifold, and SM its unit tangent bundle. The geodesic flow $G\colon SM \times R \to SM$ on SM is defined by $G(v,t) = c'(t)$, where $c(s) = \exp(sv)$. $G(v,t)$ will be written as $G_t(v)$.

The unit tangent bundle possesses a Borel measure determined by the riemannian structure of its base manifold. It is called the Liouville measure, and it is invariant with respect to the geodesic flow.

Birkhoff's Ergodic Theorem applies to measure spaces with a measure invariant flow. In our case, we have

Birkhoff's Ergodic Theorem. *Let M be a complete riemannian manifold, $G\colon SM \times R \to SM$ be the geodesic flow, and $f\colon SM \to R$ be a function whose positive or negative part is integrable with respect to the Liouville measure μ. Then the following hold:*

(1) *The following limit exists for almost all v in SM:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(G_t v)\, dt.$$

(2) *If A is a flow invariant subset of SM of finite measure, then*

$$\int_A f(v)\, d\mu(v) = \int_A \lim_{T\to\infty} \frac{1}{T} \int_0^T f(G_t v)\, dt\, d\mu(v).$$

Letting α be any real number, the above statement is also true when

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(G_t v)\, dt$$

is replaced by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(G_{k\alpha} v).$$

3. On the density of conjugate points along geodesics

Let M be a complete riemannian manifold. Any unit tangent vector v on M determines a unit speed geodesic $c_v\colon [0,\infty) \to M$ via $c_v'(0) = v$, and then two elements of $[0,+\infty]$, namely,

$$\underline{\psi}(v) = \liminf_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]} \end{pmatrix},$$

$$\overline{\psi}(v) = \limsup_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]} \end{pmatrix}.$$

Writing n for the dimension of M, we immediately have

Proposition 1. *If β is a positive number for which* $\mathrm{Ric}(c_v') \le \beta$, *then*

$$\underline{\psi}(v) \ge \frac{1}{\pi\sqrt{\beta(n-1)}} \liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(c'(t))\, dt.$$

It is clear that if c_v gives rise to no conjugate points of $c_v(0)$, then $\underline{\psi}(v) = \overline{\psi}(v) = 0$. The converse is not true. The paraboloid $z = x^2 + y^2$ has many unit tangent vectors that serve as counterexamples.

However, the following is true.

Proposition 2. *Let M be a complete riemannian manifold with finite volume. Then the set of unit vectors v for which $\underline{\psi}(v) = 0$ and the set of those for which $c_v(0)$ has no conjugate points along c_v differ by a set of measure zero.*

With regard to $\underline{\psi}$ and $\overline{\psi}$, it is natural to ask whether they are really different. To show that they are the same, it suffices to show that $\underline{\psi} \ge \overline{\psi}$. Along these lines, it is true that $(n-1)\underline{\psi} \ge \overline{\psi}$ almost everywhere for M a complete riemannian manifold of dimension n. This follows from

Proposition 3. *Let M be a complete riemannian manifold. Then the following is a well-defined element of* $[0,+\infty]$ *for almost all v in SM:*

$$\lim_{T\to\infty} \frac{1}{T} \begin{pmatrix} \textit{the number of points conjugate to} \\ c_v(0) \textit{ along } c_v|_{[0,T]}, \\ \textit{counted according to multiplicity} \end{pmatrix}.$$

4. Proofs of the propositions and Theorem 2

First, we give some notation:

$c_v\colon [0,\infty) \to M$ is the geodesic $c_v(t) = \exp(tv)$ for v a unit tangent vector of a riemannian manifold M.

Z denotes the set of unit vectors v for which $c_v(0)$ has no conjugate points along c_v.

Z' denotes the set of unit vectors v for which c_v has finitely many conjugate points along c_v.

Z'' denotes the set of unit vectors v for which $\overline{\psi}(v) = 0$.

Z''' denotes the set of unit vectors v for which $\underline{\psi}(v) = 0$.

We have $Z \subseteq Z' \subseteq Z'' \subseteq Z'''$. Note that Z', Z'', Z''' each are invariant with respect to the geodesic flow. μ will denote the Liouville measure on SM.

Proof of Proposition 2. We are required to show that Z and Z''' differ by a set of measure zero. It suffices to show that $\mu(SM-Z''') \geq \mu(SM-Z)$.

For each positive integer j, define $f_j\colon SM \to R$ by

$$f_j = \begin{cases} 1, & \text{if } c_v(0) \text{ has a conjugate point along } c_v|_{[0,j]}, \\ 0, & \text{otherwise.} \end{cases}$$

Also, define $f\colon SM \to R$ by

$$f = \begin{cases} 1, & \text{if } c_v(0) \text{ has a conjugate point along } c_v, \\ 0, & \text{otherwise.} \end{cases}$$

Then f_j approaches f from below, as j goes to infinity. It follows that

$$\lim_{j\to\infty} \int_{SM} f_j = \int_{SM} f.$$

By Lemma 2,

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{f_j(G_{jk}v)}{jN} \leq \underline{\psi}(v)$$

for all unit vectors v for which the above limit exists. For v in Z''', we then have

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{f_j(G_{jk}v)}{N} = 0.$$

We can now conclude that

$$\begin{aligned}\mu(SM-Z) &= \int_{SM} f = \lim_{j\to\infty}\int_{SM} f_j \\ &= \lim_{j\to\infty}\int_{SM}\lim_{N\to\infty}\sum_{k=0}^{N-1}\frac{f_j(G_{jk}v)}{N}\,d\mu(v) \\ &= \lim_{j\to\infty}\int_{SM-Z'''}\lim_{N\to\infty}\sum_{k=0}^{N-1}\frac{f_j(G_{jk}v)}{N}\,d\mu(v) \\ &\le \int_{SM-Z'''} 1 = \mu(SM-Z''').\end{aligned}$$

Proof of Theorem 2. Suppose that

$$\int_0^L \operatorname{Ric}(c'(t))\,dt \ge \pi(n-1)^{1/2}\sqrt{\max_{t\in[0,L]}(0,\ \operatorname{Ric}(c'(t)))}$$

and $\operatorname{Ric}(c')$ is not identically zero.

To show that $c(0)$ has a conjugate point along c, it suffices to find an admissible vector field $W\colon [0,L]\to TM$ on c, which is not identically zero and for which $I_c(W,W)\le 0$.

Letting $\beta = \max_{t\in[0,L]}\operatorname{Ric}(c'(t))$ we have that β is positive. Let $y = (\pi/2)\sqrt{(n-1)/\beta}$ so that $0 < y \le L/2$. Define $v\colon [0,L]\to R$ by

$$v(t) = \begin{cases} \sin(\pi t/2y), & \text{if } 0\le t\le y, \\ 1, & \text{if } y\le t\le L-y, \\ \sin(\pi(L-t)/2y), & \text{if } L-y\le t\le L.\end{cases}$$

Let E be a parallel unit vector field on c that is orthogonal to c, and let $V = vE$. Then

$$\begin{aligned}I(V,V) &= \int_0^L \left\langle \frac{DV}{dt}, \frac{DV}{dt}\right\rangle - \langle R(V,c'(t))c'(t), V\rangle\,dt \\ &= \int_0^y v'(t)^2 + (1-v(t)^2)\langle R(E,c'(t))c'(t),E\rangle\,dt \\ &\quad + \int_{L-y}^L v'(t)^2 + (1-v(t)^2)\langle R(E,c'(t))c'(t),E\rangle\,dt \\ &\quad - \int_0^L \langle R(E,c'(t))c'(t),E\rangle\,dt.\end{aligned}$$

Let $E_1, \cdots, E_{n-1}$ be mutually orthogonal parallel unit vector fields on c that are orthogonal to c. Let $V_i = vE_i$ for $i = 1$ to $n-1$. We then have

$$\begin{aligned}\sum_{i=1}^{n-1} I(V_i, V_i) &= \int_0^y (n-1)v'(t)^2 + (1 - v(t)^2)\mathrm{Ric}(c'(t))\,dt \\ &\quad + \int_{L-y}^{L} (n-1)v'(t)^2 + (1 - v(t)^2)\mathrm{Ric}(c'(t))\,dt \\ &\quad - \int_0^L \mathrm{Ric}(c'(t))\,dt \\ &\le \int_0^y (n-1)v'(t)^2 + (1 - v(t)^2)\beta\,dt \\ &\quad + \int_{L-y}^{L} (n-1)v'(t)^2 + (1 - v(t)^2)\beta\,dt - \pi(n-1)^{1/2}\sqrt{\beta} \\ &= 0.\end{aligned}$$

We now move onto the equality condition. Suppose the first conjugate point to $c(0)$ along c is $c(L)$. Since $\sum_{i=1}^{n-1} I(V_i, V_i) \le 0$ and $I(V_i, V_i) \ge 0$ for $i = 1$ to $n-1$, each V_i is a Jacobi field. This means that v is C^∞, so that $y = L/2$, giving us $v(t) = \sin(\pi t/2y)$. The fact that $V = vE$ is a Jacobi field now tells us that

$$\langle R(c'(t), E)E, c'(t)\rangle = (\pi/2y)^2$$

for all t in $[0, L]$. Using $y = L/2$ once more, we can conclude this proof with

$$(\pi/2y)^2 = \pi^2/L^2.$$

Proof of Proposition 1. By Theorem 2 and Lemma 2, if $c_v(0)$ has exactly N conjugate points along $c_v|_{[0,T]}$, then

$$\int_0^T \mathrm{Ric}(c_v'(t))\,dt \le (N+1)\pi(n-1)^{1/2}\sqrt{\beta}$$

so that

$$\frac{1}{\pi(n-1)^{1/2}\sqrt{\beta}}\frac{1}{T}\int_0^T \mathrm{Ric}(c_v'(t))\,dt \le \frac{(N+1)}{T}.$$

Proof of Proposition 3. For v a unit tangent vector, define

$$\underline{\phi}(v) = \liminf_{T\to\infty} \frac{1}{T}\begin{pmatrix}\text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \\ \text{counted according to multiplicity}\end{pmatrix},$$

$$\overline{\phi}(v) = \limsup_{T\to\infty} \frac{1}{T}\begin{pmatrix}\text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \\ \text{counted according to multiplicity}\end{pmatrix}.$$

It suffices to show that $\overline{\phi}(v) \leq \underline{\phi}(v)$ for almost all v.

For j a positive integer, define $g_j \colon SM \to R$ by

$$g_j(v) = \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,j]}, \\ \text{counted according to multiplicity} \end{pmatrix},$$

and $\hat{g}_j$ by

$$\hat{g}_j = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} g_j(G_{jk}v).$$

By Birkhoff's Ergodic Theorem, this is well defined almost everywhere.

Whenever $\hat{g}_j(v)$ exists, we have in consequence of Lemma 3,

$$\underline{\phi}(v) \geq \frac{1}{j}\hat{g}_j(v) - \frac{(n-1)}{j}, \qquad \overline{\phi}(v) \leq \frac{1}{j}\hat{g}_j(v) + \frac{(n-1)}{j}.$$

For almost all v, we then have

$$\overline{\phi}(v) \leq \underline{\phi}(v) + \frac{2(n-1)}{j}.$$

Letting j go to infinity, we are done.

5. Proofs of the remaining theorems

Proof of Theorem 5(1). By Proposition 2, it suffices to prove $\int_{Z'} \mathrm{Ric} \leq 0$.

By Birkhoff's Ergodic Theorem,

$$\int_{Z'} \mathrm{Ric} = \int_{Z'} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(G_t v)\, dt\, d\mu(v).$$

Using Ambrose's criterion for conjugate points and Lemma 2, we obtain that for v in Z',

$$\liminf_{T\to\infty} \int_0^T \mathrm{Ric}(c_v'(t))\, dt < +\infty$$

from which it follows that

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(c_v'(t))\, dt \leq 0.$$

In terms of the geodesic flow,

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(G_t v)\, dt \leq 0.$$

Combining this with

$$\int_{Z'} \mathrm{Ric} = \int_{Z'} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(G_t v)\, dt\, d\mu(v)$$

we are done.

Proof of Theorem 5(2). By Proposition 2, it suffices to prove

$$\int_{SM-Z'''} \mathrm{Ric} \le \pi(n-1)^{1/2}\sqrt{\sup(0, \mathrm{Ric})} \int_{SM} \underline{\psi}.$$

Let β be any positive upper bound to the Ricci curvature. Birkhoff's Ergodic Theorem and Proposition 1 now give

$$\begin{aligned}\int_{SM-Z'''} \mathrm{Ric} &\le \int_{SM-Z'''} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(G_t v)\, dt\, d\mu(v) \\ &= \int_{SM-Z'''} \liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Ric}(G_t v)\, dt\, d\mu(v) \\ &\le \int_{SM-Z'''} \pi\sqrt{\beta(n-1)}\underline{\psi}\, d\mu(v) = \pi\sqrt{\beta(n-1)} \int_{SM} \underline{\psi}.\end{aligned}$$

Proof of Theorem 1. By Theorem 5,

$$\int_{SM} \mathrm{Ric} \le \pi(n-1)^{1/2}\sqrt{\sup(0, \mathrm{Ric})} \int_{SM} \underline{\psi}.$$

Using

$$\int_{SM} \mathrm{Ric} = \frac{\mathrm{vol}(S^{n-1}, can)}{n} \int_M \mathrm{Scal}$$

we are done.

Proof of Theorem 4. Suppose that

$$\int_0^L F(t)\, dt \ge \pi\sqrt{\max_{t\in[0,L]}(0, F(t))}$$

and that F is not identically zero.

Following [6], to prove the existence of T in $(0, L]$ satisfying $z(T) = 0$, it suffices to find a continuous piecewise differentiable function $\phi\colon [0, L] \to R$ such that $\phi(0) = \phi(L) = 0$ (and ϕ not identically zero) for which

$$\int_0^L (\phi'(t))^2 - F(t)(\phi(t))^2\, dt \le 0.$$

Letting $\beta = \max_{t\in[0,L]} F(t)$ and $y = \pi/2\sqrt{\beta}$ so that $0 < y \le L/2$ we define $v\colon [0, L] \to R$ by

$$v(t) = \begin{cases} \sin(\pi t/2y), & \text{if } 0 \le t \le y, \\ 1, & \text{if } y \le t \le L - y, \\ \sin(\pi(L-t)/2y), & \text{if } L - y \le t \le L. \end{cases}$$

The argument can be completed by following the proof of Theorem 2.

References

[1] W. Ambrose, *A theorem of Myers*, Duke Math. J. **24** (1957) 345–348.

[2] A. Avez, *Variétés riemannienes sans points focaux*, C. R. Acad. Sci. Paris Ser. A-B **270** (1970) A188–A191.

[3] J. Cheeger & D. Ebin, *Comparison theorems in riemannian geometry*, North-Holland, Amsterdam, 1975.

[4] L. W. Green, *A theorem of E. Hopf*, Michigan Math. J. **5** (1958) 31–34.

[5] ——, *Auf Wiedersehensflächen*, Ann. of Math. **78** (1963) 289–299.

[6] W. Leighton, *An introduction to the theory of ordinary differential equations*, Wadsworth, Belmont, CA, 1976.

[7] S. B. Myers, *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8** (1941) 401–404.

[8] K. Petersen, *Ergodic theory*, Cambridge University Press, 1983.

University of Oklahoma