J. DIFFERENTIAL GEOMETRY
29 (1989) 361-372

RIGIDITY OF NONNEGATIVELY CURVED COMPACT QUATERNIONIC-KÄHLER MANIFOLDS

BENNETT CHOW & DEANE YANG

0. Introduction

We prove the following (see §1 for definitions).

Theorem 1. *Let M be a compact quaternionic-Kähler manifold with positive (nonnegative) quaternionic bisectional curvature. Then M is $\mathbb{H}P^n$ with the standard metric (a quaternionic symmetric space).*

This generalizes a result of M. Berger [3], who obtained the same conclusion assuming that M has positive sectional curvature. Also, since quaternionic-Kähler manifolds are Einstein (but not Kähler!), Theorem 1 is analogous to results of Goldberg-Kobayashi [6] and Mok-Zhong [14] which state that a compact Kähler-Einstein manifold with positive (nonnegative) bisectional curvature is isometric to $\mathbb{C}P^n$ (a hermitian symmetric space).

The idea of the proof is to study the twistor space $\mathfrak{Z}$, which is the space of all almost complex structures on M compatible with the quaternionic structure. On $\mathfrak{Z}$ there are a natural almost complex structure and hermitian metric. S. Salamon [16] showed that the complex structure on $\mathfrak{Z}$ is integrable, and if M has positive scalar curvature (suitably normalized), then the metric on $\mathfrak{Z}$ is Kähler-Einstein with positive scalar curvature. For example, if M is $\mathbb{H}P^n$, then $\mathfrak{Z}$ is $\mathbb{C}P^{2n+1}$ with the Fubini-Study metric.

Theorem 1 then follows from:

Theorem 2. *Let M be a quaternionic-Kähler manifold with positive scalar curvature.*

(a) *M has positive (nonnegative) quaternionic bisectional curvature if and only if $\mathfrak{Z}$ has positive (nonnegative) holomorphic bisectional curvature.*

(b) *M is a symmetric space if $\mathfrak{Z}$ is locally symmetric.*

Proof of Theorem 1. By Theorem 2, $\mathfrak{Z}$ is a Kähler-Einstein manifold with nonnegative holomorphic bisectional curvature. By the theorem of Mok-Zhong [14], $\mathfrak{Z}$ must be a symmetric space. Thus, so is M.

When the bisectional curvature of $\mathfrak{Z}$ is positive, the theorem of Goldberg-Kobayashi [6] implies that it is isometric to $\mathbb{C}P^{2n+1}$ with the standard metric.

Received June 18, 1986 and, in revised form, October 12, 1987.

By Wolf's classification of quaternionic symmetric spaces [19] and their twistor spaces, it follows that M is $\mathbb{H}P^n$ with the standard metric. q.e.d.

In §§1–2 we recall the definition of a quaternionic-Kähler manifold and the construction of the twistor space $\mathfrak{Z}$; in §§3–6 the method of moving frames on a Riemannian submersion and Alekseevksii's characterizations of the curvature of a quaternionic-Kähler manifold are used to compute the curvature of $\mathfrak{Z}$. Theorem 2 then follows easily. In §7 we apply Theorem 1 to the classification of Riemannian manifolds with nonnegative curvature operator.

We would like to thank the following people for the help they have given us: R. L. Bryant, H. D. Cao, R. S. Hamilton, G. Huisken, M. Micallef, P. M. Wong, and S. T. Yau. We thank the referee for pointing out errors and omissions in an earlier version of this paper.

1. Preliminaries

The setup here follows that of [16].

Let $\mathbb{H}$ be the quaternions and identify $\mathbb{R}^{4n} = \mathbb{H}^n$. Let $\mathbb{H}$ act on the right; this defines an antihomomorphism

$$\lambda\colon \{\text{unit quaternions}\} \to \mathrm{SO}(4n),$$

where $\mathrm{SO}(4n)$ acts on the left. Denote the image by $\mathrm{Sp}(1)$. Define

$$\mathrm{Sp}(n) = \{A \in \mathrm{SO}(4n) \mid AB = BA, B \in \mathrm{Sp}(1)\},$$

and $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ to be the product of the two groups in $\mathrm{SO}(4n)$. Abstractly,

$$\mathrm{Sp}(n)\,\mathrm{Sp}(1) = (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2.$$

We shall use the isomorphism

$$\mathbb{R}^{4n} \to \mathbb{H}^n,$$
$$(v^a, v^{n+a}, v^{2n+a}, v^{3n+a}) \mapsto (v^a + iv^{n+a} + jv^{2n+a} + kv^{3n+a}),$$

where $1 \le a \le n$. Using this, a matrix A in the Lie algebra $\mathrm{sp}(n)\,\mathrm{sp}(1)$ is of the form

$$A = \begin{bmatrix} A_0 & -A_1 - a_1 & -A_2 - a_2 & -A_3 - a_3 \\ A_1 + a_1 & A_0 & -A_3 + a_3 & A_2 - a_2 \\ A_2 + a_2 & A_3 - a_3 & A_0 & -A_1 + a_1 \\ A_3 + a_3 & -A_2 + a_2 & A_1 - a_1 & A_0 \end{bmatrix},$$

where $A_0 = -{}^tA_0, A_\mu ={}^t A_\mu$, $1 \le \mu \le 4$, are $n \times n$ matrices, and a_1, a_2, a_3 are scalar multiples of the identity matrix.

Definition. Let $n \ge 2$. A smooth, oriented $4n$-dimensional riemannian manifold M is *quaternionic-Kähler* if its holonomy group lies in $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \subset \mathrm{SO}(4n)$.

Observe that the definition is meaningless when $n = 1$ because $\mathrm{Sp}(1)\,\mathrm{Sp}(1) = \mathrm{SO}(4n)$.

M is quaternionic-Kähler if and only if its bundle $\mathcal{F}M$ of oriented orthonormal frames can be reduced to a principal $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-bundle $\mathcal{P}$ so that the Levi-Civita connection on FM drops down to $\mathcal{P}$. In other words, given any section (e_A) of $\mathcal{P}$, the associated $\mathrm{so}(4n)$-valued connection 1-form $[\omega^A_B]$ is really $\mathrm{sp}(n)\,\mathrm{sp}(1)$-valued.

Recall that on an almost complex manifold, there is a natural action of $\mathbb{C}$ on the tangent space. A quaternionic(-Kähler) manifold does not have a canonical $\mathbb{H}$-action. Given a frame $(e_A) \in \mathcal{P}_x$, $x \in M$, we can identify

$$\begin{aligned} T_xM &= \mathbb{H}^n, \\ v^A e_A &\mapsto [v^a + iv^{n+a} + jv^{2n+a} + kv^{3n+a}], \end{aligned} \tag{1.2}$$

and have $\mathbb{H}$ act on the right. A 4-plane in T_xM is *quaternionic* if it corresponds to a quaternionic line in $\mathbb{H}^n$. This does not depend on the frame (e_A) chosen.

Recall that the Riemann curvature on a Riemannian manifold M defines a quadrilinear map

$$R\colon T_*M \times T_*M \times T_*M \times T_*M \to \mathbb{R}.$$

Given a 2-plane σ in the tangent space, the *sectional curvature of* σ is defined to be $R(X,Y,X,Y)$, where X, Y form an orthonormal basis of σ. If M has an almost complex structure J, then given two complex lines σ and σ' in the tangent space, the *holomorphic bisectional curvature of* σ *and* σ' is defined to be

$$K_{\mathbb{C}}(\sigma,\sigma') = R(X,JX,Y,JY),$$

where $|X| = |Y| = 1$; X and JX span σ; Y and JY span σ'.

Now assume M is quaternionic-Kähler. Given $x \in M$, let π and π' be quaternionic lines in T_xM. Fix a unit tangent vector $X \in \pi$ and a frame $(e_A) \in \mathcal{P}$, and identify $T_xM \cong \mathbb{H}^n$ as in (2.1). Let $I, J, K\colon T_xM \to T_xM$ denote the right actions of $i, j, k \in \mathbb{H}$. The *quaternionic bisectional curvature of* π *and* π' is defined to be

$$K_{\mathbb{H}}(\pi,\pi') = \inf_{\substack{X\in\pi,\, Y\in\pi',\, |X|=|Y|=1 \\ (e_A)\in\mathcal{P}_x}} [R(X,JX,Y,JY) + 2(|X\cdot IY|^2 + |X\cdot KY|^2)].$$

This definition is, of course, rigged to make Theorem 2 work. On the other hand, it is clear that positive quaternionic bisectional curvature is a somewhat weaker assumption than positive sectional curvature.

2. The twistor space $\mathfrak{Z}$

We can also identify

$$\mathbb{H}^n \cong \mathbb{C}^{2n},$$
$$[v^a + iv^{n+a} + jv^{2n+a} + kv^{3n+a}] \mapsto [v^a + jv^{2n+a}, v^{n+a} + jv^{3n+a}],$$

where the action of $\mathbb{C}$ corresponds to multiplication on the right by elements of the form $u+jv \in \mathbb{H}$. Therefore, if we fix a frame $(e_A) \in \mathcal{P}_x$, the isomorphism (1.2) induces an almost complex structure

$$J\colon T_xM \to T_xM,$$

corresponding to multiplication on the right by j. Any almost complex structure on M which can be obtained in this way is said to be compatible with the quaternionic structure on M. It depends, of course, on the isomorphism (1.2) and therefore the frame $(e_A) \in \mathcal{P}$.

Two frames (e^A) and $(R_A^B e_B) \in \mathcal{P}_x$ determine the same almost complex structure if and only if the matrix $[R_B^A] \in \mathrm{U}(2n) \cap \mathrm{Sp}(n)\,\mathrm{Sp}(1)$. Therefore, the space of all possible compatible almost complex structures is

$$\mathfrak{Z} = \mathcal{P}/(\mathrm{U}(2n) \cap \mathrm{Sp}(n)\,\mathrm{Sp}(1)).$$

The space $\mathfrak{Z}$ is a fiber bundle over M with fiber

$$\begin{aligned}\mathfrak{Z}_x &= \mathcal{P}_x/(\mathrm{U}(2n) \cap \mathrm{Sp}(n)\,\mathrm{Sp}(1)) \cong \mathrm{Sp}(n)\,\mathrm{Sp}(1)/(\mathrm{U}(2n) \cap \mathrm{Sp}(n)\,\mathrm{Sp}(1)) \\ &= \mathrm{Sp}(1)/\,\mathrm{U}(1) = \mathbb{C}P^1,\end{aligned}$$

and is called the *twistor space* of M.

The twistor space $\mathfrak{Z}$ has a natural almost complex structure J and compatible Riemannian metric (i.e., $X \cdot Y = JX \cdot JY$), which will be described shortly. The following was proved by Salamon [16].

Theorem 3. *The natural almost complex structure of a twistor space $\mathfrak{Z}$ of a quaternionic-Kähler manifold M is integrable. Moreover, if the manifold M has positive scalar curvature, then the metric in $\mathfrak{Z}$ is Kähler-Einstein.*

We will use moving frames to prove this theorem and to find the curvature of $\mathfrak{Z}$ in terms of the curvature of M.

3. Moving frames on $\mathfrak{Z}$

Recall that on $\mathcal{F}M$ there is a basis of canonical 1-forms ω^A, $\omega^A{}_B = -\omega^B{}_A$, $1 \le A$, $B \le 4n$, defined as follows: $\omega^1, \cdots, \omega^{4n}$ at $(x, e_A) \in \mathcal{F}M$ is the dual coframe of (e_A) pulled back to $\mathcal{F}M$. The matrix $[\omega^A{}_B]$ is the unique so$(4n)$-valued 1-form on $\mathcal{F}M$ which is invariant under the action of SO$(4n)$ and such that $\nabla e_A = e_B\omega^B{}_A$ for any section (e_A). The 1-forms satisfy the following structure equations:

$$d\omega^A + \omega^A{}_B \wedge \omega^B = 0; \qquad d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = \Omega^A{}_B,$$

where $[\Omega^A{}_B]$ is a skew-symmetric matrix of 2-forms giving, at $(x, e_A) \in \mathcal{F}M$, the curvature 2-form of M at x with respect to the orthonormal frame (e_A). Expanding the curvature forms in terms of the coframe, we have

$$\Omega^A{}_B = \tfrac{1}{2} R_{ABCD}\omega^C \wedge \omega^D.$$

The symmetries of the curvature tensor imply that (see [11])

$$\tfrac{1}{4} R_{ABCD}(\omega^A \wedge \omega^B) \circ (\omega^C \wedge \omega^D) \in \mathrm{Sym}^2\left(\bigwedge^2 T_x^* M\right) = \mathrm{Sym}^2(\mathrm{so}(4n)).$$

If M is quaternionic-Kähler, the canonical 1-forms can be restricted to $\mathcal{P}$ so that $[\omega^A{}_B]$ is an $\mathrm{sp}(n)\,\mathrm{sp}(1)$-valued 1-form, and $[\Omega^A{}_B]$ is an $\mathrm{sp}(n)\,\mathrm{sp}(1)$-valued 2-form. In particular,

$$[R_{ABCD}] \in \mathrm{Sym}^2(\mathrm{sp}(n)\,\mathrm{sp}(1)).$$

It will be convenient to use the following vector- and matrix-valued notation. (Superscripts will usually denote row indices and subscripts column indices. Therefore, vectors are to be viewed as column vectors.)

$$X = [X^\mu]; \quad X^\mu = [\omega^{\mu n + a}], \qquad 0 \le \mu \le 3,\ 1 \le a \le n,$$

$$\Gamma = [\omega^A_B] = \begin{bmatrix} \Gamma_0 & -\Gamma_1 - \alpha_1 & -\Gamma_2 - \alpha_2 & -\Gamma_3 - \alpha_3 \\ \Gamma_1 + \alpha_1 & \Gamma_0 & -\Gamma_3 + \alpha_3 & \Gamma_2 - \alpha_2 \\ \Gamma_2 + \alpha_2 & \Gamma_3 - \alpha_3 & \Gamma_0 & -\Gamma_1 + \alpha_1 \\ \Gamma_3 + \alpha_3 & -\Gamma_2 + \alpha_2 & \Gamma_1 - \alpha_1 & \Gamma_0 \end{bmatrix}, \tag{3.1}$$

where $\Gamma_0 = -{}^t\Gamma_0$, $\Gamma_\mu = {}^t\Gamma_\mu$, $1 \le \mu \le 3$, and $\alpha_1, \alpha_2, \alpha_3$ are scalar multiples of the identity matrix. The structure equations then take the form

$$dX + \Gamma \wedge X = 0; \qquad d\Gamma + \Gamma \wedge \Gamma = \Omega, \tag{3.2}$$

where

$$\Omega = \begin{bmatrix} \Omega^0{}_0 & \Omega^0{}_1 & \Omega^0{}_2 & \Omega^0{}_3 \\ \Omega^1{}_0 & \Omega^0{}_0 & \Omega^1{}_2 & \Omega^1{}_3 \\ \Omega^2{}_0 & \Omega^2{}_1 & \Omega^0{}_0 & \Omega^2{}_3 \\ \Omega^3{}_0 & \Omega^3{}_1 & \Omega^3{}_2 & \Omega^0{}_0 \end{bmatrix},$$

$\Omega^0{}_0 = -{}^t\Omega^0{}_0$, $\Omega^\mu{}_\nu = {}^t\Omega^\mu{}_\nu = -\Omega^\nu{}_\mu$, $0 \le \mu < \nu \le 3$, and, if (μ, ν, η) is a cyclic permutation of (1,2,3), then $\Omega^\mu{}_0 + \Omega^\nu{}_\eta$ is a multiple of the identity matrix.

The canonical 1-forms on $\mathcal{F}M$ and $\mathcal{P}$ define natural Riemannian metrics such that the 1-forms are orthonormal. These metrics are such that the projections $\mathcal{F}M \to M$ and $\mathcal{P} \to M$ are Riemannian submersions and such that the fibers have the standard bi-invariant metrics defined on the corresponding Lie groups. The twistor space $\mathfrak{Z}$ then has a natural metric such that the projections $\mathfrak{Z} \to M$ and $\mathcal{P} \to \mathfrak{Z}$ are both Riemannian submersions.

An orthonormal coframe on $\mathfrak{Z}$ can be obtained by choosing a local section of $\mathcal{P}$ over $\mathfrak{Z}$ and pulling back the canonical 1-forms ω^A, $1 \le A \le 4n$, α_1, α_3, where $\alpha_1, \alpha_2, \alpha_3$ are as defined in (3.1).

The almost complex structure on $\mathfrak{Z}$ can be specified by designating the following as a basis of $(1,0)$-forms:

$$\begin{aligned} \varsigma^0 &= \alpha_1 + i\alpha_3; \\ \varsigma^a &= \omega^a + i\omega^{2n+a}; \\ \varsigma^{n+a} &= \omega^{n+a} + i\omega^{3n+a}; \qquad 1 \le a \le n. \end{aligned}$$

Our task here is to compute the structure equations on $\mathfrak{Z}$, i.e., the exterior derivatives of ς^p, $1 \le p \le 2n$. We shall confirm Theorem 3, find the $\mathrm{u}(2n)$-valued connection $(1,0)$-form of the Kähler metric on $\mathfrak{Z}$, and finally compute the curvature of $\mathfrak{Z}$.

Again, it is convenient to introduce vector-valued notation:

$$Z^1 = [\varsigma^a] = X^0 + iX^2; \qquad Z^2 = [\varsigma^{n+a}] = X^1 + iX^3.$$

Computing dZ^1 and dZ^2 is straightforward using (3.2), and we obtain:

$$\begin{aligned} dZ^1 + \overline{Z}^2 \wedge \varsigma^0 + [\Gamma_0 + i(\Gamma_2 + \alpha_2)] \wedge Z^1 + (-\Gamma_1 + i\Gamma_3) \wedge Z^2 &= 0; \\ dZ^2 - \overline{Z}^1 \wedge \varsigma^0 + (\Gamma_1 + i\Gamma_3) \wedge Z^1 + [\Gamma_0 + i(\Gamma_2 - \alpha_2)] \wedge Z^2 &= 0. \end{aligned}$$

Computing $d\varsigma^0$ is somewhat harder, because it involves the curvature of M. Before carrying this out, it is useful to recall some facts about the curvature of M.

4. The curvature of a quaternionic-Kähler manifold

The following is due to Alekseevskii [1], and a proof is given in [16].

Theorem 4. *A quaternionic-Kähler manifold is Einstein, and its Riemann curvature is of the form*

$$R = (S/\widetilde{S})\widetilde{R} + R';$$

where $\widetilde{R}$ is the curvature of $\mathbb{H}P^n$, $\widetilde{S}$ is the scalar curvature of $\mathbb{H}P^n$, S is the (constant) scalar curvature of M, and

$$R' \in \mathrm{Sym}^2(\mathrm{sp}(n)) \subset \mathrm{Sym}^2\left(\bigwedge^2 T^*M\right).$$

We shall denote the corresponding splitting of the curvature 2-form on M by

$$\Omega = (S/\widetilde{S})\widetilde{\Omega} + \Omega'.$$

Notice that Ω' is an sp(n)-valued 2-form and therefore has the form of (1.1) with $a_1 = a_2 = a_3 = 0$.

We shall also need the explicit formula of $\widetilde{\Omega}$. To compute the curvature of $\mathbf{H}P^n$, observe that the principal bundle $\mathcal{P}$ for $\mathbf{H}P^n$ is $\mathrm{Sp}(n+1)$,

$$\mathbf{H}P^n = \mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\,\mathrm{Sp}(1)),$$

and the structure equations are given by the Maurer-Cartan equations of $\mathrm{Sp}(n+1)$. A straightforward calculation shows that the curvature of $\mathbf{H}P^n$ is given by the following:

$$\widetilde{\Omega}^{\mu}{}_{\mu} = \sum_{\nu=0}^{3} X^{\nu} \wedge X^{\nu}, \qquad 0 \le \mu \le 3.$$

(As always, let (μ, ν, η) be any cyclic permutation of $(1, 2, 3)$.)

$$\begin{aligned} \widetilde{\Omega}^{0}{}_{\mu} &= X^0 \wedge {}^tX^{\mu} - X^{\mu} \wedge {}^tX^0 + X^{\nu} \wedge {}^tX^{\eta} - X^{\eta} \wedge {}^tX^{\nu} \\ &\quad + 2({}^tX^0 \wedge X^{\mu} - {}^tX^{\nu} \wedge X^{\eta}); \\ \Omega^{\nu}{}_{\eta} &= X^0 \wedge {}^tX^{\mu} - X^{\mu} \wedge {}^tX^0 + X^{\nu} \wedge {}^tX^{\eta} - X^{\eta} \wedge {}^tX^{\nu} \\ &\quad - 2({}^tX^0 \wedge X^{\mu} - {}^tX^{\nu} \wedge X^{\eta}). \end{aligned}$$

5. The connection on $\mathfrak{Z}$

The exterior derivatives of α_1, α_2, and α_3 can now be computed. As before, let (μ, ν, η) be a cyclic permutation of $(1, 2, 3)$. The structure equations on M tell us that

$$\begin{aligned} \Omega^{\mu}{}_{0} &= d(\Gamma_{\mu} + \alpha_{\mu}) + (\Gamma_{\mu} + \alpha_{\mu}) \wedge \Gamma_0 + \Gamma_0 \wedge (\Gamma_{\mu} + \alpha_{\mu}) \\ &\quad + (\Gamma_{\nu} - \alpha_{\nu}) \wedge (\Gamma_{\eta} + \alpha_{\eta}) - (\Gamma_{\nu} - \alpha_{\nu}) \wedge (\Gamma_{\eta} + \alpha_{\eta}); \\ \Omega^{\eta}{}_{\nu} &= d(-\Gamma_{\mu} + \alpha_{\mu}) + (-\Gamma_{\mu} + \alpha_{\mu}) \wedge \Gamma_0 + \Gamma_0 \wedge (-\Gamma_{\mu} + \alpha_{\mu}) \\ &\quad - (\Gamma_{\eta} + \alpha_{\eta}) \wedge (\Gamma_{\nu} + \alpha_{\nu}) + (\Gamma_{\nu} - \alpha_{\nu}) \wedge (\Gamma_{\eta} - \alpha_{\eta}). \end{aligned}$$

Adding these two equations together, we get

$$\Omega^{\mu}{}_{0} + \Omega^{\eta}{}_{\nu} = 2d\alpha_{\mu} - 4\alpha_{\eta} \wedge \alpha_{\nu}.$$

Now observe that

$$\begin{aligned} \Omega^{\mu}{}_{0} + \Omega^{\eta}{}_{\nu} &= \widetilde{\Omega}^{\mu}{}_{0} + \widetilde{\Omega}^{\eta}{}_{\nu} + \Omega'^{\mu}{}_{0} + \Omega'^{\eta}{}_{\nu} = \widetilde{\Omega}^{\mu}{}_{0} + \widetilde{\Omega}^{\eta}{}_{\nu} \\ &= 4(S/\widetilde{S})(-{}^tX^0 \wedge X^{\mu} + {}^tX^{\eta} \wedge X^{\nu}). \end{aligned}$$

Therefore,

$$d\alpha_\mu - 2\alpha_\eta \wedge \alpha_\nu + 2(S/\widetilde{S})(X^0 \wedge {}^tX^\mu + X^\eta \wedge {}^tX^\nu) = 0. \tag{5.1}$$

Using (5.1) we find that ς^0 satisfies the following equation:

$$d\varsigma^0 + 2i\alpha_2 \wedge \varsigma^0 + 2(S/\widetilde{S})^t Z^1 \wedge Z^2 = 0.$$

The structure equations for ς^0, Z^1, Z^2 are therefore the following:

$$d\begin{bmatrix} \varsigma^0 \\ Z^1 \\ Z^2 \end{bmatrix} = -\begin{bmatrix} 2i\alpha_2 & -(S/\widetilde{S})^t Z^2 & (S/\widetilde{S})^t Z^1 \\ \overline{Z} & \Gamma_0 + i(\Gamma_2 + \alpha_2) & -\Gamma_1 + i\Gamma_3 \\ -\overline{Z}^1 & \Gamma_1 + i\Gamma_3 & \Gamma_0 - i(\Gamma_2 - \alpha_2) \end{bmatrix} \wedge \begin{bmatrix} \varsigma^0 \\ Z^1 \\ Z^2 \end{bmatrix}.$$

Since there is no $(0,2)$ term on the right-hand side, the almost complex structure defined by specifying ς^0, Z^1, Z^2 as a basis of (1,0) form is integrable. On the other hand, these (1,0)-forms form a unitary coframe for a Kähler metric if and only if the connection matrix is skew-hermitian. Since the scalar curvature S of M is a constant and assumed to be positive, the matrix can always be made skew-hermitian by scaling either the metric of the fiber of $\mathfrak{Z}$ or the metric of M by a constant. It is more convenient for us to rescale the metric of M so that the scalar curvature of M is equal to $\widetilde{S}$.

6. The curvature of the twistor space $\mathfrak{Z}$

The curvature of $\mathfrak{Z}$ can now be computed from the connection forms, using the structure equations of M and the exterior derivatives of α_1, α_2, and α_3. We omit the details of the straightforward calculation and give the answer. The curvature of $\mathfrak{Z}$ is given by the following skew-hermitian matrix of 2-forms, which we shall denote $\mathbf{\Omega}$:

$$\begin{bmatrix} \begin{matrix} 2\varsigma^0 \wedge \bar\varsigma^0 + {}^tZ^1 \wedge \overline{Z}^1 \\ + {}^tZ^2 \wedge \overline{Z}^2 \end{matrix} & \varsigma^0 \wedge \overline{Z}^1 & \varsigma^0 \wedge \overline{Z}^2 \\ Z^1 \wedge \bar\varsigma^0 & \begin{matrix} \Omega^0{}_0 + i\Omega^2{}_0 - \overline{Z}^2 \wedge {}^t Z^2 \\ + \varsigma^0 \wedge \bar\varsigma^0 \end{matrix} & \begin{matrix} -\frac{1}{2}[\Omega^1{}_0 + \Omega^3{}_2 - i(\Omega^3{}_0 + \Omega^2{}_1)] \\ + \overline{Z}^2 \wedge {}^tZ^1 \end{matrix} \\ Z^2 \wedge \bar\varsigma^0 & \begin{matrix} \frac{1}{2}[\Omega^1{}_0 + \Omega^3{}_2 + i(\Omega^3{}_0 + \Omega^2{}_1)] \\ + \overline{Z}^1 \wedge {}^tZ^2 \end{matrix} & \begin{matrix} \Omega^0{}_0 + i\Omega^3{}_1 - \overline{Z}^1 \wedge {}^tZ^1 \\ + \varsigma^0 \wedge \bar\varsigma^0 \end{matrix} \end{bmatrix}$$

Observe that

$$\begin{aligned} \operatorname{Ric}(\mathfrak{Z}) &= \operatorname{tr}\mathbf{\Omega} = 2[(n+1)\varsigma^0 \wedge \bar\varsigma^0 + {}^tZ^1 \wedge \overline{Z}^1 + {}^tZ^2 \wedge \overline{Z}^2] + i(\operatorname{tr}\Omega^2{}_0 + \Omega^3{}_1) \\ &= 2(n+1)(\varsigma^0 \wedge \bar\varsigma^0 + {}^tZ^1 \wedge \overline{Z}^1 + {}^tZ^2 \wedge \overline{Z}^2), \end{aligned}$$

proving that $\mathfrak{Z}$ is Kähler-Einstein.

Let ε_0, ε_a, ε_{n+a}, $1 \le a \le n$, be the holomorphic tangent vectors on $\mathfrak{Z}$ which form the dual basis to ς^0, Z^1, Z^2. Denote $E_1 = [\varepsilon_a]$, $E_2 = [\varepsilon_{n+a}]$. Any holomorphic tangent vector V is of the form

$$V = v^0\varepsilon_0 + V^1E_1 + V^2E_2,$$

where $V^1 = [v^a]$ and $V^2 = [v^{n+a}]$.

Let σ and σ' be complex lines in $T_*\mathfrak{Z}$ which are spanned by unit holomorphic tangent vectors V and W, respectively. Let π and π' be the quaternionic lines which contain the projections of σ and σ' in T_*M. Then the holomorphic bisectional curvature of σ and σ' is

$$\begin{aligned} K_{\mathbf{C}}(\sigma,\sigma') &= \tfrac{1}{2}{}^t\overline{W}\Omega(V,\overline{V})W \\ &= \tfrac{1}{2}[|v^0|^2|W|^2 + |w^0|^2|V|^2 + v^0\bar{w}^0({}^t\overline{V}^1W^1 +{}^t \overline{V}^2W^2) \\ &\qquad + \bar{v}^0w^0({}^tV^1\overline{W}^1 +{}^t V^2\overline{W}^2) + |{}^tV^2W^1 -{}^t V^1W^2|^2 \\ &\qquad\qquad +{}^t \pi_*\overline{W}\Omega(\pi_*V,\pi_*\overline{V})\pi u_*W] \\ &\geq \tfrac{1}{2}[|v^0|^2|W|^2 + |w^0|^2|V|^2 - 2|v^0||w^0||V||W|] + K_{\mathbf{H}}(\pi,\pi') \\ &\geq K_{\mathbf{H}}(\pi,\pi'), \end{aligned}$$

with equality holding if and only if V and W are horizontal, i.e., if and only if $v^0 = w^0 = 0$, and V, W, J chosen so that the infinum is achieved in the definition of quaternionic bisectional curvature. It follows that the holomorphic bisectional curvature of $\mathfrak{Z}$ is positive (nonnegative) if and only if the quaternionic bisectional curvature of M is positive (nonnegative).

Observe that the inequality is somewhat surprising since it goes in the opposite direction of the O'Neill inequality [5, p. 66] which states that for a Riemannian submersion the sectional curvature of the base manifold is greater than or equal to that of the total space. There is no contradiction, however, because the inequality above applies only to the sectional curvature of a holomorphic 2-plane (by setting $W = V$) and equality holds in that case.

Also, from the explicit description of the connection and curvature of $\mathfrak{Z}$, it is a simple—but tedious—matter to check that if the covariant derivative of the curvature of $\mathfrak{Z}$ vanishes, then so does that of M. Moreover, $\mathfrak{Z}$ is a Kähler manifold with positive Ricci curvature and by a theorem of Kobayashi [10], it is simply connected. Therefore, if $\mathfrak{Z}$ is locally symmetric, both $\mathfrak{Z}$ and M are symmetric spaces.

This completes the proof of Theorem 2.

7. Compact manifolds with nonnegative curvature operator

First, recall that the Riemann curvature tensor of a Riemannian manifold is a section of $\mathrm{Sym}^2(\bigwedge^2 T_*M)$ and therefore defines a bilinear form on $\bigwedge^2 T_*M$. We say that the curvature operator is positive (nonnegative) if the bilinear form is positive (semi)definite.

Hamilton has conjectured that any compact n-manifold with positive curvature operator is the quotient of the n-sphere by a discrete subgroup of

SO$(n+1)$. He proved this in dimensions 3 and 4, [8], [9]. More recently, Micallef-Moore [12] proved that a simply connected compact Riemannian manifold with positive curvature operator is homeomorphic to a sphere.

Theorem 1, along with Berger's classification of holonomy groups, the Mori-Siu-Yau solution to the Frankel conjecture, and a theorem of R. S. Hamilton, can be used to extend these results to classify manifolds with nonnegative curvature operator. A similar classification for compact Kähler manifolds with nonnegative curvature operator has been obtained by Cao-Chow [4]. This case was further generalized by Mok [13], who proved that any compact Kähler manifold with nonnegative holomorphic bisectional curvature is biholomorphic to a symmetric space.

We observe the following:

Theorem 5. *Let (M, g) be a compact Riemannian manifold with nonnegative curvature operator, and $(\widetilde{M}, \tilde{g})$ its universal cover. Then*

$$(\widetilde{M}, \tilde{g}) = (\mathbb{R}^p, g_{\text{flat}}) \times (M_1, g_1) \times \cdots \times (M_k, g_k),$$

where (M_i, g_i), $i = 1, \cdots, k$, is one of the following:

(1) *a compact symmetric space,*

(2) *a Riemannian manifold with positive curvature operator homeomorphic to a sphere,*

(3) *a Kähler manifold biholomorphic to a complex projective space.*

Proof. Let $H \in \mathrm{SO}(N)$ be the holonomy of $\widetilde{M}$. By a theorem of deRham [11, p. 1897], if H can be written as a product of smaller subgroups of SO(N), then $\widetilde{M}$ splits into a corresponding Riemannian product of manifolds, with each factor having irreducible holonomy. If the holonomy of a factor is trivial, then it must be Euclidean space with the standard flat metric. Otherwise, it is a nonflat Riemannian manifold with nonnegative curvature operator. In particular, its Ricci curvature is bounded from below by a positive constant, and by Myers' theorem, it must be compact. The theorem then follows from

Theorem 5′. *Let M be a simply connected, compact manifold with irreducible holonomy and nonnegative curvature operator. Then one of the following must hold:*

(1) *M is (isometric to) a symmetric space.*

(2) *M has positive curvature operator and is homeomorphic to a sphere.*

(3) *M is a Kähler manifold biholomorphic to complex projective space.*

Proof. Recall the following theorem of Hamilton [9]:

Theorem. *Let M be a compact Riemannian manifold with nonnegative curvature operator. Let $g(t)$ be a solution to*

$$\partial_t g(t) = -2\,\mathrm{Ric}(g(t)), \qquad g(0) = g_0,$$

where g_0 is the original metric on M. Then the curvature operator of $g(t)$ is also nonnegative, and for some $t > 0$ the image of the curvature operator in $\bigwedge^2 T_*M$ is invariant under parallel translation and, in particular, must be isomorphic to the Lie algebra of the holonomy of M.

Assume that M is not isometric to a symmetric space. Use the metric $g(t)$ described in Hamilton's theorem, instead of g_0. Then M. Berger [2] has classified all possible irreducible holonomy groups. In particular, if M is not symmetric and has positive Ricci curvature—as a nonflat manifold with nonnegative curvature operator must—the holonomy H is either equal to $\mathrm{SO}(N)$, $\mathrm{U}(N/2)$, or $\mathrm{Sp}(N/4)\,\mathrm{Sp}(1)$. Moreover, by Hamilton's theorem, the curvature operator must be positive when restricted to the Lie algebra of H sitting inside $\bigwedge^2 T_*M = \mathrm{so}(N)$.

If M has holonomy equal to $\mathrm{SO}(N)$, then the curvature operator must be positive. By the theorem of Micallef-Moore [12], M must be homeomorphic to a sphere.

If M has holonomy equal to $\mathrm{U}(N/2)$, then M is a Kähler manifold with positive holomorphic bisectional curvature and, by the Mori-Siu-Yau theorem [5], [6], is biholomorphic to a complex projective space.

If M has holonomy equal to $\mathrm{Sp}(N/4)\,\mathrm{Sp}(1)$, then it is a quaternionic-Kähler manifold with positive quaternionic bisectional curvature. By Theorem 1, it must be symmetric, contradicting our assumption.

Remark. G. Huisken and M. Micallef kindly pointed out to us that Theorems 5 and 5′ could also be proved using a result of Tachibana [18], which states that an Einstein metric with nonnegative curvature operator is locally symmetric.

Bibliography

[1] D. V. Alekseevskii, *Riemannian spaces with exceptional holonomy groups*, Funct. Anal. Appl. **2** (1968) 97–105.

[2] M. Berger, *Sur les groupes d'holonomie homogene des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France **83** (1955) 279–300.

[3] ——, *Trois remarques sur les variétés riemanniennes à courbure positive*, C. R. Acad. Sci. Paris. **263** (1966) 76–78.

[4] H. D. Cao & B. Chow, *Compact Kähler manifolds with nonnegative curvature operator*, Invent. Math. **83** (1985) 553–556.

[5] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, American-Elsevier, New York, 1975.

[6] S. I. Goldberg & S. Kobayashi, *Holomorphic bisectional curvature*, J. Differential Geometry **1** (1967) 225–233.

[7] A. Gray, *Compact Kähler manifolds with nonnegative sectional curvature*, Invent. Math. **41** (1977) 33–43.

[8] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) 255–306.

[9] ——, *Four-manifolds with positive curvature operator*, J. Differential Geometry **24** (1986) 153–179.

[10] S. Kobayashi, *On compact Kähler manifolds with positive Ricci tensor*, Ann. of Math. (2) **74** (1961) 571–574.

[11] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.

[12] M. Micallef & J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, preprint.

[13] N. Mok, *The uniformisation theorem for compact Kaehler manifolds of nonnegative holomorphic bisectional curvature*, preprint.

[14] N. Mok & J. Q. Zhong, *Curvature characterization of compact Hermitian symmetric spaces*, J. Differential Geometry **23** (1986) 15–68.

[15] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979) 593–606.

[16] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982) 143–171.

[17] Y. T. Siu & S. T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980) 189–204.

[18] S. Tachibana, *A theorem on Riemannian manifolds of positive curvature operator*, Proc. Japan Acad. Ser. A. Math. Sci. **50** (1974) 301–302.

[19] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965) 1033–1047.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RICE UNIVERSITY