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# INTRINSIC CR NORMAL COORDINATES AND THE CR YAMABE PROBLEM

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## 1. Introduction

There is a deep analogy between the geometry of strictly pseudoconvex CR manifolds and that of conformal Riemannian manifolds. A CR manifold carries a natural hermitian metric on its holomorphic tangent bundle — the *Levi form* — which, like the metric on a conformal manifold, is determined only up to multiplication by a smooth function. The multiple is fixed by choosing a *contact form* — a real one-form annihilating the holomorphic tangent bundle. A CR manifold together with a choice of contact form is called a *pseudohermitian manifold.*

The simplest scalar invariant of a pseudohermitian manifold is the *pseudohermitian scalar curvature*, which we denote $R$, defined independently by S. Webster [15] and N. Tanaka [14]. The *CR Yamabe problem* is: *Given a compact, strictly pseudoconvex CR manifold, find a choice of contact form for which the pseudohermitian scalar curvature is constant.* In [6]–[8] we posed this problem and gave a sufficient condition for its solvability. The purpose of this paper is to show that "most" compact strictly pseudoconvex CR manifolds satisfy the sufficient condition, so that the CR Yamabe problem can almost always be solved. The precise statement of our result is Theorem A below.

Solutions to the CR Yamabe problem on a $2n+1$-dimensional CR manifold $M$ are critical points of the functional

$$\mathscr{Y}_M(\theta) = \frac{\int_M R\theta \wedge d\theta^n}{\left(\int_M \theta \wedge d\theta^n\right)^{2/p}}, \qquad p = 2 + \frac{2}{n}, \tag{1.1}$$

over the set of contact forms $\theta$ associated to the CR structure of $M$. In [7] we defined an invariant

$$\lambda(M) = \inf_\theta \mathscr{Y}_M(\theta),$$

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and showed that if $M$ is compact and strictly pseudoconvex, the CR Yamabe problem has a solution provided $\lambda(M) < \Lambda \equiv \lambda(S^{2n+1})$, where $S^{2n+1}$ is the sphere in $\mathbb{C}^{n+1}$ with its standard CR structure. Then in [8] we showed that the critical constant $\Lambda$ is realized by the "standard" contact form $\hat{\theta} = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ on $S^{2n+1}$, and thus $\Lambda = \mathscr{Y}_{S^{2n+1}}(\hat{\theta}) = pn^2\pi = 2\pi n(n+1)$.

The main result of this paper is

**Theorem A.** *Suppose $M$ is a compact, strictly pseudoconvex, $2n+1$-dimensional CR manifold. If $n \geq 2$ and $M$ is not locally CR equivalent to $S^{2n+1}$, then $\lambda(M) < \Lambda$, and thus the CR Yamabe problem can be solved on $M$.*

This is analogous to the result of T. Aubin [1] for the Riemannian version of the Yamabe problem: Every compact Riemannian manifold of dimension $\geq 6$ which is not locally conformally flat possesses a conformal metric of constant scalar curvature. Aubin's result is limited to dimension $\geq 6$ because these are the dimensions in which the local conformal geometry contains enough information to solve the problem. In the remaining cases the problem becomes a global one, which was solved by R. Schoen in [13] (see also [11]). Our Theorem A likewise covers the cases in which only local information is required. In fact, in terms of the parabolic dilations described below, a $2n+1$-dimensional CR manifold has "homogeneous dimension" $2n+2$, and the limitation $n \geq 2$ is the same as $2n+2 \geq 6$. Thus the analogy is closer than might appear at first glance.

To illustrate our method of proof, let us recall the much simpler proof of the fact that $\lambda(M) \leq \Lambda$ for *every* compact strictly pseudoconvex $M$. The key idea is that the sphere possesses a one-parameter family of extremal contact forms that concentrate near a point. To see this, it is easiest to use as a model not the sphere but the *Heisenberg group* $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, with coordinates $(z,t)$ and holomorphic tangent bundle spanned by the vector fields $Z_\alpha = \partial/\partial z^\alpha + i\overline{z}^\alpha \partial/\partial t$. The *Cayley transform* gives a CR equivalence between $\mathbb{H}^n$ and the sphere minus a point, which allows us to think of the standard spherical contact form $\hat{\theta}$ as a contact form on $\mathbb{H}^n$.

The Heisenberg group carries a natural family of *parabolic dilations*: for $s > 0$, the map $\delta_s(z,t) = (sz, s^2t)$ is a CR automorphism of $\mathbb{H}^n$. These dilations give rise to a family of extremal contact forms $\hat{\theta}^\varepsilon = \delta^*_{1/\varepsilon}\hat{\theta}$ on $\mathbb{H}^n$ which become more and more concentrated near the origin as $\varepsilon \to 0$. Since an arbitrary CR structure can be closely approximated near a point by the Heisenberg group via suitable "normal coordinates", one expects that the Yamabe functional $\mathscr{Y}_M$ should be closely approximated by $\mathscr{Y}_{\mathbb{H}^n}$ for contact forms supported very near the base point. This is indeed the case: in [7] we showed that for suitably transplanted contact forms $\theta^\varepsilon$ on $M$, $\mathscr{Y}_M(\theta^\varepsilon) \to \Lambda$ as $\varepsilon \to 0$, thus proving that $\lambda(M) \le \Lambda$.

In order to prove the strict inequality $\lambda(M) < \Lambda$, we need a much more precise asymptotic expression for $\mathscr{Y}_M(\theta^\varepsilon)$ as $\varepsilon \to 0$. For this it is necessary to refine our notion of normal coordinates considerably. In solving the Riemannian version of the problem, Aubin [1] used geodesic normal coordinates. In [11] it was shown that his argument can be simplified by using "conformal normal coordinates" instead, in which the conformal factor has been strategically chosen to simplify the Yamabe functional near a point. In the CR case, we find that such a normalization is needed just to make the calculation tractable. The main technical contribution of this paper is a new intrinsic construction of CR normal coordinates for an abstract CR manifold, in terms of which the asymptotic expansion of $\mathscr{Y}_M(\theta^\varepsilon)$ can be calculated explicitly in terms of CR invariants of $M$ at the base point. (H.-S. Luk [12] has given another intrinsic construction of CR normal coordinates; we have chosen a different approach because we need coordinates more closely related to pseudohermitian invariants.)

Our construction of normal coordinates proceeds in two stages. The first is to choose an arbitrary contact form $\theta$ on $M$ and construct, for any base point $q$ and any holomorphic frame at $q$, canonical coordinates in a neighborhood of $q$. Our construction is reminiscent of geodesic normal coordinates, in which each line through the origin in the tangent space $T_qM$ is mapped to the geodesic in $M$ tangent to that line. Since the pseudohermitian connection constructed by S. Webster [15] and N. Tanaka [14] determines pseudohermitian-invariant geodesics, we could in fact follow the same procedure on $M$. This was first observed by C. Stanton, who showed in an unpublished note how to calculate the Taylor series of the pseudohermitian structure in exponential coordinates, using a method of Cartan. However, this approach is not practical for our purposes: radial lines are the orbits of the *standard* dilations in $T_qM$ (i.e. multiplication by positive reals), while the natural homogeneity of our problem is that of the *parabolic* dilations of the Heisenberg group mentioned above. It is these dilations we wish to use as the basis for a coordinate system.

In §2 we show how to map the orbits of the parabolic dilations into $M$ in a canonical way, as the solutions to a modified geodesic equation (see Theorem 2.1). The resulting curves are called *parabolic geodesics*; they induce a natural map from $T_qM$ into $M$ called the *parabolic exponential map*. By means of this map we define a family of natural charts near $q$ called *pseudohermitian normal coordinates.*

The second step in our construction of CR normal coordinates is to analyze the effect of a change in contact form. The asymptotic expansion of $\mathscr{Y}(\theta^\varepsilon)$

will ultimately be expressed in terms of pseudohermitian curvature and torsion invariants, so in order to make the calculation as easy as possible we attempt to simplify these invariants at $q$ as much as possible. In §3 we show that $\theta$ can be chosen in a neighborhood of $q$ so that the pseudohermitian Ricci and torsion tensors and certain combinations of their covariant derivatives vanish at $q$. Once the one-jet of $\theta$ is fixed, its Taylor series at $q$ is completely determined by this condition.

Our construction is inspired by a similar construction of normal coordinates for a conformal Riemannian manifold due to C. R. Graham [5]. Graham showed that any metric can be changed conformally so that the Ricci tensor and all its symmetrized covariant derivatives vanish at a given point. In the CR case, we must replace the Ricci tensor by a more complicated tensor constructed from the pseudohermitian Ricci and torsion. (The alternate normalization used in [11], in which the volume element of the metric is approximated to arbitrarily high order by the Euclidean volume, does not seem to have a useful analogue in the CR case.)

By choosing pseudohermitian normal coordinates for such a normalized contact form, we obtain an intrinsically defined "CR normal coordinate chart" near $q \in M$. The set of all such charts is parametrized by the same finite-dimensional Lie group that parametrizes the extrinsic normal coordinates defined by Chern and Moser [3].

Having completed these preliminary constructions, we proceed in §§4 and 5 to define the "test forms" $\theta^\varepsilon$ and to compute an asymptotic formula for $\mathscr{Y}_M(\theta^\varepsilon)$ as $\varepsilon \to 0$. In §4 we use simple invariant theory to show that, if $\theta^\varepsilon$ is defined in terms of the CR normal coordinates of §§2 and 3, the asymptotic expression takes a particularly nice form:

$$\mathscr{Y}_M(\theta^\varepsilon) = \begin{cases} \Lambda(1 - c(n)|S(q)|^2\varepsilon^4) + O(\varepsilon^5) & \text{for } n \geq 3, \\ \Lambda(1 - c(2)|S(q)|^2\varepsilon^4 \log \frac{1}{\varepsilon}) + O(\varepsilon^4) & \text{for } n = 2. \end{cases}$$

Here $S(q)$ is the Chern curvature tensor of $M$ evaluated at $q$ [3]. Since this formula shows that a priori $|S(q)|^2$ is the only invariant that will appear in the final expression, it allows us to ignore most of the terms that arise in the computation of the asymptotic expansion.

Finally, in §5 we make use of this simplification to compute the exact value of the constant $c(n)$ and show that it is strictly positive. Since, when $n \geq 2$, $S$ is identically zero precisely when $M$ is locally CR equivalent to the sphere [3], under the hypotheses of Theorem A there is a point $q \in M$ where $S(q) \neq 0$. This implies that for $\varepsilon$ small enough we can achieve $\mathscr{Y}_M(\theta^\varepsilon) < \Lambda$, thus proving the theorem.

We will use the notation and terminology of [8], which we review briefly here. For our purposes a *CR manifold* (always assumed to be of hypersurface type and nondegenerate) is a real $2n+1$-dimensional manifold $M$ together with a distinguished $n$-dimensional subbundle $\mathscr{H} \subset \mathbb{C}TM$, the *holomorphic tangent bundle*, satisfying $\mathscr{H} \cap \overline{\mathscr{H}} = 0$ and $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. We write $H = \operatorname{Re} \mathscr{H} \oplus \overline{\mathscr{H}} \subset TM$. If $\theta$ is a nonvanishing real one-form annihilating $H$, we assume $\theta \wedge d\theta^n \neq 0$ ($\theta$ is a contact form). The *Levi form* of $\theta$ is the nondegenerate hermitian form defined on $\mathscr{H}$ by $L_\theta(X, \overline{Y}) = -2i d\theta(X \wedge \overline{Y})$ for $X, Y \in \mathscr{H}$; it is determined up to a conformal multiple by $\theta$. If $L_\theta$ is positive definite, $M$ is *strictly pseudoconvex.*

A contact form $\theta$ determines a *characteristic vector field* $T$, defined by $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. If $\{W_\alpha\}$ is any local frame for $\mathscr{H}$, the *admissible coframe* dual to $\{W_\alpha\}$ is the collection of (1,0)-forms $\{\theta^\beta\}$ defined by $\theta^\beta(W_\alpha) = \delta_\alpha^\beta$ and $\theta^\beta(W_{\overline{\alpha}}) = \theta^\beta(T) = 0$. (Unless otherwise noted, we will always let Greek indices run from 1 to $n$ and assume summation over repeated indices.) Thus, writing $W_{\overline{\beta}} = \overline{W_\beta}$, $\{T, W_\alpha, W_{\overline{\beta}}\}$ forms a frame for $\mathbb{C}TM$, with dual coframe $\{\theta, \theta^\alpha, \theta^{\overline{\beta}}\}$. In terms of such a frame, we can write $d\theta = i h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}}$ and $L_\theta(X^\alpha W_\alpha, Y^{\overline{\beta}} W_{\overline{\beta}}) = h_{\alpha\overline{\beta}} X^\alpha Y^{\overline{\beta}}$ for some hermitian matrix of functions $h_{\alpha\overline{\beta}}$. We will use $h_{\alpha\overline{\beta}}$ and its inverse $h^{\alpha\overline{\beta}}$ to lower and raise indices.

According to [15] or [14], a pseudohermitian structure induces a natural linear connection on $M$, which we denote by $\nabla$ and call the *pseudohermitian connection.* Using Webster's notation (see also [9]), the connection is expressed in terms of a holomorphic frame by $\nabla W_\alpha = \omega_\alpha{}^\beta \otimes W_\beta$, $\nabla T = 0$, where the one-forms $\omega_\alpha{}^\beta$ satisfy

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha{}^\beta + A^\beta{}_{\overline{\alpha}} \theta \wedge \theta^{\overline{\alpha}}, \quad A_{\alpha\beta} = A_{\beta\alpha}, \quad \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} = dh_{\alpha\overline{\beta}}. \tag{1.2}$$

The tensor with components $A_{\alpha\beta}$ is called the *pseudohermitian torsion.* The connection forms also satisfy

$$\begin{aligned} d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta = R_\alpha{}^\beta{}_{\rho\overline{\sigma}} \theta^\rho \wedge \theta^{\overline{\sigma}} + A_{\alpha\gamma,}{}^\beta \theta^\gamma \wedge \theta - A^{\overline{\gamma}\beta}{}_{,\alpha} \theta^{\overline{\gamma}} \wedge \theta \\ + i A_{\overline{\gamma}}{}^\beta \theta_\alpha \wedge \theta^{\overline{\gamma}} - i A_{\alpha\gamma} \theta^\gamma \wedge \theta^\beta, \end{aligned} \tag{1.3}$$

(cf. [10, (2.2), (2.4)]), where the *pseudohermitian curvature* $R_\alpha{}^\beta{}_{\rho\overline{\sigma}}$ has the symmetries

$$R_{\alpha\overline{\beta}\rho\overline{\sigma}} = R_{\rho\overline{\beta}\alpha\overline{\sigma}} = \overline{R}_{\beta\overline{\alpha}\sigma\overline{\rho}}.$$

Contractions of the curvature yield the *pseudohermitian Ricci* $R_{\rho\overline{\sigma}} = R_\alpha{}^\alpha{}_{\rho\overline{\sigma}}$ and the *pseudohermitian scalar curvature* $R = R_\rho{}^\rho$.

We will denote the components of pseudohermitian covariant derivatives of a tensor by indices preceded by a comma, as in $A_{\alpha\beta,\overline{\gamma}\rho}$; a zero index indicates covariant differentiation with respect to $T$. The *sublaplacian* is the real differential operator defined on functions by

$$\Delta_b f = -(f_{,\alpha}{}^{\alpha} + f_{,\overline{\beta}}{}^{\overline{\beta}}).$$

We would like to thank Victor Guillemin for showing us a quick proof of Proposition 5.3.

## 2. Pseudohermitian normal coordinates

A pseudohermitian structure on a manifold $M$ induces natural parabolic dilations on any tangent space $T_qM$ analogous to those on the Heisenberg group. In this section we will show how to map the orbits of these dilations to pseudohermitian-invariant curves in $M$, called *parabolic geodesics.* The resulting *parabolic exponential map* is a local diffeomorphism from $T_qM$ into $M$, naturally induced by the pseudohermitian structure. Then any choice of orthonormal frame for $\mathscr{H}_q$ gives an identification of $T_qM$ with $\mathbb{H}^n$; composing this identification with the parabolic exponential map yields *pseudohermitian normal coordinates* near $q$. We will show how to use these coordinates to compute the Taylor series of the pseudohermitian structure explicitly in terms of pseudohermitian curvature and torsion invariants.

To see how to incorporate the parabolic dilations into our exponential map, let us first examine the model case of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, with coordinates $(z,t)$. We consider $\mathbb{H}^n$ to be a pseudohermitian manifold with holomorphic tangent bundle $\mathscr{H}$ spanned by the vector fields

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\overline{z}\frac{\partial}{\partial t}, \qquad \alpha = 1, \cdots, n,$$

and standard contact form

$$\Theta = dt + iz^\alpha d\overline{z}^\alpha - i\overline{z}^\alpha dz^\alpha.$$

This pseudohermitian structure is left-invariant under the group law on $\mathbb{H}^n$ (cf. [7]). (We will have no occasion to use the group structure here.) With these choices, the characteristic vector field of $\Theta$ is $\partial/\partial t$, the admissible coframe dual to $\{Z_\alpha\}$ is $\{dz^\alpha\}$, and the Levi form is given by $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$. The natural parabolic dilations on $\mathbb{H}^n$ are the CR automorphisms $\delta_s \colon \mathbb{H}^n \to \mathbb{H}^n$ defined by $\delta_s(z,t) = (sz, s^2t)$ for $s > 0$. The infinitesimal generator of this $\mathbb{R}^+$-action on $\mathbb{H}^n$ is the vector field

$$P_{(z,t)} = z^\alpha \frac{\partial}{\partial z^\alpha} + \overline{z}^\alpha \frac{\partial}{\partial \overline{z}^\alpha} + 2t\frac{\partial}{\partial t} = z^\alpha Z_\alpha + \overline{z}^\alpha Z_{\overline{\alpha}} + 2tT.$$

A function or tensor $\omega$ on $\mathbb{H}^n$ is homogeneous of degree $m$ with respect to the dilations if and only if its Lie derivative with respect to $P$ satisfies $L_P\omega = m\omega$. For example, the natural distance function $\rho(z,t) = (|z|^4 + t^2)^{1/4}$ is homogeneous of degree 1.

The orbits of the dilations (except for the degenerate orbits where $z = 0$ or $t = 0$) lie on parabolas through 0. For fixed $(W, c) \in \mathbb{H}^n$, consider the curve $\gamma\colon \mathbb{R} \to \mathbb{H}^n$ given by $\gamma(s) = (sW, s^2c)$. Its image is the (possibly degenerate) parabola containing the orbits of $(W, c)$ and $(-W, c)$. Its tangent vector at 0 is $(W, 0)$, and for $s \neq 0$,

$$\dot\gamma(s) = s^{-1}P_{\gamma(s)}.$$

Using the fact that the pseudohermitian connection on $\mathbb{H}^n$ satisfies $\nabla Z_\alpha = \nabla T = 0$, one can compute that $\gamma$ satisfies the ordinary differential equation

$$\nabla_{\dot\gamma}\dot\gamma = 2cT. \tag{2.1}$$

On a manifold $M$, a pseudohermitian structure yields a natural splitting $TM = H \oplus \mathbb{R}T$. This splitting in turn determines a natural family of parabolic dilations on any tangent space $T_qM$ analogous to those on the Heisenberg group, by setting $\delta_s(W + cT) = sW + s^2cT$ for $W \in H$, $c \in \mathbb{R}$. The curves in $T_qM$ given by $\sigma_{W,c}(s) = sW + s^2cT$ are parabolas analogous to the curves $\gamma$ described above. The key to the construction of our parabolic exponential map is to use equation (2.1), which makes sense on any pseudohermitian manifold, to map these parabolas into $M$, in the same way that the classical exponential map sends radial lines to geodesics. This is carried out in the following theorem.

**Theorem 2.1.** *Let $M$ be a nondegenerate pseudohermitian manifold and $q \in M$. For any $W \in H_q$ and $c \in \mathbb{R}$, let $\gamma = \gamma_{W,c}$ denote the solution to the ordinary differential equation* (2.1) *on $M$ with initial conditions $\gamma(0) = q$ and $\dot\gamma(0) = W$. We call $\gamma$ the parabolic geodesic determined by $W$ and $c$. Define the parabolic exponential map $\Psi\colon T_qM \to M$ by*

$$\Psi(W + cT) = \gamma_{W,c}(1), \tag{2.2}$$

*where defined. Then $\Psi$ maps a neighborhood of* 0 *in $T_qM$ diffeomorphically to a neighborhood of $q$ in $M$, and sends $\sigma_{W,c}$ to $\gamma_{W,c}$.*

*Proof.* For any $(W, c)$, $\gamma_{W,c}(s)$ is uniquely defined for $s$ small enough. We begin by showing that the curves $\gamma_{W,c}$ satisfy the following rescaling law:

$$\gamma_{rW,r^2c}(s) = \gamma_{W,c}(rs), \tag{2.3}$$

whenever either side is defined. Fix $r \in \mathbb{R}$, and set $\tilde\gamma(s) = \gamma_{W,c}(rs)$. Then $\dot{\tilde\gamma}(s) = r\dot\gamma_{W,c}(rs)$, and so $\tilde\gamma$ satisfies

$$\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = 2r^2cT.$$

Since $\dot{\tilde{\gamma}}(0) = rW$, this shows that $\tilde{\gamma} = \gamma_{rW,r^2c}$, which is (2.3). It follows immediately that $\Psi$ maps $\sigma_{W,c}$ to $\gamma_{W,c}$ wherever it is defined.

Now choose a coordinate neighborhood $N \subset M$, with coordinates $\{x^i\}$ centered at $q$. Let $\{\xi^i\}$ be the fiber coordinates on $TN$ given by $\xi^i(V) = dx^i(V)$ for $V \in TN$, and let $T^k = \xi^k(T)$ denote the component functions of the vector field $T$. Let $\Gamma_{ij}^k$ denote the Christoffel symbols of the pseudohermitian connection in these coordinates, and define a vector field $X$ on $\mathbb{R} \times TN$ by

$$X_{(c,(x,\xi))} = \left(0, \xi^k \frac{\partial}{\partial x^k} - \Gamma_{ij}^k(x)\xi^k \xi^j \frac{\partial}{\partial \xi^k} + 2cT^k(x)\frac{\partial}{\partial \xi^k}\right).$$

Let $\Phi\colon \mathbb{R} \times (\mathbb{R} \times TN) \to \mathbb{R} \times TN$ be the local flow determined by $X$, which by standard ODE theory is defined and smooth in a neighborhood of the origin. Fix $c \in \mathbb{R}$ and $W \in H_q \subset T_qN$. If $\pi\colon \mathbb{R} \times TN \to N$ denotes the projection on the second factor followed by the natural projection $TN \to N$, then the curve $\gamma(s) = \pi \circ \Phi(s, c, (0, W))$ in $N$ satisfies

$$\ddot{\gamma}^k(s) = -\Gamma_{ij}^k(\gamma(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) + 2cT^k(\gamma(s)),$$

which is equivalent to (2.1). Since $\gamma(0) = q$ and $\dot{\gamma}(0) = W$, by uniqueness we must have $\gamma = \gamma_{W,c}$. Therefore

$$\gamma_{W,c}(s) = \pi \circ \Phi(s, c, (0, W)), \tag{2.4}$$

where either side is defined.

The theory of ordinary differential equations implies that there exists $\varepsilon > 0$ such that $\Phi(\varepsilon, c, (0, W))$ is defined for all $(W, c)$ in a neighborhood of $(0, 0)$. Therefore, by (2.3) and (2.4),

$$\Psi(W + cT) = \gamma_{W,c}(1) = \gamma_{W/\varepsilon, c/\varepsilon^2}(\varepsilon) = \pi \circ \Phi(\varepsilon, c/\varepsilon^2, (0, W/\varepsilon))$$

is defined and smooth for $(W, c)$ in some smaller neighborhood of the origin.

To show that $\Psi$ is a diffeomorphism near 0, we will show that its differential $\Psi_*$ at 0 is the identity mapping on $T_qM$. If $W \in H_q$, (2.3) implies

$$\Psi_*(W) = \left.\frac{d}{ds}\right|_0 \Psi(sW) = \left.\frac{d}{ds}\right|_0 \gamma_{W,0}(s) = W.$$

To compute $\Psi_*(T)$, let $s \mapsto \tau(s)$ be the integral curve of $T$ beginning at $q$. Fix $c \in \mathbb{R}$, and set $\beta(r) = \tau(r^2c)$. Then $\dot{\beta}(r) = 2rcT_{\beta(r)}$ and $\dot{\beta}(0) = 0$. Since $\nabla T = 0$, we have

$$\nabla_{\dot{\beta}}\dot{\beta}(r) = \frac{d}{dr}(2rc)T = 2cT,$$

and thus by uniqueness $\beta(r) = \gamma_{0,c}(r)$. Setting $r = 1$, we get $\gamma_{0,c}(1) = \tau(c)$, and therefore

$$\Psi_*(T) = \left.\frac{d}{ds}\right|_0 \Psi(sT) = \left.\frac{d}{ds}\right|_0 \gamma_{0,s}(1) = \left.\frac{d}{ds}\right|_0 \tau(s) = T.$$

Since $TM = H \oplus \mathbf{R}T$, this shows that $\Psi_*$ is the identity. q.e.d.

For computational purposes, it will be convenient to have a holomorphic frame in a neighborhood of $q$ which is parallel along each curve $\gamma_{W,c}$. Choose any holomorphic frame $\{W_\alpha|_q\}$ at $q$, and extend it to a neighborhood of $q$ by parallel translation along the curves $\gamma_{W,c}$. Since every point in some punctured neighborhood of $q$ is on a unique curve $\gamma_{W,c}$, this defines a frame $\{W_\alpha\}$ uniquely near $q$. The following lemma shows that the resulting frame is smooth.

**Lemma 2.2.** *Suppose $X$ is a vector field defined in a neighborhood of $q$ in $M$ which is parallel along each curve $\gamma_{W,c}$. Then $X$ is smooth near $q$.*

*Proof.* Choose any coordinates $\{x^i\}$ centered at $q$ and write $X = X^j\partial/\partial x^j$. Writing $\xi^j(s,W,c) = X^j(\gamma_{W,c}(s))$ for $W \in H_q$ and $c \in \mathbf{R}$, the differential equation $\nabla_{\dot\gamma}X = 0$ becomes

$$\frac{\partial}{\partial s}\xi^j(s,W,c) = -\Gamma^j_{kl}(\gamma_{W,c}(s))\dot\gamma^k_{W,c}(s)\xi^l(s,W,c),$$

with the initial condition $\xi^j(0,W,c) = X^j(0)$. Since solutions to ODE's depend smoothly on parameters, $\xi^j$ is a smooth function of $(s,W,c)$ where it is defined. Therefore $X^j \circ \Psi(W + cT) = X^j(\gamma_{W,c}(1)) = \xi^j(1,W,c)$ is smooth in a neighborhood of $0 \in T_qM$, and hence $X^j$ itself is smooth. q.e.d.

Since the pseudohermitian connection is compatible with the complex structure, if $W_\alpha|_q \in \mathscr{H}$, then the parallel extension $W_\alpha$ is a section of $\mathscr{H}$. Let $\{\theta^\alpha\}$ denote the dual admissible coframe to $\{W_\alpha\}$. Since $\nabla T = 0$ it follows that $\theta^\alpha$ is parallel along each $\gamma_{W,c}$. We can then write $d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$ for some matrix of functions $h_{\alpha\overline{\beta}}$. Because $\nabla(d\theta) = 0$, $h_{\alpha\overline{\beta}}$ is a constant matrix for this coframe.

Suppose $M$ is strictly pseudoconvex. We define a *special frame* to be a holomorphic frame $\{W_\alpha\}$ which is parallel along each curve $\gamma_{W,c}$, and for which $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$ (as on the Heisenberg group); we call the dual admissible coframe a *special coframe.* We summarize the preceding results as follows:

**Proposition 2.3.** *Any holomorphic frame at $q \in M$ for which $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$ can be extended smoothly to a special frame $\{W_\alpha\}$ in a neighborhood of $q$. The dual special coframe $\{\theta^\alpha\}$ is parallel along each curve $\gamma_{W,c}$, and satisfies $d\theta = 2i\theta^\alpha \wedge \theta^{\overline{\alpha}}$. Any two such extensions agree on their common domain.*

Now choose a special frame $\{W_\alpha\}$ near $q$, and let $\{\theta^\alpha\}$ be the dual special coframe. The coframe determines an isomorphism $\lambda\colon T_qM \to \mathbf{H}^n$ by $(z^\alpha, t) = \lambda(V) = (\theta^\alpha(V), \theta(V))$. This in turn determines a coordinate chart $\lambda\circ\Psi^{-1}$ in a neighborhood of $q$. We call such a chart *pseudohermitian normal coordinates* determined by $\{W_\alpha\}$. It is clear from the definition that the set of all special frames, and hence also the set of all such charts, is parametrized by the group $\mathrm{U}(n)$. In the remainder of this section, we will show how to compute the Taylor series of $\theta$ and a special coframe $\{\theta^\alpha\}$ in pseudohermitian normal coordinates.

Identifying a neighborhood of $q \in M$ with an open set in $\mathbb{H}^n$ by means of a pseudohermitian normal coordinate chart, we can consider $\theta$ and $\theta^\alpha$ as one-forms on (a subset of) $\mathbb{H}^n$. If $\varphi$ is any tensor field on $\mathbb{H}^n$, let us denote by $\varphi_{(m)}$ the part of its Taylor series that is homogeneous of degree $m$ in terms of the parabolic dilations. Thus $\varphi_{(m)}$ is a tensor field with polynomial coefficients, and $\varphi - \sum \varphi_{(m)}$ can be made to vanish to arbitrarily high order at 0. As mentioned above, each term $\varphi_{(m)}$ satisfies $L_P\varphi_{(m)} = m\varphi_{(m)}$, and therefore if $\varphi$ is a differential form,

$$\varphi_{(m)} = \frac{1}{m}(L_P\varphi)_{(m)} = \frac{1}{m}(P \lrcorner d\varphi + d(P \lrcorner \varphi))_{(m)}. \tag{2.5}$$

In order to use this relation to compute the homogeneous parts of $\theta$ and $\theta^\alpha$, we will need the following lemma. The simple relationship between the vector field $P$ and the forms $\{\theta, \theta^\alpha, \omega_\beta{}^\alpha\}$ expressed in this lemma is the reason why pseudohermitian normal coordinates are valuable for the computations we plan to do in §§4 and 5.

**Lemma 2.4.** *Let $\{W_\alpha\}$ be a special frame near $q \in M$ with dual special coframe $\{\theta^\alpha\}$; let $\omega_\alpha{}^\beta$ denote the associated Webster connection forms and $(z,t)$ the associated pseudohermitian normal coordinates. Let $P$ be the vector field defined in these coordinates by*

$$P_{(z,t)} = z^\alpha \frac{\partial}{\partial z^\alpha} + \overline{z}^\alpha \frac{\partial}{\partial \overline{z}^\alpha} + 2t\frac{\partial}{\partial t}.$$

*Then*

$$\text{(a) } \theta(P) = 2t; \quad \text{(b) } \theta^\alpha(P) = z^\alpha; \quad \text{(c) } \omega_\beta{}^\alpha(P) = 0.$$

*In particular, $P = z^\alpha W_\alpha + z^{\overline{\alpha}} W_{\overline{\alpha}} + 2tT$.*

*Proof.* It suffices to show that (a)–(c) hold along each curve $\gamma_{W,c}$. Fix $W \in H_q$ and $c \in \mathbb{R}$, and write $W = w^\alpha W_\alpha|_q$. In these coordinates, the curve $\gamma = \gamma_{W,c}$ is given explicitly by

$$(z^\alpha, t) = \gamma(s) = (sw^\alpha, s^2c).$$

Thus by explicit computation $\dot\gamma(s) = s^{-1}P_{\gamma(s)}$ for $s \neq 0$. Along $\gamma$, since $\nabla\theta = 0$, we have

$$\frac{d}{ds}\theta(\dot\gamma(s)) = \theta(\nabla_{\dot\gamma}\dot\gamma(s)) = \theta(2cT) = 2c.$$

Since $\theta(\dot\gamma(0)) = 0$, this implies $\theta(\dot\gamma(s)) = 2cs$. Therefore $\theta(P) = \theta(s\dot\gamma(s)) = 2s^2c = 2t$, which proves (a). Similarly, using $\nabla_{\dot\gamma}\theta^\alpha = 0$,

$$\frac{d}{ds}\theta^\alpha(\dot\gamma(s)) = \theta^\alpha(\nabla_{\dot\gamma}\dot\gamma(s)) = \theta^\alpha(2cT) = 0.$$

At the origin we have $\theta^\alpha(\dot\gamma(0)) = \theta^\alpha(W) = w^\alpha$, and so $\theta^\alpha(\dot\gamma(s)) = w^\alpha$ all along $\gamma$. This implies $\theta^\alpha(P) = \theta^\alpha(s\dot\gamma(s)) = sw^\alpha = z^\alpha$, which is (b). To prove (c), just note that $\nabla_P\theta^\alpha = s\nabla_{\dot\gamma}\theta^\alpha = 0$ along $\gamma$, which is equivalent to ${\omega_\beta}^\alpha(P)\theta^\beta = 0$. Since the forms $\theta^\beta$ are independent, (c) follows. q.e.d.

The following proposition will enable us to compute the homogeneous parts of $\theta$ and $\theta^\alpha$ inductively in terms of the pseudohermitian curvature and torsion and their covariant derivatives at $q$.

**Proposition 2.5.** *Let $\{W_\alpha\}$ be a special frame and $\{\theta^\alpha\}$ the dual special coframe. Then in pseudohermitian normal coordinates*

(a) $\theta_{(2)} = \Theta$; $\theta_{(3)} = 0$; $\theta_{(m)} = \frac{2}{m}(iz^\alpha\theta^{\overline{\alpha}} - iz^{\overline{\alpha}}\theta^\alpha)_{(m)}$, $m \geq 4$;

(b) $\theta^\alpha_{(1)} = dz^\alpha$; $\theta^\alpha_{(2)} = 0$; $\theta^\alpha_{(m)} = \frac{1}{m}(z^\beta{\omega_\beta}^\alpha + tA_{\overline{\alpha}\overline{\beta}}\theta^{\overline{\beta}} - \frac{1}{2}z^{\overline{\beta}}A_{\overline{\alpha}\overline{\beta}}\theta)_{(m)}$, $m \geq 3$;

(c) ${\omega_\beta}^\alpha{}_{(1)} = 0$; ${\omega_\beta}^\alpha{}_{(m)} = \frac{1}{m}(R_\beta{}^\alpha{}_{\rho\overline{\sigma}}(z^\rho\theta^{\overline{\sigma}} - z^{\overline{\sigma}}\theta^\rho) + \frac{1}{2}A_{\beta\gamma,\overline{\alpha}}(z^\gamma\theta - 2t\theta^\gamma)$
$- \frac{1}{2}A_{\overline{\alpha}\overline{\gamma},\beta}(z^{\overline{\gamma}}\theta - 2t\theta^{\overline{\gamma}}) + iA_{\overline{\alpha}\overline{\gamma}}(z^{\overline{\beta}}\theta^{\overline{\gamma}} - z^{\overline{\gamma}}\theta^{\overline{\beta}})$
$- iA_{\beta\gamma}(z^\gamma\theta^\alpha - z^\alpha\theta^\gamma))_{(m)}$, $m \geq 2$.

*Proof.* Using $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$, the structure equation (1.3) for the pseudohermitian connection can be written

$$d{\omega_\beta}^\alpha - {\omega_\beta}^\gamma \wedge {\omega_\gamma}^\alpha = R_\beta{}^\alpha{}_{\rho\overline{\sigma}}\theta^\rho \wedge \theta^{\overline{\sigma}} + \tfrac{1}{2}A_{\beta\gamma,\overline{\alpha}}\theta^\gamma \wedge \theta - \tfrac{1}{2}A_{\overline{\alpha}\overline{\gamma},\beta}\theta^{\overline{\gamma}} \wedge \theta + iA_{\overline{\alpha}\overline{\gamma}}\theta^{\overline{\beta}} \wedge \theta^{\overline{\gamma}} - iA_{\beta\gamma}\theta^\gamma \wedge \theta^\alpha.$$

Inserting this into (2.5) and using Lemma 2.4 yield (c).

Similarly, using (1.2), equation (2.5) for $\theta^\alpha$ becomes

$$\begin{aligned}\theta^\alpha_{(m)} &= \tfrac{1}{m}(P \lrcorner (\theta^\beta \wedge {\omega_\beta}^\alpha + \tfrac{1}{2}A_{\overline{\alpha}\overline{\beta}}\theta \wedge \theta^{\overline{\beta}}) + d(P \lrcorner \theta^\alpha))_{(m)}\\ &= \tfrac{1}{m}(z^\beta{\omega_\beta}^\alpha + tA_{\overline{\alpha}\overline{\beta}}\theta^{\overline{\beta}} - \tfrac{1}{2}z^{\overline{\beta}}A_{\overline{\alpha}\overline{\beta}}\theta + dz^\alpha)_{(m)}.\end{aligned}$$

Comparing homogeneous terms of order 1, 2, and $m \geq 3$ gives (b). Finally,

$$\begin{aligned}\theta_{(m)} &= \tfrac{1}{m}(P \lrcorner (2i\theta^\alpha \wedge \theta^{\overline{\alpha}}) + d(P \lrcorner \theta))_{(m)}\\ &= \tfrac{1}{m}(2iz^\alpha\theta^{\overline{\alpha}} - 2iz^{\overline{\alpha}}\theta^\alpha + 2dt)_{(m)},\end{aligned}$$

from which (a) follows. q.e.d.

Observe that the map which sends a pair of points $(q_1, q_2)$ near the diagonal in $M \times M$ to $\Psi_{q_1}(q_2)$ is smooth in both arguments, since solutions to ordinary differential equations vary smoothly with parameters. Therefore Proposition 2.5 implies, in particular, that these pseudohermitian normal coordinates are also normal coordinates in the sense of Folland and Stein [4]. Of course, the coordinates defined here approximate $M$ by $\mathbf{H}^n$ in a much more precise way than required for Folland-Stein coordinates.

## 3. CR normal coordinates

To compute the asymptotic expansion of the CR Yamabe functional, we will need to calculate the Taylor series of a contact form $\theta$ and a special coframe $\{\theta^\alpha\}$ to high order at a point $q \in M$ in terms of the pseudohermitian curvature and torsion. Since the problem is CR-invariant, we lose no generality by first judiciously choosing the contact form $\theta$ so as to simplify the curvature and torsion at $q$ as much as possible. In this section we determine exactly how far these can be simplified by a choice of contact form. In particular, we show that a certain tensor $Q$ constructed from the pseudohermitian Ricci and torsion tensors can be made to vanish at $q$, together with its symmetrized covariant derivatives of all orders.

Let $\theta$ be any contact form for $M$, and let $(z, t)$ be pseudohermitian normal coordinates for $\theta$ centered at $q$. Write $Z_\alpha = \partial/\partial z^\alpha + i z^{\overline{\alpha}} \partial/\partial t$ in these coordinates, and $\mathscr{L}_0 = -\frac{1}{2}(Z_\alpha Z_{\overline{\alpha}} + Z_{\overline{\alpha}} Z_\alpha)$.

The Heisenberg dilations in pseudohermitian normal coordinates give us a notion of homogeneity of tensor (or vector) fields on $M$. For example, $dz^\alpha$ is homogeneous of degree 1, and $Z_\alpha$ of degree $-1$. If $\varphi$ is a smooth tensor field defined in a neighborhood of $q$, we say $\varphi \in \mathscr{O}_m$ if all the terms in the Taylor series of $\varphi$ in normal coordinates are homogeneous tensors of degree $\geq m$. If $\varphi \in \mathscr{O}_M$ is a differential form, then $d\varphi \in \mathscr{O}_m$ as well. Thus if $u$ is a function in $\mathscr{O}_m$ and $V$ a vector field in $\mathscr{O}_k$, then $Vu = du(V) \in \mathscr{O}_{m+k}$.

Although $\mathscr{O}_m$ is defined in terms of a specific choice of coordinates, it is easy to check that any other pseudohermitian normal coordinates $(\tilde{z}, \tilde{t})$ satisfy $\tilde{z}^\alpha \in \mathscr{O}_1$ and $\tilde{t} \in \mathscr{O}_2$, and multiplication maps $\mathscr{O}_m \times \mathscr{O}_k$ to $\mathscr{O}_{m+k}$, so the set $\mathscr{O}_m$ is actually CR-invariant.

We let $\mathscr{P}_m$ denote the vector space of polynomials in $(z, t)$ (considered either as polynomials on $\mathbf{H}^n$ or as functions on a neighborhood of $q$ in $M$) that are homogeneous of degree $m$ in terms of parabolic dilations.

Let $\{W_\alpha\}$ be a special frame near $q$ and $\{\theta^\alpha\}$ the dual special coframe. It follows from Proposition 2.5 that in pseudohermitian normal coordinates

$$W_\alpha = Z_\alpha + \mathscr{O}_1, \qquad T = \frac{\partial}{\partial t} + \mathscr{O}_0.$$

We will have to deal with a number of complicated expressions involving many derivatives. In order to bring some sense of order to the forest of indices, we begin with a brief catalog of notation conventions.

**Notation.** We adopt the following index conventions:

$$\alpha, \beta, \gamma, \delta, \varepsilon, \rho, \sigma \in \{1, \cdots, n\},$$
$$a, b, c \in \{1, \cdots, 2n\}, \qquad j, k, l \in \{0, \cdots, 2n\}.$$

We write $x = (t, z, \overline{z})$, with $x^0 = t$, $x^\alpha = z^\alpha$, $x^{\overline{\alpha}} = \overline{z^\alpha}$, and $\overline{\alpha} = \alpha + n$. Denote $\theta^0 = \theta$, $W_0 = T$, and $Z_0 = \partial/\partial t$. The order of $j$ is defined to be $o(j) = 2$ if $j = 0$, and $o(j) = 1$ otherwise. For a multi-index $J = (j_1, \cdots, j_s)$, we denote $\# J = s$, $o(J) = o(j_1) + \cdots + o(j_s)$, $x^J = x^{j_1} \cdots x^{j_s}$, $Z_J = Z_{j_s} \cdots Z_{j_1}$, and $\partial_J = \partial^s / \partial x^{j_s} \cdots \partial x^{j_1}$.

The symmetrization of an $r$-tensor with components $B_J = B_{j_1 \cdots j_r}$ is the symmetric tensor with components

$$B_{\langle J \rangle} = \frac{1}{r!} \sum_{\sigma \in S_r} B_{\sigma J},$$

where $S_r$ denotes the symmetric group on $r$ elements and $\sigma J = (j_{\sigma 1}, \cdots, j_{\sigma r})$. (The coefficient is chosen so that $B_{\langle J \rangle} = B_J$ if $B$ is symmetric.)

**Definition.** On a pseudohermitian manifold $(M, \theta)$, let $Q = Q_{jk} \theta^j \cdot \theta^k$ denote the (real) symmetric tensor whose components with respect to any admissible coframe are

$$Q_{\alpha\beta} = \overline{Q_{\overline{\alpha}\overline{\beta}}} = (n+2) i A_{\alpha\beta}, \qquad Q_{\alpha\overline{\beta}} = Q_{\overline{\beta}\alpha} = R_{\alpha\overline{\beta}},$$
$$Q_{0\alpha} = Q_{\alpha 0} = \overline{Q_{0\overline{\alpha}}} = \overline{Q_{\overline{\alpha}0}} = 4 A_{\alpha\beta,}{}^{\beta} + \frac{2i}{n+1} R_{,\alpha},$$
$$Q_{00} = \frac{16}{n} \operatorname{Im} A_{\alpha\beta,}{}^{\beta\alpha} - \frac{4}{n(n+1)} \Delta_b R.$$

The main result of this section is the following theorem.

**Theorem 3.1.** *Let $M$ be a strictly pseudoconvex CR manifold. For any integer $N \geq 2$, there exists a choice of contact form $\theta$ such that all symmetrized covariant derivatives of $Q$ with total order $\leq N$ vanish at $q$; that is,*

$$Q_{\langle jk,L \rangle} = 0 \quad \text{if } o(jkL) \leq N. \tag{3.1}$$

*Writing $\theta = e^{2u}\overline{\theta}$ for some fixed contact form $\overline{\theta}$, the one-jet of $u$ can be chosen arbitrarily; once it is fixed, the Taylor series of $u$ at $q$ is uniquely determined by this condition.*

We begin with a sequence of lemmas examining the way in which $Q$ transforms under a change of contact form.

**Lemma 3.2.** *If $\varphi$ is a tensor in $\mathscr{O}_m$, the components of its covariant derivatives with respect to a special frame satisfy $\varphi_{J,K} = Z_K\varphi_J + \mathscr{O}_{m-o(JK)+2}$.*

*Proof.* Covariant derivatives of an $r$-tensor $\varphi = \varphi_J\theta^J$ are related to ordinary derivatives by

$$\varphi_{J,k} = W_k\varphi_J - \sum_{i=1}^{r} \omega_{j_i}{}^l(W_k)\varphi_{j_1\cdots j_{i-1}lj_{i+1}\cdots j_r}, \tag{3.2}$$

where we understand $\omega_j{}^l = 0$ unless $1 \le j, l \le n$ or $n+1 \le j, l \le 2n$. Observe that $W_k = Z_k + \mathscr{O}_{2-o(k)}$ and Proposition 2.5 implies $\omega_j{}^l \in \mathscr{O}_2$. Moreover $\varphi_J \in \mathscr{O}_{m-o(J)}$, and this remains true when $j_i$ is replaced by $l \neq 0$; when $l = 0$ the summand vanishes. For first derivatives therefore,

$$\varphi_{J,k} = Z_k\varphi_J + \mathscr{O}_{m-o(Jk)+2}.$$

The general case follows easily by induction.

**Lemma 3.3.** *With respect to a special frame, if $u \in \mathscr{O}_m$,*

$$\begin{aligned}
u_{,ab} &= Z_bZ_au + \mathscr{O}_m, \qquad \Delta_b u = \mathscr{L}_0 u + \mathscr{O}_m,\\
(\Delta_b u)_{,\alpha} &= -Z_\alpha Z_{\overline{\beta}}Z_\beta u + niZ_\alpha Z_0 u + \mathscr{O}_{m-1},\\
u_{,\alpha\beta}{}^{\beta} &= \tfrac{1}{2}Z_\alpha Z_{\overline{\beta}}Z_\beta u + iZ_\alpha Z_0 u + \mathscr{O}_{m-1},\\
\Delta_b^2 u &= \operatorname{Re} Z_{\overline{\alpha}}Z_\alpha Z_{\overline{\beta}}Z_\beta u + n^2 Z_0Z_0u + \mathscr{O}_{m-2},\\
u_{,\alpha\beta}{}^{\beta\alpha} &= \tfrac{1}{4}Z_{\overline{\alpha}}Z_\alpha Z_{\overline{\beta}}Z_\beta u + \tfrac{i}{2}Z_{\overline{\alpha}}Z_\alpha Z_0 u + \mathscr{O}_{m-2}.
\end{aligned}$$

*Proof.* These follow from Lemma 3.2 and the commutation relations for $\{Z_j\}$. q.e.d.

Next we wish to examine how covariant derivatives of a tensor transform when we change the background pseudohermitian structure. Suppose we are given a contact form $\theta$, a holomorphic frame $\{W_\alpha\}$, and its dual admissible coframe $\{\theta^\alpha\}$. If $\tilde\theta = e^{2u}\theta$ is another contact form, one can check easily that the characteristic field transforms by

$$\tilde T = e^{-2u}(T - 2iu^{,\alpha}W_\alpha + 2iu^{,\overline{\beta}}W_{\overline{\beta}}),$$

and therefore the coframe $\{\tilde\theta^\alpha = \theta^\alpha + 2iu^{,\alpha}\theta\}$ is admissible for $\tilde\theta$ and dual to the original frame $\{W_\alpha\}$.

**Lemma 3.4.** *Suppose $\tilde\theta = e^{2u}\theta$ and let $\theta^\alpha$, $\tilde\theta^\alpha$ be as above. The pseudohermitian connection forms $\tilde\omega_\beta{}^\alpha$ determined by $\tilde\theta$ and $\tilde\theta^\alpha$ are*

$$\tilde\omega_\beta{}^\alpha = \omega_\beta{}^\alpha + 2(u_{,\beta}\theta^\alpha - u^{,\alpha}\theta_\beta) + 2\delta_\beta^\alpha u_{,\gamma}\theta^\gamma + i(2u^{,\alpha}{}_\beta + 4u_{,\beta}u^{,\alpha} + 4\delta_\beta^\alpha u_{,\gamma}u^{,\gamma})\theta.$$

*In particular, if $u \in \mathscr{O}_m$, then $\tilde\omega_\beta{}^\alpha = \omega_\beta{}^\alpha + \mathscr{O}_m$.*

*Proof.* Consider the coframe $\{\hat\theta^\alpha = e^u(\theta^\alpha + 2iu^{,\alpha}\theta)\}$, which is also admissible for $\tilde\theta$. In [9], it was shown that the pseudohermitian connection forms for $\{\tilde\theta, \hat\theta^\alpha\}$ are

$$\begin{aligned}\hat\omega_\beta{}^\alpha = \omega_\beta{}^\alpha &+ 2(u_{,\beta}\theta^\alpha - u^{,\alpha}\theta_\beta) + \delta_\beta^\alpha(u_{,\gamma}\theta^\gamma - u^{,\gamma}\theta_\gamma)\\ &+ i(u^{,\alpha}{}_\beta + u_{,\beta}{}^\alpha + 4u_{,\beta}u^{,\alpha} + 4\delta_\beta^\alpha u_{,\gamma}u^{,\gamma})\theta.\end{aligned}$$

The transformation law for connection forms under a change of coframe shows that $\tilde\omega_\beta{}^\alpha = \hat\omega_\beta{}^\alpha + \delta_\beta^\alpha\, du$. Using the facts that $du = u_{,\gamma}\theta^\gamma + u_{,\bar\gamma}\theta^{\bar\gamma} + u_{,0}\theta$ and $u_{,\beta}{}^\alpha = u^{,\alpha}{}_\beta + iu_{,0}\delta_\beta^\alpha$ (cf. [10]), the lemma follows.

**Lemma 3.5.** *Suppose $\tilde\theta = e^{2u}\theta$ with $u \in \mathscr{O}_m$, $\varphi$ is an s-tensor field, and let $\nabla^r\varphi$ and $\tilde\nabla^r\varphi$ denote the rth pseudohermitian covariant derivatives of $\varphi$, computed with respect to $\theta$ and $\tilde\theta$, respectively. Let $J$ and $K$ be multi-indices with $\#J = s$ and $\#K = r$, and let $\varphi_{J,K}$ and $\tilde\varphi_{J,K}$ denote the corresponding components of $\nabla^r\varphi$ and $\tilde\nabla^r\varphi$ in terms of the coframes $\{\theta,\theta^\alpha\}$ and $\{\tilde\theta,\tilde\theta^\alpha\}$ defined above. Then*

$$\tilde\varphi_{J,K} = \varphi_{J,K} + \mathscr{O}_{m-o(K)-1}.$$

*If $J$ is a multi-index with no zero entries, then*

$$\tilde\varphi_{J,K} = \varphi_{J,K} + \mathscr{O}_{m-o(K)}.$$

*Proof.* Consider first the case $r = 0$, and write $\varphi = \varphi_J\theta^J = \tilde\varphi_J\tilde\theta^J$. Since we are not changing $W_\alpha$, it is clear that $\tilde\varphi_J = \varphi_J = \varphi(W_{j_1}, \cdots, W_{j_s})$ if $J$ has no zero entries. On the other hand, since $\tilde T = T + \mathscr{O}_m T + \mathscr{O}_{m-1}(W_\alpha, W_{\bar\alpha})$, the components of $\varphi$ containing zero entries satisfy $\tilde\varphi_J = \varphi_J + \mathscr{O}_{m-1}$.

For $r = 1$, Lemma 3.4 shows that replacing $\omega_\alpha{}^\beta$ by $\tilde\omega_\alpha{}^\beta$ in formula (3.2) can make an error of order at most $\mathscr{O}_{m-o(k)}$. Similarly, replacing $W_k$ by $\tilde W_k$ and $\varphi_J$ by $\tilde\varphi_J$ results in an overall error of at most $\mathscr{O}_{m-o(k)-1}$ (or $\mathscr{O}_{m-o(k)}$ if $J$ has no zero entries). The lemma now follows by induction.

**Lemma 3.6.** *With $\tilde\theta = e^{2u}\theta$, $u \in \mathscr{O}_m$, $m \geq 2$, the pseudohermitian Ricci and torsion tensors satisfy the following approximate transformation laws, computed with respect to a fixed special frame $\{W_\alpha\}$ for $\theta$:*

$$\begin{aligned}
&\tilde A_{\alpha\beta} = A_{\alpha\beta} + 2iu_{,\alpha\beta} + \mathscr{O}_m,\\
&\tilde R_{\alpha\bar\beta} = R_{\alpha\bar\beta} - (n+2)(u_{,\alpha\bar\beta} + u_{,\bar\beta\alpha}) - (u_{,\bar\gamma}{}^{\bar\gamma} + u_{,\gamma}{}^\gamma)h_{\alpha\bar\beta} + \mathscr{O}_m,\\
&\tilde R = R + 2(n+1)\Delta_b u + \mathscr{O}_m,\\
&\tilde A_{\alpha\beta,}{}^\beta = A_{\alpha\beta,}{}^\beta + 2iu_{,\alpha\beta}{}^\beta + \mathscr{O}_{m-1},\\
&\tilde A_{\alpha\beta,}{}^{\beta\alpha} = A_{\alpha\beta,}{}^{\beta\alpha} + 2iu_{,\alpha\beta}{}^{\beta\alpha} + \mathscr{O}_{m-2},\\
&\tilde R_{,\alpha} = R_{,\alpha} + 2(n+1)(\Delta_b u)_{,\alpha} + \mathscr{O}_{m-1},\\
&\tilde\Delta_b\tilde R = \Delta_b R + 2(n+1)\Delta_b^2 u + \mathscr{O}_{m-2}.
\end{aligned}$$

*Proof.* The first two formulas follow immediately from the transformation laws for $A_{\alpha\beta}$ and $R_{\alpha\overline{\beta}}$ given in Lemma 2.4 of [10], and the third follows from the second since $\tilde{h}^{\alpha\overline{\beta}} = e^{-2u}h^{\alpha\overline{\beta}} + \mathscr{O}_m$. Now Lemma 3.5 shows that up to errors of the order indicated, it makes no difference whether we compute covariant derivatives with respect to $\theta$ or $\tilde{\theta}$. Therefore the remaining formulas follow from differentiating the first and third.

**Lemma 3.7.** *Suppose $\tilde{\theta} = e^{2u}\theta$ with $u \in \mathscr{O}_m$, $m \geq 2$. The components of the tensor $Q$, with respect to the fixed special frame $\{W_\alpha\}$, transform as follows:*

$$\begin{aligned}
\tilde{Q}_{\alpha\overline{\beta}} &= Q_{\alpha\overline{\beta}} - (n+2)(Z_\alpha Z_{\overline{\beta}}u + Z_{\overline{\beta}}Z_\alpha u) + \mathscr{L}_0 u h_{\alpha\overline{\beta}} + \mathscr{O}_m,\\
\tilde{Q}_{\alpha\beta} &= Q_{\alpha\beta} - 2(n+2)Z_\alpha Z_\beta u + \mathscr{O}_m,\\
\tilde{Q}_{0\alpha} &= Q_{0\alpha} - 4(n+2)Z_0 Z_\alpha u + \mathscr{O}_{m-1},\\
\tilde{Q}_{00} &= Q_{00} - 8(n+2)Z_0 Z_0 u + \mathscr{O}_{m-2}.
\end{aligned}$$

*Proof.* From the definition of $Q$ and Lemma 3.6,

$$\begin{aligned}
\tilde{Q}_{\alpha\overline{\beta}} - Q_{\alpha\overline{\beta}} &= -(n+2)(u_{,\alpha\overline{\beta}} + u_{,\overline{\beta}\alpha}) - (u_{,\overline{\gamma}}{}^{\overline{\gamma}} + u_{,\gamma}{}^{\gamma})h_{\alpha\overline{\beta}} + \mathscr{O}_m,\\
\tilde{Q}_{\alpha\beta} - Q_{\alpha\beta} &= -2(n+2)u_{,\alpha\beta} + \mathscr{O}_m,\\
\tilde{Q}_{0\alpha} - Q_{0\alpha} &= 8iu_{,\alpha\beta}{}^{\beta} + 4i(\Delta_b u)_{,\alpha} + \mathscr{O}_{m-1},\\
\tilde{Q}_{00} - Q_{00} &= \frac{16}{n}\operatorname{Re} 2u_{,\alpha\beta}{}^{\beta\alpha} - \frac{8}{n}\Delta_b^2 u + \mathscr{O}_{m-2}.
\end{aligned}$$

The result then follows from Lemma 3.3.

**Lemma 3.8.** *Suppose $\tilde{\theta} = e^{2u}\theta$, with $u \in \mathscr{O}_m$, $m \geq 2$. Let $\Psi$ and $\tilde{\Psi}$ denote the parabolic exponential maps at $q \in M$ associated with $\theta$ and $\tilde{\theta}$, respectively. Then $\tilde{\Psi} - \Psi \in \mathscr{O}_{m+1}$ (considered as functions on $T_qM$ with its induced CR structure).*

*Proof.* Let $x = (t, z, \overline{z})$ denote $\theta$-pseudohermitian normal coordinates near $q$ in $M$. Using the parabolic exponential map of $\theta$ to identify a neighborhood of $0 \in T_qM$ with a neighborhood of $q \in M$, we can use the same coordinates on $T_qM$, so that $\Psi^j(x) = x^j$. We can write $\tilde{\Psi}^j(x) = x^j + f^j(x)$ for some smooth functions $f^0, \cdots, f^{2n}$. Then $f^j \in \mathscr{O}_{m+1}$ if and only if, for any $(W, c) \in \mathbf{C}^n \times \mathbf{R}$, $f^j(s^2c, sW, s\overline{W}) = O(s^{m+1})$ as $s \to 0$.

Fix such a $(W, c)$, and let $\gamma(s)$ and $\tilde{\gamma}(s)$ denote the solutions to $\nabla_{\dot{\gamma}}\dot{\gamma} = 2cT$ and $\tilde{\nabla}_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}} = 2c\tilde{T}$ respectively, both with initial tangent vector $W^\alpha\partial_\alpha + W^{\overline{\alpha}}\partial_{\overline{\alpha}}$ at $q$. Note that $\gamma(s) = (s^2c, sW, s\overline{W})$ in $x$-coordinates, and $\tilde{\gamma}^j(s) = \tilde{\Psi}^j(s^2, c, sW, s\overline{W})$. It suffices therefore to show that $\tilde{\gamma}^j(s) - \gamma^j(s) = O(s^{m+1})$.

Let $\Gamma^l_{jk}$ and $\tilde{\Gamma}^l_{jk}$ denote the coefficients of the connections $\nabla$ and $\tilde{\nabla}$ with respect to $x$-coordinates. Then the curve $\sigma(s) = \tilde{\gamma}(s) - \gamma(s)$ satisfies

$$\begin{aligned}\ddot{\sigma}^l(s) = &-\tilde{\Gamma}^l_{jk}(\tilde{\gamma}(s))\dot{\tilde{\gamma}}^j(s)\dot{\tilde{\gamma}}^k(s) + \Gamma^l_{jk}(\gamma(s))\dot{\gamma}^j(s)\dot{\gamma}^k(s)\\ &+ 2c\tilde{T}^l(\tilde{\gamma}(s)) - 2cT^l(\gamma(s)),\end{aligned}$$

with initial conditions $\sigma(0) = \dot{\sigma}(0) = 0$. Therefore

$$\begin{aligned}|\ddot{\sigma}^l(s)| \le\, &|\tilde{\Gamma}^l_{jk}(\tilde{\gamma}(s))\dot{\tilde{\gamma}}^j(s)(\dot{\gamma}^k(s) - \dot{\tilde{\gamma}}^k(s))|\\ &+ |\tilde{\Gamma}^l_{jk}(\tilde{\gamma}(s))(\dot{\gamma}^j(s) - \dot{\tilde{\gamma}}^j(s))\dot{\gamma}^k(s)|\\ &+ |(\tilde{\Gamma}^l_{jk}(\gamma(s)) - \tilde{\Gamma}^l_{jk}(\tilde{\gamma}(s)))\dot{\gamma}^j(s)\dot{\gamma}^k(s)|\\ &+ |(\Gamma^l_{jk} - \tilde{\Gamma}^l_{jk})(\gamma(s))\dot{\gamma}^j(s)\dot{\gamma}^k(s)|\\ &+ |2c\tilde{T}^l(\gamma(s)) - 2c\tilde{T}^l(\tilde{\gamma}(s))| + |2c(T^l - \tilde{T}^l)(\gamma(s))|.\end{aligned}$$

To estimate these terms, let us write

$$B^l_{jk} = \tilde{\Gamma}^l_{jk} - \Gamma^l_{jk} = dx^l(\tilde{\nabla}_{\partial_j}\partial_k - \nabla_{\partial_j}\partial_k).$$

From Lemma 3.4 (and the fact that the difference between two connections transforms as a tensor) it follows that for each $k$, $l$, the 1-form $B^l_k = B^l_{jk}\,dx^j$ is in $\mathscr{O}_m$. Therefore

$$B^l_{jk}(\gamma(s))\dot{\gamma}^j(s) = B^l_k(\dot{\gamma}(s)) = O(s^{m-1}).$$

It is easy to verify that $T - \tilde{T} \in \mathscr{O}_{m-2}$, and so $T^l - \tilde{T}^l \in \mathscr{O}_{m-1}$ and

$$(T^l - \tilde{T}^l)(\gamma(s)) = O(s^{m-1}).$$

Let $\varphi$ denote the nonnegative function

$$\varphi(s) = \sum_j (|\sigma^j(s)|^2 + |\dot{\sigma}^j(s)|^2).$$

Using the above relations and Lipschitz estimates for $\tilde{\Gamma}^l_{jk}$ and $\tilde{T}^l$, for small $s > 0$ we have

$$\begin{aligned}|\ddot{\sigma}^l(s)| &\le C\sum_j |\dot{\sigma}^j(s)| + C\sum_j |\sigma^j(s)| + Cs^{m-1}\\ &\le C(\varphi(s)^{1/2} + s^{m-1}).\end{aligned}$$

Applying this to the smooth function $\varphi$,

$$\begin{aligned}|\dot{\varphi}| &= |2\sum_j(\sigma^j\dot{\sigma}^j + \dot{\sigma}^j\ddot{\sigma}^j)| \le 2\sum_j |\sigma^j|\,|\dot{\sigma}^j| + C\sum_j |\dot{\sigma}^j|(\varphi^{1/2} + s^{m-1})\\ &\le C(\varphi + \varphi^{1/2}s^{m-1}).\end{aligned}$$

One can check directly that the ordinary differential equation $\dot{y} = C(y+y^{1/2}s^{m-1})$ has a unique solution with initial value $y(0)=0$, and this solution satisfies $y = O(s^{2m})$. Therefore, by a standard comparison theorem for ordinary differential equations, we conclude that $\varphi = O(s^{2m})$. In particular, this means that $\dot{\sigma}^j = O(s^m)$, and therefore $\sigma^j = O(s^{m+1})$. q.e.d.

We remark that the above estimates can be refined somewhat to show that $\tilde{t} = t + \mathscr{O}_{m+2}$. We will not need this additional information, so we do not prove it here.

Now let $P = Z^\alpha \partial_\alpha + z^{\overline{\alpha}}\partial_{\overline{\alpha}} + 2t\partial_t$ be the infinitesimal generator of the parabolic dilations in pseudohermitian normal coordinates. To prove Theorem 3.1, we introduce the auxiliary scalar function $S$, defined near $q$ by

$$S = Q(P - tT, P - tT).$$

It is easy to see that $S$ is determined by the pseudohermitian structure and the point $q$, independently of choice of frame. With respect to a special frame and pseudohermitian normal coordinates, $P - tT = x^j W_j$ by Lemma 2.4, and so this can be written

$$S = \sum_{j,k} x^j x^k Q_{jk}.$$

We will compute the transformation law for $S$ under a change in contact form $\tilde{\theta} = e^{2u}\theta$, where $u \in \mathscr{P}_m$, $m \geq 2$.

By Lemma 3.8, the pseudohermitian normal coordinates $\tilde{x}$ associated with $\tilde{\theta}$, considered as functions of the original pseudohermitian normal coordinates, satisfy $\tilde{x}(x) = x + \mathscr{O}_{m+1}$, and thus $\tilde{P} = P + \mathscr{O}_{m-1}$ and $\tilde{t}\tilde{T} = tT + \mathscr{O}_{m-1}$. Therefore, since $\tilde{Q} \in \mathscr{O}_2$,

$$\begin{aligned}\tilde{S} &= \tilde{Q}(\tilde{P} - \tilde{t}\tilde{T}, \tilde{P} - \tilde{t}\tilde{T}) = \tilde{Q}(P - tT, P - tT) + \mathscr{O}_{m+1}\\ &= \sum_{j,k} x^j x^k \tilde{Q}_{jk} + \mathscr{O}_{m+1}.\end{aligned}$$

From Lemma 3.7, therefore, it follows that $S_{(m)}$ satisfies

$$\tilde{S}_{(m)} = S_{(m)} + 4|z|^2\mathscr{L}_0 u - 2(n+2)(x^a x^b Z_a Z_b u + 4x^0 x^a Z_0 Z_a u + 4x^0 x^0 Z_0 Z_0 u).$$

Since $u \in \mathscr{P}_m$, $Pu = mu$ and

$$\begin{aligned}m^2 u = P^2 u &= (x^a Z_a + 2x^0 Z_0)^2 u\\ &= x^a x^b Z_a Z_b u + 4x^0 x^a Z_0 Z_a u + 4x^0 x^0 Z_0 Z_0 u + 2x^0 Z_0 u + Pu.\end{aligned}$$

Thus, writing $L_m = 4|z|^2\mathscr{L}_0 + 4(n+2)x^0 Z_0 - 2(n+2)m(m-1)$, we have

$$\tilde{S}_{(m)} = S_{(m)} + L_m u. \tag{3.3}$$

The following lemma shows when the operator $L_m$ is solvable.

**Lemma 3.9.** *Let $L_m = 4|z|^2\mathscr{L}_0 + 4(n+2)x^0Z_0 - 2(n+2)m(m-1)$. If $m \geq 3$, $L_m$ is invertible on $\mathscr{P}_m$. On $\mathscr{P}_2$, $L_2$ has one-dimensional kernel spanned by the function $u(z,t) = t$, and is invertible on the subspace consisting of polynomials independent of $t$.*

*Proof.* Let $\mathscr{R}_m \subset \mathscr{P}_m$ denote the subspace consisting of polynomials independent of $t$. $\mathscr{P}_m$ decomposes as follows:

$$\mathscr{P}_m = \mathscr{R}_m \oplus t\mathscr{R}_{m-2} \oplus \cdots \oplus t^k\mathscr{R}_{m-2k},$$

where $k = [m/2]$. Let $\pi_s\colon \mathscr{P}_m \to t^s\mathscr{R}_{m-2s}$ denote the projection onto the $s$th term. If $u \in \mathscr{R}_m$, then $4\mathscr{L}_0u(z,t) = -4\partial_\alpha\partial_{\overline{\alpha}}u(z) = \Delta_0u(z)$, where $\Delta_0$ is the Euclidean Laplacian on $\mathbb{C}^n$. It is easy to verify (see Lemma 5.4 of [11]) that $|z|^2\Delta_0$ has no positive eigenvalues on $\mathscr{R}_m$.

On $\mathscr{R}_2$, $L_2 = |z|^2\Delta_0 - 4(n+2)$ is invertible. On the other hand, direct calculation shows that $L_2t = 0$. Since $\mathscr{P}_2 = \mathscr{R}_2 \oplus \mathbb{R}t$, this proves the lemma in the case $m = 2$.

For $m \geq 3$, suppose $L_mu = 0$ for some nontrivial $u \in \mathscr{P}_m$. Let $s$ be the largest integer such that $\pi_su \neq 0$, and write $\pi_su(z,t) = t^sv(z)$, $v \in \mathscr{R}_{m-2s}$. Then a straightforward computation shows that

$$\begin{aligned} 0 = \pi_sL_mu(z,t) &= |z|^2t^s\Delta_0v(z) + 4(n+2)st^sv(z) - 2(n+2)m(m-1)t^sv(z) \\ &= t^s(|z|^2\Delta_0 + (n+2)(4s - 2m(m-1)))v(z). \end{aligned}$$

Since $s \leq m/2$, this implies that $v = 0$, which is a contradiction. Thus $L_m$ is invertible on $\mathscr{P}_m$. q.e.d.

Next we will relate the function $S$ to the symmetrized covariant derivatives of $Q$. In order to do so, we will need the following version of Taylor's theorem for pseudohermitian normal coordinates.

**Lemma 3.10.** *Let $F$ be a smooth function defined near $q$. Then in pseudohermitian normal coordinates, for any $m$,*

$$F_{(m)}(x) = \sum_{o(K)=m} \frac{1}{(\#K)!}x^KZ_KF(q). \tag{3.4}$$

*Proof.* By the classical version of Taylor's theorem,

$$F_{(m)}(x) = \sum_{o(K)=m} \frac{1}{(\#K)!}x^K\partial_KF(q).$$

Consider the operator $Z_K$. Since $Z_0 = \partial_0$ commutes with $Z_j$ for all $j$, we can factor out the 0-derivatives and write $Z_K = Z_0^lZ_A = \partial_0^lZ_A$, where $A$ is a multi-index with no zero entries. Therefore it suffices to prove that

$$\sum_{\#A=m} x^AZ_AF(q) = \sum_{\#A=m} x^A\partial_AF(q)$$

for all $m$. This is proved by induction on $m$.

For $m = 1$, we just compute:

$$\sum_a x^a Z_a F(q) = \sum_\alpha z^\alpha(\partial_\alpha + iz^{\overline{\alpha}}\partial_0)F(q) + \sum_{\overline{\alpha}} z^{\overline{\alpha}}(\partial_{\overline{\alpha}} - iz^\alpha\partial_0)F(q)$$
$$= \sum_a x^a \partial_a F(q).$$

Suppose $m > 1$. By direct computation $[\partial_{\overline{\beta}}, Z_\alpha] = -[\partial_\alpha, Z_{\overline{\beta}}] = i\delta_{\alpha\overline{\beta}}\partial_0$, and $[\partial_\alpha, Z_\beta] = [\partial_{\overline{\alpha}}, Z_{\overline{\beta}}] = 0$. It follows that

$$\sum_{a,b} x^a x^b \partial_a Z_b F(q) = \sum_{a,b} x^a x^b Z_b \partial_a F(q).$$

Using the inductive hypothesis and this relation, we have

$$\begin{aligned}
\sum_{\#A=m} x^A Z_A F(q) &= \sum_a x^a \sum_{\#B=m-1} x^B Z_B(Z_a F)(q) \\
&= \sum_a x^a \sum_{\#B=m-1} x^B \partial_B(Z_a F)(q) \\
&= \sum_{\#B=m-1} x^B \sum_a x^a Z_a(\partial_B F)(q) \\
&= \sum_{\#B=m-1} x^B \sum_a x^a \partial_a(\partial_B F)(q) \\
&= \sum_{\#A=m} x^A \partial_A F(q). \quad \text{q.e.d.}
\end{aligned}$$

Now applying this lemma to $Q_{jk}$, we can write the $m$th order homogeneous part of $S$ as

$$S_{(m)}(x) = \sum_{o(jkL)=m} \frac{1}{(\#L)!} x^j x^k x^L Z_L Q_{jk}(q).$$

If $\tilde{\theta} = e^{2u}\theta$ with $u \in \mathscr{P}_m$, then by Lemma 3.5 the covariant derivative $\tilde{Q}_{jk,L}(q)$ for $o(jkL) = m$ can be computed with respect to the connection determined by $\theta$ instead of $\tilde{\theta}$, since the error is at worst of order $\mathscr{O}_{m-o(L)-1}$ and therefore vanishes at $q$. By Lemma 3.2, since $\tilde{Q} - Q \in \mathscr{O}_m$,

$$\tilde{Q}_{jk,L}(q) - Q_{jk,L}(q) = Z_L \tilde{Q}_{jk}(q) - Z_L Q_{jk}(q),$$

and so using (3.3) we have

$$(3.5) \quad L_m u = \tilde{S}_{(m)} - S_{(m)} = \sum_{o(jkL)=m} \frac{1}{(\#L)!} x^j x^k x^L (\tilde{Q}_{jk,L}(q) - Q_{jk,L}(q)).$$

The main ingredient in the proof of Theorem 3.1 is the following lemma.

**Lemma 3.11.** *Let $q \in M$, let $\theta$ be any contact form on $M$, and let $(z,t)$ be pseudohermitian normal coordinates for $\theta$ at $q$. For any $m \geq 2$, there is a polynomial $u \in \mathscr{P}_m$ in $(z,t)$ such that $\tilde{\theta} = e^{2u}\theta$ satisfies*

$$\tilde{Q}_{\langle jk,L\rangle}(q) = 0 \quad \text{if } o(jkL) = m. \tag{3.6}$$

*If $m \geq 3$, $u$ is unique. If $m = 2$, there is a unique choice of $u \in \mathscr{R}_2$.*

*Proof.* By Lemma 3.9, if $m \geq 3$ there exists a unique $u \in \mathscr{P}_m$ such that

$$L_m u = -\sum_{o(jkL)=m} \frac{1}{(\#L)!} x^j x^k x^L Q_{jk,L}(q).$$

If $m = 2$, note that the right-hand side above is independent of $t$; thus there is a unique such $u \in \mathscr{R}_2$. Therefore if we put $\tilde{\theta} = e^{2u}\theta$, from (3.5) it follows that

$$\sum_{o(jkL)=m} \frac{1}{(\#L)!} x^j x^k x^L \tilde{Q}_{jk,L}(q) = 0.$$

For any fixed multi-index $jkL$ of order $m$, the coefficient of $x^j x^k x^L$ in this polynomial is a multiple of $\tilde{Q}_{\langle jk,L\rangle}(q)$, so this choice of $u$ satisfies the conclusion of the lemma. q.e.d.

*Proof of Theorem* 3.1. Simply apply the preceding Lemma repeatedly, noting that when $\tilde{\theta} = e^{2u}\theta$ with $u \in \mathscr{P}_m$, the same argument shows that $\tilde{Q}_{\langle jk,L\rangle}(q) = Q_{\langle jk,L\rangle}(q)$ is unchanged when $o(jkL) < m$. If the one-jet of $u$ is chosen arbitrarily, this construction then inductively determines the higher terms in the Taylor series of $u$ uniquely. q.e.d.

For our application of Theorem 3.1 in the next section, we will need to know what the normalization (3.1) means explicitly up to order 4.

**Proposition 3.12.** *Suppose $\theta$ is a contact form satisfying* (3.1) *for $N = 4$. Then the following relations hold at $q$ :*

(a) $R = 0;\quad R_{\alpha\overline{\beta}} = 0;\quad A_{\alpha\beta} = 0;$

(b) $A_{\alpha\beta,\gamma} = 0;$

(c) $R_{,\alpha} = A_{\alpha\beta,}{}^{\beta} = R_{\alpha\overline{\beta}}{}^{\overline{\beta}} = 0;$

(d) $R_{\alpha\overline{\beta},}{}^{\alpha\overline{\beta}} = A_{\alpha\beta,}{}^{\alpha\beta} = \Delta_b R = R_{,0} = 0.$

*Proof.* First we note the following Bianchi identities satisfied by the pseudohermitian Ricci and torsion [10]:

$$A_{\alpha\beta,\gamma} = A_{\alpha\gamma,\beta}, \tag{3.7}$$

$$R_{,\alpha} - R_{\alpha\overline{\beta},}{}^{\overline{\beta}} = (n-1)iA_{\alpha\beta,}{}^{\beta}, \tag{3.8}$$

$$R_{,0} = A_{\alpha\beta,}{}^{\beta\alpha} + A_{\overline{\alpha\beta},}{}^{\overline{\beta\alpha}}. \tag{3.9}$$

Differentiating (3.8), contracting, and adding to its conjugate, we obtain

$$-\Delta_b R - 2\operatorname{Re} R_{\alpha\overline{\beta},}{}^{\overline{\beta}\alpha} = -2(n-1)\operatorname{Im} A_{\alpha\beta,}{}^{\beta\alpha}. \tag{3.10}$$

Writing out the symmetrized derivatives of $Q$ at $q$, we have for $m = 2$:

$$0 = Q_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}}, \qquad 0 = Q_{\alpha\beta} = (n+2)iA_{\alpha\beta},$$

which is (a). For $m = 3$,

$$0 = Q_{0\alpha} = 4A_{\alpha\beta,}{}^{\beta} + \frac{2i}{n+1}R_{,\alpha}, \tag{3.11}$$

$$0 = Q_{\alpha\beta,\overline{\gamma}} + Q_{\beta\overline{\gamma},\alpha} + Q_{\overline{\gamma}\alpha,\beta} = (n+2)iA_{\alpha\beta,\overline{\gamma}} + R_{\beta\overline{\gamma},\alpha} + R_{\alpha\overline{\gamma},\beta}, \tag{3.12}$$

$$0 = Q_{\alpha\beta,\gamma} + Q_{\beta\gamma,\alpha} + Q_{\gamma\alpha,\beta} = (n+2)i(A_{\alpha\beta,\gamma} + A_{\beta\gamma,\alpha} + A_{\gamma\alpha,\beta}). \tag{3.13}$$

Together with (3.7), this proves (b). Contracting (3.12) on the indices $\beta, \overline{\gamma}$ yields

$$0 = (n+2)iA_{\alpha\beta,}{}^{\beta} + R_{,\alpha} + R_{\alpha\overline{\beta},}{}^{\overline{\beta}}. \tag{3.14}$$

Combining (3.8), (3.11), and (3.14), we obtain (c).

Finally, for $m = 4$,

$$0 = Q_{00} = \frac{16}{n}\operatorname{Im} A_{\alpha\beta,}{}^{\beta\alpha} - \frac{4}{n(n+1)}\Delta_b R, \tag{3.15}$$

$$0 = Q_{\alpha\overline{\gamma},0} + Q_{0\alpha,\overline{\gamma}} + Q_{0\overline{\gamma},\alpha}, \tag{3.16}$$

$$\begin{aligned} 0 = {} & Q_{\alpha\beta,\overline{\rho\gamma}} + Q_{\alpha\beta,\overline{\gamma\rho}} + Q_{\overline{\rho\gamma},\alpha\beta} + Q_{\overline{\rho\gamma},\beta\alpha} + Q_{\alpha\overline{\rho},\beta\overline{\gamma}} + Q_{\alpha\overline{\rho},\overline{\gamma}\beta} \\ & + Q_{\alpha\overline{\gamma},\beta\overline{\rho}} + Q_{\alpha\overline{\gamma},\overline{\rho}\beta} + Q_{\beta\overline{\rho},\alpha\overline{\gamma}} + Q_{\beta\overline{\rho},\overline{\gamma}\alpha} + Q_{\beta\overline{\gamma},\alpha\overline{\rho}} + Q_{\beta\overline{\gamma},\overline{\rho}\alpha}. \end{aligned} \tag{3.17}$$

Contracting (3.16), we get

$$\begin{aligned} 0 &= Q_{\alpha}{}^{\alpha}{}_{,0} + Q_{0\alpha,}{}^{\alpha} + Q_{0\overline{\alpha},}{}^{\overline{\alpha}} \\ &= R_{,0} + 4A_{\alpha\beta,}{}^{\beta\alpha} + 4A_{\overline{\alpha\beta},}{}^{\overline{\beta\alpha}} + \frac{2i}{n+1}(R_{,\alpha}{}^{\alpha} - R_{,\overline{\alpha}}{}^{\overline{\alpha}}) \\ &= \frac{1-n}{1+n}R_{,0} + 8\operatorname{Re} A_{\alpha\beta,}{}^{\beta\alpha}. \end{aligned}$$

Combining this with (3.9), we obtain $R_{,0} = \operatorname{Re} A_{\alpha\beta,}{}^{\beta\alpha} = 0$.

Finally, contracting (3.17) on the indices $\alpha, \overline{\rho}$ and again on $\beta, \overline{\sigma}$, we get

(3.18)

$$\begin{aligned} 0 &= 2Q_{\alpha\beta,}{}^{\beta\alpha} + 2Q_{\overline{\rho\gamma},}{}^{\overline{\gamma\rho}} + 2Q_{\alpha}{}^{\alpha}{}_{,\beta}{}^{\beta} + 2Q_{\alpha}{}^{\alpha}{}_{,\overline{\gamma}}{}^{\overline{\gamma}} + 2Q_{\alpha\overline{\gamma},}{}^{\alpha\overline{\gamma}} + 2Q_{\alpha\overline{\gamma},}{}^{\overline{\gamma}\alpha} \\ &= (n+2)i(A_{\alpha\beta,}{}^{\beta\alpha} - A_{\overline{\rho\gamma},}{}^{\overline{\gamma\rho}}) + 2(R_{,\beta}{}^{\beta} + R_{,\overline{\gamma}}{}^{\overline{\gamma}}) \\ &\quad + 2(R_{\alpha\overline{\gamma},}{}^{\alpha\overline{\gamma}} + R_{\alpha\overline{\gamma},}{}^{\overline{\gamma}\alpha}) \\ &= -2(n+2)\operatorname{Im} A_{\alpha\beta,}{}^{\beta\alpha} - 2\Delta_b R + 4\operatorname{Re} R_{\alpha\overline{\gamma},}{}^{\overline{\gamma}\alpha}. \end{aligned}$$

Combining (3.10), (3.15), and (3.18) yields $\Delta_b R = \operatorname{Re} R_{\alpha\overline{\gamma},}{}^{\overline{\gamma}\alpha} = \operatorname{Im} A_{\alpha\beta,}{}^{\beta\alpha} = 0$. The proof is completed by applying the commutation relation for second covariant derivatives of $R_{\alpha\overline{\gamma}}$ (cf. [10]):

$$\begin{aligned}\operatorname{Im} R_{\alpha\overline{\gamma},}{}^{\overline{\gamma}\alpha} &= \frac{1}{2i}(R_{\alpha\overline{\gamma},}{}^{\overline{\gamma}\alpha} - R_{\alpha\overline{\gamma},}{}^{\alpha\overline{\gamma}}) \\ &= \frac{1}{2i}(ih^{\alpha\overline{\gamma}}R_{\alpha\overline{\gamma},0} + R_{\rho\overline{\gamma}}R_{\alpha}{}^{\rho\overline{\gamma}\alpha} - R_{\alpha\overline{\rho}}R_{\overline{\gamma}}{}^{\overline{\rho}\alpha\overline{\gamma}}) = \frac{1}{2}R_{,0} = 0. \quad \text{q.e.d.}\end{aligned}$$

Now let $G$ denote the group of CR automorphisms of the sphere $S^{2n+1}$ that fix a point (see [3] or [2]). By means of the Cayley transform, we can think of $G$ as a group of local CR automorphisms of $\mathbb{H}^n$ that fix the origin. The group $G$ decomposes as a semidirect product $G = \mathrm{U}(n) \ltimes \mathbb{R}^+ \ltimes H^\infty$, where $\mathrm{U}(n)$ is the unitary group acting in the $z$ variables, $\mathbb{R}^+$ is the group of parabolic dilations, and $H^\infty$ is isomorphic to the Heisenberg group, acting as "translations at infinity". The group $G$ acts simply transitively on the set of CR normal coordinate charts at $q \in M$ in the following way. If we choose a normal coordinate chart and use it to identify a neighborhood of $q \in M$ with a neighborhood of $0 \in \mathbb{H}^n$, then each element of $G$ induces a local diffeomorphism of $M$ fixing $q$. These are not in general CR automorphisms; however, it is easy to check that the action by an element of $G$ is a CR automorphism of *$M$ to first order.* The subgroup $\mathrm{U}(n)$ changes the orthonormal frame for $\mathscr{H}_q$, and $\mathbb{R}^+$ multiplies the contact form $\theta$ at $q$ by a scale factor. The subgroup $\mathbb{R}^+ \ltimes H^\infty$ acts transitively on the set of one-jets of contact forms $\theta$ for $M$ at $q$. Since a CR normal coordinate chart is determined uniquely by fixing the one-jet of a contact form $\theta$ and then choosing a $\theta$-orthonormal frame for $\mathscr{H}_q$, each element of $G$ in this way uniquely determines a new normal coordinate chart. Thus the set of intrinsic CR normal coordinate charts defined here is parametrized by $G$, just as are the extrinsic normal coordinates defined by Chern and Moser [3].

## 4. The asymptotic expansion of the Yamabe functional

Let $\mathscr{Y}_M$, given by (1.1), denote the Yamabe functional on a $(2n+1)$-dimensional strictly pseudoconvex CR manifold $M$. As shown in [7], if we fix a background contact form $\theta$, the Yamabe invariant $\lambda(M)$ can be expressed as the infimum of

$$\mathscr{Y}_\theta(f) \equiv \mathscr{Y}_M(f^{p-2}\theta) = \frac{\int_M (p|df|_\theta^2 + Rf^2)\theta\wedge d\theta^n}{(\int_M f^p\theta\wedge d\theta^n)^{2/p}}$$

over nonnegative functions $f \in C^\infty(M)$, where $|df|_\theta^2 = f_{,\beta}f^{,\beta} + f_{,\overline{\beta}}f^{,\overline{\beta}} = 2f_{,\beta}f^{,\beta}$. In this section we will construct a family of test functions $f^\varepsilon$ for

$\mathscr{Y}_\theta$, by transplanting extremal functions from the Heisenberg group to $M$ by means of the normal coordinates developed in §§2 and 3.

It was shown in [8] that the function $\Phi(z,t) = |w+i|^{-n}$ (where $w = t+i|z|^2$) is an extremal for the Yamabe functional $\mathscr{Y}_\Theta$ on the Heisenberg group. For each $\varepsilon > 0$, $\Phi^\varepsilon \equiv \varepsilon^{-n}\delta^*_{1/\varepsilon}\Phi = \varepsilon^n|w+i\varepsilon^2|^{-n}$ is also an extremal, normalized so that $\int_{\mathbf{H}^n} |\Phi^\varepsilon|^p \Theta \wedge d\Theta^n$ is a constant independent of $\varepsilon$. This mass of $|\Phi^\varepsilon|^p$ is concentrated closer and closer to the origin as $\varepsilon$ tends to 0.

Suppose that $(z,t)$ are pseudohermitian normal coordinates for some contact form $\theta$ near $q \in M$, defined for $|w| < 2\kappa$ for some $\kappa > 0$. Define a test function

$$f^\varepsilon(z,t) = \psi(w)\Phi^\varepsilon(z,t),$$

where $\psi \in C_0^\infty(\mathbf{C})$ is supported in the set $\{|w| < 2\kappa\}$, and $\psi(w) = 1$ for $|w| < \kappa$. Our goal is the following theorem, from which Theorem A follows as in the introduction.

**Theorem 4.1.** *Let $\theta$ be a contact form near $q \in M$ which satisfies the normalization of Proposition* 3.12. *As $\varepsilon \to 0$, $\mathscr{Y}_\theta(f^\varepsilon)$ satisfies the asymptotic formula*

$$\mathscr{Y}_\theta(f^\varepsilon) = \begin{cases} pn^2\pi(1 - c(n)|S(q)|^2\varepsilon^4) + O(\varepsilon^5) & \text{for } n \geq 3, \\ pn^2\pi(1 - c(2)|S(q)|^2\varepsilon^4 \log\frac{1}{\varepsilon}) + O(\varepsilon^4) & \text{for } n = 2, \end{cases}$$

*with $c(n) > 0$. Thus if the Chern tensor $S$ does not vanish at $q$, there exists $\varepsilon > 0$ such that $\mathscr{Y}_\theta(f^\varepsilon) < \Lambda = pn^2\pi$, and the CR Yamabe problem can be solved on $M$.*

The first step is to prove

**Proposition 4.2.** *With the notations of Theorem* 4.1, *there are dimensional constants $a_k(n)$, $b_k(n)$ such that for $n \geq 3$,*

$$\int_M |df^\varepsilon|^2_\theta \theta \wedge d\theta^n = a_0(n) + a_4(n)|S(q)|^2\varepsilon^4 + O(\varepsilon^5), \tag{4.1}$$

$$\int_M R|f^\varepsilon|^2 \theta \wedge d\theta^n = O(\varepsilon^5), \tag{4.2}$$

$$\int_M |f^\varepsilon|^p \theta \wedge d\theta^n = b_0(n) + b_4(n)|S(q)|^2\varepsilon^4 + O(\varepsilon^5). \tag{4.3}$$

*For $n = 2$, we have instead*

$$\int_M |df^\varepsilon|^2_\theta \theta \wedge d\theta^2 = a_0(2) + a_4(2)|S(q)|^2\varepsilon^4 \log\frac{1}{\varepsilon} + O(\varepsilon^4), \tag{4.4}$$

$$\int_M R|f^\varepsilon|^2 \theta \wedge d\theta^2 = O(\varepsilon^4), \tag{4.5}$$

$$\int_M |f^\varepsilon|^p \theta \wedge d\theta^2 = b_0(2) + O(\varepsilon^4). \tag{4.6}$$

Once we have shown that the expansions take the form shown in (4.1) through (4.6), we will see that we can ignore many terms when we make the explicit calculations of the constants. These calculations will be carried out in §5. (The constant $c(n)$ is of course readily expressed in terms of $a_k(n)$ and $b_k(n)$.)

All of the Taylor expansions we will need are expressed in terms of the ones given in Proposition 2.5. Thus let $\{W_\alpha\}$ be a special frame and $\{\theta^\alpha\}$ the dual special coframe. In addition to $\theta$ and $\theta^\alpha$, we will need to examine the Taylor series of $W_j$, which we write as

$$W_j = s_j^k Z_k = s_j^\beta Z_\beta + s_j^{\overline{\beta}} Z_{\overline{\beta}} + s_j^0 Z_0. \tag{4.7}$$

(Here as in §3 we write $W_0 = T$, $Z_0 = \partial/\partial t$, $\overline{\alpha} = \alpha + n$, and sum $k = 0, 1, \cdots, 2n$.) Taking terms of homogeneous degree $\leq -o(j)$ in (4.7) and recalling that $W_{\alpha(-1)} = Z_\alpha$ and $W_{0(-2)} = Z_0$, we find

$$s^\beta_{\alpha(0)} = \delta^\beta_\alpha, \qquad s^0_{0(0)} = 1, \tag{4.8}$$

$$s^{\overline{\beta}}_{\alpha(0)} = s^0_{\alpha(0)} = s^0_{\alpha(1)} = 0. \tag{4.9}$$

If we apply $\theta^l$ to (4.7) and consider terms of homogeneity $m + o(l) - o(j)$ for $m > 0$, we obtain

$$s^l_{j(m+o(l)-o(j))} = -\sum_{i\geq 2} s^k_{j(m+o(k)-o(j)-i)} \theta^l_{(o(l)+i)}(Z_k). \tag{4.10}$$

(Note that the sum begins with $i = 2$, because $\theta^l_{(o(l)+1)} = 0$.)

Terms in these expansions can be assigned a *weight* as follows. Suppose that $F$ is a homogeneous polynomial in $x = (t, z, \overline{z})$ whose coefficients are polynomial expressions of curvature, torsion, and their covariant derivatives at $q$, and of the Levi form at $q$. We define the weight $w(F)$ recursively as follows:

(a) $w(A_{\alpha\beta,J}(q)) = w(R_{\alpha\overline{\beta}\gamma\overline{\rho},J}(q)) = 2 + o(J)$,

(b) $w(F_1F_2) = w(F_1) + w(F_2)$,

(c) $w(h_{\alpha\overline{\beta}}(q)) = w(h^{\alpha\overline{\beta}}(q)) = w(c) = 0$,

(d) if $w(F_J) = m$ for all $J$, then $w(\sum F_J x^J) = m$.

Here $c$ denotes a constant independent of the choice of pseudohermitian structure. For consistency in part (d) we need to use the convention that the constant 0 has weight $m$ for every $m$. Note that we have not defined and will never need to consider the weight of a sum of two terms of different weights. In particular, $R_{\rho\overline{\gamma}}(q) = h^{\alpha\overline{\beta}}(q)R_{\alpha\overline{\beta}\rho\overline{\gamma}}(q)$ and $R(q) = h^{\alpha\overline{\beta}}(q)h^{\rho\overline{\gamma}}(q)R_{\alpha\overline{\beta}\rho\overline{\gamma}}(q)$

have weight 2, whereas $A_{\alpha\beta,0}(q)z^\alpha z^\beta$ and $R_{\alpha\overline{\beta}\rho\overline{\gamma}}(q)R^{\alpha\overline{\beta}\rho\overline{\gamma}}(q)$ have weight 4. This last expression will concern us the most.

According to Webster's formula [15, (3.8)] the Chern tensor $S_{\beta\rho}{}^{\alpha}{}_{\overline{\gamma}}(q)$ equals $R_\beta{}^\alpha{}_{\rho\overline{\gamma}}(q)$ provided $R_{\alpha\overline{\beta}}(q) = 0$. Thus if $\theta$ is normalized as in Proposition 3.12, the square of the Chern tensor at $q$ is given by

$$|S(q)|^2 = R_{\alpha\overline{\beta}\rho\overline{\gamma}}(q)R^{\alpha\overline{\beta}\rho\overline{\gamma}}(q). \tag{4.11}$$

By classical invariant theory for the unitary group, the only pseudohermitian-invariant scalars of weight $\leq 4$ must be complete contractions of tensor products of the pseudohermitian curvature and torsion and their covariant derivatives at $q$. Since all the indices must be contracted in pairs, there are no such invariants of odd weight. It is easy to verify that the only invariants of weight 0 and 2 are dimensional constants and multiples of the scalar curvature $R(q)$, respectively. When the weight is 4, the pseudohermitian curvature or torsion can be differentiated at most by order 2; thus the curvature tensor can only appear squared, as in $R_{\alpha\overline{\beta}\rho\overline{\gamma}}(q)R^{\alpha\overline{\beta}\rho\overline{\gamma}}(q) = |S(q)|^2$, or with at least one pair of indices contracted, as in $R_{\alpha\overline{\beta},}{}^{\overline{\beta}\alpha}(q)$ or $R_{,0}(q)$. If $\theta$ satisfies the normalization of Proposition 3.12, however, the torsion and Ricci tensors vanish at $q$, as do all scalars formed by complete contraction of their second derivatives; thus $|S(q)|^2$ is the only nontrivial invariant of weight $\leq 4$.

We can extend the notion of weight to tensors by saying that a tensor is of weight $m$ if its components relative to the bases $\{\Theta, dz^\alpha, dz^{\overline{\alpha}}\}$, $\{Z_0, Z_\alpha, Z_{\overline{\alpha}}\}$ are polynomials of weight $m$.

**Lemma 4.3.** *$\theta_{(m)}$ and $d\theta_{(m)}$ have weight $m-2$; $W_{\alpha(m)}$ has weight $m+1$; $R_{(m)}$ has weight $m+2$.*

*Proof.* We will prove by induction on $k$ that the following all have weight $k$:

$$\theta_{(k+2)}, \quad d\theta_{(k+2)}, \quad \theta^\alpha_{(k+1)}, \quad \omega_\alpha{}^\beta{}_{(k)}, \quad W_{j(k-o(j))},$$
$$s^l_{j(k+o(l)-o(j))}, \quad A_{\alpha\beta,J(k-2-o(J))}, \quad R_{\alpha\overline{\beta}\rho\overline{\sigma},J(k-2-o(J))}.$$

The induction hypothesis follows immediately from Proposition 2.5, (4.8), and (4.9) for $k \leq 0$. Suppose that it is true for all $k \leq m$. Proposition 2.5 and formula (4.10) show that when $k = m+1$ it is valid for $\theta$, $\theta^\alpha$, $\omega_\alpha{}^\beta$, and $s^l_j$. Since $d\theta = 2i\theta^\alpha \wedge \theta^{\overline{\alpha}}$ and $W_j$ is given by (4.7), we also find that $d\theta_{(k+2)}$ and $W_{j(k-o)(j))}$ have weight $k$ for $k \leq m+1$. It remains only to check the induction step for curvature and torsion.

We begin with $A_{\alpha\beta}$. We need to show that $A_{\alpha\beta(m-1)}$ has weight $m+1$; using Lemma 3.10, it suffices to show $Z_J A_{\alpha\beta}(q)$ has weight $m+1$ for all multi-indices $J$ such that $o(J) = m - 1$. Denote

$$(4.12)\qquad \begin{aligned} P_{\alpha\beta,J,j} &= \omega_\alpha{}^\gamma(W_j)A_{\gamma\beta,J} + \omega_\beta{}^\gamma(W_j)A_{\alpha\gamma,J} \\ &\quad + \sum_{i=1}^{r} \omega_{j_i}{}^l(W_j)A_{\alpha\beta,j_1\cdots j_{i-1}lj_{i+1}\cdots j_r} \end{aligned}$$

for $J = (j_1, \cdots, j_r)$. For any subset $A = (i_1, \cdots, i_s) \subset \{1, \cdots, r\}$, denote $J[A] = (j_{i_1}, \cdots, j_{i_s})$. Formula (3.2) implies

$$(4.13)\qquad A_{\alpha\beta,Jj} = W_j A_{\alpha\beta,J} - P_{\alpha\beta,J,j}.$$

By induction on $\#J$, we can then deduce that

$$(4.14)\qquad A_{\alpha\beta,J} = W_J A_{\alpha\beta} - \sum_{A,B,C} \alpha_{A,B,C} W_{J[A]} P_{\alpha\beta,J[B],J[C]},$$

where the sum runs over all disjoint partitions $A, B, C$ of $\{1, \cdots, \#J\}$ with $\#C = 1$. The coefficients $\alpha_{A,B,C}$ are constants of weight 0.

For any $k \le m + 1$, consider

$$(4.15)\qquad [\omega_\alpha{}^\gamma(W_j)A_{\gamma\beta,J}]_{(k-o(Jj)-2)} = \sum_{k_1+k_2=k-o(Jj)-2} [\omega_\alpha{}^\gamma(W_j)]_{(k_1)} A_{\gamma\beta,J(k_2)}.$$

Notice that the summand is zero unless $k_1 \ge 0$. Thus $k_2 \le k - o(Jj) - 2 \le m - o(J) - 2$. Therefore, by the induction hypothesis, $A_{\gamma\beta,J(k_2)}$ has weight $k_2 + o(J) + 2$. Similarly, $k_2 \ge 0$ implies $k_1 \le k - o(Jj) - 2 \le m - 1 - o(Jj)$. Now

$$[\omega_\alpha{}^\gamma(W_j)]_{(k_1)} = \sum_{l_1+l_2=k_1} \omega_\alpha{}^\gamma{}_{(l_1)}(W_{j(l_2)}),$$

and $l_1 \ge 1$ implies $l_2 \le k_1 - 1 \le m - 2 - o(J) - o(j)$, so by the induction hypothesis $W_{j(l_2)}$ has weight $l_2 + o(j)$. Since $l_2 \ge -o(j)$, $l_1 \le k_1 + o(j) \le m - 1 - o(J)$; hence $\omega_\alpha{}^\gamma{}_{(l_1)}$ has weight $l_1$. Thus $[\omega_\alpha{}^\gamma(W_j)]_{(k_1)}$ has weight $l_1 + l_2 + o(j) = k_1 + o(j)$ and (4.15) has weight $(k_2 + o(J) + 2) + (k_1 + o(j)) = k$. Almost identical reasoning on the other terms (with the extra observation that $\omega_j{}^l = 0$ for $l = 0$) shows that $P_{\alpha\beta,J,j(k-o(Jj)-2)}$ has weight $k$ for all $k \le m+1$.

If we evaluate (4.14) at $q$ with $o(J) = m - 1$, we see that $W_J A_{\alpha\beta}(q) = A_{\alpha\beta,J}(q)$ plus a sum of terms of weight $m+1$. (This is because the expansions of $W_j$ and $P_{\alpha\beta,J,j}$ in homogeneous terms have appropriate weight $k$ for $k \le m+1$.) Moreover, $A_{\alpha\beta,J}(q)$ has weight $m+1$ by definition, so $W_J A_{\alpha\beta}(q)$ has weight $m + 1$. Next,

$$W_J A_{\alpha\beta}(q) = Z_J A_{\alpha\beta}(q) + \sum_{\substack{k_0+\cdots+k_r=0 \\ k_0<m-1}} W_{j_r(k_r)} \cdots W_{j_1(k_1)} A_{\alpha\beta(k_0)}.$$

(The term $Z_J A_{\alpha\beta}(q)$ would appear in the sum if we allowed $k_0 = m - 1$, $k_i = -o(j_i)$, $i = 1, \cdots, r$.) The limitations $k_i \ge -o(j_i)$, $i = 1, \cdots, r$, and $k_0 \ge 0$

imply $k_i \leq m - 1 - o(j_1)$, so that the induction hypothesis applies to each summand. It follows that $Z_J A_{\alpha\beta}(q)$ has weight $m+1$ whenever $o(J) = m-1$. Therefore we have proved the induction step for $A_{\alpha\beta}$. The induction step for $A_{\alpha\beta,J}$ now follows by a routine induction on $o(J)$ using (4.13). The proof of the induction step for $R_{\alpha\overline{\beta}\rho\overline{\sigma},J}$ is similar, and this concludes the proof of Lemma 4.3.

*Proof of Proposition* 4.2. Lemma 4.3 implies that

$$\int_M |f^\varepsilon|^p \theta \wedge d\theta^n = \int_{\mathbf{H}^n} |\psi|^p |\Phi^\varepsilon|^p (1 + v_1 + v_2 + v_3 + v_4 + O(\rho^5))\Theta \wedge d\Theta^n,$$

where $v_j$ is a homogeneous polynomial of degree $j$ and weight $j$. By changing to polar coordinates in the $z$ variable, it is easy to check that if $|\varphi| \leq CF(\rho)$ then

$$\int_{a<\rho<b} \varphi\Theta \wedge d\Theta^n = O\left(\int_a^b F(\rho)\rho^{2n+1}\, d\rho\right).$$

Thus if we replace $(z,t)$ by $\delta_\varepsilon(z,t) = (\varepsilon z, \varepsilon^2 t)$ and note that $\delta_\varepsilon^* \Phi^\varepsilon = \varepsilon^{-n}\Phi$, $\delta_\varepsilon^*\Theta \wedge d\Theta^n = \varepsilon^{2n+2}\Theta \wedge d\Theta^n$, and $\Phi \leq C(1+\rho)^{-2n}$, we find

$$\begin{aligned}
\int_M |f^\varepsilon|^p \theta \wedge d\theta^n &= \int_{\rho<\kappa/\varepsilon} |\Phi|^p (1 + \varepsilon v_1 + \cdots + \varepsilon^4 v_4 + O(\varepsilon^5\rho^5))\Theta \wedge d\Theta^n \\
&\quad + O\left(\int_{\kappa/\varepsilon<\rho<2\kappa/\varepsilon} |\Phi|^p \Theta \wedge d\Theta^n\right) \\
&= \int_{\mathbf{H}^n} |\Phi|^p (1 + \varepsilon v_1 + \cdots + \varepsilon^4 v_4)\Theta \wedge d\Theta^n \\
&\quad + O\left(\int_{\kappa/\varepsilon}^\infty \sum_{i=0}^4 \varepsilon^i \rho^i (1+\rho)^{-4n-4}\rho^{2n+1}\, d\rho\right) \\
&\quad + O\left(\int_0^{\kappa/\varepsilon} \varepsilon^5\rho^5(1+\rho)^{-4n-4}\rho^{2n+1}\, d\rho\right) \\
&\quad + O\left(\int_{\kappa/\varepsilon}^{2\kappa/\varepsilon} (1+\rho)^{-4n-4}\rho^{2n+1}\, d\rho\right) \\
&= \int_{\mathbf{H}^n} |\Phi|^p (1 + \varepsilon v_1 + \cdots + \varepsilon^4 v_4)\Theta \wedge d\Theta^n + O(\varepsilon^5)
\end{aligned}$$

for all $n \geq 2$.

The coefficient of $\varepsilon^j$, $\int |\Phi|^p v_j \Theta \wedge d\Theta^n$, is a pseudohermitian-invariant scalar of weight $j$. However, as mentioned above, $|S(q)|^2$ is the only nontrivial scalar of weight $\leq 4$. This proves (4.3) and (4.6).

For (4.1) and (4.4), observe that $|df^\varepsilon|_\theta^2 = 2(f^\varepsilon)_{,\beta}(f^\varepsilon)^{,\beta} = W_\beta f^\varepsilon W_{\overline{\beta}} f^\varepsilon = s^j_\beta s^k_{\overline{\beta}} Z_j f^\varepsilon Z_k f^\varepsilon$. Thus Lemma 4.3 implies

$$\int_M |df^\varepsilon|_\theta^2 \theta \wedge d\theta^n = \int_{\rho<\kappa} (v_0^{jk} + \cdots + v_4^{jk} + O(\rho^{3+o(jk)})) Z_j \Phi^\varepsilon Z_k \Phi^\varepsilon \Theta \wedge d\Theta^n$$
$$+ O\left( \int_{\kappa<\rho<2\kappa} (|Z_j \Phi^\varepsilon| \, |Z_k \Phi^\varepsilon| + |Z_j \Phi^\varepsilon| \, |\Phi^\varepsilon| + |\Phi^\varepsilon|^2) \Theta \wedge d\Theta^n \right),$$

where

$$v_m^{jk} = \sum_{m_1+m_2=m} s^j_{\beta(m_1+o(j)-1)} s^k_{\overline{\beta}(m_2+o(k)-1)}$$

is a homogeneous polynomial of degree $m+o(jk)-2$ and weight $m$. Now note that $\delta_\varepsilon^*(Z_j \Phi^\varepsilon) = \varepsilon^{-n-o(j)} Z_j \Phi$ and $|Z_j \Phi| \le C(1+\rho)^{-2n-o(j)}$, $j = 0, 1, \cdots, 2n$. Changing variables, we have

$$\int_M |df^\varepsilon|_\theta^2 \theta \wedge d\theta^n = \int_{\rho<\kappa/\varepsilon} \sum_{m=0}^4 \varepsilon^m v_m^{jk} Z_j \Phi Z_k \Phi \Theta \wedge d\Theta^n$$
$$+ O\left( \int_0^{\kappa/\varepsilon} \sum_{i=2}^4 \varepsilon^5 \rho^{3+i} (1+\rho)^{-4n-i} \rho^{2n+1} \, d\rho \right)$$
$$+ O\left( \int_{\kappa/\varepsilon}^{2\kappa/\varepsilon} \sum_{i=0}^4 \varepsilon^{2-i} (1+\rho)^{-4n-i} \rho^{2n+1} \, d\rho \right).$$

When $n \ge 3$, the last two integrals are $O(\varepsilon^5)$. Hence

$$\int_M |df^\varepsilon|_\theta^2 \theta \wedge d\theta^n = \int_{\mathbf{H}^n} \sum_{m=0}^4 \varepsilon^m v_m^{jk} Z_j \Phi Z_k \Phi \Theta \wedge d\Theta^n$$
$$+ O\left( \int_{\kappa/\varepsilon}^\infty \sum_{m=0}^4 \sum_{i=2}^4 \varepsilon^m \rho^{m+i-2} (1+\rho)^{-4n-i} \rho^{2n+1} \, d\rho \right)$$
$$+ O(\varepsilon^5)$$
$$= \int_{\mathbf{H}^n} \sum_{m=0}^4 \varepsilon^m v_m^{jk} Z_j \Phi Z_k \Phi \Theta \wedge d\Theta^n + O(\varepsilon^5),$$

and (4.1) now follows in the same way as (4.2).

When $n = 2$, we have instead

$$\int_M |df^\varepsilon|_\theta^2 \theta \wedge d\theta^n = \int_{\mathbf{H}^n} \sum_{m=0}^3 \varepsilon^m v_m^{jk} Z_j \Phi Z_k \Phi \Theta \wedge d\Theta^2$$
$$+ \int_{\rho<\kappa/\varepsilon} \varepsilon^4 v_4^{jk} Z_j \Phi Z_k \Phi \Theta \wedge d\Theta^2 + O(\varepsilon^4).$$

Note that $v_4^{jk} Z_j \Phi Z_k \Phi = F + O(\rho^{-7})$ as $\rho \to \infty$, where $F$ is a homogeneous function of degree $-6$. Let $\varsigma$ be coordinates on the set $S = \{(z,t) : \rho = 1\}$. Lebesgue measure can be written in "polar coordinates" as $\rho^5\, d\rho\, d\sigma(\varsigma)$ for some measure $d\sigma$ on $S$, and hence

$$\begin{aligned}\int_{\rho<\kappa\varepsilon} v_4^{jk} Z_j \Phi Z_k \Phi \Theta \wedge d\Theta^2 &= \int_S F\, d\sigma \int_1^{\kappa/\varepsilon} \rho^{-1}\, d\rho + O(1) \\ &= \int_S F\, d\sigma \log \frac{1}{\varepsilon} + O(1).\end{aligned}$$

We can then deduce (4.4) in a way similar to (4.1) and (4.3).

Finally,

$$\begin{aligned}&\int_M R|f^\varepsilon|^2 \theta \wedge d\theta^n \\ &\quad = \int_{\mathbf{H}^n} (R_{(0)} + R_{(1)} + R_{(2)} + O(\rho^3))(1 + v_1 + v_2 + O(\rho^3))|f^\varepsilon|^2 \Theta \wedge d\Theta^n,\end{aligned}$$

where $v_k$ is as above and $R_{(k)}$ is a homogeneous polynomial of degree $k$ and weight $k+2$. The same reasoning as above implies that for $n \geq 3$,

$$\int_M R|f^\varepsilon|^2 \theta \wedge d\theta^n = c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + O(\varepsilon^5),$$

with

$$c_2 = \int_{\mathbf{H}^n} R_{(0)}|\Phi|^2 \Theta \wedge d\Theta^n, \quad c_3 = \int_{\mathbf{H}^n} (R_{(1)} + R_{(0)} v_1)|\Phi|^2 \Theta \wedge d\Theta^n,$$

$$c_4 = \int_{\mathbf{H}^n} (R_{(2)} + R_{(1)} v_1 + R_{(0)} v_2)|\Phi|^2 \Theta \wedge d\Theta^n.$$

The only coefficient that can be nonzero is the term $c_4$ of weight 4. Our normalization implies $R_{(0)} = R_{(1)} = 0$ by Proposition 3.12. Note that $R_{(2)} = \frac{1}{2} x^a x^b Z_a Z_b R(q)$ by Lemma 3.10, and $Z_a Z_b R(q) = R_{,ab}(q)$ by Lemma 3.3. But the only scalars that can be formed from $R_{,ab}(q)$ by contraction are $R_{,0}(q)$ and $\Delta_b R(q)$, which are also zero. In all, $c_2 = c_3 = c_4 = 0$ and we have proved (4.2). Similarly, when $n = 2$ the coefficients in the expansion on $\varepsilon^2$, $\varepsilon^3$, $\varepsilon^4 \log \frac{1}{\varepsilon}$ vanish, proving (4.5) and completing the proof of Proposition 4.2.

We close this section by remarking that Lemma 4.3 also permits us to give a geometric characterization of the pseudohermitian scalar curvature in terms of the volumes of balls. We define the ball $B(q,s)$ of radius $s$ centered at $q$ as the image under the parabolic exponential map $\Psi\colon T_q M \to M$ of a natural ball $\{\rho < s\}$ on $T_q M$.

**Proposition 4.4.** *There are positive dimensional constants $a_n$ and $b_n$ such that as $s$ tends to 0,*

$$\int_{B(q,s)} \theta \wedge d\theta^n = a_n s^{2n+2} - b_n R(q) s^{2n+4} + O(s^{2n+5}).$$

*Proof.* In pseudohermitian normal coordinates $(z,t)$, $B(q,s) = \{|z|^4 + t^2 < s^4\}$. With $v_j$ as above,

$$\int_{|z|^4+t^2<s^4} \theta \wedge d\theta^n = \int_{|z|^4+t^2<s^4} (1 + v_1 + v_2 + O(\rho^3))\Theta \wedge d\Theta^n$$
$$= c_0 s^{2n+2} + c_1 s^{2n+3} + c_2 s^{2n+4} + O(s^{2n+5}),$$

where $c_j$ is a scalar of weight $j$. Since there are no scalars of weight 1, and constant multiples of $R(q)$ are the only scalars of weight 2, we have $c_1 = 0$ and $c_2 = b_n R(q)$. It is a routine matter to calculate these constants. Denote

$$\beta(n) = \int_{-1}^{1} (1-t^2)^{n/2}\, dt = \frac{2^{n+1}\Gamma(1+n/2)^2}{\Gamma(n+1)}.$$

By Corollary 5.4 applied to the characteristic function of $B(q,s)$, we have

$$\int_{|z|^4+t^2<s^4} \Theta \wedge d\Theta^n = (4\pi)^n \beta(n) s^{2n+2}.$$

The calculation in §5 (see (5.2) and Lemma 5.1) implies

$$v_2 \Theta \wedge d\Theta^n = (\theta \wedge d\theta^n)_{(2n+4)} = -\tfrac{1}{3} R_{\gamma\bar\rho}(q) z^\gamma z^{\bar\rho} \Theta \wedge d\Theta^n + \text{ torsion terms.}$$

Because there are no scalar invariants of weight 2 involving torsion, the torsion terms must have mean value zero. Thus using Proposition 5.3 for $m = 1$ and (5.3), we obtain

$$\int_{|z|^4+t^2<s^4} v_2 \Theta \wedge d\Theta^n = -\frac{1}{3} \int_{|z|^4+t^2<s^4} R_{\gamma\bar\rho}(q) z^\gamma z^{\bar\rho} \Theta \wedge d\Theta^n$$
$$= -\frac{2R(q)(4\pi)^n \beta(n+1) s^{2n+4}}{3(n+1)}.$$

Thus $a_n = (4\pi)^n \beta(n)$ and $b_n = 2(4\pi)^n \beta(n+1)/3(n+1)$.

## 5. Explicit evaluation of constants

Because the asymptotic expansion involves only the square of the curvature tensor, we know that no terms involving $A_{\alpha\beta}$, $R_{\alpha\bar\beta}$, $R$ or any of their derivatives, and no terms involving derivatives of $R_{\alpha\bar\beta\rho\bar\sigma}$ contribute to the computation of the constants in (4.1)–(4.6). We will use the notation $A \equiv B$ to signify equality of $A$ and $B$ modulo terms of this kind and modulo terms of weight $> 4$. We will also use the notations

$$E = R_{\alpha}{}^{\beta}{}_{\rho\bar\sigma}(q) R_{\gamma}{}^{\rho}{}_{\delta\bar\varepsilon}(q) z^{\bar\sigma} z^{\gamma} z^{\alpha} z^{\bar\beta} z^{\delta} z^{\bar\varepsilon},$$
$$D_1 = R_{\alpha}{}^{\beta}{}_{\rho\bar\sigma}(q) R_{\gamma}{}^{\rho}{}_{\beta\bar\varepsilon}(q) z^{\alpha} z^{\bar\sigma} z^{\gamma} z^{\bar\varepsilon},$$
$$D_2 = R_{\alpha}{}^{\beta}{}_{\rho\bar\sigma}(q) R_{\bar\delta}{}^{\bar\sigma}{}_{\bar\gamma\beta}(q) z^{\bar\gamma} z^{\alpha} z^{\rho} z^{\bar\delta}.$$

We begin with an algebraic lemma.

**Lemma 5.1.** *Let* $\omega = m_{\alpha\beta} dz^\alpha \wedge dz^\beta + 2i m_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}} + m_{\overline{\alpha\beta}} dz^{\overline{\alpha}} \wedge dz^{\overline{\beta}}$ *be a real two-form. Denote* $\operatorname{Tr}\omega = \delta^{\alpha\overline{\beta}} m_{\alpha\overline{\beta}}$. *Then if* $n \geq 2$,

$$n\Theta \wedge \omega \wedge d\Theta^{n-1} = (\operatorname{Tr}\omega)\Theta \wedge d\Theta^n,$$

$$\begin{aligned} n(n-1)\Theta \wedge \omega^2 \wedge d\Theta^{n-2} = {} & [(\delta^{\alpha\overline{\beta}}\delta^{\rho\overline{\sigma}} - \delta^{\alpha\overline{\sigma}}\delta^{\rho\overline{\beta}}) m_{\alpha\overline{\beta}} m_{\rho\overline{\sigma}} \\ & + \tfrac{1}{2}(\delta^{\alpha\overline{\rho}}\delta^{\beta\overline{\sigma}} - \delta^{\alpha\overline{\sigma}}\delta^{\beta\overline{\rho}}) m_{\alpha\beta} m_{\overline{\rho\sigma}}]\Theta \wedge d\Theta^n. \end{aligned}$$

*Proof.* The first formula is left to the reader. For the second observe that

$$d\Theta^2 = -4 dz^\alpha \wedge dz^{\overline{\alpha}} \wedge dz^\beta \wedge dz^{\overline{\beta}}$$

has $n(n-1)$ nonzero terms — the ones for which $\alpha \neq \beta$. Now fix $\alpha$ and $\beta$, $\alpha \neq \beta$. Since the $n(n-1)$ terms in the sum above are similar to each other,

$$-4n(n-1)\Theta \wedge dz^\alpha \wedge dz^{\overline{\alpha}} \wedge dz^\beta \wedge dz^{\overline{\beta}} \wedge d\Theta^{n-2} = \Theta \wedge d\Theta^n,$$

where *in this line only* we are *not* using the summation convention. It follows that

$$\begin{aligned} & -4n(n-1)\Theta \wedge c_{\alpha\overline{\beta}\rho\overline{\sigma}} dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^\rho \wedge dz^{\overline{\sigma}} \wedge d\Theta^{n-2} \\ & \qquad = (\delta^{\alpha\overline{\beta}}\delta^{\rho\overline{\sigma}} - \delta^{\alpha\overline{\sigma}}\delta^{\rho\overline{\beta}}) c_{\alpha\overline{\beta}\rho\overline{\sigma}} \Theta \wedge d\Theta^n. \end{aligned} \tag{5.1}$$

Finally, the only terms of $\omega^2$ that can contribute to $\Theta \wedge \omega^2 \wedge d\Theta^{n-2}$ are ones with an equal number of barred and unbarred indices, so

$$\begin{aligned} \Theta \wedge \omega^2 \wedge d\Theta^{n-2} = \Theta \wedge ( & -4 m_{\alpha\overline{\beta}} m_{\rho\overline{\sigma}} dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^\rho \wedge dz^{\overline{\sigma}} \\ & + 2 m_{\alpha\beta} m_{\overline{\rho\sigma}} dz^\alpha \wedge dz^\beta \wedge dz^{\overline{\rho}} \wedge dz^{\overline{\sigma}}) \wedge d\Theta^{n-2}. \end{aligned}$$

If we now use (5.1) we obtain the second formula in the lemma.

**Lemma 5.2.** *If* $f^\varepsilon$ *is the test function of* §4 *and* $\delta_\varepsilon$ *denotes a parabolic dilation in the normal coordinates of* §3, *then*

(a) $\delta_\varepsilon^*(|f^\varepsilon|^p \theta \wedge d\theta^n) \equiv (\delta_\varepsilon^*\psi)^p |w+i|^{-2n-2}(1 - \frac{1}{90}\varepsilon^4(D_1 + D_2))\Theta \wedge d\Theta^n$.

(b)
$$\begin{aligned} \delta_\varepsilon^*(|df^\varepsilon|_\theta^2 \theta \wedge d\theta^n) \equiv n^2(\delta_\varepsilon^*\psi)^2 [ & |z|^2 |w+i|^{-2n-2} \\ & - \tfrac{1}{90}\varepsilon^4(D_1 + D_2)|z|^2|w+i|^{-2n-2} \\ & + \tfrac{17}{180}\varepsilon^4 E t^2 |w+i|^{-2n-4}]\Theta \wedge d\Theta^n. \end{aligned}$$

*Proof.* Let $E_\beta$, $E_\beta^\alpha$, $E_{\overline{\beta}}^\alpha$ denote any tensors satisfying $E_\beta z^\beta = E_\beta^\alpha z^{\overline{\alpha}} z^\beta = E_{\overline{\beta}}^\alpha z^{\overline{\alpha}} z^{\overline{\beta}} = E$. The tensor $E_\beta^\alpha$ in one occurrence may be different from that in another. We also denote

$$U_\beta = R_\gamma{}^\alpha{}_{\beta\overline{\sigma}}(q) z^\gamma z^{\overline{\sigma}} z^{\overline{\alpha}} = R_{\overline{\gamma}}{}^{\overline{\alpha}}{}_{\overline{\rho}\beta}(q) z^{\overline{\gamma}} z^{\overline{\rho}} z^\alpha.$$

Note that $U_\beta U_{\overline{\beta}} = E$.

Let us continue with the calculations begun in Proposition 2.5. Recall that

$$\theta^{\alpha}_{(1)} = dz^{\alpha}, \quad \theta_{(2)} = \Theta; \quad \omega_{\beta}{}^{\alpha}{}_{(1)} = 0, \quad \theta^{\alpha}_{(2)} = 0, \quad \theta_{(3)} = 0.$$

We have

$$\omega_{\beta(m)}^{\alpha} \equiv \frac{1}{m} R_{\beta}{}^{\alpha}{}_{\rho\overline{\sigma}}(q)[z^{\rho}\theta^{\overline{\sigma}}_{(m-1)} - z^{\overline{\sigma}}\theta^{\rho}_{(m-1)}], \qquad m \geq 2,$$

$$\theta^{\alpha}_{(m)} \equiv \frac{1}{m} z^{\beta}\omega_{\beta}{}^{\alpha}{}_{(m-1)}, \qquad m \geq 3,$$

$$\theta_{(m)} = \frac{2}{m}[iz^{\alpha}\theta^{\overline{\alpha}}_{(m-1)} - iz^{\overline{\alpha}}\theta^{\alpha}_{(m-1)}], \qquad m \geq 4.$$

Thus,

$$\omega_{\beta}{}^{\alpha}{}_{(2)} \equiv \tfrac{1}{2} R_{\beta}{}^{\alpha}{}_{\rho\overline{\sigma}}(q)(z^{\rho}dz^{\overline{\sigma}} - z^{\overline{\sigma}}dz^{\rho}),$$

$$\theta^{\alpha}_{(3)} \equiv \tfrac{1}{6} z^{\beta} R_{\beta}{}^{\alpha}{}_{\rho\overline{\sigma}}(q)(z^{\rho}dz^{\overline{\sigma}} - z^{\overline{\sigma}}dz^{\rho}), \qquad \theta_{(4)} \equiv \tfrac{i}{6}(U_{\sigma}dz^{\sigma} - U_{\overline{\sigma}}dz^{\overline{\sigma}});$$

$$\omega_{\beta}{}^{\alpha}{}_{(3)} \equiv 0, \quad \theta^{\alpha}_{(4)} \equiv 0, \quad \theta_{(5)} \equiv 0;$$

$$\begin{aligned}\omega_{\beta(4)}^{\alpha} \equiv \tfrac{1}{24} R_{\beta}{}^{\alpha}{}_{\rho\overline{\sigma}}(q)[&z^{\rho} R_{\overline{\varepsilon}}{}^{\overline{\sigma}}{}_{\overline{\gamma}\delta}(q) z^{\overline{\varepsilon}}(z^{\overline{\gamma}}dz^{\delta} - z^{\delta}dz^{\overline{\gamma}}) \\ &-z^{\overline{\sigma}} R_{\varepsilon}{}^{\rho}{}_{\delta\overline{\gamma}}(q) z^{\varepsilon}(z^{\delta}dz^{\overline{\gamma}} - z^{\overline{\gamma}}dz^{\delta})],\end{aligned}$$

$$\theta^{\alpha}_{(5)} \equiv \tfrac{1}{5} z^{\beta}\omega_{\beta}{}^{\alpha}{}_{(4)}, \quad \theta_{(6)} \equiv \tfrac{i}{90}(E_{\overline{\alpha}}dz^{\overline{\alpha}} - E_{\alpha}dz^{\alpha}).$$

Also, since $d\theta = 2i\theta^{\alpha} \wedge \theta^{\overline{\alpha}}$, we have

$$d\theta_{(2)} = d\theta, \qquad d\theta_{(3)} = 0,$$

$$\begin{aligned} d\theta_{(4)} &= 2idz^{\alpha} \wedge \theta^{\overline{\alpha}}_{(3)} + 2i\theta^{\alpha}_{(3)} \wedge dz^{\overline{\alpha}} \\ &\equiv m_{\alpha\beta}dz^{\alpha} \wedge dz^{\beta} + 2im_{\alpha\overline{\beta}}dz^{\alpha} \wedge dz^{\overline{\beta}} + m_{\overline{\alpha\beta}}dz^{\overline{\alpha}} \wedge dz^{\overline{\beta}}, \end{aligned}$$

with $m_{\alpha\beta} = \tfrac{i}{3} R_{\overline{\gamma}}{}^{\overline{\alpha}}{}_{\overline{\varepsilon}\beta}(q) z^{\overline{\varepsilon}} z^{\overline{\gamma}}$, and $m_{\alpha\overline{\beta}} = -\tfrac{1}{3} R_{\gamma}{}^{\beta}{}_{\alpha\overline{\varepsilon}}(q) z^{\gamma} z^{\overline{\varepsilon}}$. Finally,

$$d\theta_{(5)} \equiv 0, \qquad d\theta_{(6)} \equiv 2idz^{\alpha} \wedge \theta^{\overline{\alpha}}_{(5)} + 2i\theta^{\alpha}_{(5)} \wedge dz^{\overline{\alpha}} + 2i\theta^{\alpha}_{(3)} \wedge \theta^{\overline{\alpha}}_{(3)}.$$

Clearly $(\theta \wedge d\theta^{n})_{(2n+2)} = \Theta \wedge d\Theta^{n}$. Only even-degree terms in $\theta$ are nonzero; therefore the terms in $\theta \wedge d\theta^{n}$ of degree $2n+3$ and $2n+5$ are equivalent to zero. Moreover

$$(\theta \wedge d\theta^{n})_{(2n+4)} = \theta_{(4)} \wedge d\Theta^{n} + n\Theta \wedge d\theta_{(4)} \wedge d\Theta^{n-1}.$$

Now $\theta_{(4)} \wedge d\Theta^{n} \equiv 0$ because $\theta_{(4)}$ contains no $dt$ term. Lemma 5.1 and

$$\text{(5.2)} \qquad \operatorname{Tr}(d\theta_{(4)}) \equiv -\tfrac{1}{3}\delta^{\alpha\overline{\beta}} R_{\gamma}{}^{\beta}{}_{\alpha\overline{\varepsilon}}(q) z^{\gamma} z^{\overline{\varepsilon}} = -\tfrac{1}{3} R_{\gamma\overline{\varepsilon}}(q) z^{\gamma} z^{\overline{\varepsilon}} \equiv 0$$

imply $n\Theta \wedge d\theta_{(4)} \wedge d\Theta^{n-1} \equiv 0$. Thus $(\theta \wedge d\theta^{n})_{(2n+4)} \equiv 0$.

Now we must examine

$$(\theta \wedge d\theta^n)_{(2n+6)} \equiv \theta_{(6)} \wedge d\Theta^n + n\theta_{(4)} \wedge d\theta_{(4)} \wedge d\Theta^{n-1} + n\Theta \wedge d\theta_{(6)} \wedge d\Theta^{n-1} + \tfrac{1}{2}n(n-1)\Theta \wedge (d\theta_{(4)})^2 \wedge d\Theta^{n-2}.$$

Because $\theta_{(4)}$, $d\theta_{(4)}$, $d\Theta$, and $\theta_{(6)}$ have no $dt$ term, the first two terms in the sum are equivalent to 0. It is easy to check that

$$\operatorname{Tr} 2i\theta^{\alpha}_{(3)} \wedge \theta^{\overline{\alpha}}_{(3)} \equiv \tfrac{1}{36}(D_1 - D_2),$$
$$\operatorname{Tr} 2i dz^{\alpha} \wedge \theta^{\overline{\alpha}}_{(5)} \equiv \operatorname{Tr} 2i\theta^{\alpha}_{(5)} \wedge dz^{\overline{\alpha}} \equiv \tfrac{1}{120}(D_1 + D_2).$$

Hence, $\operatorname{Tr} d\theta_{(6)} \equiv \frac{2}{45}D_1 - \frac{1}{90}D_2$. Lemma 5.1 then gives

$$n\Theta \wedge d\theta_{(6)} \wedge d\Theta^{n-1} \equiv (\tfrac{2}{45}D_1 - \tfrac{1}{90}D_2)\Theta \wedge d\Theta^n,$$
$$n(n-1)\Theta \wedge (d\theta_{(4)})^2 \wedge d\Theta^{n-2} \equiv -\tfrac{1}{9}D_1\Theta \wedge d\Theta^n.$$

In all, $(\theta \wedge d\theta)_{(2n+6)} \equiv -\frac{1}{90}(D_1 + D_2)\Theta \wedge d\Theta^n$. Therefore

$$\delta^*_{\varepsilon}(\theta \wedge d\theta) \equiv \varepsilon^{2n+2}(1 - \tfrac{1}{90}\varepsilon^4(D_1 + D_2))\Theta \wedge d\Theta^n.$$

Also, $\delta^*_{\varepsilon}|\Phi^{\varepsilon}|^p = \varepsilon^{-2n-2}|w+i|^{-2n-2}$. This proves Lemma 5.2(a).

From (4.8)–(4.10) we have

$$s^{\alpha}_{\beta(0)} = \delta^{\alpha}_{\beta}; \qquad s^{\overline{\alpha}}_{\beta(0)} = s^{0}_{\beta(0)} = s^{\alpha}_{\beta(1)} = s^{\overline{\alpha}}_{\beta(1)} = s^{0}_{\beta(1)} = s^{0}_{\beta(2)} = 0;$$
$$s^{\alpha}_{\beta(2)} = -\theta^{\alpha}_{(3)}(Z_\beta) \equiv \tfrac{1}{6} z^{\gamma} R_{\gamma}{}^{\alpha}{}_{\beta\overline{\sigma}}(q) z^{\overline{\sigma}},$$
$$s^{\overline{\alpha}}_{\beta(2)} = -\theta^{\overline{\alpha}}_{(3)}(Z_\beta) \equiv -\tfrac{1}{6} z^{\overline{\gamma}} R_{\overline{\gamma}}{}^{\overline{\alpha}}{}_{\overline{\rho}\beta}(q) z^{\overline{\rho}};$$
$$s^{\alpha}_{\beta(3)} \equiv 0, \qquad s^{\overline{\alpha}}_{\beta(3)} \equiv 0.$$

It is easy to check that

$$\theta^{\alpha}_{(5)}(Z_\beta) \equiv \tfrac{1}{60}E^{\alpha}_{\beta}, \quad s^{\gamma}_{\beta(2)}\theta^{\alpha}_{(3)}(Z_\gamma) = -\tfrac{1}{36}E^{\alpha}_{\beta}, \quad s^{\overline{\gamma}}_{\beta(2)}\theta^{\alpha}_{(3)}(Z_{\overline{\gamma}}) \equiv -\tfrac{1}{36}E^{\alpha}_{\beta}.$$

Hence

$$s^{\alpha}_{\beta(4)} \equiv -(\theta^{\alpha}_{(5)}(Z_\beta) + s^{\gamma}_{\beta(2)}\theta^{\alpha}_{(3)}(Z_\gamma) + s^{\overline{\gamma}}_{\beta(2)}\theta^{\alpha}_{(3)}(Z_{\overline{\gamma}})) \equiv \tfrac{7}{180}E^{\alpha}_{\beta}.$$

Similarly, $s^{\overline{\alpha}}_{\beta(4)} \equiv -\frac{7}{180}E^{\overline{\alpha}}_{\beta}$. Finally,

$$s^{0}_{\beta(3)} = -\theta_{(4)}(Z_\beta) \equiv -\tfrac{i}{6}U_\beta; \qquad s^{0}_{\beta(4)} \equiv -\theta_{(5)}(Z_\alpha) \equiv 0;$$
$$s^{0}_{\beta(5)} \equiv -(s^{\gamma}_{\beta(2)}\theta_{(4)}(Z_\gamma) + s^{\overline{\gamma}}_{\beta(2)}\theta_{(4)}(Z_{\overline{\gamma}}) + \theta_{(6)}(Z_\beta)) \equiv -\tfrac{2i}{45}E_\beta.$$

Recalling that $\Phi = |w+i|^{-n}$ and $\Phi^{\varepsilon} = \varepsilon^n|w + i\varepsilon^2|^{-n}$, we have

$$\delta^*_{\varepsilon} Z_\alpha \Phi^{\varepsilon} = -in\varepsilon^{-n-1}|w+i|^{-n-2}(\overline{w} - i)z^{\overline{\alpha}},$$
$$\delta^*_{\varepsilon} Z_0 \Phi^{\varepsilon} = -nt\varepsilon^{-n-2}|w+i|^{-n-2}.$$

Thus

$$\begin{aligned}
\delta_\varepsilon^*(W_\beta f^\varepsilon) &= \delta_\varepsilon^*(s_\beta^\alpha Z_\alpha f^\varepsilon + s_\beta^{\overline{\alpha}} Z_{\overline{\alpha}} f^\varepsilon + s_\beta^0 Z_0 f^\varepsilon) \\
&\equiv (\delta_\varepsilon^*\psi)\delta_\varepsilon^*(s_\beta^\alpha Z_\alpha \Phi^\varepsilon + s_\beta^{\overline{\alpha}} Z_{\overline{\alpha}} \Phi^\varepsilon + s_\beta^0 Z_0 \Phi^\varepsilon) \\
&\equiv (\delta_\varepsilon^*\psi)\delta_\varepsilon^*(Z_\beta \Phi^\varepsilon + s_{\beta(2)}^\alpha Z_\alpha \Phi^\varepsilon + s_{\beta(2)}^{\overline{\alpha}} Z_{\overline{\alpha}} \Phi^\varepsilon \\
&\qquad + s_{\beta(3)}^0 Z_0 \Phi^\varepsilon + s_{\beta(4)}^\alpha Z_\alpha \Phi^\varepsilon + s_{\beta(4)}^{\overline{\alpha}} Z_{\overline{\alpha}} \Phi^\varepsilon + s_{\beta(5)}^0 Z_0 \Phi^\varepsilon) \\
&\equiv (\delta_\varepsilon^*\psi)[-in\varepsilon^{-n-1}|w+i|^{-n-2}(\overline{w}-i)z^{\overline{\beta}} \\
&\qquad + \varepsilon^{-n+1}\tfrac{1}{6} z^\gamma R_\gamma{}^\alpha{}_{\beta\overline{\sigma}}(q) z^{\overline{\sigma}}(-in)|w+i|^{-n-2}(\overline{w}_i) z^{\overline{\alpha}} \\
&\qquad - \varepsilon^{-n+1}\tfrac{1}{6} z^{\overline{\gamma}} R_{\overline{\gamma}}{}^{\overline{\alpha}}{}_{\overline{\rho}\beta}(q) z^{\overline{\rho}}(in)|w+i|^{-n-2}(w+i) z^\alpha \\
&\qquad + \varepsilon^{-n+1}(-\tfrac{i}{6}U_\beta)(-nt)|w+i|^{-n-2} \\
&\qquad + \varepsilon^{-n+3}\tfrac{7}{180}E_\beta^\alpha(-in)|w+i|^{-n-2}(\overline{w}-i)z^{\overline{\alpha}} \\
&\qquad - \varepsilon^{-n+3}\tfrac{7}{180}E_\beta^{\overline{\alpha}}(in)|w+i|^{-n-2}(w+i)z^\alpha \\
&\qquad - \varepsilon^{-n+3}\tfrac{2i}{45}E_\beta(-nt)|w+i|^{-n-2}] \\
&\equiv -in(\delta_\varepsilon^*\psi)|w+i|^{-n-2}\varepsilon^{-n-1}[(\overline{w}-i)z^{\overline{\beta}} + \tfrac{1}{6}\varepsilon^2 t U_\beta + \tfrac{1}{30}\varepsilon^4 t E_\beta].
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta_\varepsilon^*(|df^\varepsilon|_\theta^2) &= \delta_\varepsilon^*(W_\beta f^\varepsilon W_{\overline{\beta}} f^\varepsilon) \\
&\equiv n^2(\delta_\varepsilon^*\psi)^2\varepsilon^{-2n-2}[|z|^2|w+i|^{-2n-2} + \tfrac{1}{3}\varepsilon^2 t^2 z^\beta U_\beta |w+i|^{-2n-4} \\
&\qquad\qquad + \tfrac{17}{180}\varepsilon^4 t^2 E|w+i|^{-2n-4}].
\end{aligned}$$

Multiplying this by the earlier formula for $\delta_\varepsilon^*(\theta\wedge d\theta^n)$ we find the formula in Lemma 5.2(b). The term $z^\beta U_\beta$ does not contribute to the final result because its integral only gives rise to the scalar $R(q)$ of weight 2, which is zero. (We have already implicitly ruled out this term on more general grounds since it contributes to the coefficient on $\varepsilon^2$ in (4.1).) q.e.d.

Let $d\mu$ denote Lebesgue measure on $\mathbb{C}^n$. Let $d\nu$ be the uniform measure on $S^{2n-1} = \{z \in \mathbb{C}^n : |z| = 1\}$, normalized so that if $z = r\varsigma$, $\varsigma \in S^{2n-1}$, represent polar coordinates for $z \in \mathbb{C}^n$, then $d\mu(z) = r^{2n-1}dr\, d\nu(\varsigma)$. Since

$$\begin{aligned}
\Theta\wedge d\Theta^n &= dt\wedge(2i dz^\alpha \wedge dz^{\overline{\alpha}})^n = 2^n n! dt \wedge (i dz^1\wedge dz^{\overline{1}})\wedge\cdots\wedge(i dz^n \wedge dz^{\overline{n}}) \\
&= 4^n n! dt\wedge dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n,
\end{aligned}$$

we have

$$\int_{\mathbf{H}^n}\varphi(z,t)\Theta\wedge d\Theta^n = 4^n n!\int_{\mathbb{C}^n}\int_{-\infty}^{\infty}\varphi(z,t)\,dt\,d\mu(z), \tag{5.3}$$

whenever $\varphi$ is integrable.

**Proposition 5.3.** *Let $A = (\alpha_1, \cdots, \alpha_m)$, $B = (\beta_1, \cdots, \beta_m)$ be multi-indices with $1 \leq \alpha_i \leq n$, $1 \leq \beta_i \leq n$. Let $\delta(A, B) = 1$ if $A = B$ and $0$ otherwise. Then*

$$\int_{S^{2n-1}} z^{\alpha_1} \cdots z^{\alpha_m} \overline{z}^{\beta_1} \cdots \overline{z}^{\beta_m} \, d\nu(z) = \frac{2\pi^n}{(n+m-1)!} \sum_{\sigma \in S_m} \delta(A, \sigma B).$$

*Proof.* Denote $c_{2k+1} = \int_0^\infty r^{2k+1} e^{-r^2} \, dr$. It is easy to check by integration by parts that $c_{2k+1} = k!/2$. Let $p(z)$ be a homogeneous polynomial on $\mathbb{C}^n$ of even degree $2d$. Then

$$\begin{aligned} \int_{\mathbb{C}^n} p(z) e^{-|z|^2} \, d\mu(z) &= \int_0^\infty \int_{S^{2n-1}} p(r\varsigma) e^{-r^2} \, d\nu(\varsigma) r^{2n-1} \, dr \\ &= c_{2d+2n-1} \int_{S^{2n-1}} p(\varsigma) \, d\nu(\varsigma). \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{C}^n} |z^1|^{2m_1} \cdots |z^n|^{2m_n} e^{-|z|^2} \, d\mu(z) &= \prod_{j=1}^n \int_{\mathbb{C}} |z^j|^{2m_j} e^{-|z^j|^2} \, d\mu(z^j) \\ &= \prod_{j=1}^n 2\pi \int_0^\infty r^{2m_j+1} e^{-r^2} \, dr \\ &= (2\pi)^n \prod_{j=1}^n c_{2m_j+1} = \pi^n m_1! \cdots m_n!. \end{aligned}$$

Hence

$$\begin{aligned} \int_{S^{2n-1}} |z^1(\varsigma)|^{2m_1} \cdots |z^n(\varsigma)|^{2m_n} \, d\nu(\varsigma) &= \frac{\pi^n m_1! \cdots m_n!}{c_{2d+2n-1}} \\ &= \frac{2\pi^n m_1! \cdots m_n!}{(n + m_1 + \cdots + m_n - 1)!}. \end{aligned}$$

Observe that if $m = m_1 + \cdots + m_n$ and the entries of $A$ and $B$ both have exactly $m_j$ entries equal to $j$, then

$$\sum_{\sigma \in S_m} \delta(A, \sigma B) = m_1! \cdots m_n!$$

and the proposition is proved.

**Corollary 5.4.** *Suppose that $\varphi$ is a function of $|z|$ and $t$. Then the following formulas hold whenever either side is integrable:*

$$\int_{\mathbf{H}^n} \varphi\Theta \wedge d\Theta^n = (4\pi)^n 2n \int_{-\infty}^{\infty}\int_0^{\infty} \varphi(r,t) r^{2n-1}\, dr\, dt,$$

$$\int_{\mathbf{H}^n} \varphi D_1\Theta \wedge d\Theta^n = (4\pi)^n \frac{8|S(q)|^2}{n+1} \int_{-\infty}^{\infty}\int_0^{\infty} \varphi(r,t) r^{2n+3}\, dr\, dt,$$

$$\int_{\mathbf{H}^n} \varphi D_2\Theta \wedge d\Theta^n = 2\int_{\mathbf{H}^n} \varphi D_1\Theta \wedge d\Theta^n,$$

$$\int_{\mathbf{H}^n} \varphi E\Theta \wedge d\Theta^n = (4\pi)^n \frac{16|S(q)|^2}{(n+1)(n+2)} \int_{-\infty}^{\infty}\int_0^{\infty} \varphi(r,t) r^{2n+5}\, dr\, dt.$$

*Proof.* These follow from (5.3) and Proposition 5.3 in the cases $m = 0, 2, 2,$ and 3, respectively, along with the observation that

$$4|S(q)|^2 = R_{\alpha}{}^{\beta}{}_{\rho\bar\sigma}(q) R_{\bar\alpha}{}^{\bar\beta}{}_{\bar\rho\sigma}(q). \quad \text{q.e.d.}$$

Denote

$$N_1(\alpha,\beta,\gamma) = \frac{\Gamma(\frac{1}{2}(\beta+1))\Gamma(\alpha-\gamma-\frac{1}{2}\beta-\frac{3}{2})\Gamma(\frac{1}{2}(\gamma+1))\Gamma(\frac{1}{2}(\alpha-\gamma-1))}{2\Gamma(\alpha-\gamma-1)\Gamma(\alpha/2)}.$$

(We will only consider this function in the range where all the arguments of the gamma function are positive.) Let $\delta = 2\alpha - 2\gamma - \beta - 3 > 0$. Because $\Gamma(1+z) = z\Gamma(z)$,

$$\lim_{\delta\to 0^+} \delta\Gamma\left(\frac{\delta}{2}\right) = \lim_{\delta\to 0^+} 2\Gamma\left(1+\frac{\delta}{2}\right) = 2.$$

It follows that if $\alpha \to \alpha_0$, $\beta \to \beta_0$, $\gamma \to \gamma_0$ in such a way that $\delta \to 0^+$, then

$$N_2(\alpha_0,\beta_0,\gamma_0) = \lim_{\delta\to 0^+} \delta N_1(\alpha,\beta,\gamma)$$

exists and

$$N_2(\alpha_0,\beta_0,\gamma_0) = \frac{\Gamma((\beta_0+1)/2)\Gamma((\gamma_0+1)/2)\Gamma((\alpha_0-\gamma_0-1)/2)}{\Gamma(\alpha_0-\gamma_0-1)\Gamma(\alpha_0/2)}.$$

**Lemma 5.5.** *Suppose that $\alpha$, $\gamma+1$, $\beta+1$, and $\alpha-\gamma-1$ are positive real numbers.*

(a) *If $2\alpha - 2\gamma - \beta > 3$, then*

$$\int_{-\infty}^{\infty}\int_0^{\infty} |t + i(1+r^2)|^{-\alpha} r^\beta |t|^\gamma\, dr\, dt = N_1(\alpha,\beta,\gamma).$$

(b) *If $2\alpha - 2\gamma - \beta = 3$, then as $\varepsilon \to 0^+$,*

$$\int_{r^4+t^2<\varepsilon^{-4}} |t + i(1+r^2)|^{-\alpha} r^\beta |t|^\gamma\, dr\, dt = N_2(\alpha,\beta,\gamma)\log\frac{1}{\varepsilon} + O(1).$$

*Consequently, if $\psi$ is the cut-off function in the definition of $f^\varepsilon$, then*

$$\int_{-\infty}^{\infty}\int_0^{\infty} \psi(\varepsilon^2(t+ir^2))|t+i(1+r^2)|^{-\alpha}r^\beta|t|^\gamma\,dr\,dt = N_2(\alpha,\beta,\gamma)\log\frac{1}{\varepsilon}+O(1).$$

*Proof.* Euler's formula [16, p. 254] implies

$$\int_0^1 x^{m-1}(1-x)^{k-1}\,dx = \Gamma(k)\Gamma(m)/\Gamma(m+k)$$

for all positive real numbers $k$ and $m$. We deduce that

$$\int_0^\infty b^\gamma(a^2+b^2)^{-\alpha/2}\,db = \frac{\Gamma((\gamma+1)/2)\Gamma((\alpha-\gamma-1)/2)}{2\Gamma(\alpha/2)}a^{\gamma-\alpha+1}. \tag{5.4}$$

In fact, let $b = as$ and $x = (1+s^2)^{-1}$. Then $s = x^{-1/2}(1-x)^{1/2}$ and

$$\begin{aligned}\int_0^\infty b^\gamma(a^2+b^2)^{-\alpha/2}\,db &= a^{\gamma-\alpha+1}\int_0^\infty s^\gamma(1+s^2)^{-\alpha/2}\,ds\\ &= \frac{1}{2}a^{\gamma-\alpha+1}\int_0^1 x^{(\alpha-\gamma-3)/2}(1-x)^{(\gamma-1)/2}\,dx,\end{aligned}$$

and Euler's formula gives (5.4). Next substituting $a = r^2+1$ and $b = |t|$ in (5.4) gives

$$\begin{aligned}&\int_{-\infty}^\infty\int_0^\infty |t+i(1+r^2)|^{-\alpha}r^\beta|t|^\gamma\,dr\,dt\\ &\qquad= \frac{\Gamma((\gamma+1)/2)\Gamma((\alpha-\gamma-1)/2)}{\Gamma(\alpha/2)}\int_0^\infty (1+r^2)^{\gamma-\alpha+1}r^\beta\,dr,\end{aligned}$$

and a second application of (5.4) with $a = 1$, $b = r$ yields Lemma 5.5(a).

For part (b) consider polar coordinates $\rho = (r^4+t^2)^{1/4}$ and $s$ a parameter on the curve $S = \{(r,t)\colon r^4+t^2 = 1\}$. Let $dv(s)$ be the measure on $S$ such that $drdt = \rho^2 dv(s)d\rho$. Suppose that $F_\delta(\rho,s)$ is a family of functions satisfying

$$F_\delta(\rho,s) = \begin{cases} F_0(s)\rho^{-3-\delta}+O(\rho^{-4}) & \text{for } \rho\ge 1,\\ O(1) & \text{for } 0\le\rho\le 1,\end{cases}$$

*uniformly* for $0\le\delta\le 1$. Then as $\delta\to 0^+$,

$$\begin{aligned}\int_0^\infty\int_S F_\delta(\rho,s)\,dv(s)\rho^2\,d\rho &= \int_1^\infty\int_S F_0(s)\,dv(s)\rho^{-1-\delta}\,d\rho+O(1)\\ &= \delta^{-1}\int_S F_0(s)\,dv(s)+O(1).\end{aligned}$$

Also as $\varepsilon\to 0^+$,

$$\begin{aligned}\int_0^{\varepsilon^{-1}}\int_S F_0(\rho,s)\,dv(s)\rho^2\,d\rho &= \int_1^{\varepsilon^{-1}}\int_S F_0(s)\,dv(s)\rho^{-1}d\rho+O(1)\\ &= \left(\log\frac{1}{\varepsilon}\right)\int_S F_0(s)\,dv(s)+O(1).\end{aligned}$$

Let $\delta = 2\alpha - 2\gamma - \beta - 3$. It is easy to see that the integrand in (a) satisfies the hypothesis on $F_\delta(\rho, s)$ and part (b) follows. q.e.d.

From Lemma 5.2, Corollary 5.4, and Lemma 5.5, it follows that

$$\begin{aligned}
\int |f^\varepsilon|^p \theta \wedge d\theta^n &= \int_{\mathbf{H}^n} |w+i|^{-2n-2}\left(1 - \frac{\varepsilon^4}{90}(D_1 + D_2)\right)\Theta \wedge d\Theta^n + O(\varepsilon^5) \\
&= (4\pi)^n \left[2n \int_{-\infty}^{\infty}\int_0^\infty |t + i(1+r^2)|^{-2n-2} r^{2n-1}\, dr\, dt \right. \\
&\quad \left. - \frac{4|S(q)|^2}{15(n+1)}\varepsilon^4 \int_{-\infty}^{\infty}\int_0^\infty |t+i(1+r^2)|^{-2n-2} r^{2n+3}\, dr\, dt\right] \\
&\quad + O(\varepsilon^5) \\
&= (4\pi)^n (2n) N_1(2n+2, 2n-1, 0) \\
&\quad \cdot \left[1 - \frac{2}{15}\varepsilon^4 |S(q)|^2 \frac{1}{n(n+1)} \frac{N_1(2n+2, 2n+3, 0)}{N_1(2n+2, 2n-1, 0)}\right] + O(\varepsilon^5).
\end{aligned}$$

Note that $N_1(2n+2, 2n+3, 0)/N_1(2n+2, 2n-1, 0) = (n+1)/(n-1)$. It is also easy to check by induction that

$$N_1(2n+2, 2n-1, 0) = 4^{-n}\pi/(2n).$$

Thus

$$\int |f^\varepsilon|^p \theta \wedge d\theta^n = \pi^{n+1}\left(1 - \frac{2}{15n(n-1)}|S(q)|^2\varepsilon^4\right) + O(\varepsilon^5),$$

$$\left(\int |f^\varepsilon|^p \theta \wedge d\theta^n\right)^{-2/p} = \pi^{-n}\left(1 + \frac{2}{15(n-1)(n+1)}|S(q)|^2 \varepsilon^4\right) + O(\varepsilon^5).$$

We also have

(5.5)

$$\begin{aligned}
&\int |df^\varepsilon|_\theta^2 \theta \wedge d\theta^n \\
&= n^2(4\pi)^n \left[2n \int_{-\infty}^{\infty}\int_0^\infty \psi(\varepsilon^2 w)|t + i(1+r^2)|^{-2n-2} r^{2n+1}\, dr\, dt \right. \\
&\quad - \frac{1}{30}\frac{8|S(q)|^2}{n+1}\varepsilon^4 \int_{-\infty}^{\infty}\int_0^\infty \psi(\varepsilon^2 w)|t+i(1+r^2)|^{-2n-2} r^{2n+5}\, dr\, dt \\
&\quad \left. + \frac{17}{180}\frac{16|S(q)|^2}{(n+1)(n+2)}\varepsilon^4 \int_{-\infty}^{\infty}\int_0^\infty \psi(\varepsilon^2 w)|t+i(1+r^2)|^{-2n-4} t^2 r^{2n+5}\, dr\, dt\right] \\
&\quad + \begin{cases} O(\varepsilon^5) & \text{when } n \geq 3, \\ O(\varepsilon^4) & \text{when } n = 2. \end{cases}
\end{aligned}$$

When $n \geq 3$, we can replace the factor $\psi(\varepsilon^2 w)$ by 1 with an error of magnitude $O(\varepsilon^5)$, so

$$\begin{aligned}
&\int |df^\varepsilon|_\theta^2 \theta \wedge d\theta^n \\
&= n^2(4\pi)^n 2nN_1(2n+2,2n+1,0)\left[1 - \frac{2}{15}\frac{|S(q)|^2}{n(n+1)}\frac{N_1(2n+2,2n+5,0)}{N_1(2n+2,2n+1,0)}\varepsilon^4\right. \\
&\qquad\qquad\left. + \frac{34}{45}\frac{|S(q)|^2}{n(n+1)(n+2)}\frac{N_1(2n+4,2n+5,2)}{N_1(2n+2,2n+1,0)}\varepsilon^4\right] \\
&\quad + O(\varepsilon^5) \\
&= n^2\pi^{n+1}\left[1 - \frac{2(n+2)|S(q)|^2}{15n(n-1)(n-2)}\varepsilon^4 + \frac{17|S(q)|^2}{45(n+1)n(n-1)(n-2)}\varepsilon^4\right] + O(\varepsilon^5) \\
&= n^2\pi^{n+1}\left[1 - \frac{(6n^2+18n-5)|S(q)|^2}{45(n+1)n(n-1)(n-2)}\varepsilon^4\right] + O(\varepsilon^5).
\end{aligned}$$

Therefore, for all $n \geq 3$,

$$\mathscr{Y}_\theta(f^\varepsilon) = pn^2\pi[1 - c(n)|S(q)|^2\varepsilon^4] + O(\varepsilon^5)$$

with $c(n) = (30n-5)/[45(n+1)n(n-1)(n-2)]$.

In the case $n = 2$, we conclude from (5.5) and Lemma 5.5 that we can evaluate the constant in the asymptotic expansion as $\varepsilon \to 0$ by taking the limit as $n \to 2^+$ and treating $n$ as a continuous variable. In fact, if $\delta = 2n-4$, the coefficients arising from (5.5) are

$$\begin{aligned}
\lim_{n\to 2^+} \delta N_1(2n+2,2n+5,0) &= N_2(6,9,0), \\
\lim_{n\to 2^+} \delta N_1(2n+4,2n+5,2) &= N_2(8,9,2).
\end{aligned}$$

It follows that

$$\begin{gathered}
c(2) = \lim_{n\to 2^+}(2n-4)c(n) = \frac{11}{27}, \\
\mathscr{Y}_\theta(f^\varepsilon) = 12\pi\left(1 - c(2)|S(q)|^2\varepsilon^4 \log\frac{1}{\varepsilon}\right) + O(\varepsilon^4)
\end{gathered}$$

when $n = 2$. Thus Theorem A is proved.

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