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EINSTEIN MANIFOLDS OF DIMENSION FIVE WITH SMALL FIRST EIGENVALUE OF THE DIRAC OPERATOR

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1. Introduction

Let M^n be a compact Einstein spin manifold with positive scalar curvature $R > 0$ and denote by $D\colon \Gamma(S) \to \Gamma(S)$ the Dirac operator acting on sections of the spinor bundle. If λ_1 is the first eigenvalue of this operator we have

$$\lambda_1^2 \geq \frac{1}{4}\frac{n \cdot R}{n-1}$$

(see e.g. [4]). Thus, there arises the interesting problem to classify all those Einstein spaces where the lower bound $\pm\frac{1}{2}\sqrt{\frac{(n \cdot R)}{(n-1)}}$ actually is an eigenvalue of the Dirac operator. The corresponding eigenspinor ψ satisfies the stronger equation

$$\nabla_X \psi = \mp\frac{1}{2}\sqrt{\frac{R}{n(n-1)}} X \cdot \psi$$

(see e.g. [4]) and these spinors are sometimes called Killing spinors (see e.g. [9], [16]). In case $n = 4$ the only possible manifold is $M^4 = S^4$ (see e.g. [5]).

In dimension six each solution of the equation $D\psi = \frac{1}{2}\sqrt{(6 \cdot R)/5}\psi$ defines a (nonintegrable) almost complex structure (see e.g. [8]). Furthermore, the assumption that $\pm\frac{1}{2}\sqrt{(n \cdot R)/(n-1)}$ is an eigenvalue of the Dirac operator imposes algebraic conditions on the Weyl tensor of the space (see e.g. [5]) as well as on the covariant derivative of the curvature tensor and the harmonic forms on M^n (see e.g. [9]). On the other hand, in the dimensions 5,6,7 examples of Einstein spaces different from the sphere are known for which $\pm\frac{1}{2}\sqrt{(n \cdot R)/(n-1)}$ is an eigenvalue of the Dirac operator (see e.g. [4], [7], [17]). Moreover, if M^n is a Kähler manifold, K.D. Kirchberg proved the stronger inequality $\lambda_1^2 \geq \frac{1}{4}(n+2)R/n$ (see e.g. [12]) and solved in the complex dimension $n/2 = 3$ the corresponding classification problem (see e.g. [13]); the only possible Einstein-Kähler spaces of complex dimension three realizing $\sqrt{\frac{2}{3}R}$ as an eigenvalue of the Dirac operator are $P^3(\mathbf{C})$ and $F(1,2)$

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with their canonical metrics. The aim of this paper is to study the above mentioned classification problem in the case of 5-dimensional real Einstein spaces. First of all we prove that any solution of the equation $D\psi = \pm\frac{1}{4}\sqrt{5R}\psi$ defines an Einstein-Sasaki structure on M^5. Conversely, if M^5 is a simply-connected Einstein-Sasaki space then the equation under consideration has a nontrivial solution. In the next step we classify all regular contact metric structures arising from a nontrivial solution of the equation $D\psi = \frac{1}{4}\sqrt{5R}\psi$. The regularity assumption implies that M^5 is a fiber bundle over a four-dimensional Einstein-Kähler manifold X^4 with positive scalar curvature. Therefore, we know the possible X^4 $(= S^2 \times S^2, P^2(\mathbf{C})$ or the del Pezzo surfaces $P_{k'}$ $3 \le k \le 8)$ as well as the topological type of the fibration $\pi\colon M^5 \to X^4$. In particular, if M^5 is a simply-connected, compact 5-dimensional Einstein spin manifold such that $D\psi - \frac{1}{4}\sqrt{5R}\psi$ admits a nontrivial solution and the corresponding Sasaki structure is regular, then M^5 is isometric to the sphere S^5, or to the Stiefel manifold $V_{4,2}$ with the Einstein metric considered in [11], [4], or M^5 is the simply-connected S^1-bundle with Chern class $c_1^* = c_1(P_k)$ over one of the del Pezzo surfaces P_k $(3 \le k \le 8)$. In the last case M^5 is diffeomorphic to the connected sum $M^5 \approx (S^2 \times S^3)\# \cdots \#(S^2 \times S^3)$ and there is a one-to-one correspondence between Killing spinors on M^5 and Einstein-Kähler metrics on the del Pezzo surface P_k. The existence of Einstein-Kähler structures on P_k has been recently proved by Tian and Yau (see [21], [22]).

2. Einstein-Sasaki manifolds in dimension 5

We introduce some notation concerning contact structures. A general reference is [3]. A contact metric structure on a manifold M^5 consists of a 1-form η, a vector field ξ, a (1,1)-tensor φ and a Riemannian metric g such that the following conditions are satisfied:

(a) $\eta \wedge (d\eta)^2 \neq 0$.

(b) $\eta(\xi) = 1,\ \varphi(\xi) = 0$.

(c) $\varphi^2 = -Id + \eta \otimes \xi$.

(d) $g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X)\eta(Y)$.

(e) $d\eta(X,Y) = 2g(X,\varphi(Y))$ with $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta[X,Y]$.

Formal consequences of conditions (b) and (d) are the equations $\eta(X) = g(X,\xi), \varphi(\xi) = 0$.

In case ξ is a Killing vector field we call the given structure on M^5 a K-contact structure. This is equivalent to

(f) $\nabla_X \xi = -\varphi(X)$.

A Sasaki manifold is a K-contact structure satisfying the integrability condition

$$[\varphi, \varphi] + d\eta\,\xi = 0$$

or, equivalently,

(g) $(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X.$

The curvature tensor of a Sasaki manifold commutes with φ and has the following special property:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

In particular, if M^5 is a 5-dimensional Einstein-Sasaki manifold we obtain for the scalar curvature the value $R = 20$, and the Weyl tensor W satisfies $W(X, Y)\xi = 0$. Denote by $T^h \subset T(M^5)$ the bundle of all vectors orthogonal to ξ. According to $W(X, Y)\xi = 0$ we can consider the Weyl tensor of M^5 as a linear transformation

$$W\colon \bigwedge^2(T^h) \to \bigwedge^2(T^h).$$

T^h is an oriented 4-dimensional bundle and, consequently, we have the algebraic Hodge operator $*\colon \bigwedge^2(T^h) \to \bigwedge^2(T^h)$, obviously different from the Hodge operator of M^5.

Proposition 1. *Let $(M^5; \varphi, \xi, \eta, g)$ be a 5-dimensional Einstein-Sasaki manifold. Denote by $W\colon \bigwedge^2(T^h) \to \bigwedge^2(T^h)$ the Weyl tensor on the horizontal bundle. Then W is anti-selfdual with respect to the algebraic Hodge operator of the bundle T^h, i.e. $*W = -W$.*

Proof. We fix an orthonormal basis $e_1, e_2 = \varphi(e_1)$, $e_3, e_4 = \varphi(e_3)$ in T^h. By the rule $\varphi(X \wedge Y) = \varphi(X) \wedge \varphi(Y)$, φ acts on $\bigwedge^2(T^h) = \bigwedge^2_+(T^h) \oplus \bigwedge^2_-(T^h)$ and we see immediately that in the basis $\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3\}$ of $\bigwedge^2_+(T^h)$ the matrix representation of φ is given by

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since the curvature tensor commutes with the transformation φ in a Sasaki manifold, the Weyl tensor $W\colon \bigwedge^2(T^h) \to \bigwedge^2(T^h)$ also commutes with φ. Consequently, we obtain for $W_+\colon \bigwedge^2_+(T^h) \to \bigwedge^2_+(T^h)$ the matrix representation

$$W_+ = \begin{pmatrix} A & 0 & 0 \\ 0 & B & D \\ 0 & D & C \end{pmatrix}$$

with

$$A = W_{1212} + 2W_{1234} + W_{3434}, \qquad B = W_{1313} - 2W_{1324} + W_{2424},$$
$$C = W_{1414} + 2W_{1423} + W_{2323}, \qquad D = -2(W_{2414} + W_{2423}).$$

We prove $A = B = C = D = 0$, In fact, since M^5 is an Einstein space with scalar curvature $R = 20$, we have

$$W_{1212} = R_{1212} + 1, \quad W_{3434} = R_{3434} + 1, \quad W_{1234} = R_{1234},$$

and taking into account $R_{1551} = 1$ $(e_5 = \xi)$ we obtain

$$\begin{aligned} A &= R_{1212} + R_{3434} + 2R_{1234} + 2 \\ &= (-R_{1221} - R_{1331} - R_{1441} - R_{1551}) \\ &\quad + (-R_{4114} - R_{4224} - R_{4334} - R_{4554}) \\ &\quad + R_{1331} + R_{1441} + R_{4114} + R_{4224} + 2R_{1234} + 4 \\ &= -R_{11} - R_{44} + 2(R_{1331} + R_{1441} + R_{1234}) + 4 \\ &= -8 + 2(R_{1331} + R_{1441} + R_{1234}) + 4. \end{aligned}$$

The Muskal-Okumara lemma (see e.g. [3, p. 93]) now yields

$$R_{1234} + R_{1331} + R_{1441} = -d\eta(e_3, e_4)g(e_1, e_1) = -2g(e_3, \varphi(e_4)) = 2$$

and we finally have $A = 0$. In the same way we prove $B = C = 0$. Finally, we calculate D—using once again the Einstein equation and the Muskal-Okumara formula—

$$D = -2(W_{2414} + W_{2423}) = -2(R_{2414} + R_{2423}) = 0.$$

3. The SU(2)-reduction defined by a nonvanishing spinor

Consider the group Spin(5) and its complex spinor representation $\kappa\colon \mathrm{Spin}(5) \to \mathrm{GL}(\Delta_5)$. Spin(5) acts transitively on the 7-dimensional sphere $S(\Delta_5) = \{\psi \in \Delta_5 \colon |\psi| = 1\}$. The isotropy group $\hat{H}(\psi)$ of a fixed spinor $\psi \neq 0$ is a subgroup $\hat{H}(\psi) \subset \mathrm{Spin}(5)$ which projects one-to-one onto a subgroup $H(\psi) \subset \mathrm{SO}(5)$ which is conjugate to $\mathrm{SU}(2) \subset \mathrm{SO}(5)$. We fix an orthonormal basis $e_1, \cdots, e_5$ in R^5 and identify Δ_5 with $\mathbf{C}^2 \otimes \mathbf{C}^2$. Let us introduce the basis $u(\varepsilon_1, \varepsilon_2)$ in Δ_5 (see e.g. [4]):

$$u(\varepsilon_1, \varepsilon_2) = u(\varepsilon_1) \otimes (\varepsilon_2), \qquad \text{with } u(1) = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \; u(-1) = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Denote by g_1, g_2 and T the matrices

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = g_1 g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Clifford multiplication of a vector by a spinor is then defined by

$$e_1 = I \otimes g_1, \quad e_2 = I \otimes g_2, \quad e_3 = ig_1 \otimes T,$$
$$e_4 = ig_2 \otimes T, \quad e_5 = -iT \otimes T.$$

The Lie algebra $\hat{\mathfrak{h}}$ of the isotropy group of the spinor $\psi_0 = u(1,1)$ is given by

$$\hat{\mathfrak{h}} = \{\alpha \in \mathrm{spin}(5)\colon \alpha \cdot u(1,1) = 0\}$$
$$= \left\{ \sum_{1\le i<j\le 5} w_{ij}e_ie_j\colon \begin{matrix} w_{12}+w_{34}=0 & w_{14}+w_{23}=0 \\ w_{13}-w_{24}=0 & w_{15}=w_{25}=w_{35}=w_{45}=0 \end{matrix} \right\}.$$

Using this concrete realization of the spin-representation one immediately proves

Lemma 1. (a) *Let $\psi_1, \psi_2 \in S(\Delta_5)$ be two orthogonal spinors of length one and suppose that for the corresponding Lie algebras $\hat{\mathfrak{h}}(\psi_1) \cap \hat{\mathfrak{h}}(\psi_2) \neq \{0\}$. Then for each vector $X \in R^5$ it holds that*

$$(\psi_1, X \cdot \psi_2) = 0,$$

where $X \cdot \psi_2$ denotes the Clifford multiplication of the vector X by the spinor ψ_2.

(b) *For each spinor $\psi \neq 0$ there exists a unique vector $\xi \in R^5$ of length one such that $\xi \cdot \psi = i\psi$.*

Denote by $\pi\colon Q \to M^5$ the frame bundle of the oriented Riemannian manifold (M^5, g) and let $\pi\colon P \to M^5$ be a spin-structure. If $\psi \in \Gamma(S)$ is a section of length one in the spinor-bundle $S = P \times_\kappa \Delta_5$, then we consider

$$P^0 = \{p \in P\colon \psi(\pi(p)) = [p, u(1,1)]\}.$$

Since Spin(5) acts transitively on $S(\Delta_5)$ with isotropy group $\widehat{H}(\psi_0) = \mathrm{SU}(2)$, P^0 is a SU(2)-principal fiber bundle over M^5. Denote by $\lambda\colon P \to Q$ the two-fold covering of the spin structure over the frame bundle. Then $\lambda\,|_{P^0}\colon P^0 \to \lambda(P^0) = Q^0$ is bijective and, consequently, we obtain an SU(2)-reduction $Q^0 \subset Q$ of the frame bundle Q. We now investigate the topological type of this reduction in the case that M^5 is simply-connected. The classifying space of the group $\mathrm{SU}(2) = \mathrm{Sp}(1)$ is $P^\infty(H)$, a CW-complex of the type $e^0 \cup e^4 \cup e^8 \cup \cdots$. Since M^5 is a 5-dimensional CW-complex we see that the isomorphy classes of SU(2)-bundles over M^5 correspond to the set $[M^5, P^\infty(H)] = [M^5, S^4]$. Using the classification theorem of Steenrod (see e.g. [18]) we obtain

$$[M^5; S^4] = \frac{H^5(M^5; Z_2)}{\mathrm{Sq}^2 \mu_* H^3(M^5; Z)},$$

where $\mu_*\colon H^3(M^5; Z) \to H^3(M^5; Z_2)$ is the Z_2-reduction and Sq^2 denotes the second Steenrod square. Since M^5 is a spin-manifold its second Stiefel-Whitney class vanishes and, consequently, (look, for example, into the Wu-formula!) $\mathrm{Sq}^2 = 0$. Therefore, on a 5-dimensional, compact, simply-connected spin-manifold M^5 there are precisely two SU(2)-principal fiber bundles:

$$[M^5, S^4] = H^5(M^5; Z_2) = Z_2.$$

Theorem 1. *Let M^5 be a 5-dimensional, compact simply-connected spin-manifold with a nowhere vanishing spinor field $\psi \in \Gamma(S)$. Then the following conditions are equivalent:*

(1) *Q^0 is the trivial* $\mathrm{SU}(2)$*-principal fiber bundle.*

(2) *The subbundle $T^h = Q^0 \times_{\mathrm{SU}(2)} R^4 \subset TM^5$ is trivial.*

(3) *M^5 is parallelizable.*

(4) $\dim H_2(M^5; Z_2) \equiv 1 \bmod 2$.

On the other hand Q^0 is a nontrivial $\mathrm{SU}(2)$*-principal fiber bundle if and only if* $\dim H_2(M^5; Z_2) \equiv 0 \bmod 2$.

Proof. The implications (1)⇒(2)⇒(3) are trivial, (3) ⇒ (4) follows from classical results concerning vector fields on spin-manifolds (see [20]). Suppose now that $\dim H_2(M^5; Z_2) \equiv 1 \bmod 2$ and fix a point $m_0 \in M^5$. The space $M^5 \backslash \{m_0\}$ has the homotopy type of a 4-dimensional CW-complex and $\pi_1(M^5) = 0$ implies $H^4(M^5 \backslash \{m_0\}; Z) = 0$. Using the Hopf Classification Theorem we obtain

$$[M^5 \backslash \{m_0\}; P^\infty(H)] = [M^5 \backslash \{m_0\}; S^4] = H^4(M^5 \backslash \{m_0\}; Z) = 0.$$

This means that the bundle Q^0 is trivial over $M^5 \backslash \{m_0\}$. Consider a section $X^* = (X_1, \cdots, X_5)$ in Q^0 over $M^5 \backslash \{m_0\}$. The index $\mathrm{Ind}(X^*)$ is an element of $\pi_4(\mathrm{SU}(2)) = Z_2$. Furthermore, if $\mathrm{Ind}(X^*) = 0$ then Q^0 is a trivial bundle over M^5. We calculate the index of X^* in the following way: Look at the pair (X_1, X_2) of vector fields on $M^5 \backslash \{m_0\}$ and its index $\mathrm{Ind}(X_1, X_2) \in \pi_4(V_{5,2}) = Z_2$. An easy homotopy argument shows that the map $f\colon \mathrm{SU}(2) \to \mathrm{SO}(4) \to \mathrm{SO}(5) \to V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(2)$ induces an isomorphism $f_\#\colon \pi_4(\mathrm{SU}(2)) \to \pi_4(V_{5,2})$. Consequently, $\mathrm{Ind}(X^*)$ vanishes in $\pi_4(\mathrm{SU}(2))$ if and only if $\mathrm{Ind}(X_1, X_2)$ vanishes in $\pi_4(V_{5,2})$. Now the index of a pair of vector fields with isolated singularities is well known (see e.g. [20]):

$$\begin{aligned}\mathrm{Ind}(X_1, X_2) &= \sum_{i=0}^{2} \dim H_i(M^5; Z_2) \\ &= 1 + \dim H_2(M^5; Z_2) \quad \bmod 2.\end{aligned}$$

This proves the implication (4) ⇒ (1).

Remark. Using similar techniques one can show that in case the $\mathrm{SU}(2)$-reduction $Q^0 \subset Q$ is nontrivial it does *not* admit a reduction to the subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$.

4. The Einstein-Sasaki structure defined by a Killing spinor

Let $\psi \in \Gamma(S)$ be an eigenspinor of the Dirac operator corresponding to the eigenvalue $\pm\frac{1}{4}\sqrt{5R}$ on a compact, 5-dimensional Einstein spin-manifold M^5

with positive scalar curvature R,

$$D\psi = \pm\frac{1}{4}\sqrt{5R}\psi.$$

Then ψ satisfies a stronger equation, namely

$$\nabla_X\psi = \mp\frac{\sqrt{R}}{4\sqrt{5}}X\cdot\psi,$$

where $X\cdot\psi$ denotes the Clifford multiplication of the vector X by the spinor ψ (see e.g. [4]). Such spinor fields are sometimes called Killing spinors (see e.g. [9]). It is well known that the length $|\psi|$ of ψ is constant.

Denote by $E_\pm \subset L^2(S)$ the eigenspace of the Dirac operator corresponding to the eigenvalues $\pm\frac{1}{4}\sqrt{5R}$, respectively.

Proposition 2. *If M^5 is not conformally flat then* $\dim E_\pm \le 1$.

Proof. Suppose we have two solutions ψ_1, ψ_2 satisfying

$$\nabla_X\psi_i = -\frac{1}{4\sqrt{5}}\sqrt{R}X\cdot\psi_i \qquad (i=1,2).$$

Without loss of generality we can assume that $(\psi_1,\psi_2)\equiv 0$ since $X(\psi_1,\psi_2) = (\nabla_X\psi_1,\psi_2)+(\psi_1,\nabla_X\psi_2) = 0$.

Fix a point $m_0 \in M^5$ such that the Weyl tensor does not vanish at m_0. Then we have for any 2-form $\eta^2 \in \Lambda^2$

$$W(\eta^2)\cdot\psi_1 = 0 = W(\eta^2)\cdot\psi_2,$$

where $W\colon \bigwedge^2(TM^5) \to \bigwedge^2(TM^5)$ is the Weyl tensor (see e.g. [5]). Since $W \ne 0$ at m_0 we apply Lemma 1 and conclude $(\psi_1, X\cdot\psi_2) = 0$ for any vector $X \in T_{m_0}(M^5)$. Consider a local frame $s = (s_1,\cdots,s_5)$ in the SU(2)-bundle $Q^0 \subset Q$ corresponding to ψ_1 as well as the section s^* in the reduction P^0 of the spin-structure P. Then we have (locally) $\psi_1 = [s^*, u(1,1)]$ and $(\psi_1, X\cdot\psi_2) = 0$ for each vector X implies $\psi_2 = [s^*, z\cdot u(-1,-1)]$ with a complex valued function z. Consequently, we obtain

$$\nabla_X\psi_1 = \frac{1}{2}\sum_{i<j} w_{ij}(X)e_ie_ju(1,1) = -\frac{1}{4\sqrt{5}}\sqrt{R}X\cdot u(1,1),$$

$$\begin{aligned}\nabla_X\psi_2 &= dz(X)\cdot u(-1,-1) + \frac{1}{2}\sum_{i<j} w_{ij}(X)e_ie_ju(-1,-1)\\ &= -\frac{1}{4\sqrt{5}}\sqrt{R}X\cdot u(-1,-1),\end{aligned}$$

where w_{ij} are the connection forms of the Riemannian manifold M^5 with respect to the frame s. Using the formulas for the Clifford multiplication given above we conclude in particular ($X = s_1$)

$$- w_{15}(s_1) + iw_{25}(s_1) = i\frac{1}{2\sqrt{5}}\sqrt{R},$$
$$- w_{15}(s_1) - iw_{25}(s_1) = i\frac{1}{2\sqrt{5}}\sqrt{R},$$

thus a contradiction.

Remark. Consider a Killing spinor ψ with

$$\nabla_X\psi = \lambda X \cdot \psi \qquad \left(\lambda = \pm\frac{\sqrt{R}}{4\sqrt{5}}\right)$$

and the corresponding SU(2)-reduction Q^0 of the frame bundle Q. If s is a local section in Q^0 we have

$$\frac{1}{2}\sum_{i<j} w_{ij}(X)e_ie_ju(1,1) = \lambda X \cdot u(1,1).$$

Denote by $\sigma^1, \cdots, \sigma^5$ the dual frame to $s_1, \cdots, s_5$. Then an algebraic calculation yields the following formulas:

$$w_{12} + w_{34} = 2\lambda\sigma^5, \quad w_{13} - w_{24} = 0, \quad w_{14} + w_{23} = 0,$$
$$w_{15} = -2\lambda\sigma^2, \quad w_{25} = 2\lambda\sigma^1, \quad w_{35} = -2\lambda\sigma^4, \quad w_{45} = 2\lambda\sigma^3.$$

We consider now an Einstein space (M^5, g) such that $R = 20$ as well as a Killing spinor ψ satisfying $\nabla_X\psi = -\frac{1}{2}X \cdot \psi$. According to Lemma 1 there exists a unique vector field ξ of length one such that $\xi \cdot \psi = i\psi$. Furthermore, we define a 1-form η by $\eta(X) = (X \cdot \psi, \psi)/i$ and a (1,1)-tensor $\varphi := -\nabla\xi$.

Theorem 2. *Let (M^5, g) be an Einstein space with scalar curvature $R =$ 20 and Killing spinor ψ. Then $(M^5; \varphi, \xi, \eta, g)$ is an Einstein-Sasaki manifold.*

Proof. We must check the conditions (a)–(g) defining a Sasaki structure in our situation. For the local calculations we choose a frame s in the SU(2)-reduction. We have

$$\begin{aligned} d\eta(X,Y) &= \frac{1}{i}\{X(Y\psi,\psi) - Y(X\psi,\psi) - ([X,Y]\psi,\psi)\} \\ &= \frac{1}{i}\{(Y\nabla_X\psi,\psi) + (Y\psi,\nabla_X\psi) - (X\nabla_Y\psi,\psi) - (X\psi,\nabla_Y\psi)\} \\ &= -\frac{1}{i}((YX - XY)\psi,\psi) \end{aligned}$$

and, consequently,

$$d\eta = 2(\sigma^1 \wedge \sigma^2 + \sigma^3 \wedge \sigma^4).$$

This implies immediately

$$\eta \wedge d\eta \wedge d\eta = 8dM^5.$$

The equation $\eta(\xi) = 1$ follows directly from the definition of ξ and η. We differentiate the equation $\xi \cdot \psi = i\psi$ and obtain

$$(\nabla_X \xi) \cdot \psi + \xi \nabla_X \psi = i\nabla_X \psi$$
$$-\varphi(X) \cdot \psi - \frac{1}{2}\xi X\psi = -\frac{i}{2}X \cdot \psi.$$

In particular we have $\varphi(X)\psi = iX \cdot \psi$ for each X orthogonal to ξ. Replacing X by $\varphi(X)$ we have

$$-\varphi^2(X) \cdot \psi - \frac{1}{2}\xi\varphi(X)\psi = -\frac{i}{2}\varphi(X) \cdot \psi.$$

Combining the last two equations we obtain

$$-\varphi^2(X) \cdot \psi - \frac{1}{2}(X + i\xi X)\psi = 0.$$

If X is parallel to ξ it follows that $\varphi^2(X) \cdot \psi = 0$ and, consequently, $\varphi^2(X) = 0$. If X is orthogonal to ξ we have $\frac{1}{2}(X+i\xi X)\psi = \frac{1}{2}(X-iX\xi)\psi = \frac{1}{2}(X-i^2X)\psi = X \cdot \psi$ and

$$\{\varphi^2(X) + X\} \cdot \psi = 0.$$

The last formula implies $\varphi^2(X) = -X$ in case X is orthogonal to ξ. Summing up we proved $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$.

We prove now that ξ is a Killing vector field, i.e. φ is antisymmetric. We already know

$$\varphi(X) \cdot \psi + \frac{1}{2}\xi X\psi = \frac{i}{2}X \cdot \psi.$$

We multiply by $Y \cdot \psi$ from the right and left side:

$$(\varphi(X)\psi, Y \cdot \psi) + \frac{1}{2}(\xi X\psi, Y\psi) = \frac{i}{2}(X\psi, Y\psi),$$
$$(Y\psi, \varphi(X)\psi) + \frac{1}{2}(Y\psi, \xi X\psi) = -\frac{i}{2}(Y\psi, X\psi).$$

Taking into account $Y \cdot \varphi(X) + \varphi(X) \cdot Y = -2g(Y, \varphi(X))$ we obtain

$$2g(Y, \varphi(X))|\psi|^2 + \mathrm{Re}((\xi X\psi, Y\psi)) = -\mathrm{Im}(X\psi, Y\psi).$$

Finally we remark that the real part of $(\xi X\psi, Y\psi)$ and the imaginary part of $(X\psi, Y\psi)$ are antisymmetric in X and Y. It follows that

$$g(Y, \varphi(X)) = -g(X, \varphi(Y)),$$

i.e. ξ is a Killing vector field.

The equation $g(\varphi(X),\varphi(Y)) = g(X,Y) - \eta(X)\eta(Y)$ is now a formal consequence of some formulas we already proved:

$$\begin{aligned} g(\varphi(X),\varphi(Y)) &= -g(\varphi^2(X),Y) \\ &= -g(-X+\eta(X)\xi,Y) = g(X,Y) - \eta(X)g(\xi,Y) \\ &= g(X,Y) - \eta(X)\eta(Y). \end{aligned}$$

We prove the property $d\eta(X,Y) = 2g(X,\varphi(Y))$—using the fact that ξ is a Killing field—as follows:

$$\begin{aligned} d\eta(X,Y) &= X\eta(Y) - Y\eta(X) - \eta[X,Y] \\ &= Xg(\xi,Y) - Yg(\xi,X) - g(\xi,[X,Y]) = g(\nabla_X\xi,Y) - g(\nabla_Y\xi,X) \\ &= -g(\varphi(X),Y) + g(X,\varphi(Y)) = 2g(X,\varphi(Y)). \end{aligned}$$

It remains to prove the integrability condition $(\nabla_Y\varphi)(X) = g(X,Y)\xi - \eta(X)Y$. We again start with $\varphi(X)\cdot\psi = \frac{1}{2}(iX - \xi X)\cdot\psi$ and differentiate this equation:

$$\begin{aligned} \nabla_Y(\varphi(X))\cdot\psi - \tfrac{1}{2}\varphi(X)Y\psi &= \tfrac{1}{2}(i\nabla_Y X - \nabla_Y\xi\cdot X - \xi\nabla_Y X)\cdot\psi \\ &\quad + \tfrac{1}{2}(iX - \xi X)(-\tfrac{1}{2}Y\psi). \end{aligned}$$

On the other hand we have

$$\varphi(\nabla_Y X)\psi = \tfrac{1}{2}(i\nabla_Y X - \xi\nabla_Y X)\psi.$$

This implies

$$(\nabla_Y\varphi)(X)\cdot\psi = \frac{1}{2}\left\{\varphi(X)Y + \varphi(Y)X + \frac{\xi XY - iXY}{2}\right\}\psi.$$

First of all we consider the case that X and Y are orthogonal to ξ. Then $(\xi XY - iXY)\psi = 0$ and $\varphi(X)\cdot\psi = \frac{1}{2}(iX - \xi X)\psi = iX\psi$. In this situation we have

$$\begin{aligned} (\nabla_Y\varphi)(X)\psi &= \tfrac{1}{2}\{\varphi(X)Y + \varphi(Y)X\}\psi \\ &= \frac{1}{2}\{-Y\varphi(X) - 2g(Y,\varphi(X)) - X\varphi(Y) - 2g(X,\varphi(Y))\}\psi \\ &= \frac{1}{2}\{-iYX - iXY\}\psi = g(X,Y)\xi\cdot\psi \end{aligned}$$

and finally $(\nabla_Y\varphi)(X) = g(X,Y)\xi$.

The second case we want to consider is $X = \xi$. Then

$$\begin{aligned} (\nabla_Y\varphi)(X) &= \nabla_Y(\varphi(\xi)) - \varphi(\nabla_Y\xi) = \varphi^2(Y) = -Y + \eta(Y)\xi \\ &= g(\xi,Y)\xi - g(\xi,X)Y = g(X,Y)\xi - \eta(X)Y. \end{aligned}$$

If $Y = \xi$ we have

$$\{\xi XY - iXY\}\psi = \{-X\xi\xi - iX\xi\}\psi = \{X + X\}\psi = 2X\cdot\psi,$$

(X orthogonal to ξ) and it follows that

$$(\nabla_Y\varphi)(X)\cdot\psi = \tfrac{1}{2}\{\varphi(X)\xi + X\}\psi = \tfrac{1}{2}\{i\varphi(X) + X\}\psi = \tfrac{1}{2}\{i^2X + X\}\psi = 0.$$

The last equation implies

$$(\nabla_\xi\varphi)(X) = 0 = g(\xi, X)\xi - \eta(X)\xi$$

for each X orthogonal to ξ. Last but not least we consider the case $X = Y = \xi$. Then we have

$$\begin{aligned}(\nabla_\xi\varphi)(\xi) &= \nabla_\xi(\varphi(\xi)) - \varphi(\nabla_\xi\xi)\\ &= 0 - \varphi^2(\xi) = 0 = g(\xi,\xi)\xi - \eta(\xi)\xi\end{aligned}$$

and the integrability condition is proved.

Remark 1. The existence of a Killing spinor ψ imposes algebraic conditions on the Weyl tensor W, namely $W(\eta^2)\cdot\psi = 0$ for any 2-forms η^2. In the case of dimension five this implies

$$\sum_{1\le i<j\le 5} W_{ij}e_ie_ju(1,1) = 0.$$

Taking into account the structure of the Lie algebra $\hat{h}$ described in §3 we conclude

$$W_{12} + W_{34} = 0, \quad W_{13} - W_{24} = 0, \quad W_{14} + W_{23} = 0, \quad W_{i5} = 0,$$

and this is precisely the anti-selfduality condition for the Weyl tensor

$$W\colon \bigwedge^2(T^h) \to \bigwedge^2(T^h),$$

which is satisfied automatically in any Einstein-Sasaki space (Proposition 1).

Remark 2. Using the properties of the Sasaki structure we have in particular for the Lie-derivative:

$$\mathscr{L}_\xi\eta = 0, \quad \mathscr{L}_\xi(d\eta) = 0, \quad \mathscr{L}_\xi\varphi = 0.$$

Remark 3. Obviously, if we start with a spinor satisfying $\nabla_X\psi = \frac{1}{2}X\cdot\psi$ we obtain in the same way an Einstein-Sasaki structure.

5. A simply-connected Einstein-Sasaki manifold admits a Killing spinor

Theorem 3. *Let $(M^5;\varphi,\xi,\eta,g)$ be a simply-connected Einstein-Sasaki manifold, with spin-structure. Then the equations $\nabla_X\psi = \pm\frac{1}{2}X\cdot\psi$ have nontrivial solutions.*

Proof. Consider the subbundle E of the spinor bundle S defined by

$$E = \{\psi \in S \colon \xi\psi = i\psi, \{2\varphi(X) + \xi X - iX\}\psi = 0 \quad \text{for each vector } X \in TM^5\}.$$

Using the algebraic description of Δ_5 given above it is easy to see that E is a 1-dimensional complex subbundle of S. We introduce a covariant derivative $\tilde{\nabla}\colon \Gamma(E) \to \Gamma(T^* \otimes E)$ in E by the formula

$$\tilde{\nabla}_X\psi = \nabla_X\psi + \tfrac{1}{2}X \cdot \psi.$$

First of all we must prove that $\tilde{\nabla}_X\psi$ is a section in E if ψ belongs to $\Gamma(E)$.
Suppose that $\xi\psi = i\psi$ and $\{2\varphi(X) + \xi X - iX\}\psi = 0$. Then

$$\nabla_Y\xi \cdot \psi + \xi\nabla_Y\psi = i\nabla_Y\psi,$$
$$\nabla_Y\xi\psi + \xi\left(\nabla_Y\psi + \tfrac{1}{2}Y\psi\right) - \tfrac{1}{2}\xi Y\psi = i(\nabla_Y\psi + \tfrac{1}{2}Y\psi) - \tfrac{1}{2}iY\psi,$$
$$\tfrac{1}{2}(2\nabla_Y\xi - \xi Y + iY)\psi + \xi(\tilde{\nabla}_Y\psi) = i(\tilde{\nabla}_Y\psi).$$

Since we have a Sasaki structure it holds that $\nabla_Y\xi = -\varphi(Y)$. ψ is a section in E. This implies

$$\xi(\tilde{\nabla}_Y\psi) = i(\tilde{\nabla}_Y\psi).$$

In the same way we prove the second algebraic condition for $\tilde{\nabla}_Y\psi$. We differentiate the equation

$$\{2\varphi(X) + \xi X - iX\}\psi = 0$$

with respect to Y and we use the Sasaki conditions $\varphi = -\nabla\xi$, $(\nabla_Y\varphi)(X) = g(X,Y)\xi - \eta(X)Y$. After some obvious calculations we obtain

$$\left\{2g(X,Y)\xi - 2\eta(X)Y - \varphi(Y)X - \varphi(X)Y - \frac{\xi XY - iXY}{2}\right\}\psi$$
$$+ \{2\varphi(X) + \xi X - iX\}\tilde{\nabla}_Y\psi = 0.$$

The first term vanishes. Consider for example the case that X and Y are orthogonal to ξ. Then we have $\{\xi XY - iXY\}\psi = 0$ with respect to $\xi\psi = i\psi$ and, consequently, the first term reduces to

$$\begin{aligned}&\{2g(X,Y)\xi - \varphi(Y)X - \varphi(X)Y\}\psi\\ &\quad = \{2g(X,Y)i + (2g(\varphi(Y),X) + X\varphi(Y)) + (2g(\varphi(X),Y) + Y\varphi(X))\}\psi\\ &\quad = \{2g(X,Y)i + X\varphi(Y) + Y\varphi(X)\}\psi.\end{aligned}$$

Since ψ is a section in E, we have

$$\{2\varphi(X) + \xi X - iX\}\psi = 0.$$

If X is orthogonal to ξ we obtain

$$\varphi(X)\cdot\psi = iX\cdot\psi.$$

The first term mentioned above thus eventually reduces to

$$\{2g(X,Y)i + iXY + iYX\}\psi = 2i\{g(X,Y) - g(X,Y)\}\psi = 0.$$

We handle the cases where X or Y is parallel to ξ in the same way. Then we obtain

$$\{2\varphi(X) + \xi X - iX\}\tilde{\nabla}_Y\psi = 0,$$

i.e. $\tilde{\nabla}_Y\psi$ is a section in E.

The calculation of the curvature tensor $\tilde{R}$ of the connection $\tilde{\nabla}$ in the bundle E yields the formula

$$\begin{aligned}\tilde{R}(X,Y)\psi &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\psi + \frac{1}{4}(XY - YX)\psi \\ &= \frac{1}{4}\left(\sum_{i,j} R_{XYij}e_ie_j + XY - YX\right)\psi = \frac{1}{4}\sum_{i,j} W_{XYij}e_ie_j\cdot\psi\end{aligned}$$

with the Weyl tensor W. Here we use the formula

$$W_{ijke} = R_{ijke} + (\delta_{ik}\delta_{je} - \delta_{ie}\delta_{jk})$$

valid in a 5-dimensional Einstein space with scalar curvature $R = 20$. Since M^5 is an Einstein-Sasaki manifold, we have $W(\xi, X) = 0$ and we obtain

$$\tilde{R}(X,Y)\psi = \frac{1}{4}\sum_{i=1}^{4} e_i\cdot W(X,Y)e_i\cdot\psi,$$

where $\{e_1, e_2, e_3, e_4\}$ is a frame in T^h orthogonal to ξ. A simple algebraic calculation—using Proposition 1, i.e. $*W = -W$ in $\bigwedge^2(T^h)$—now shows

$$\tilde{R}(X,Y)\psi = 0, \qquad \psi\in\Gamma(E).$$

Consequently, $(E,\tilde{\nabla})$ is a flat 1-dimensional bundle over a simply-connected manifold M^5. Thus there exists a $\tilde{\nabla}$-parallel section ψ in E, i.e. a spinor field satisfying the equation $\nabla_X\psi = -\frac{1}{2}X\cdot\psi$.

Remark. The same procedure allows us to construct a solution of the equation $\nabla_X\psi = +\frac{1}{2}X\cdot\psi$.

Corollary. *In case M^5 is simply-connected we have* $\dim E_+ = \dim E_-$, *where $E_\pm \subset L^2(S)$ is the eigenspace of the Dirac operator corresponding to the eigenvalue $\pm\frac{1}{4}\sqrt{5R}$.*

6. The classification of compact Einstein spin-manifolds admitting a Killing spinor with regular contact structure

A Sasaki manifold $(M^5; \varphi, \xi, \eta, g)$ is called regular if all integral curves of ξ are closed and have the same length L (see e.g. [3]). In this situation we have an S^1-action on M^5 and the orbit space is a 4-dimensional manifold X^4. The projection $\pi\colon M^5 \to X^4$ is a principal S^1-bundle and $2\pi i\eta/L\colon TM^5 \to R\cdot i = \mathfrak{S}^1$ is a connection in this bundle. Since $\mathscr{L}_\xi g = 0$ and $\mathscr{L}_\xi \varphi = 0$, X^4 admits a Riemannian metric and an almost complex structure which is integrable (see e.g. [3]). Denote by Ω the Kähler form of X^4. Then

$$\pi^*\Omega(X,Y) = g(X, \varphi(Y)) = \tfrac{1}{2} d\eta(X,Y)$$

and we conclude $d\Omega = 0$, i.e. X^4 is a Kähler manifold. Suppose now in addition that M^5 is an Einstein-Sasaki space. The O'Neill formulas yield that X^4 is an Einstein-Kähler manifold with scalar curvature $\mathfrak{R} = \frac{6}{5}R = 24$. Consequently, X^4 is analytically isomorphic to $S^2 \times S^2$, $P^2(\mathbf{C})$ or to one of the del Pezzo surfaces P_k $(3 \leq k \leq 8; P_k$ is the surface obtained by blowing up k points in general position in $P^2(\mathbf{C})$, see e.g. [2]). Next we study the topological type of the S^1-fiber bundle $\pi\colon M^5 \to X^4$. The curvature form of the connection $2\pi i\eta/L$ is $\Omega^* = (2\pi i/L)d\eta$. Consequently, the Chern class $c_1^* \in H^2(X^4; R)$ is given by $c_1^* = \Omega^*/2\pi i = d\eta/L$. On the other hand, since X^4 is an Einstein-Kähler manifold its Chern class is given by the Ricci form

$$c_1 = \Omega_{\mathrm{Ric}} = \frac{1}{2\pi}\ \frac{R}{4}\Omega = \frac{3}{\pi}\Omega = \frac{3}{2\pi}d\eta = \frac{3L}{2\pi}c_1^*$$

and we obtain the relation

$$c_1 = \frac{3L}{2\pi}c_1^*$$

between the Chern class c_1 of X^4 and the Chern class c_1^* of the S^1-bundle $\pi\colon M^5 \to X^4$. X^4 is simply connected. We now apply the Thom-Gysin sequence of the fibration $\pi\colon M^5 \to X^4$ and conclude:

(a) $H^1(M^5; Z) = 0$ (since $c_1^* \neq 0$).

(b) $H^4(M^5; Z) = H^4(X^4; Z)/c_1^* \cup H^2(X^4; Z)$.

(c) $0 = w_2(M^5) = \pi^* w_2(X^4)$. If $w_2(X^4) \neq 0$ then $c_1^* \equiv w_2(X^4) \equiv c_1 \bmod 2$.

In case $w_2(X^4) \neq 0$ the spin structure of M^5 implies an additional condition, namely

$$\frac{1}{2}\left(1 - \frac{2\pi}{3L}\right) c_1(X^4) \in H^2(X; Z).$$

(d) The Killing spinor ψ on M^5 defines an SU(2)-reduction Q^0 of the frame bundle. Consequently, we have an isomorphism

$$\pi^* T_{\mathbf{C}} X^4 = T^h = Q^0 \times_{\mathrm{SU}(2)} \mathbf{C}^2$$

of 2-dimensional complex vector bundles. This isomorphism yields $\pi^* c_1(X^4) = 0$ because the first Chern class of any SU(2)-bundle vanishes. The Thom-Gysin sequence imposes a further restriction: $c_1/c_1^* \in Z$.

We now classify all possible Einstein spaces M^5.

First case: $X^4 = P^2(\mathbf{C})$. If X^4 is analytically isomorphic to $P^2(\mathbf{C})$ and admits an Einstein-Kähler metric then X^4 is analytically isometric to $P^2(\mathbf{C})$ (see e.g. [15]). The cohomology algebra $H^*(P^2(\mathbf{C}))$ is isomorphic to $Z[\alpha]/(\alpha^3)$ and the first Chern class is given by the $c_1 = 3\alpha$, $\alpha \in H^2(P^2(\mathbf{C}))$. Using the restrictions (c) and (d) stated above we have two possibilities for the Chern class $c_1^* = \alpha, 3\alpha$ with $\pi_1(M^5) = H^4(M^5) = 0$, Z_3 and $L = 2\pi, 2\pi/3$. Since we know the curvature tensor of $P^2(\mathbf{C})$ as well as the curvature form $\Omega^* = (2\pi i/L)d\eta = 4\pi i\Omega/L$ of the Riemannian submersion $\pi\colon M^5 \to X^4$ we can apply the O'Neill formulas again and conclude that M^5 is conformally flat. Consequently, M^5 is isometric to S^5 in case $c_1^* = \alpha$ and isometric to S^5/Z_3 in case $c_1^* = 3\alpha$. $P^2(\mathbf{C})$ is a homogeneous Einstein-Kähler manifold. A simple geometric argument shows that we can lift the isometries of $P^2(\mathbf{C})$ to isometries of M^5, i.e. $M^5 = S^5/Z_3$ is the homogeneous space of constant curvature one and fundamental group $\pi_1(M^5) = Z_3$.

Second case: $X^4 = S^2 \times S^2$. Suppose that X^4 is analytically isomorphic to $S^2 \times S^2 = \check{G}_{4,2} = Q_2 =$ the Klein quadric in $P^3(\mathbf{C})$. Moreover, X^4 has an Einstein-Kähler metric with positive scalar curvature. Then the Lie algebra $\mathfrak{h}$ of all holomorphic vector fields on X^4 is the complexification of the Lie algebra $\mathfrak{i}$ of all Killing vector fields (see [14]) and we conclude that $\dim_R \mathfrak{i} = \dim_{\mathbf{C}} \mathfrak{h} = 6$, i.e. X^4 admits a 6-dimensional group of isometries. We now apply a result of L. Berard Bergery (see e.g. [1]) stating in our situation that X^4 is a symmetric Einstein-Kähler structure on $S^2 \times S^2$. Consequently, X^4 is analytically isometric to $S^2 \times S^2$. The cohomology algebra of $S^2 \times S^2$ is $H^*(S^2 \times S^2) = \Lambda(\alpha, \beta)$ and its first Chern class is given by $c_1 = 2(\alpha+\beta)$. We again have two possibilities $c_1^* = (\alpha+\beta)$, $2(\alpha+\beta)$ with $\pi_1(M^5) = H^4(M^5) = 0$, Z_2 and $L = 4\pi/3, 2\pi/3$.

Now we study the geometry of the Riemannian submersion $\pi\colon M^5 \to X^4$ and conclude that M^5 is isometric to the Stiefel manifold $V_{4,2}$ or to $V_{4,2} \mid Z_2$ with the Einstein metric considered in [11]. The calculation in [4] shows that this space admits a nontrivial Killing spinor.

Third case: $X^4 = P_k$. If X^4 is analytically isomorphic to a del Pezzo surface P_k $(3 \leq k \leq 8)$ there is only one possibility for M^5, namely the simply-connected S^1-fiber bundle over P_k. Indeed, the cohomology algebra of P_k is generated by elements $\alpha, E_1, \cdots, E_k \in H^2(P_k)$ and the first Chern class is given by

$$c_1(P_k) = 3\alpha + E_1 + \cdots + E_k$$

(see e.g. [2]). Using the restriction for c_1^* given above we see that there remains only one possibility,

$$c_1^* = 3\alpha + E_1 + \cdots + E_k$$

with $\pi_1(M^5) = H^4(M^5) = H^4(P_k)/c_1^* \cup H^2(P_k) = 0$.

Summing up we proved the following

Theorem 4. *Let (M^5, g) be an Einstein space with Killing spinor ψ and scalar curvature $R = 20$. Suppose in addition that the associated contact structure is regular. Then there are three possibilities:*

(1) *M^5 is isometric to S^5 or S^5/Z_3 with the homogeneous metric of constant curvature.*

(2) *M^5 is isometric to the Stiefel manifold $V_{4,2}$ or $V_{4,2}/Z_2$ with the Einstein metric considered in* [11],[4].

(3) *M^5 is diffeomorphic to the simply-connected S^1-fiber bundle with Chern class $c_1^* = c_1(P_k)$ over a del Pezzo surface P_k $(3 \leq k \leq 8)$.*

Remark. S. Sulanke (see [19]) classified all spaces S^5/Γ of constant curvature with a Killing spinor. It turned out that except for the case S^5/Z_3 all other examples defined a nonregular contact structure. The integral curves of ξ are all closed but have different length. It seems to be interesting, using higher-dimensional Seifert-fibrations, to classify all Einstein spaces with Killing spinors such that the integral curves are closed, but with different length. The orbit space X^4 in this case is smooth except for a finite number of points.

References

[1] L. Berard Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein* (preprint, no. 839), Inst. Élie Cartan, 1981.

[2] A. Besse, *Geometric riemannienne en dimension* 4, Cedic, Paris, 1981.

[3] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., Vol. 509, Springer, Berlin, 1976.

[4] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. **97** (1980) 117–146.

[5] ——, *A remark on the first eigenvalue of the Dirac operator on* 4*-dimensional manifolds*, Math. Nachr. **102** (1981) 53–56.

[6] Th. Friedrich & R. Grunewald, *On Einstein metrics on the twistor space of a four-dimensional Riemannian manifold*, Math. Nachr. **123** (1985) 55–60.

[7] ——, *On the first eigenvalue of the Dirac operator on* 6*-dimensional manifolds*, Ann. Global Anal. Geom. **3** (1985) 265–273.

[8] R. Grunewald, *Untersuchungen über Einstein-Metriken auf* 6*-dimensionalen Mannigfaltigkeiten*, Diss. A, Humboldt Universität, Berlin, 1987.

[9] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986) 151–162.

[10] ——, *Caracterisation de la sphere par les premieres valeurs propres de l'operateur de Dirac en dimension* 3,4,7 *et* 8, C.R. Acad. Sci. Paris Sér. I Math. **303** (1986) 417–419.

[11] G. R. Jensen, *Imbeddings of Stiefel manifolds into Grassmannians*, Duke Math. J. **42** (1975) 397–407.

[12] K. D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds with positive scalar curvature*, Ann. Global Anal. Geom. **4** (1986) 291–326.

[13] ——, *Compact six-dimensional Kähler spin manifolds of positive scalar curvature with the smallest possible first eigenvalue of the Dirac operator*, Math. Ann., to appear.

[14] S. Kobayashi, *Transformation groups in differential geometry*, Springer, Berlin, 1972.

[15] A. Lascoux & M. Berger, *Variétés Kähleriennes Compactes*, Lecture Notes in Math., Vol. 154, Springer, Berlin, 1970.

[16] A. Lichnerowicz, *Variétés spinorielles et universalite de l'inegalite de Hijazi*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987) 227–231.

[17] B. E. Nilsson & C. N. Pope, *Scalar and Dirac eigenfunctions on the squashed seven-sphere*, Phys. Lett. B **133** (1983) 67–71.

[18] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[19] S. Sulanke, *Der Erste Eigenwert des Dirac-Operators auf* S^5/Γ, Math. Nachr. **99** (1980) 259–271.

[20] E. Thomas, *The index of a tangent 2-field*, Comment. Math. Helv. **42** (1967) 86–110.

[21] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$, Invent. Math. **89** (1987) 225–246.

[22] G. Tian & S. T. Yau, Kähler-Einstein metrics on complex surfaces with $C_1 > 0$, Comm. Math. Phys. **112** (1987) 175–203.

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