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PROPER FREDHOLM SUBMANIFOLDS OF HILBERT SPACE

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0. Introduction

The first step in the study of submanifolds of Euclidean spaces is to find enough local invariants and their relations so that they determine the submanifolds uniquely up to rigid motion. This is well known in classical differential geometry [7]. In fact, the first fundamental form I, the second fundamental form II and the induced normal connection are the basic local invariants, and they are related by the Gauss, Codazzi and Ricci equations. The shape operator A_v of an immersed submanifold M in $\mathbb{R}^n$ in the normal direction v at x is the selfadjoint operator on TM_x corresponding to the second fundamental form $\mathrm{II} \cdot v$. The eigenvalues of A_v are called the principal curvatures of M in the normal direction v. The Ricci equation implies that the normal curvature (the curvature of the normal connection) Ω^ν measures the commutativity of the shape operators, i.e., $\Omega^\nu(u,v) = [A_u, A_v]$. So if the normal curvature is zero, that is, if the normal bundle $\nu(M)$ is flat, then $\{A_v | v \in \nu(M)_x\}$ is a commuting family of selfadjoint operators, and locally there exists a parallel orthonormal normal frame field on M. It follows that many results of hypersurfaces can be generalized to submanifolds with flat normal bundles.

One natural type of problem is to determine all submanifolds of $\mathbb{R}^n$ which, in various senses, have simple local invariants. As a by-product of such investigations one often obtains many geometrically interesting examples of Riemannian manifolds. A special case of the above is the problem of finding all isoparametric submanifolds ([31], [30], [9], [41]), i.e., submanifolds with zero normal curvature and constant principal curvatures along any parallel normal field. It is not surprising that group theory provides examples. In fact, the principal orbits of the adjoint action of a simple Lie group on its Lie algebra (or more generally the principal orbits of the isotropy representations of symmetric spaces) are models for such manifolds. But they are still far from being completely classified.

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Another type of natural question concerns special functions on M defined by symmetric functions of the principal curvatures. Setting such functions equal to zero gives geometrically natural partial differential equations (for example the minimal submanifold equations and constant mean curvature equations). Problems of this type often lead to interesting interplay between geometry, analysis, and topology. Often submanifolds with local invariants provide the most easily found solutions of such equations.

As is well known, the behavior of the shape operators and the homology of M are closely related via the Morse inequalities and the Morse index theorem [29]. For if f_a denotes the Euclidean distance function on M (that is, f_a is the restriction of $\|x-a\|^2$ to the submanifold M), then q is a critical point of f_a if and only if $(a-q)$ is normal to TM_q, and the Hessian of f_a at a critical point q is $(\mathrm{I}-A_{(a-q)})$. One beautiful application in this direction is the rich theory of tight and taut immersions ([12], [27], [8], [11]). Once again isoparametric submanifolds provide numerous examples of tight and taut immersions ([10], [21]).

A Hilbert manifold is a differentiable manifold locally modeled on a Hilbert space. The foundation work on Hilbert manifolds was done in the 1960's. For example, standard differential calculus works the same way as in the finite dimension [28], Smale [40] developed the differential topology for Fredholm maps between Banach manifolds, Palais and Smale ([33], [39]) developed the Morse theory on Hilbert manifolds. Some basic notions of Riemannian geometry for Hilbert manifolds could also be carried over from the finite dimension theory, for example the Levi-Civita connection and the Riemann curvature tensor are defined. But the main motivations at that time came from the calculus of variations, and one applied the infinite dimensional theory mainly to the manifolds of maps between finite dimensional Riemannian manifolds.

Probably the major reason that an independent theory of infinite dimensional Riemannian geometry did not flourish in the 1960's was a lack of geometrically interesting examples. One way to obtain such examples is to find interesting submanifolds of Hilbert spaces. The three basic local invariants and their related equations can be easily generalized to submanifolds of Hilbert spaces. But the spectral theory of the shape operators is complicated, and infinite dimensional differential topology and Morse theory cannot be applied easily to these submanifolds without further restrictions. Thus in order to generalize the above theory of submanifolds of $\mathbb{R}^n$ to submanifolds of Hilbert space, one must find a suitable class of submanifolds to which infinite dimensional differential topology and Morse theory can be applied.

A submanifold M of V is called *proper Fredholm* (PF) if the end point map Y of M is Fredholm, and the restriction of Y to the unit disk normal bundle is a proper map. (Here Y is the map from $\nu(M)$ to V defined by $Y(v) = x + v$, if $v \in \nu(M)_x$). The main purpose of this paper is to show that there is a satisfactory generalization for the first and third type of questions raised above for PF submanifolds of Hilbert space. Since the trace of a compact operator need not be convergent, there are technical difficulties in making sense of many questions of the second type. But we believe that these may eventually be overcome.

In §1, we review the basic notions of Riemannian Hilbert manifolds and submanifolds. In §2, we prove that a PF submanifold has a natural Fredholm structure (induced from the immersion), and we show that the shape operators are compact and that every Euclidean distance function f_a satisfies Condition C of Palais and Smale. In §3, we study the geometry of PF submanifolds M with flat normal bundle, especially the focal structure of M, the curvature distributions, and the curvature normals. In §4 we study a family of PF submanifolds, which arise from infinite dimensional Lie theory. Let G be a simple, connected, compact Lie group, and $\mathscr{G}$ its Lie algebra. Let ξ be the trivial principal G-bundle over S^1, $V = H^0(S^1, \mathscr{G})$ the space of H^0-connections of ξ, and $\hat{G} = H^1(S^1, G)$ the H^1 gauge group of ξ. Then $\hat{G}$ acts isometrically on V by gauge transformations. In §4 we describe in detail the submanifold geometry of the principal orbits of $\hat{G}$ on V. In fact these orbits are isoparametric, i.e., they have flat normal bundle, and the shape operators along any parallel normal field are all conjugate. In §5 we extend the definition of tautness to PF submanifolds. Starting from §6, we assume M is isoparametric, and prove that the finite dimensional isoparametric theory can be generalized to this infinite dimensional setting. Although an isoparametric submanifold of a Hilbert space need not be an orbit of a group action, we prove in §6 that we can associate to each isoparametric submanifold M a marked affine Dynkin diagram, such that the corresponding affine Weyl group W acts on M by diffeomorphisms, and on the normal plane $q + \nu(M)_q$ by rigid motions. In §7 we prove that every nondegenerate Euclidean distance function f_a on M is a perfect Morse function, that M is taut, and that the set of critical points of f_a is a W-orbit. It follows that the homology of M can be computed explicitly from its marked Dynkin diagram. In §8 we determine the possible marked affine Dynkin diagrams for isoparametric submanifolds.

Given a parallel normal field v on M, we define the parallel set, M_v, to be $\{x + v(x) | x \in M\}$. In §9, we show that each M_v is a smooth PF submanifold (its codimension may be larger than that of M), and the set $\mathscr{F}$ of parallel sets of M forms a singular foliation of V. If f is a smooth function on V such that f is constant on each leaf of $\mathscr{F}$, then $f|(q + \nu(M)_q)$ is a smooth

W-invariant function. In §10 we prove that the converse is true; this is the geometric analogue of the Chevalley restriction theorem.

Finally in an appendix we review some basic facts concerning Coxeter groups. We also prove a necessary and sufficient condition for a group of rigid motions of $\mathbb{R}^k$ generated by reflections in affine hyperplanes to be a Coxeter group.

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1. Basic properties of Riemannian Hilbert manifolds

Let M be a smooth infinite dimensional Hilbert manifold modeled on a separable Hilbert space $(V, (\ ,\ ))$. The bracket operation for vector fields and the exterior derivative for p-forms are defined to be the same as they are for finite dimensional smooth manifolds [28]. Moreover for $\omega \in C^\infty(\bigwedge^p T^*M)$ and $X_0, \cdots, X_p \in C^\infty(TM)$, we have

(1)
$$\begin{aligned} d\omega(X_0, \cdots, X_p) = &\sum (-1)^i X_i \omega(X_0, \cdots, \hat{X}_i, \cdots, X_p) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p). \end{aligned}$$

A Riemannian metric for M is a smooth section g of $S^2(T^*M)$ such that $g(x)$ is an inner product for TM_x equivalent to the inner product $(\ ,\ )$ on V for all x in M. Then (M, g) is called a Riemannian Hilbert manifold. It is well known that there exists a unique torsion free connection ∇ compatible with the metric g, called the Levi-Civita connection. If M is of finite dimension, then ∇ is characterized by

$$\begin{aligned} 2(\nabla_X Y, Z) = X(Y, Z) + Y(Z, X) - Z(X, Y) + ([X, Y], Z) \\ + ([Z, X], Y) - ([Y, Z], X). \end{aligned} \tag{2}$$

Note that the right-hand side of (2) defines a continuous linear functional of the Hilbert space TM_x. Since TM_x^* is isomorphic to TM_x via the metric $g(x)$, (2) also defines a unique element in TM_x, and the argument for a unique compatible, torsion free connection is also valid for infinite dimensional Riemannian Hilbert manifolds. Similarly the following definition of Riemann curvature tensor is valid for the infinite dimensional case:

$$(R(X, Y)(Z), U) = ((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(Z), U). \tag{3}$$

Let M, X be Hilbert manifolds, g a Riemannian metric on X, and ∇ the Levi-Civita connection of g. A smooth map $f\colon M \to X$ is called an *immersion* (or M is called an *immersed submanifold* of X) if df_x is injective and $df_x(TM_x)$ is a closed linear subspace of TX_x. Next we define the two fundamental forms and the normal connection for submanifolds. The restriction of $g(x)$ to $df_x(TM_x)$ defines a Riemannian metric on M. This induced metric I on M is called *the first fundamental form* of M. Let $\nu(M)$ be the normal bundle of M in X, i.e., $\nu(M) = (TM)^{\perp}$, and let v be a local cross section of $\nu(M)$. Then $A_{v(x)}(u) = -(\nabla_u v)(x)^{TM_x}$ defines a selfadjoint linear operator on TM_x, and $A_{v(x)}$ only depends on $v(x)$. A_v is called the *shape operator* of M with respect to the normal vector v. *The second fundamental form* II of M is a section of $S^2(T^*M)\otimes\nu(M) \approx L(S^2(TM), \nu(M))$ defined by $g(\mathrm{II}(u_1.u_2), v) = g(A_v(u_1), u_2)$. The normal connection ∇^{ν} is the induced connection on $\nu(M)$ by ∇, i.e., $\nabla^{\nu}(v) = (\nabla u)^{\nu(M)}$, the orthogonal projection of ∇v to $\nu(M)$.

Next we want to use the method of moving frames to study the local geometry of Riemannian Hilbert manifolds. Let $\{e_i\}$ be a local orthonormal frame field on a Riemannian Hilbert manifold (X, g), i.e., the e_i are smooth vector fields defined on an open neighborhood U of X such that $\{e_i(x)\}$ is an orthonormal basis for the Hilbert space $(TX_x, g(x))$ at each x in U. Let $\{w_i\}$ denote the dual coframe of $\{e_i\}$. Note that the bracket operation on vector fields, the exterior differentiations on differential forms, connections and Riemann curvature are well defined on Hilbert manifolds, so we can express them locally in terms of the frame field $\{e_i\}$ and coframe field $\{\omega_i\}$. Suppose $[e_i, e_j] = \sum c_{ijk}e_k$. There exist uniquely one-forms ω_{ij} such that

$$\nabla e_i = \sum \omega_{ij} \otimes e_j.$$

Then it follows from (1) and (2) that

$$d\omega_i = \sum \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0. \tag{4}$$

In fact ω_{ij} is uniquely determined by (3), and

$$\omega_{ij} = \sum r_{ijk}\omega_k, \quad \text{where } 2r_{ijk} = -c_{ijk} + c_{jki} + c_{kij}.$$

Using (1), (3) then becomes

$$\Omega_{ij} = -d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = \tfrac{1}{2}\sum R_{ijkl}\omega_k \wedge \omega_l, \tag{5}$$

where $R_{ijkl} = (\nabla_{e_k}\nabla_{e_l} - \nabla_{e_l}\nabla_{e_k} - \nabla_{[e_k,e_l]})(e_i), (e_j)$. X has constant sectional curvature c if

$$\Omega_{ij} = c\omega_i \wedge \omega_j, \quad \text{or equivalently} \quad R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

It is easily seen that an infinite dimensional Hilbert space with the constant metric has zero sectional curvature.

The Levi-Civita connection ∇ extends uniquely to any tensor bundles over X by requiring that ∇ commute with tensor products and contractions. For example if

$$T = \sum t_{ijkl} e_i \otimes e_j \otimes e_k,$$

then $\nabla T = \sum t_{ijkl}\omega_l \otimes e_i \otimes e_j \otimes e_k$, where

$$\sum t_{ijkl}\omega_l = dt_{ijk} + \sum t_{mjk}\omega_{mi} + \sum t_{imk}\omega_{mj} + \sum t_{ijm}\omega_{mk}.$$

In particular if $u\colon X \to R$ is a smooth function, then $du = \sum u_i\omega_i$, and the Hessian of u

$$\nabla^2 u = \sum u_{ij}\omega_j, \quad \text{where } \sum u_{ij}\omega_j = du_i + \sum u_j\omega_{ji}.$$

Suppose M is an immersed submanifold of (X, g). Since locally f is an embedding, in order to study the local submanifold geometry of M we may identify x in M with $f(x)$ in X. *In this paper we will assume that all submanifolds have finite codimension.* Suppose M has codimension k_0 in X. Let $\{e_i | i \in \mathbb{N}\}$ be a local orthonormal frame field defined in a neighborhood U of X such that, when restricted to M, $\{e_i | i > k_0\}$ is a local tangent frame field and $\{e_i | i \le k_0\}$ is a local normal frame field. Henceforth we will adopt the following index convention:

$$1 \le \alpha, \beta, \gamma \le k_0, \quad i, j, k > k_0, \quad 1 \le A, B, C < \infty.$$

Let $\{\omega_A\}$ be the dual coframe of $\{e_A\}$, ω_{AB} the Levi-Civita connection, and $\hat{\Omega}$ the Riemann tensor of (X, g). Then we have

$$d\omega_A = \sum \omega_{AB} \wedge \omega_B, \tag{6}$$

$$d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB} - \hat{\Omega}_{AB}. \tag{7}$$

Restricting ω_α and $d\omega_\alpha$ to M, we have

$$\omega_\alpha = 0 \quad \text{and} \quad \sum \omega_{\alpha i} \wedge \omega_i = 0. \tag{8}$$

Let $\omega_{i\alpha} = \sum h_{i\alpha j}\omega_j$. Then $\sum h_{i\alpha j}\omega_j \wedge \omega_i = 0$, and since $\{\omega_i \wedge \omega_j | i < j\}$ is a basis for $\bigwedge^2 T^*M$, we have $h_{i\alpha j} = h_{j\alpha i}$. So *the first and second fundamental forms* of M in X are

$$\mathrm{I} = \sum_i \omega_i \otimes \omega_i,$$

$$\mathrm{II} = \sum_{i,\alpha} \omega_{i\alpha} \otimes \omega_i = \sum_{i,j,\alpha} h_{i\alpha j}\omega_i \otimes \omega_j \otimes e_\alpha.$$

The *shape operator* A_{e_α} is given by

$$A_{e_\alpha}(e_i) = \sum h_{i\alpha j} e_j.$$

The spectrum of the shape operator A_v will be called the *principal curvature spectrum* of M at x in the normal direction v.

2. Proper Fredholm submanifolds of a Hilbert space

Although the elementary part of infinite dimensional Riemannian geometry of submanifolds works in the same way as the finite dimension case, many of the deeper results are not true in general. Recall that the spectral theory of the shape operators and the Morse theory of the Euclidean distance functions of submanifolds of $\mathbb{R}^n$ are closely related, and they play essential roles in the study of the geometry and topology of submanifolds of $\mathbb{R}^n$. But these theories are not true without some restrictions in the infinite dimensional setting. One of the main goals of this section is to find a class of submanifolds of Hilbert spaces for which the techniques of infinite dimensional geometry and topology can be applied. Roughly speaking we study submanifolds of a Hilbert space with proper, Fredholm end point maps. In fact, properness of the end point map allows us to apply the infinite dimensional Morse theory, and Fredholm property implies that the shape operators are compact.

In the 1960's Smale [40] developed the differential topology for Fredholm maps between Banach manifolds. We will restrict ourselves to Hilbert manifolds. Let V, W be Hilbert spaces, and M, N Hilbert manifolds. A bounded linear map $T\colon V \to W$ is *Fredholm* if $\ker T$ and $\operatorname{coker} T$ are of finite dimension. It is then a well-known, easy consequence of the closed graph theorem that $T(V)$ is closed in W. A differentiable map $f\colon M \to N$ is *Fredholm* if df_x is Fredholm for all x in M. Two bounded linear operators $S\colon V \to V$ and $T\colon W \to W$ are orthogonally equivalent if there exists a linear isometry φ from V onto W such that $S = \varphi^{-1}T\varphi$.

2.1. Definition. Let V be a Hilbert space. The *end point map* Y of an immersed submanifold $f\colon M \to V$ is the restriction of the exponential map of V to $\nu(M)$, i.e., $Y\colon \nu(M) \to V$ with $Y(x,e) = f(x) + e$.

Suppose $\{e_\alpha\}$ is an orthonormal normal frame field defined on an open neighborhood U of M. Then it is easily seen that

$$\begin{aligned} &U \times \mathbb{R}^{k_0} \simeq \nu(M) \restriction U, \text{ via } (x,z) \to \left(x, \sum z_\alpha e_\alpha(x)\right), \\ (9)\qquad &Y = X + \sum z_\alpha e_\alpha, \\ &dY_{(x,e)}(u,t) = (u - A_e(u), t + s(u)), \end{aligned}$$

where $s(u)_\alpha = \sum z_\beta \omega_{\beta\alpha}(u)$ and $e = \sum z_\alpha e_\alpha(x)$. So for a hypersurface we have

$$dY_{(x,e)}(u,t) = ((\mathrm{I} - A_e)(u), t). \qquad (10)$$

It follows from (9) that we have:

2.2. Proposition. *The end point map Y of an immersed submanifold M of V is Fredholm if and only if* $(\mathrm{I} - A_v)$ *is Fredholm for all normal vectors v of M.*

2.3. Definition. An immersed finite codimension submanifold M of V is *proper Fredholm* (PF) if the restriction of the end point map Y to a disk normal bundle of M of any radius r is proper and Fredholm.

2.4. Remark. If $V = \mathbb{R}^n$, then an immersed submanifold M of V is PF if and only if the immersion is proper.

2.5. Remark. If M is a PF submanifold of V, and M is contained in the sphere of radius r with center x_0 in V, then $v(x) = (x_0 - x)$ is a normal field on M with length r, and $Y(x, v(x)) = x_0$. Since Y is proper on the r-disk normal bundle, M is compact. Hence M is of finite dimension.

2.6. Examples. Any finite codimension linear subspace of V is a PF submanifold. The hypersurface M of V defined by $\{x \in V | \langle \varphi(x), x\rangle = 1\}$ is PF if $\varphi\colon V \to V$ is a selfadjoint, injective compact linear operator. To see this we note that $v(x) = \varphi(x)/\|\varphi(x)\|$ is a unit normal field to M, and $A_{v(x)}(u) = -(\varphi(u))^T/\|\varphi(x)\|$ is a compact operator on TM_x. So it follows from (10) that the end point map Y is Fredholm. Next assume that $x_n \in M$, $\lambda_n\varphi(x_n)$ is bounded, and $(x_n + \lambda_n\varphi(x_n)) \to y$. Then x_n is bounded, and $(x_n + \lambda_n\varphi(x_n), x_n) = \|x_n\|^2 + \lambda_n$ is bounded, which implies that λ_n is bounded. Since φ is compact and $\{\lambda_n x_n\}$ is bounded, $\varphi(\lambda_n x_n)$ has a convergent subsequence. So $\{x_n\}$ has a convergent subsequence.

2.7. Proposition. *Suppose M is a PF submanifold of V. Let $x \in M$, $v \in \nu(M)_x$, and let A_v denote the shaper operator with respect to v. Then:*

(i) *A_v has no residual spectrum,*

(ii) *the eigenspace corresponding to a nonzero eigenvalue of A_v is of finite dimension,*

(iii) *the only possible point in the continuous spectrum of A_v is* 0,

(iv) *A_v is compact.*

Proof. Since A_v is selfadjoint, it has no residual spectrum. Note that the eigenspace of A_v with respect to a nonzero eigenvalue λ is

$$\mathrm{Ker}(\lambda \mathrm{I} - A_v) = \mathrm{Ker}(\mathrm{I} - (1/\lambda)A_v) = \mathrm{Ker}(\mathrm{I} - A_{v/\lambda}).$$

So (ii) follows from Proposition 2.2. Now suppose $\lambda \neq 0$, $\mathrm{Ker}(A_v - \lambda I) = 0$, and $\mathrm{Im}(A_v - \lambda \mathrm{I})$ is dense in TM_x. Then it follows again from Proposition 2.2 that $A_v - \lambda\mathrm{I}$ is invertible, and (iii) is proved. To prove (iv) it suffices to prove that if λ_i is a sequence of distinct real numbers in the discrete spectrum of A_v and $\lambda_i \to \lambda$, then $\lambda = 0$. But if $\lambda \neq 0$, then the selfadjoint Fredholm operator $P = \mathrm{I} - A_{v/\lambda}$ induces an isomorphism $\tilde{P}$ on $V/\mathrm{Ker}(P)$. So $\tilde{P}^{-1}$

is bounded. Letting δ denote $\|\tilde{P}^{-1}\|$, we have $|(1-(\lambda_i/\lambda))^{-1}| \le \delta$. Hence $|\lambda - \lambda_i|/|\lambda| \ge 1/\delta > 0$, which contradicts to the fact that $\lambda_i \to \lambda$. q.e.d.

Recall that a Fredholm structure [16] of a manifold M consists of an open cover $\{U_\alpha\}$ of M and a smooth tangent frame field $\{e_i^\alpha\}$ on each U_α such that for $x \in U_\alpha \cap U_\beta$ the linear operator $g_{\alpha\beta}(x)$ defined by $g_{\alpha\beta}(x)(e_i^\alpha(x)) = e_i^\beta(x)$ is of the form identity plus a compact operator. Then we have the following:

2.8. Proposition. *An immersed PF submanifold of V has a natural equivalence class of Fredholm structure given by the immersion.*

Proof. Since TM is parallelizable [26], there exists a global tangent frame field $\{\xi_i\}$. It follows from (9) that $(\mathrm{I} - A_v)$ is an isomorphism if and only if v is a regular point of the end point map Y. Let $\mathscr{N}$ be the collection of local normal fields v of M such that $(x, v(x))$ is a regular point for Y for all x in the domain of v, and let U_v be the domain of v in $\mathscr{N}$. Since $(x, 0)$ is regular for Y and the set of regular points of Y is open, $\{U_v | v \in \mathscr{N}\}$ is an open cover of M. Then it follows from 2.7 that $e_i^v = (\mathrm{I} - A_v)(\xi_i)$ gives a Fredholm structure. Using the well-known theorem of Kuiper that $\mathrm{GL}(\infty)$ is contractible [26], we conclude that the equivalence class of Fredholm structure is independent of the choice of ξ_i. q.e.d.

In the rest of this section we explain the relations between the focal structure and the critical point theory of the Euclidean distance function of PF submanifolds of a Hilbert space. It follows from (9) that (x, e) is a regular point of Y (i.e., $dY_{(x,e)}$ is an isomorphism) if and only if $(\mathrm{I} - A_e)$ is an isomorphism. Moreover, the dimensions of $\mathrm{Ker}(\mathrm{I} - A_e)$ and $\mathrm{Ker}(dY_{(x,e)})$ are finite and equal. Hence the definition of focal points and multiplicities [29] can be generalized to PF submanifolds.

2.9. Definition. A point $a = Y(x, e)$ in V is called a nonfocal point for a PF submanifold M of V with respect to x if $dY_{(x,e)}$ is an isomorphism. If $m = \dim(\ker(dY_{(x,e)})) > 0$, then a is called a focal point of multiplicity m for M with respect to x. The focal set Σ of M in V is the set of all critical values of the end point map Y.

Applying the Sard-Smale transversality theorem [40] for Fredholm maps to the end point map Y of M, we have:

2.10. Proposition. *The set of nonfocal points of a PF submanifold M of V is open and dense in V.*

The structure of the focal set of M is closely related to the critical point theory of the Euclidean distance functions by the following:

2.11. Proposition. *Let $M \subset V$ be a PF submanifold, $a \in V$, and define $f_a \colon M \to \mathbb{R}$ by $f_a(x) = \|x - a\|^2$. Then we have*

(i) $\nabla f_a(x) = 2(x-a)^T$, *the projection of $(x-a)$ onto TM_x, so in particular x_0 is a critical point of f_a if and only if $(x_0 - a) \in \nu(M)_{x_0}$,*

(ii) $\frac{1}{2}\nabla^2 f_a(x) = \mathrm{I} - A_{(a-x)}$ *at the critical point* x,

(iii) f_a *is nondegenerate if and only if* a *is a nonfocal point.*

Suppose x_0 is a nondegenerate critical point of f_a, and V_λ is the eigenspace of A_{a-x_0} with respect to eigenvalue λ. Then

$$\operatorname{index}(f_a, x_0) = \Sigma\{\dim(V_\lambda) | \lambda > 1\},$$

and from 2.11 we have

2.12. Corollary. *Let* M *be a PF submanifold of* V. *Then the index of any critical point of* f_a *is finite.*

2.13. Corollary (*Morse Index Theorem*). *Let* M *be a PF submanifold of a Hilbert space* V, $x \in M$, $e \in \nu(M)_x$, *and* $a = x + e$. *Then* a *is nonfocal with respect to* x *if and only if* x *is a nondegenerate critical point of* f_a. *Moreover, the index* f_a *at* x *is equal to the number of focal points of* M *with respect to* x *on the segment joining* x *to* a, *each counted with its multiplicity.*

2.14. Corollary. *If* M *is a PF submanifold of* V, *then* f_a *is nondegenerate for all* a *in an open dense subset of* V.

Morse theory relates the homology of a smooth manifold to the critical point structure of certain smooth functions. This theory was successfully extended to infinite dimensional Hilbert manifolds in the 1960's by Palais and Smale ([33], [39]) for the class of smooth functions which satisfy the following Condition C.

2.15. Definition. A smooth function f on a Riemannian Hilbert manifold M satisfies *Condition* C if any sequence $\{x_n\}$ in M, such that $\|f(x_n)\|$ is bounded and that $\|\nabla f(x_n)\| \to 0$, has a convergent subsequence in M.

2.16. Proposition. *Let* M *be a PF submanifold of a Hilbert space* V, *and* $a \in V$. *Then the map* $f_a \colon M \to \mathbb{R}$ *defined by* $f_a(x) = \|x - a\|^2$ *satisfies condition* C.

Proof. We will write f for f_a. Suppose $|f(x_n)| \leq c$ and $\|\nabla f(x_n)\| \to 0$. Let u_n be the orthogonal projection of $(x_n - a)$ onto TM_{x_n}, and v_n the projection of $(x_n - a)$ onto $\nu(M)_{x_n}$. Since $\|x_n - a\|^2 \leq c$ and $u_n \to 0$, $\{v_n\}$ is bounded (say by r). So $(x_n, -v_n)$ is a sequence in the r-disk normal bundle of M, and

$$Y(x_n, -v_n) = x_n - v_n = (x_n - a) - v_n + a = u_n + a \to a.$$

Since M is a PF submanifold, $(x_n, -v_n)$ has a convergent subsequence, which implies that x_n has a convergent subsequence in M.

2.17. Remark. Let M be a submanifold of V (not necessarily PF). Then the condition that all f_a satisfy Condition C is equivalent to the condition that the restriction of the end point map to the unit disk normal bundle is proper.

The above statements (from 2.11 to 2.17) concerning the Euclidean distance functions f_a hold for a PF immersion $\varphi\colon M \to V$, where f_a is defined by $f_a(x) = \|\varphi(x) - a\|^2$.

3. PF submanifolds with flat normal bundles

Let M be a PF submanifold in a Hilbert space V, and $\{e_i | i \in \mathbb{N}\}$ a local orthonormal frame field defined in a neighborhood U of V such that, when restricted to M, $\{e_i | i > k_0\}$ is a local tangent frame field and $\{e_\alpha | \alpha \leq k_0\}$ is a local normal frame field. We continue to use the following index convention:

$$1 \leq \alpha, \beta, \gamma \leq k_0, \quad i, j, k > k_0, \quad 1 \leq A, B, C < \infty.$$

Let $\{\omega_A\}$ be the dual coframe of $\{e_A\}$, and ω_{AB} the Levi-Civita connection for V. Since V has zero sectional curvature, (6) and (7) of §1 give the structure equations for V:

$$d\omega_A = \sum \omega_{AB} \wedge \omega_B, \tag{11}$$

$$d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB}. \tag{12}$$

Restricting e_A to M we have $\omega_\alpha = 0$ and $d\omega_\alpha = \sum \omega_{\alpha j} \wedge \omega_j$. Let $\omega_{i\alpha} = \sum h_{i\alpha j}\omega_j$. Then the two fundamental forms are:

$$\mathrm{I} = \sum (\omega_i)^2,$$

$$\mathrm{II} = \sum \omega_{i\alpha} \otimes \omega_i \otimes e_\alpha = \sum h_{i\alpha j}\omega_i \otimes \omega_j \otimes e_\alpha.$$

Restricting (11) to M we have

$$d\omega_i = \sum \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{13}$$

i.e., (ω_{ij}) is the Levi-Civita connection for the induced metric of M, and

$$d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \sum \omega_{i\alpha} \wedge \omega_{\alpha j}, \tag{14}$$

$$d\omega_{i\alpha} = \sum \omega_{ij} \wedge \omega_{j\alpha} + \sum \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{15}$$

$$d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \sum \omega_{\alpha i} \wedge \omega_{i\beta}. \tag{16}$$

(14)–(16) are called the *Gauss, Codazzi*, and *Ricci* equations, respectively. It follows from (14) that the Riemann tensor Ω_{ij} for the induced metric on M is

$$\Omega_{ij} = \sum \omega_{i\alpha} \wedge \omega_{j\alpha} = \tfrac{1}{2} \sum R_{ijkl}\omega_k \wedge \omega_l.$$

Let $\hat{\nabla}$ denote the Levi-Civita connection of V. Then the induced connection on the normal bundle $\nu(M)$ is defined as follows:

$$\nabla^\nu e_\alpha = \text{the orthogonal projection of } \hat{\nabla} e_\alpha \text{ onto } \nu(M) = \sum \omega_{\alpha\beta} \otimes e_\beta.$$

A normal field v is *parallel* if $\nabla^\nu v = 0$. The *normal curvature* $\Omega^\nu_{\alpha\beta}$ of M in X is the curvature of the normal connection ∇^ν, i.e.,

$$-\Omega^\nu_{\alpha\beta} = d\omega_{\alpha\beta} - \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = -\sum \omega_{i\alpha} \wedge \omega_{i\beta} = -\tfrac{1}{2} \sum R^\nu_{\alpha\beta kl} \omega_k \wedge \omega_l.$$

Then (14)–(16) give

$$R_{ijkl} = \sum (h_{i\alpha k} h_{j\alpha l} - h_{j\alpha k} h_{i\alpha l}), \tag{17}$$

$$d\omega_{i\alpha} = \sum \omega_{ij} \wedge \omega_{j\alpha} + \sum \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{18}$$

$$R^\nu_{\alpha\beta ij} = \sum (h_{k\alpha i} h_{k\beta j} - h_{k\alpha j} h_{k\beta i}). \tag{19}$$

Identifying TM^* with TM via the metric, we can rewrite (19) as

$$[A_u, A_v] = \Omega^\nu(u, v), \tag{20}$$

i.e., the normal curvature of M measures the commutability of the shape operators.

If $\Omega^\nu = 0$, then $d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}$. So locally there exists an orthonormal parallel normal frame field $\tilde{e}_\alpha$.

3.1. Definition. The normal bundle $\nu(M)$ is *flat* if $\Omega^\nu_{\alpha\beta} = 0$, and $\nu(M)$ is *globally flat* if there exists a global orthonormal parallel normal frame field on M.

As a consequence of the Ricci equation (20), we have

3.2. Proposition. *Suppose M is a submanifold of a Hilbert space V and $\nu(M)$ is flat. Then for $x \in M$, all the shape operators at x of M commute.*

Note that the proof of the fundamental theorem of hypersurfaces in $\mathbb{R}^n$ is based on the Frobenius theorem [24], so it generalizes easily to arbitrary codimension submanifolds of $\mathbb{R}^n$. The Frobenius theorem also holds for Hilbert manifolds. So we have the following fundamental theorem:

3.3. Theorem. *Let (M, g) be a Riemannian Hilbert manifold, ∇ its Levi-Civita connection, ξ a trivial Hilbert vector bundle of rank k on M, and $A \colon \xi \to L(TM, TM)$ a bundle morphism covering the identity map such that $\{A(v) | v \in \xi_x\}$ consists of commuting, bounded selfadjoint operators. Let $\{e_i | i \in I\}$ be a local orthonormal tangent frame field defined on a neighborhood U of M for the metric g, $\{\omega_i | i \in I\}$ its dual coframe, and $\{e_\alpha | 1 \leq \alpha \leq k\}$ an orthonormal frame field for ξ. Let ω_{ij} be the Levi-Civita connection 1-form determined by ω_i, and define $\omega_{i\alpha}$ by $A(e_\alpha) = \sum \omega_{i\alpha} \otimes e_i$. Set $\omega_{\alpha\beta} = 0$, $\omega_{\alpha i} = -\omega_{i\alpha}$. Then given x_0 in U and an orthogonal basis $\{u_i\}$ for the Hilbert* *space V, there exist a neighborhood $\mathscr{O}$ of x_0 in U and an immersion $\varphi\colon \mathscr{O} \to V$ such that g is the induced metric, $\xi|\mathscr{O}$ is isomorphic to the normal bundle of $\varphi(\mathscr{O})$ in V, $\varphi(\mathscr{O})$ has flat normal bundle, and $A(v)$ is the shape operator with respect to the normal vector v.*

As a consequence of 2.7, 3.2 and the fact that compact operators have eigen-decompositions, we have

3.4. Corollary. *Suppose M is a PF submanifold with flat normal bundle in V. Then locally there exist finite rank continuous distributions $\{E_i | i \in I\}$ such that $TM_x = \bigoplus\{E_i(x) | i \in I\}$ is a common eigen-decomposition of A_v for all v in $\nu(M)_x$.*

Since A_v is linear for $v \in V$, there exist local continuous sections λ_i of $\nu(M)^*$ such that

$$A_v(u_i) = \lambda_i(v) u_i \quad \text{for all } u_i \text{ in } E_i.$$

Identifying $\nu(M)^*$ with $\nu(M)$ by the induced inner product from V of the fibers, we obtain continuous sections v_i of $\nu(M)$ such that

$$A_v(u_i) = \langle v, v_i \rangle u_i \quad \text{for all } u_i \text{ in } E_i.$$

These E_i's, λ_i's, and v_i's are called the *curvature distributions, principal curvatures*, and *curvature normals* for M respectively. Although they are not smooth everywhere, they are smooth on an open dense subset of M.

3.5. Proposition. *Let M be a PF submanifold in V with flat normal bundle, and v_i its curvature normals. Then given $q \in M$, there exists a positive constant c such that $\|v_i(q)\| \le c$ for all i.*

Proof. Let F denote the continuous function defined on the unit sphere S^{k-1} of the normal plane $\nu(M)_q$ by $F(v) = \|A_v\|$. Since S^{k-1} is compact, there is a constant $c > 0$ such that $F(v) \le c$. Since the eigenvalues of A_v are of the form $\langle v, v_i \rangle$, we have $|\langle v, v_i \rangle| \le c$ for all unit vectors v. q.e.d.

The set of focal points Σ of PF submanifolds in general can be rather complicated. However if M has flat normal bundle, then Σ is rather simple.

3.6. Proposition. *Let M be a PF submanifold of V with flat normal bundle, ν_q the affine k-plane $(q + \nu(M)_q)$ in V, and Σ_q the set of focal points for M with respect to q. Then*

(i) *$\Sigma_q = \bigcup\{l_i(q) | i \in I\}$, where $l_i(q)$ is the hyperplane in ν_q defined by $\{q + v | v \in \nu(M)_q$ and $\langle v, v_i(q) \rangle = 1\}$,*

(ii) *$\mathscr{H} = \{l_i(q) | i \in I\}$ is locally finite, i.e., given any point $p \in \nu_q$ there is an open neighborhood U of p such that $\{i \in I | l_i(q) \cap U \ne \varnothing\}$ is finite.*

Proof. Since $\nu(M)$ is flat, it follows from (9) that we have

$$dY_{(q,e)}(u, z) = (u - A_e(u), z).$$

Hence $x = (q+e) \in \Sigma_q$ if and only if 1 is an eigenvalue of A_e. So there exists $i \in I$ such that $1 = \langle v_i, e\rangle$, i.e., $x \in l_i(q)$. Since A_e is compact and the eigenvalues of A_e are $\{(e, v_i) | i \in I\}$, the set $J(x) = \{i \in I | x \in l_i(q)\}$ is finite and there exists $\delta > 0$ such that $|1 - \langle e, v_i\rangle| > \delta$ for all $j \notin J(x)$. Because $d(x, l_j) = |1 - \langle e, v_j\rangle|/\|v_j\| > \delta/c$, where c is the upper bound for $\|v_i\|$ as in 3.5, we conclude that $B(x, \delta/c)$ meets only finitely many l_i's (in fact it only intersects $l_i(q)$ for $i \in J(x)$). q.e.d.

Using essentially the same argument as for submanifolds of $\mathbb{R}^n$ with flat normal bundles [43], we have

3.7. Proposition. *Let M be a PF submanifold of V with flat normal bundle. If all the curvature distributions E_i are smooth on an open subset U of M, then the following hold.*

(i) *$E_i \restriction U$ is integrable.*

(ii) *Suppose* $\operatorname{rank}(E_i) = m_i > 1$. *Then the leaf S_i of E_i through x is contained in a m_i-plane if $v_i = 0$, and in a standard m_i-sphere of radius $1/\|v_i\|$ if $v_i \neq 0$.*

(iii) *The curvature normal v_i is parallel on S_i.*

We note that if M has flat normal bundle and the multiplicities of the shape operators $A_{v(x)}$ along any parallel normal field v are independent of $x \in M$, then the curvature distributions E_i are smooth.

4. Examples

In this section we apply the submanifold geometry in §§2 and 3 to an interesting family of PF submanifolds of a Hilbert space. They arise as the principal orbits of the action of the gauge group of a trivial principal G-bundle ξ over S^1 on the space of connections of ξ. In fact we show that these submanifolds have flat normal bundles, and the shape operators along any parallel normal field are orthogonally equivalent. In particular they have zero normal curvature and the principal spectrum along any parallel normal field is discrete and constant.

First we review and set some terminology for the manifolds of maps. Let (M, g) be a compact Riemannian manifold. Then, for all k,

$$(u, v)_k = \int_M ((\mathrm{I} + \Delta)^{k/2} u(x), v(x))\, dx$$

defines an inner product on $C^\infty(M, \mathbb{R}^m)$. Let $H^k(M, \mathbb{R}^m)$ denote the completion of $C^\infty(M, \mathbb{R}^m)$ with respect to $(\ ,\ )_k$. If N is a complete Riemannian manifold isometrically embedded in the Euclidean space $\mathbb{R}^m$, then it is well known (see for example [34]) that

$$H^k(M,N) = \{u \in H^k(M,\mathbb{R}^m) | u(M) \subset N\}$$

is a Hilbert manifold for $2k > n = \dim(M)$. In particular, $H^s(S^1,N)$ is a Hilbert manifold if $s > 1/2$.

Let G be a simple compact connected Lie group, T a maximal torus of G, and $\mathscr{G},\mathscr{T}$ the corresponding Lie algebras. Then the killing form makes $\mathscr{G}$ a Euclidean space. Let ξ denote the product principal G-bundle on S^1. The Hilbert group $\hat{G} = H^1(S^1,G)$ is the gauge group and the Hilbert space $V = H^0(S^1,\mathscr{G})$ is the space of H^0-connections of ξ. $\hat{G}$ acts on V by the gauge transformations $g \cdot u = gug^{-1} + g'g^{-1}$, which is an affine isometry of V. Let $\hat{x}$ denote the constant map in $H^0(S^1,\mathscr{G})$ with value $x \in \mathscr{G}$, and $\mathscr{T}^0 = \{\hat{t} | t \in \mathscr{T}\}$. Given $u \in H^0(S^1,\mathscr{G})$ it follows from the theory of ordinary differential equations that there is a unique $f \in H^1(S^1,G)$ such that $f(0) = e$ (the identity in G), and $u = f'f^{-1}$. We define the holonomy map $\Phi\colon H^0(S^1,\mathscr{G}) \to G$ by $\Phi(u) = f(2\pi)$. The following three statements are proved by Segal [38]:

(i) Let $s \in G$ and $a \in \mathscr{T}$ be such that $s\Phi(u)s^{-1} = \exp(2\pi a)$, and let $h(t) = \exp(ta)sf^{-1}(t)$. Then $h \in H^1(S^1,G)$ (in fact $h(0) = h(2\pi) = s$), and $h \cdot u = \hat{a}$. Hence every $\hat{G}$-orbit meets $\mathscr{T}^0$.

(ii) $\Phi(g \cdot \hat{a}) = g(0)\Phi(u)g(0)^{-1}$ for all $g \in H^1(S^1,G)$ and $a \in \mathscr{T}$.

(iii) Let W be the Weyl group of G, and $W(\mathscr{T}^0) = N(\mathscr{T}^0)/Z(\mathscr{T}^0)$, where $N(\mathscr{T}^0) = \{g \in \hat{G} | g \cdot \mathscr{T}^0 \subset \mathscr{T}^0\}$ and $Z(\mathscr{T}^0) = \{g \in \hat{G} | g \cdot \hat{t} = \hat{t}$ for all $t \in \mathscr{T}\}$ are the normalizer and centralizer of $\mathscr{T}^0$ respectively. Then $V/\hat{G} \approx \mathscr{T}^0/W(\mathscr{T}^0) \approx G/\mathrm{Ad}(G) \approx T/W$, and $W(\mathscr{T}^0)$ is the semidirect product of W and the lattice group $\Lambda = \{t \in \mathscr{T} \,|\, \exp(t) = e\}$ under the natural action of W, i.e.,

$$(w_1,\lambda_1) \cdot (w_2,\lambda_2) = (w_1w_2, \lambda_2 + w_2 \cdot \lambda_1)$$

for $w_i \in W$ and $\lambda_i \in \Lambda$. Moreover $W(\mathscr{T}^0)$ is the Coxeter group generated by reflections in the hyperplanes $\alpha(t) + n = 0$ in $\mathscr{T}^0$, where α is a root of G and $n \in \mathbb{Z}$.

In the following we study the geometry of a principal $\hat{G}$-orbit $M = \hat{G} \cdot \hat{t}_0$ in V, where $t_0 \in \mathscr{T}$. If G is of rank k, then the codimension of M in V is k,

$$TM_{\hat{t}_0} = \{[u,\hat{t}_0] + u' | u \in H^1(S^1,\mathscr{G})\} \quad \text{and} \quad \nu(M)_{\hat{t}_0} = \mathscr{T}^0,$$

where $[u,v](\theta) = [u(\theta), v(\theta)]$. Moreover

$$TM_{g\cdot\hat{t}_0} = gTM_{\hat{t}_0}g^{-1} \quad \text{and} \quad \nu(M)_{g\cdot\hat{t}_0} = g\nu(M)_{\hat{t}_0}g^{-1}.$$

Given $t \in \mathscr{T}$, we define

$$\tilde{t}(g \cdot \hat{t}_0) = dg_{\hat{t}_0}(\hat{t}) = g\hat{t}g^{-1}.$$

Because M is a principal orbit, $\tilde{t}$ is a well-defined normal vector field on M. By a direct computation, we have

$$d\tilde{t}_{\hat{t}_0}([u,\hat{t}_0]+u')=[u,\hat{t}]\,.$$

Since $[u,\hat{t}]$ is perpendicular to $\mathscr{T}^0$, we have $\nabla^\nu(\tilde{t})=0$, i.e., $\tilde{t}$ is a parallel normal field, and the shape operator

$$A_{\hat{t}}([u,\hat{t}_0]+u')=-\,[u,\hat{t}]\,.$$

But M is homogeneous, so we have

$$A_{\tilde{t}(g\cdot\hat{t}_0)}(g([u,\hat{t}_0]+u')g^{-1})=-g\,[u,\hat{t}]\,g^{-1},$$

i.e., $A_{\tilde{t}(x)}$ and $A_{\tilde{t}(g\cdot x)}$ are orthogonally equivalent.

Next we claim that the shape operator $A_{\hat{t}}$ is compact. To see this, let Δ^+ denote the set of positive roots of $\mathscr{G}$. Then there exist x_α, y_α for all $\alpha\in\Delta^+$ such that

$$\mathscr{G}=\mathscr{T}\oplus\{\mathbb{R}x_\alpha\oplus\mathbb{R}y_\alpha|\alpha\in\Delta^+\}$$
$$[h,x_\alpha]=\alpha(h)y_\alpha,\ [h,y_\alpha]=-\alpha(h)x_\alpha\quad\text{for all } h\in\mathscr{T}\text{ and }\alpha\in\Delta^+.$$

If $\operatorname{rank}(G)=k$, and $\{t_1,\cdots,t_k\}$ is a basis of $\mathscr{T}$, then $\{x_\alpha,y_\alpha,t_i,x_\alpha\cos n\theta,$ $x_\alpha\sin n\theta,y_\alpha\cos n\theta,y_\alpha\sin n\theta,t_i\cos n\theta,t_i\sin n\theta|\alpha\in\Delta^+,\ 1\le i\le k,\ n\in\mathbb{N}\}$ forms a separable basis for V. Moreover, the orbit $M=\hat{G}\cdot\hat{t}_0$ is principal if and only if $(\alpha(t_0)+n)\neq 0$ for all $\alpha\in\Delta^+$ and $n\in\mathbb{Z}$. Using the above separable bases of V, it is easily seen that $A_{\hat{t}}$ is a compact operator whose eigenvalues are $(\alpha(t)/(\alpha(t_0)+n))$ for $\alpha\in\Delta^+$, $n\in\mathbb{Z}$, with multiplicities 2. This proves that the end point map Y of M in V is Fredholm.

It is easy to determine the focal set of M. A point $\hat{t}+\hat{t}_0$ in $\hat{t}_0+\nu(M)_{\hat{t}_0}=\mathscr{T}^0$ is a focal point with respect to $\hat{t}_0$ if and only if 1 is an eigenvalue of $A_{\hat{t}}$, i.e., there is an integer n such that $-\alpha(t)/(\alpha(t_0)+n)=1$, or equivalently $\alpha(t+t_0)\in\mathbb{Z}$. So the set of focal points with respect to $\hat{t}_0$ is the union of the reflection hyperplanes of $W(\mathscr{T}^0)$ in $\mathscr{T}^0$.

Next we will prove that the end point map Y of M restricted to the normal disk bundle of radius r is proper. Suppose

$$Y(g_n\cdot x_0,\tilde{t}_n(g_n\cdot x_0))\to u\quad\text{and}\quad\|\tilde{t}_n\|\le r.$$

Then $\{t_n\}$ is a bounded sequence in the k-dimensional Euclidean space $\mathscr{T}$, so we may assume that $t_n\to t_0$ for some $t_0\in\mathscr{T}$. Note that

$$\begin{aligned}Y(g\cdot x_0,\tilde{t}(g\cdot x_0))&=Y(g\cdot x_0,gtg^{-1})\\&=(gx_0g^{-1}+g'g^{-1})+gtg^{-1}=g\cdot(x_0+t).\end{aligned}\tag{21}$$

So it suffices to prove that the $\hat{G}$-action is proper, i.e., if $g_n\cdot u_n\to v$ and $u_n\to u$, then g_n has a convergent subsequence in G. It follows from the assumption that $g_n \cdot u \to v$, i.e., $(g_n u g_n^{-1} + g_n' g_n^{-1}) \to v$, which implies that $\|u + g_n^{-1} g_n'\|_0$ is bounded. Hence $\|g_n^{-1} g_n'\|_0$ is bounded. Since G is compact, $\|g_n\|_0$ is bounded. So $\|g_n\|_1$ is bounded. By Rellich's lemma, the inclusion map $H^1(S^1, G) \hookrightarrow H^0(S^1, G)$ is compact, hence a subsequence (still denoted by g_n) converges to g_0 in $H^0(S^1, G)$. But

$$\|g_n u g_n^{-1} - g_n' g_n^{-1} - v\|_0 \to 0,$$

so $\|g_n u - g_n' - v g_n\|_0 \to 0$. Therefore $g_n \to g_0$ in $H^1(S^1, G)$.

Let $M_{\tilde{t}}$ denote the parallel set $\{x + \tilde{t}(x) | x \in M\}$. It follows from (21) that $M_{\tilde{t}} = \hat{G} \cdot \hat{t}$, i.e., the orbit foliation of $\hat{G}$ is the same as the parallel foliation of M. In fact the $\hat{G}$-orbit foliation is completely determined by a single principal $\hat{G}$-orbit.

To summarize, we have

4.1. Theorem. *Let $M = \hat{G} \cdot \hat{t}_0$ be a principal $\hat{G}$-orbit of V. Then the following hold.*

(i) *M is a PF submanifold.*

(ii) *The codimension of M in V is $k = \operatorname{rank}(G)$.*

(iii) *$\nu(M)$ is globally flat.*

(iv) *Given any parallel normal field v on M, the shape operators $A_{v(x)}$ and $A_{v(y)}$ are orthogonally equivalent.*

(v) *The curvature distribution E_i are smooth, and* $\operatorname{rank}(E_i) = 2$.

(vi) *Associated to M there is a discrete Coxeter group $W(\mathscr{T}^0)$ which is generated by reflections in the focal hyperplanes $l_i(\hat{t}_0)$ in $(t_0 + \nu(M)_{\hat{t}_0})$.*

(vii) *The parallel sets M_v are smooth submanifolds of V for any parallel normal field on M, in fact, M_v is a $\hat{G}$-orbit.*

(viii) *If v and w are two parallel normal fields on M, then either $M_v = M_w$ or $M_v \cap M_w = \varnothing$.*

4.2. Remark. Although the trace of the shape operator A_v for the above M is divergent, it is conditionally convergent and there is a natural way to sum it, so the mean curvature and Ricci curvature of M can be *formally* defined. For example, if $G = \mathrm{SU}(2)$, then the principal orbit $M = \hat{G} \cdot \hat{t}_0$ is a hypersurface of V, $\{\lambda_n = 1/(t_0 + n) | n \in \mathbb{Z}\}$ are the principal curvatures of M, and each has multiplicity 2. Let $\{e_i, e_i'\}$ be a local orthonormal tangent frame on M such that e_i and e_i' are the principal directions for λ_i. Then $H = \sum 2\{\lambda_i | i \in \mathbb{Z}\}$ is convergent if it is summed as $\sum 2\{(\lambda_i + \lambda_{-i}) | i \in \mathbb{N}\}$. Moreover if $t_0 = 1/2$, then $H = 0$. Using (17), the Ricci curvature of M has eigenvalues μ_i with multiplicity 2 and eigenvectors e_i, e_i', where

$$\mu_i = 1/(t_0 + i)^2 + 2t_0 \sum \{1/(t_0^2 - n^2) | n \in \mathbb{N}\}.$$

There are several natural metrics on a $\hat{G}$-orbit M. The induced metric on M as a submanifold of V gives the H^1 metric. The restriction of the H^0 metric on V to the orthogonal complement of $\mathscr{G}^0$ in V defines a $\hat{G}$-invariant metric on M, which is the H^0 metric. The $H^{1/2}$ metric on M is a homogeneous Kähler metric. The Riemannian curvature of these metrics and the characteristic classes of the orbits have been studied by Freed in [18], [19].

5. Taut immersions

Let $f\colon M \to \mathbb{R}^n$ be an immersed, compact submanifold, and $\nu^1(M)$ the unit normal bundle of M. Then there is a natural volume element $d\sigma$ on $\nu^1(M)$. The total absolute curvature of M is

$$\tau(M, f) = \int_{\nu^1(M)} |\det(A_v)|\, d\sigma.$$

Chern and Lashof [12] proved that $\tau(M, f) \geq b(M)$, where $b(M) = \sum b_i(M)$ is the sum of Betti numbers of M. An immersion f is called *tight* if $\tau(M, f)$ is equal to $\inf\{\tau(M, \varphi) | \varphi\colon M \to \mathbb{R}^n$ is an immersion for some $n\}$. It is a very difficult and unsolved problem to determine what manifolds can have tight immersions. An important step in this direction is Kuiper's reformulation [25] of the problem in terms of Morse theory for height functions. Banchoff [2] began the program of finding all tight surfaces in spheres, and later this led to the study of taut immersions by Carter and West [8]. There has been a lot of beautiful theory developed for tight and taut immersions (cf. [11]). In this section we generalize the definition and some of the basic properties of taut immersions to PF submanifolds of Hilbert spaces.

Let M be a Riemannian Hilbert manifold. A smooth function $f : M \to \mathbb{R}$ is called a *Morse* function if f is nondegenerate, bounded from below and satisfies Condition C. Let

$$M_r(f) = \{x \in M | f(x) \leq r\}.$$

Then it is easily seen that there are only finitely many critical points of f in $M_r(f)$. Let

$$\mu_k(f, r) = \text{ the number of critical points of index } k \text{ on } M_r(f),$$
$$\beta_k(f, r, F) = \dim(H_k(M_r(f), F)) \quad \text{for a field } F.$$

Then the weak Morse inequalities ([33], [39]) are $\mu_k(f, r) \geq \beta_k(f, r, F)$ for all r and F.

5.1. Definition. A Morse function $f\colon M \to \mathbb{R}$ is *perfect* if there exists a field F such that $\mu_k(f, r) = \beta_k(f, r, F)$ for all r and k.

5.2. Theorem. *Let f be a Morse function. Then f is perfect if and only if there exists a field F such that the induced map on the homology*

$$i_*: H_*(M_r(f), F) \to H_*(M, F)$$

of the inclusion of $M_r(f)$ in M is injective for all r.

5.3. Definition. An immersed PF submanifold M of a Hilbert space V is *taut* if every nondegenerate Euclidean distance function f_a on M is a perfect Morse function.

5.4. Remark. If M is properly immersed in $\mathbb{R}^n$, then the above definition is the same as in [8].

5.5. Remark. Unlike the finite dimensional theory of tautness, the unit hypersphere S of an infinite dimensional Hilbert space is *not* taut. Since S is not PF by 2.5, the nondegenerate distance function f_a on S does not satisfy Condition C. We will see later that, for a simple compact connected group G, the orbits of the gauge group $H^1(S^1, G)$, acting on the space of connections $H^0(S^1, \mathscr{G})$ by gauge transformations as in §4 are taut.

Let $R(f)$ denote the set of all regular values of f, and let $C(f)$ denote the set of all critical points of f. The fact that the restriction of the end point map to the unit disk normal bundle is proper gives a uniform Condition C for the Euclidean distance functions as we see in the following two propositions.

5.6. Proposition. *Let M be an immersed PF submanifold of V, and $a \in V$. Suppose $r < s$ and $[r, s] \subset R(f_a)$. Then there exists $\delta > 0$ such that if $\|b - a\| < \delta$, then $[r, s] \subset R(f_b)$.*

Proof. If it is not true, then there exist a sequence $\{b_n\}$ in V and $\{x_n\}$ in M such that $b_n \to a$, $r \leq \|x_n - b_n\| \leq s$, and x_n is a critical point of f_{b_n}. So it follows from 2.11(i) that $x_n - b_n$ is in $\nu(M)_{x_n}$. Since the end point map Y of M restricted to the disk normal bundle of radius s is proper and $Y(x_n, b_n - x_n) = b_n \to a$, a subsequence of x_n converges to a point x_0 in M. Then it is easily seen that $r \leq \|x_0 - a\| \leq s$, and x_0 is a critical point of f_a, which is a contradiction.

5.7. Proposition. *Let M be an immersed PF submanifold of V, and $a \in V$. Suppose $r < s$ and $[r, s] \subset R(f_a)$. Then there exist $\delta_1 > 0, \delta_2 > 0$ such that if $\|b - a\| < \delta_1$ and $x \in M_s(f_b) \backslash M_r(f_b)$, then $\|\nabla f_b(x)\| \geq \delta_2$.*

Proof. By 5.6 there exists $\delta > 0$ such that $[r, s] \subset R(f_b)$ if $\|b - a\| < \delta$. Suppose no such δ_1 and δ_2 exist. Then there exist sequences b_n in V and x_n in M such that $b_n \to a$, $x_n \in M_s(f_{b_n}) \backslash M_r(f_{b_n})$, and $\|\nabla(f_{b_n})(x_n)\| \to 0$. Moreover,

$$Y(x_n, -(x_n - b_n)^\nu) = x_n - (x_n - b_n)^{\nu(M)_{x_n}} = b_n + (x_n - b_n)^{T(M)_{x_n}} \to a,$$

and $\|x_n - b_n\| \leq s$. Since M is PF, x_n has a subsequence converging to a critical point x_0 of f_a in $M_s(f_a) \backslash M_r(f_a)$, a contradiction.

5.8. Proposition. *Let M be an immersed taut submanifold of V, $a \in V$, and r a real value of f_a. Then the induced map on homology*

$$i_*\colon H_*(M_r(f_a), F) \to H_*(M, F)$$

of the inclusion of $M_r(f_a)$ in M is injective.

Proof. If a is a nonfocal point (so f_a is nondegenerate), then it follows from the definition of tautness that i_* is an injection. Now suppose a is a focal point. If $(r, r'] \subset R(f_a)$, then $M_r(f_a)$ is a deformation retract of $M_r(f_a)$. So we may assume that $r \in R(f_a)$ and $s > r$ such that $[r, s] \subset R(f_a)$. Choose $\delta_1 > 0$ and $\delta_2 > 0$ as in 5.7, and $\varepsilon > 0$ such that $\varepsilon < \min\{\delta_1, \delta_2, (s-r)/5\}$. Since the set of nonfocal points of M in V is open and dense, there exists a nonfocal point b such that $\|b - a\| < \varepsilon$. Since f_b is nondegenerate, it follows from the definition of tautness that $i_*\colon H_*(M_t(f_b), F) \to H_*(M, F)$ is injective for all t. So it suffices to prove that $M_r(f_a)$ is a deformation retract of $M_r(f_b)$. Since $\varepsilon < (s-r)/5$, there exist r_1, r_2, s_1 and s_2 such that $r_1 < s_1, r_2 < s_2, r < r_1 - \varepsilon < s_1 + \varepsilon < s$ and $r_1 < r_2 - \varepsilon < s_2 < s_2 + \varepsilon < s_1$. From triangle inequality we have

$$M_{s_2}(f_b) \backslash M_{r_2}(f_b) \subset M_{s_1}(f_a) \backslash M_{r_1}(f_a) \subset M_s(f_b) \backslash M_r(f_b).$$

Note that $\|\nabla f_a(x)\| \geq \delta_2$ if $x \in M_s(f_a) \backslash M_r(f_a)$, and $\|\nabla f_b(x)\| \geq \delta_2$ if $x \in M_s(f_b) \backslash M_r(f_b)$. Since $\varepsilon < \delta_2$, $(a-b)^T$ is the shortest side of the triangle with three sides $(x-a)^T, (x-b)^T$ and $(a-b)^T$ for all x in $M_{s_1}(f_a) \backslash M_{r_1}(f_a)$. Using the cosine formula for the triangle we have

$$\langle \nabla f_a(x), \nabla f_b(x) \rangle > (2\delta_2^2 - \varepsilon^2)/2 > \varepsilon^2/2 \quad \text{for } x \text{ in } M_{s_1}(f_a) \backslash M_{r_1}(f_a).$$

Hence the gradient flow of f_a gives a deformation retract of $M_{s_1}(f_a)$ to $M_{s_2}(f_b)$. If $[r, s] \subset R(f)$, then $M_r(f)$ is a deformation retract of $M_t(f)$ for all $t \in [r, s]$, which proves our claim.

5.9. Corollary. *If M is connected and $\varphi\colon M \to V$ is a taut immersion, then φ is an embedding.*

Proof. Since M is PF, $\varphi = Y|M \times 0$ is proper. So it suffices to prove that φ is one-to-one. Suppose $\varphi(p) = \varphi(q) = a$. If $p \neq q$, then there exists $\varepsilon > 0$ such that $(0, \varepsilon) \subset R(f_a)$, and p, q are in two different connected components of $M_\varepsilon(f_a)$. This contradicts the fact that $i_*\colon H_0(M_\varepsilon(f_a), F) \to H_0(M, F)$ is injective.

5.10. Corollary. *Suppose M is a taut submanifold of V and x_0 is an index* 0 *critical point of f_a. Then*

(i) *$f_a(x) \geq f_a(x_0)$ for all $x \in M$, i.e., $f_a(x_0)$ is the absolute minimum of f_a,*

(ii) *$f_a^{-1}(f_a(x_0))$ is connected; in particular for $f_a^{-1}(f_a(x_0)) = \{x_0\}$, an isolated critical point x_0.*

6. Geometry of isoparametric submanifolds

In this section we will study the geometry of a special class of PF submanifolds of Hilbert spaces having simple local invariants. Roughly speaking they have zero normal curvature and constant curvature spectrum. The main result of this section is that there exists an affine Coxeter group (for definition, see the appendix) acting on these submanifolds by diffeomorphism.

6.1. Definition. An immersed PF submanifold $f\colon M \to V$ of a Hilbert space $(V, (\ ,\ ))$ is called *isoparametric* if

(i) $\operatorname{codim}(M)$ is finite,

(ii) $\nu(M)$ is globally flat,

(iii) for any parallel normal field v on M, the shape operators $A_{v(x)}$ and $A_{v(y)}$ are orthogonally equivalent for all x and y in M.

The principal $\hat{G}$-orbits of the Hilbert space $H^0(S^1, \mathscr{G})$ in §4 are isoparametric. Although an isoparametric submanifold of a Hilbert space need not be an orbit of an affine isometric action, we will prove that they share many properties of the samples in §4 as in 4.1.

6.2. Definition. An immersed submanifold $f\colon M \to V$ is *full* if $f(M)$ does not lie in a hyperplane of V.

6.3. Definition. A *rank-k* isoparametric submanifold of V is a full, k-codimensional isoparametric submanifold of V.

6.4. Remark. The above definitions are the same as in [41] if $V = \mathbb{R}^n$.

6.5. Remark. It follows from 2.5 that if M is a full isoparametric submanifold of V, and M is contained in the sphere of radius r centered at c_0, then both M and V must be of finite dimension.

From the definition of isoparametric, we have:

6.6. Proposition. *If M is isoparametric in V, then*

(i) *the curvature distributions E_i's are smooth,*

(ii) *the curvature normal fields v_i's are parallel and smooth.*

Let I be the index set for the curvature distributions of M. We arrange the indices in I so that $v_0 = 0$ and $v_i \neq 0$ for $i \neq 0$. There exists an orthonormal frame e_A such that $\{e_\alpha | 1 \leq \alpha \leq k\}$ is a global parallel normal frame, and $\{e_i | i > k\}$ is a local tangent frame of M with E_j spanned by $\{e_m | \mu_{j-1} < m < (1 + u_j)\}$, where $\mu_j = k + \sum\{m_i | i \leq j\}$. Then we have

$$\operatorname{rank}(E_i) = m_i, \quad \omega_{\alpha\beta} = 0, \quad \omega_{i\alpha} = \lambda_{i\alpha}\omega_i,$$
$$\lambda_{i\alpha} = n_{j\alpha}, \quad \text{if } \mu_{j-1} < i < (1 + \mu_j), \qquad v_i = \sum n_{i\alpha} e_\alpha.$$

It is easily seen that the proofs of most results in §1 of [41] generalize directly to this infinite dimensional setting. Therefore we will restate them without proof.

6.7. Proposition. *An isoparametric immersion $f: M \to V$ is full if and only if the curvature normals $\{v_i | i \in I\}$ span $\nu(M)$.*

6.8. Proposition. *Let $\omega_{ij} = \sum r_{ijk}\omega_k$. Then $(\lambda_{i\alpha} - \lambda_{j\alpha})r_{ijk} = (\lambda_{i\alpha} - \lambda_{k\alpha})r_{ikj}$. In particular if $e_i, e_k \in E_{i_1}$, $e_j \in E_{i_2}$, and $i_1 \neq i_2$, then $r_{ijk} = 0$.*

6.9. Theorem. *If M is an immersed isoparametric submanifold of V, then the following hold:*

(1) *E_i is integrable for all $i \in I$.*

(2) *For $0 \neq i \in I$, let $S_i(x_0)$ denote the leaf of E_i through x_0. Then*

(i) *$x + (v_i(x)/\|v_i\|^2) = c_i$ is a constant vector for all x in $S_i(x_0)$,*

(ii) *$E_i(x) \oplus \mathbb{R}v_i(x) = \xi_i$ is a fixed $(m_i + 1)$-plane in V for all $x \in S_i(x_0)$,*

(iii) *$S_i(x_0)$ is the standard sphere in $c_i + \xi_i$ centered at c_i with radius $1/\|v_i\|$.*

(3) *The leaves of E_0 are affine linear subspaces of V.*

Proof. All the statements can be proved in the same manner as in 1.9 of [41], except (2)(ii). To prove this we may assume that $i = 1$ and $m_i = m$. Let $\operatorname{Gr}(m, V)$ denote the set of m-dimensional subspaces of V, and $g: S_1(x_0) \to \operatorname{Gr}(m, V)$ the map defined by

$$g(x) = (e_{k+1} \wedge e_{k+2} \wedge \cdots \wedge e_{k+m} \wedge v_1)(x).$$

Then

$$dg = \sum_i \{e_{k+1} \wedge \cdots \wedge e_{i-1} \wedge \omega_{ij} e_j \wedge \cdots \wedge e_{k+m} \wedge v_1 | j < k + m\}$$
$$+ \sum_{i,\alpha,\beta} e_{k+1} \wedge \cdots \wedge e_{i-1} \wedge \lambda_{1\beta}\omega_i e_\beta \wedge \cdots \wedge e_{k+m} \wedge \lambda_{1\alpha} e_\alpha.$$

Using 6.8, we have $\omega_{ij} = 0$ for $j > k + m$ on $S_1(x_0)$. So $dg = 0$.

6.10. Corollary. *For $0 \neq i \in I$, define $\phi_i: M \to M$ by $\phi_i(q) =$ the antipodal point of q in $S_i(q)$, where $S_i(q)$ is the leaf of E_i through q. Then $\phi_i^2 = \mathrm{id}$. In particular we have*

$$\phi_i(x) = x + 2v_i(x)/\|v_i\|^2$$

is a diffeomorphism, which is called the involution associated to E_i.

6.11. Theorem. *For $i \in I$, let ϕ_i be the diffeomorphism associated to the curvature distribution E_i as in* 6.10. *Then the following hold:*

(i) *There exists a bijection $\sigma_i\colon I \to I$ such that $\sigma_i(i) = i$ and $E_j(\phi_i(q)) = E_{\sigma_i(j)}(q)$ for all q in M. In particular $m_j = m_{\sigma_i(j)}$.*

(ii) $v_{\sigma_i(j)}(q) = (1 - 2(\langle v_i, v_{\sigma_i(j)}\rangle/\|v_i\|^2))v_j(\phi_i(q))$.

(iii) *Let R_i be the reflection of $\nu(M)_q$ in the linear hyperplane $v_i(q)^\perp$. Then $R_i(v_j) = (1 - 2(\langle v_i, v_{\sigma_i(j)}\rangle/\|v_i\|^2))^{-1}v_{\sigma_i(j)}$.*

(iv) $TM_q = TM_{\phi_i(q)}$, $q + \nu(M)_q = \phi_i(q) + \nu(M)_{\phi_i(q)}$ *for all $\phi_i \in W$.*

(v) $\phi_i^*(E_j) = E_{\sigma_i(j)}$.

Let W be the subgroup of the group of diffeomorphisms of M generated by $\{\phi_i | i \in I\}$. In the following we will prove that W is an affine Weyl group.

6.12. Theorem. *Let M be an immersed isoparametric submanifold of V, $\nu_q = q + \nu(M)_q$, and $l_i(q)$ the focal hyperplane of ν_q as in* 3.6. *Let φ_i denote the reflection in ν_q. Then*

(i) $\varphi_i(q) = \phi_i(q)$,

(ii) $\varphi_i(l_j(q)) = l_{\sigma_i(j)}(q)$, *i.e., φ_i permutes $\mathscr{H} = \{l_i(q) | i \in I\}$.*

Proof. We may assume that $\sigma_1(2) = 3$. Let l be the hyperplane $\varphi_1(l_2)$. We claim that $l = l_3$. It follows from 6.11(iii) that l is parallel to l_3, and

$$\|v_2\| = \|R_1(v_2)\| = \|v_3\|(1 - 2\langle v_1, v_3\rangle/\|v_1\|^2)^{-1}.$$

If $R_1(v_2) = rn$ with $r > 0$ and $\|n\| = 1$, then $v_3 = xn$ for some $x \in \mathbb{R}$. Let q^* be the point on l which is the closest to q. To prove the claim it suffices to show that $v_3/\|v_3\| = \overline{qq^*}$, the vector joining q to q^*. Let $\angle(u_1, u_2)$ denote the angle from the vector u_1 to u_2. Then there are the following three cases:

Case (1). $l_1 \cap l_2 \neq \varnothing$ and $\angle(v_1, v_2) = \pi - \theta$, where $0 < \theta \leq \pi/2$.

Let $p_0 \in l_1 \cap l_2$, u be the vector joining q to p_0, $\pi/2 - \alpha = \angle(u, v_2)$, and L_i be the line in l_i passing through p_0 and orthogonal to $l_1 \cap l_2$. Then $\angle(v_1, n) = \theta$ and we claim that $x \geq 0$. For if $x < 0$, then $\angle(v_1, v_3) = \pi - \theta$, so

$$\begin{aligned} R_1(v_2) = rn &= (1 - 2\langle v_1, v_3\rangle/\|v_1\|^2)^{-1}v_3 \\ &= x(1 + 2\cos\theta\|v_3\|/\|v_1\|)^{-1}n. \end{aligned} \tag{22}$$

But then $r = x(1 + 2\cos\theta\|v_3\|/\|v_1\|)^{-1}$ is less than 0, which is a contradiction. This proves that v_3 and $\overline{qq^*}$ have the same direction, and $\angle(v_1, v_3) = \angle(v_1, n) = \theta$. It follows from (22) that

$$1/\|v_2\| = (1 - 2\cos\theta\|v_3\|/\|v_1\|)/\|v_3\| = 1/\|v_3\| - 2\cos\theta/\|v_1\|.$$

Using 6.9 we have

$$\begin{aligned} d(q, l_3) &= 1/\|v_3\| = 1/\|v_2\| + 2\cos\theta/\|v_1\| \\ &= \|u\|\sin\alpha + 2\|u\|\cos\theta\,\sin(\theta - \alpha) = \|u\|\sin(2\theta - \alpha) = d(q, l). \end{aligned}$$

Therefore $l = l_3$.

Case (2). $l_1 \cap l_2 \neq \emptyset$ and $\angle(v_1, v_2) = \theta$, where $0 < \theta \leq \pi/2$.

Using the same notation as in Case (1), we claim that $\overline{qq^*}$ and v_3 again have the same direction. By analytic geometry in $\mathbb{R}^2$, we see that if $\overline{qq^*}$ has the opposite direction of n, then

$$(23) \quad 2 \cos \theta/\|v_1\| > 1/\|v_2\| = d(q, l_2), \quad \text{i.e., } (1/\|v_2\| - 2 \cos \theta/\|v_1\|) < 0.$$

If $v_3 = xn$, $x > 0$, then $\angle(v_1, v_3) = \pi - \theta$, and

$$1/\|v_2\| = 1/x(1 + (2x \cos \theta/\|v_1\|)) = 1/x + 2 \cos \theta/\|v_1\|.$$

Hence $1/\|v_2\| - 2 \cos \theta/\|v_1\| = 1/x$ is positive, which contradicts (23). This implies that $v_3 = xn$ with $x < 0$. Similarly if $\overline{qq^*}$ has the same direction as n, then $\overline{qq^*}$ and v_3 are in the same direction, and the proof that $\overline{qq^*} = v_3/\|v_3\|$ is similar to the first case.

Case (3). $l_1 \| l_2$.

Let v denote $\overline{qq^*}$, and $v_3 = xn$. Suppose v_1 and v_2 have the opposite directions. Then

$$v_2/\|v_2\| = -v_1/\|v_1\| = n, \qquad v = (2/\|v_1\| + 1/\|v_2\|)n.$$

Using (22), we have

$$\begin{aligned} R_1(v_2) &= (1 - 2\langle v_1, v_3\rangle/\|v_1\|^2)^{-1} v_3 = (1 - 2x/\|v_1\|)^{-1} xn \\ &= -v_2 = \|v_2\| n. \end{aligned}$$

So $1/x - 2/\|v_1\| = 1/\|v_2\|$, which implies that $x > 0$, and

$$d(q, l_3) = 1/\|v_3\| = 1/x = 1/\|v_2\| + 2/\|v_1\| = d(q, l).$$

A similar proof works if v_1 and v_2 are in the same direction. q.e.d.

As a consequence of 3.6(ii) and the Theorem in the appendix, we have

6.13. Theorem. *Let W^q be the subgroup of the group of isometries of the affine space $\nu_q = q + \nu(M)_q$ generated by reflections φ_i in $l_i(q)$. Then W^q is an affine Weyl group. Moreover the parallel translation map $\pi_{q,q'}\colon \nu(M)_q \to \nu(M)_{q'}$ conjugates W^q to $W^{q'}$ for any q and q' in M.*

6.14. Corollary. *Let M be an isoparametric submanifold of V. Then the subgroup W of diffeomorphisms of M generated by involutions $\{\phi_i | i \in \mathbb{N}\}$ is an affine Weyl group, and $W \approx W^q$ for all q. W is called the affine Weyl group associated to the immersed isoparametric submanifold M of V. Moreover the curvature normals $\{v_i | i \in I\}$ form a root system for W. Since $m_j = m_{\sigma_i(j)}$, we have associated to M a marked Dynkin diagram.*

6.15. Remark. If M is a principal $\hat{G}$-orbit of $H^0(S^1, \mathscr{G})$ as in §4, then the associated affine Weyl group of 6.14 is $W(\mathscr{T}^0)$ (as in §4), and all the multiplicities $m_i = 2$.

6.16. Remark. Let M be an isoparametric submanifold of a Hilbert space V. If $\dim(V)$ is finite, then there are only finitely many E_i's, so W is a finite Coxeter group (in fact W is crystallographic or a Weyl group). If $\dim(V)$ is infinite, then there are infinitely many E_i's, so W is an infinite discrete Coxeter group.

7. Homology of isoparametric submanifolds

In this section we use Morse theory to calculate the homology of isoparametric submanifolds of Hilbert spaces and prove that they are taut.

The results on Morse theory in §4 of [21] hold for functions satisfying Condition C (for details see [36]). We have shown in 2.16 that the Euclidean distance function f_a satisfies Condition C, so we can apply the infinite dimensional Morse theory to f_a. Let f be a Morse function on M, and q a critical point of f of index m. Recall that a pair (N, φ) is called a Bott-Samelson cycle for f at q if N is a smooth m-dimensional manifold, and $\varphi\colon N \to M$ is a smooth map such that $f \circ \varphi$ has a unique and nondegenerate maximum at y_0, where $\varphi(y_0) = q$. (N, φ) is $\mathscr{R}$-orientable for a ring $\mathscr{R}$ if $H^m(N, \mathscr{R}) = \mathscr{R}$. We say f is of Bott-Samelson type with respect to $\mathscr{R}$ if every critical point of f has an $\mathscr{R}$-orientable Bott-Samelson cycle. Moreover if $\{q_i | i \in I\}$ is the set of critical points of f, and (N_i, φ_i) is an $\mathscr{R}$-orientable Bott-Samelson cycle for f at q_i for $i \in I$, then $H_*(N, \mathscr{R})$ is a free module over $\mathscr{R}$ generated by the descending cells $(\varphi_i)_*([N_i])$, so that f is of linking type perfect. In [21] we obtained the homology of the finite dimensional isoparametric submanifolds by constructing the Bott-Samelson cycles for f_a. The same construction works here in the infinite dimensional setting, so we have:

7.1. Theorem. *Let M be an immersed isoparametric submanifold in a Hilbert space V with multiplicities m_i, and $a \in V$ a nonfocal point of M. Then*

(i) *f_a is of Bott-Samelson type with respect to the ring $\mathscr{R} = \mathbb{Z}$ if all the multiplicities $m_i > 1$, and with respect to $\mathscr{R} = \mathbb{Z}_2$ otherwise,*

(ii) M *is taut.*

It follows from 5.9 that we have

7.2. Corollary. *An immersed isoparametric submanifold of a Hilbert space V is embedded.*

To obtain more precise information concerning the homology groups of isoparametric submanifolds, we need to know the structure of the set of critical points of f_a. When the isoparametric submanifold M is of finite dimension, we used [41] the existence of isoparametric maps to show that if a is nonfocal, and q is a critical point of f_a, then the set of critical points of f_a is $W \cdot q$, the W-orbit through q. Thus using 2.13, we can obtain $H_*(M)$ explicitly.

Although this proof does not work when M is of infinite dimension, we can nevertheless use the tautness and some geometry to obtain similar results. First from 2.11, 2.13 and 6.11 we have

7.3. Theorem. *Let $M \subset V$ be isoparametric, W its associated affine Weyl group, and m_j the associated multiplicities. Let $a \in V$, $f_a\colon M \to \mathbb{R}$ be the smooth function defined by $f_a(x) = \|x-a\|^2$, and let $C(f_a)$ denote the set of critical points of f_a. Then the following hold:*

(i) *$x \in C(f_a)$ if and only if $(a-x) \in \nu(M)_x$.*

(ii) *If $x \in C(f_a)$, then $W \cdot x \subset C(f_a)$.*

(iii) *For x in $C(f_a)$ the index of f at x is the sum of the m_j's such that the open line segment (x,a) joining x to a meets $l_j(x)$.*

The closure of a connected component of the complement of the focal hyperplanes $l_i(x)$ in $\nu_x = x + \nu(M)_x$ is called a *Weyl chamber* for the affine Weyl group W-action on ν_x. A Weyl chamber is a simplex and a fundamental domain for W. As a consequence of 7.3 and 5.10, we have

7.4. Proposition. *Suppose M is isoparametric in V, and $q \in M$. Let Δ_q be the Weyl chamber in $\nu_q = (q + \nu(M)_q)$ containing q, and $a \in \Delta_q$. Then q is a critical point of f_a with index 0. Moreover if a is nonfocal with respect to q, then $f_a(q)$ is the absolute minimum of f, and q is also the only point on M assuming this value.*

7.5. Theorem. *Let M be an isoparametric submanifold of V, and $a \in \nu_q \cap \nu_{q'}$. Then a is nonfocal with respect to q if and only if a is nonfocal with respect to q', and $q' \in W \cdot q$.*

Proof. There are $p \in W \cdot q$, $p' \in W \cdot q'$ such that $a \in \Delta_p$ and $a \in \Delta_{p'}$. If a is nonfocal with respect to q, then it follows from 5.10 and 7.4 that $p = p'$ and a is nonfocal with respect to p'. q.e.d.

Hence we have proved:

7.6. Corollary. *Let $M \subset V$ be isoparametric, and W its associated affine Weyl group. Suppose $a \in V$ is nonfocal with respect to q in M. Then $C(f_a) = W \cdot q$.*

7.7. Corollary. *Let $M \subset V$ be isoparametric. Then $H_*(M, \mathscr{R})$ can be computed explicitly in terms of the associated affine Weyl group W and its multiplicities m_i. Here $\mathscr{R}$ is $\mathbb{Z}$ if all $m_i > 1$, and is $\mathbb{Z}_2$ otherwise.*

7.8. Corollary. *A point $a \in V$ is nonfocal with respect to $q \in M$ if and only if a is W-regular with respect to the W-action on ν_q.*

7.9. Corollary. *If f_a has one nondegenerate critical point, then f_a is nondegenerate, or equivalently if $a \in \nu_q$ is nonfocal with respect to q, then a is nonfocal with respect to M.*

8. Slice theorem and marked Dynkin diagrams for isoparametric submanifolds

In this section we determine the possible marked Dynkin diagrams of isoparametric submanifolds of an infinite dimensional Hilbert space. By using the classification of discrete Coxeter group ([14], [5]) we need only determine the possible multiplicities. This can be determined by the finite dimension theory, because we can prove an analogue of the "slice theorem", and each slice is a finite dimensional isoparametric submanifold.

Let $\nu_x = x + \nu(M)_x$, and let Δ be a Weyl chamber in ν_x. For each simplex of σ of Δ, we define the following:

$I(x,\sigma) = \{j | \sigma \subset l_j(x)\}$,

$V(x,\sigma) = \bigcap \{l_j(x) | j \in I(x,\sigma)\}$,

$\xi(x,\sigma) =$ the orthogonal complement of $V(x,\sigma)$ in ν_x through x,

$\eta(x,\sigma) = \bigoplus \{E_j(x) | j \in I(x,\sigma)\} \oplus \xi(x,\sigma)$,

$m_{x,\sigma} = \sum \{m_j | j \in I(x,\sigma)\}$,

$W_{x,\sigma} =$ the subgroup of W generated by the φ_j with $j \in I(x,\sigma)$,

$\Delta_x =$ the Weyl chamber containing x.

8.1. Slice Theorem. *Let M be a rank-k isoparametric submanifold of V, and W its associated affine Weyl group. Let σ be a simplex of a Weyl chamber Δ of ν_{x_0}, and let $\tilde{x}_0 \in \sigma$. Then $\tilde{x}_0 = x_0 + v(x_0)$ for some parallel normal field v on M. Let $\xi_\sigma, \eta_\sigma, m_\sigma$, denote $\xi(x_0,\sigma), \eta(x_0,\sigma), m_{x_0,\sigma}$ respectively. Then the following hold:*

(i) *The map $I + v$ has finite corank $(m_\sigma + k)$, so $M_v = (I + v)(M)$ is an immersed submanifold of V.*

(ii) *The connected component of the fiber $N_{x_0,v}$ of the submersion $I + v$ through x_0 is an m_σ-dimension isoparametric submanifold of the $(m_\sigma + k)$-dimensional Euclidean space $\nu(M_v)_{\tilde{x}_0}$, in fact, $N_{x_0,v} \subset \eta_\sigma$ and is of rank $(k - \dim(\sigma))$.*

(iii) *The normal plane to $N_{x_0,v}$ in η_σ at x_0 is ξ_σ, the associated Weyl group of $N_{x_0,\sigma}$ is the group W_σ generated by reflections in the hyperplanes $l_j(x_0) \cap \xi_\sigma$ of ξ_σ for $j \in I(x_0,\sigma)$, and respective multiplicities are m_j.*

(iv) *If v^* is another parallel normal field such that $(x_0 + v^*(x_0)) \in \sigma$, then $N_{x_0,v} = N_{x_0,v^*}$, so we may also denote it by $N_{x_0,\sigma}$.*

8.2. Corollary. *With the same notation as in* 8.1, *let Δ be the Weyl chamber in ν_q containing q, σ a simplex of Δ, and $a \in \sigma$. Then*

(i) *the nullity of f_a at $x \in W \cdot q$ is $m_{x,\sigma}$, and the critical submanifold at x is $N_{x,\sigma}$,*

(ii) $f_a(N_{q,\sigma})$ *is the unique minimum of* M.

Proof. Let p be the orthogonal projection of x onto $V(x,\sigma)$. Then $N_{x,\sigma}$ is contained in the sphere of radius $\|x-p\|$ centered at p in the Euclidean space $\eta(x,\sigma)$. But $(p-a)$ is perpendicular to $\eta(x,\sigma)$. So $f_a(y)=\|y-a\|^2$ is constant on $N_{x,\sigma}$, which proves (i). (ii) is a consequence of 7.4.

8.3. Corollary. *Let* M *be a rank-k isoparametric submanifold of a Hilbert space* V, $q\in M$, *and* $\nu_q=q+\nu(M)_q$. *Let* $u\colon M\to\nu_q\times\mathbf{R}$ *be the map defined by* $u(x)=(P(x),\|x\|^2)$, *where* P *is the orthogonal projection of* V *onto* ν_q. *Let* C *denote the convex hull of* $\{(x,\|x\|^2)|x\in W\cdot q\}$. *Then*

(i) $u(M)\subset C$,

(ii) *the boundary of* C, ∂C, *is contained in* $u(M)$.

Proof. Let Δ denote the Weyl chamber of W containing q. Then $\bigcup\{g(\Delta)|g\in W\}=\nu_q$. Let $\mathrm{cvx}(B)$ denote the convex hull of B. Then

$$\bigcup\{\mathrm{cvx}(W_a\cdot g(q))|g\in W \text{ and } a \text{ is a vertex of } g(\Delta)\}=\nu_q.$$

Suppose $u(x)=(t,s)$, i.e., $s=\|x\|^2$ and $x=t+w$, where $t\in\nu_q$ and $w\in(\nu_q)^\perp$. Then there are $g\in W$ and a vertex of a of $g(\Delta)$ such that t lies in the convex hull of $W_a\cdot g(q)$. Suppose r is the minimum of f_a, then

$$f_a(x)=\|x-a\|^2=\|x\|^2-2(x,a)+\|a\|^2=s-2(t,a)+\|a\|^2\geq r.$$

Since $N_{g(q),a}$ is the minimum critical level of f_a, and $P(N_{g(q),a})$ is a convex hull of $W_a\cdot g(q)$ [42], $u(N_{g(q),a})$ is contained in the hyperplane $s-2(t,a)+\|a\|^2-r=0$, and (i) is thus proved.

It is easily seen that $\partial C=\bigcup\{\mathrm{cvx}(u(W_a\cdot g(q)))|g\in W,\ a \text{ is a vertex of } g(\Delta)\}$, and (ii) follows.

8.4. Remark. If $\dim(M)$ is finite, it is proved in [42] that $u(M)=C$. If M is one of the homogeneous examples in §4, then it is proved by Atiyah and Pressley [1] that $u(M)=C$. So it is natural to conjecture that $u(M)=C$ if M is any isoparametric submanifold of a Hilbert space.

8.5. Remark. Let $\langle\ ,\ \rangle$ denote the following nondegenerate bilinear form on $\tilde V=V\oplus\mathbf{R}\oplus\mathbf{R}$:

$$\langle(x,s,t),(x',s',t')\rangle=\langle x,x'\rangle+st'+s't.$$

Then $\mathbf{H}=\{(x,s,t)|\|x\|^2+2st=-1\}$ with the induced metric is a Riemannian Hilbert manifold with constant sectional curvature -1, and $\iota\colon V\to\mathbf{H}$ defined by $\iota(x)=(x,(\|x\|^2+1)/2,-1)$ is an isometric embedding. In fact $\iota(V)$ is the intersection of $\mathbf{H}$ with the hyperplane defined by $t=-1$ in $\tilde V$. If M is a submanifold of V, then the normal plane of $\iota(M)$ in $\tilde V$ at $\iota(q)$ can be naturally identified as $\tilde\nu=\nu(M)_q\oplus\mathbf{R}\oplus\mathbf{R}$, and the map u in the above theorem is the restriction of the projection map of $\tilde V$ along $\tilde\nu$ to M.

8.6. Remark. Let G be a connected compact Lie group, $\mathscr{G}$ its Lie algebra, $V = H^0(S^1, \mathscr{G})$, and $\tilde{V}$ as above. Let $c = (0,1,0)$ and $d = (0,0,1) \in \tilde{V}$. Define

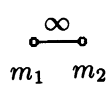
$$\begin{aligned}
&[c, v] = 0 \quad \text{for all } v \in \tilde{V},\\
&[d, u] = u'(\theta) \quad \text{for } u \in V,\\
&[u, v]\,(\theta) = [u(\theta), v(\theta)] \quad \text{for } u, v \in V.
\end{aligned}$$

Then $\tilde{V}$ is the Lie algebra of the Kac-Moody group $\tilde{G}$, and $\langle\ ,\ \rangle$ is the Killing form on $\hat{V}$ (i.e., Ad-invariant) ([22], [23]). Let $\hat{t} \in \mathscr{T}^0$, $\|t\| = 1$, and M be the $H^1(S^1, G)$-orbit through $\hat{t}$ in V. Then $\iota(M)$ is the adjoint orbit of $\tilde{G}$ on $\tilde{V}$ through $(\hat{t}, 0, -1)$.

It follows from the classification of the discrete Coxeter groups, the slice theorem, and the same proof as in the finite dimensional case [21] that we have

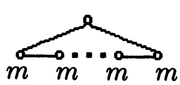
8.7. Theorem. *Let M be a rank-k isoparametric submanifold of V, and W the associated affine Weyl group. Suppose W is irreducible; then the possible marked Dynkin diagrams are as follows:*

$\tilde{A}_1$



m_1, m_2 are arbitrary,

$\tilde{A}_l$

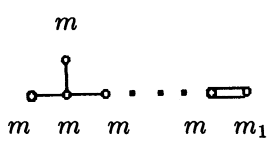


$m \in \{1, 2, 4\}$,

$\tilde{B}_2$

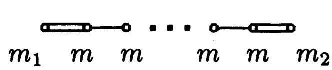
(m_1, m_2) and (m_2, m_3) satisfy $(*)$,

$\tilde{B}_l$



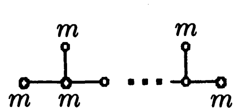
(m, m_1) satisfies $(*)$,

$\tilde{C}_l$



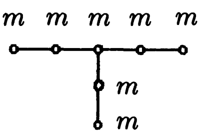
(m_1, m) and (m, m_2) satisfy $(*)$,

$\tilde{D}_l$



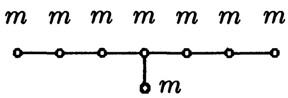
$m \in \{1, 2, 4\}$,

$\tilde{E}_6$



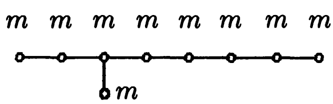
$m \in \{1, 2, 4\}$,

$\tilde{E}_7$



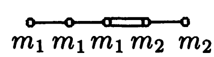
$m \in \{1, 2, 4\}$,

$\tilde{E}_8$



$m \in \{1, 2, 4\}$,

$\tilde{F}_4$



(i) $m_1 = 1,\ m_2 \in \{1, 2, 4, 8\}$,
(ii) $m_2 = 1,\ m_1 \in \{1, 2, 4\}$,
(iii) $m_1 = m_2 = 2$,

$\tilde{G}_2$

$m \in \{1, 2\}$.

The pair (m_1, m_2) *is said to satisfy* $(*)$ *if we let* $n_1 = \min\{m_1, m_2\}$ *and* $n_2 = \max\{m_1, m_2\}$. *Then* (n_1, n_2) *satisfies one of the following conditions*:

(i) 2^u *divides* $(n_1 + n_2 + 1)$, *where* $2^u = \min\{2^\sigma | n_1 < 2^\sigma,\ \sigma \in \mathbb{N}\}$,

(ii) *if* m_1 *is a power of* 2, *then* $2n_1$ *divides* $(n_2 + 1)$ *or* $3n_1 = 2(n_2 + 1)$.

8.8. Remark. Every irreducible affine Weyl group occurs in the examples of §4, and all the multiplicities of these examples are 2.

Let $(V, \langle\ ,\ \rangle)$ be a Hilbert space. In the following we will give a necessary and sufficient condition for an orbit of an affine isometric action on V to be isoparametric. A linear operator $T\colon V \to V$ is orthogonal if T preserves the inner product $\langle\ ,\ \rangle$. A diffeomorphism $\varphi\colon V \to V$ is an isometry if $d\varphi_x$ is orthogonal for all x in V. It is easily seen that the group $\mathrm{Iso}(V)$ of isometries of V is the semidirect product of the group $\mathrm{O}(V)$ of orthogonal transformations and the group V of translations. In particular, if $\varphi\colon V \to V$ is an isometry, then there exist $t_0 \in V$ and $T \in \mathrm{O}(V)$ such that $\varphi(x) = t_0 + T(x)$ for all $x \in V$.

8.9. Definition. Let G be a Hilbert Lie group. An affine representation $\rho\colon G \to \mathrm{Iso}(V)$ is called *polar* if

(i) the induced G-action on V is proper,

(ii) each orbit map $G \to V$ (mapping $g \to \rho(g)(x)$) is Fredholm,

(iii) the principal orbits have finite codimension,

(iv) the normal plane to a principal G-orbit in V meets every orbit orthogonally (such normal planes are called *sections*).

Using a similar argument as in [35], we have

8.10. Theorem. *Let $\rho\colon G \to \mathrm{Iso}(V)$ be a polar affine representation, and $M = G \cdot x_0$ a principal orbit. Then*

(i) *M is isoparametric,*

(ii) *the associated Weyl group of M as an isoparametric submanifold is $N(\nu_{x_0})/Z(\nu_{x_0})$, where $N(\nu_{x_0})$ and $Z(\nu_{x_0})$ are the normalizer and centralizer of ν_{x_0} respectively, and $\nu_{x_0} = x_0 + \nu(M)_{x_0}$,*

(iii) *a point x in ν_{x_0} is subregular (i.e., if an isotropy subgroup G_y is contained in G_x, then $G \cdot y$ must be a principal orbit) if and only if x lies in one and only one of the focal hyperplanes $l_i(x_0)$ in ν_{x_0}; moreover the multiplicity m_i is equal to* $\dim(M) - \dim(G \cdot x)$.

8.11. Theorem. *If an isoparametric submanifold M of V is a G-orbit for an affine representation $\rho\colon G \to \mathrm{Iso}(V)$, then ρ is polar, and M is a principal G-orbit.*

8.12. Corollary. *An orbit M of an affine representation ρ on V is isoparametric if and only if ρ is polar, and M is a principal orbit.*

Let T be the abelian group of all the translations of V. Then the natural action of T on V is polar affine. A polar affine representation of this type will be called of translation type. It follows from the geometric theory of finite dimensional isoparametric submanifolds that every finite dimensional polar affine representation can be written as the product of a polar affine representation of translation type and a linear polar representation. Hence it is completely classified by Dadok's theorem [15] (they are essentially the isotropy representations of symmetric spaces). However the only known infinite dimensional examples are those given in §4. This suggests the following:

8.13. Open problems and questions. (i) Classification of polar affine representations, and their marked Dynkin diagrams.

(ii) Is there an infinite dimensional isoparametric submanifold with irreducible affine Weyl group that is not an orbit of some polar affine representation (i.e., nonhomogeneous)? Since there are many finite dimensional nonhomogeneous rank-2 isoparametric submanifolds ([32], [17]) and the product of isoparametric submanifolds is isoparametric, there are many nonhomogeneous examples with reducible Coxeter groups.

9. Parallel foliations

Let M be a PF submanifold of V with flat normal bundle. In general the parallel set $M_v = \{Y(v(x)) = x + v(x),\ x \in M\}$, defined by a parallel normal field v, may be a singular set, and $\mathscr{F} = \{M_v | v$ parallel normal field on $M\}$ need not foliate V. The main result of this section is that if M is isoparametric, then each M_v is an embedded submanifold of V, and $\mathscr{F}$ gives an orbit-like singular foliation on V.

In what follows M is an isoparametric submanifold of a Hilbert space V.

9.1. Proposition. *$M \cap \nu_q = W \cdot q$, where $\nu_q = (q + \nu(M)_q)$.*

Proof. It is easily seen that $W \cdot q \subset M \cap \nu_q$. Now suppose that $b \in M \cap \nu_q$. Then $b \in \nu_b \cap \nu_q$. But b is nonfocal with respect to b, so it follows from 7.5 that we have $b \in W \cdot q$.

9.2. Proposition. *Suppose σ is a simplex of Δ_q, and σ' is a simplex of $\Delta_{q'}$. If $\sigma \cap \sigma' \neq \emptyset$, then $\sigma = \sigma'$ and $N_{q,\sigma} = N_{q',\sigma}$.*

Proof. Suppose $a \in \sigma \cap \sigma'$. Then q and q' are critical points of f_a with 0 as index, $m_{q,\sigma}, m_{q',\sigma'}$ as nullities, and $N_{q,\sigma}, N_{q',\sigma'}$ as critical submanifolds of f_a at q and q' respectively. So it follows from 5.10 that $N_{q,\sigma} = N_{q',\sigma'}$. In particular $q' \in N_{q,\sigma}$. Using 8.1, $N_{q,\sigma}$ is isoparametric in the Euclidean space of dimension $(m_\sigma + k)$. It is a result of the finite dimensional isoparametric theory that the normal parallel translation of $N_{q,\sigma}$ transforms focal set Σ_q of $N_{q,\sigma}$ to $\Sigma_{q'}$ and conjugates the Weyl group W_σ to $W_{\sigma'}$. Applying this to every subsimplex of σ, we obtain $\sigma' = \sigma$.

9.3. Proposition. *Let σ be a simplex of a Weyl chamber in ν_q, $\varphi \in W$, and $N_{x,\sigma}$ as in* 8.1. *Then $\varphi(N_{q,\sigma}) = N_{\varphi(q),\sigma}$.*

Proof. Let ϕ_i and σ_i be as in 6.11. Using 8.1, we see that $N_{q,\sigma}$ is the leaf of the distribution $\bigoplus\{E_j | j \in I(q,\sigma)\}$ through q, and $N_{\varphi(q),\sigma}$ is the leaf of the distribution $\bigoplus\{E_j | j \in I(\varphi(q),\sigma)\}$ through $\varphi(q)$. Then the proposition follows from 6.11(i).

9.4. Theorem. *Let M be an isoparametric submanifold of V, Δ the Weyl chamber in ν_q containing q, and $a \in \nu_q$ a focal point with respect to q. Then f_a is nondegenerate in the sense of Bott, and $C(f_a) = \bigcup\{N_{x,\sigma} | x \in W \cdot q\}$.*

Proof. For $x \in W \cdot q$ it follows from 8.2 that x is a critical point of f_a with nullity $m_{x,\sigma}$ and $N_{x,\sigma}$ as the critical submanifold of f_a through x. Hence $N_{x,\sigma} \subset C(f_a)$. Conversely, if $y \in C(f_a)$, then $a \in \nu_y$. By 7.5, a is a focal point with respect to y, so there exist $\varphi \in W$ such that $\varphi^{-1}(y) = y_0$, and a simplex σ' in Δ_{y_0} such that $a \in \sigma'$. Then it follows from 9.2 that $\sigma = \sigma'$ and $N_{q,\sigma} = N_{y_0,\sigma}$. Thus we have $\varphi(N_{q,\sigma}) = N_{\varphi(q),\sigma} = \varphi(N_{y_0,\sigma}) = N_{\varphi(y_0),\sigma} = N_{y,\sigma}$.

9.5. Theorem. *Let M be a codimension k isoparametric submanifold of V, and v a parallel normal field of M. Then the following hold:*

(i) *The map $I + v$ has constant, finite corank, so the image $M_v = (I + v)(M)$ is always an immersed submanifold of V with finite codimension $(k + \text{nullity}(I - A_v))$. M_v is called a parallel submanifold of M.*

(ii) *$(I + v)$ is an immersion if and only if $\langle v, v_i\rangle \neq 1$ for all $i \in I$.*

(iii) *If $(I + v)$ is an immersion, and $q^* = (I + v)(q)$, then M_v is isoparametric, M_v and M have the same normal planes, the same focal sets at q and q^*, and the same associated Weyl groups and multiplicities.*

9.6. Theorem. *Let M be an isoparametric submanifold in V, $q \in M$, and Δ_q the Weyl chamber of W on ν_q containing q. Let $v \in \nu(M)_q$, let $\tilde{v}$ denote the parallel normal vector field on M determined by $\tilde{v}(q) = v$, and let M_v denote the parallel submanifold $M_{\tilde{v}}$. Then the following hold:*

(i) *If $v \neq w$, and $q+v$ and $q+w$ are in Δ_q, then M_v and M_w are disjoint.*

(ii) *Given any y in V there exists a unique $v \in \nu(M)_q$ such that $q+v \in \Delta_q$ and $y \in M_v$.*

Proof. Suppose $(q + v), (q + w)$ are in Δ_q, and $M_v \cap M_w \neq \emptyset$. Let $a \in M_v \cap M_w$. Then there exist $x, y \in M$ such that $a = x + \tilde{v}(x) = y + \tilde{w}(y)$. Since $a \in \Delta_q$ and $\tilde{v}, \tilde{w}$ are parallel, $a \in \Delta_x$ and $a \in \Delta_y$. So x and y are critical points of f_a with index 0. If a is nonfocal, then $x = y$, which implies that $v = w$. If a is focal (suppose a is in the simplex σ), then the two critical submanifolds $N_{x,\sigma}$ and $N_{y,\sigma}$ are equal. In particular $y \in N_{x,\sigma}$. We note that $N_{x,\sigma}$ is a finite dimensional isoparametric submanifold in $\eta(\sigma) \subset a + \nu(M_v)_a$. Let $v = u_1 + u_2$, where u_2 is the orthogonal projection of v along $V(\sigma)$. Then $N_{x,\sigma}$ is contained in the sphere of radius $\|u_1\|$ and centered at $x + u_1$. So $y + \tilde{u}_1(y) = x + u_1$. Since $V(\sigma)$ is perpendicular to $N_{x,\sigma}$, $\tilde{u}_2(y) = u_2$. Therefore we have $y + \tilde{v}(y) = x + \tilde{v}(x) = a = y + \tilde{w}(y)$, which implies that $v = w$.

9.7. Corollary. *Let M be an isoparametric submanifold of V. Then $\mathscr{F} = \{M_v | q+v \in \Delta_q\}$ defines an orbit-like singular foliation on V, which will be called the isoparametric foliation of M. The leaf space of $\mathscr{F}$ is isomorphic to the orbit space ν_q/W.*

9.8. Corollary. *If $a \in \sigma \subset \Delta_q$ and $a = q + v$, then the isoparametric foliation of $N_{q,\sigma}$ in $(a + \nu(M_v)_a)$ is $\{M_u \cap (a + \nu(M_u)_a) | M_u \in \mathscr{F}\}$.*

10. The Chevalley restriction theorem

A smooth map $f = (f_1, \cdots, f_k)\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ is isoparametric if Δf_i and $\langle \nabla f_i, \nabla f_j\rangle$ are functions of $f_1, \cdots, f_k$, and $[\nabla f_i, \nabla f_j]$ is a linear combination of $\nabla f_1, \cdots, \nabla f_k$ ([9], [41]). If M^n is isoparametric in $\mathbb{R}^{n+k}$ with W as the associated Weyl group, then in [41] we proved an analogue of the Chevalley restriction theorem [44] for the adjoint action of a compact Lie group on its Lie algebra, i.e., if u is a W-invariant polynomial on the affine normal plane ν_{x_0}, then u can be extended to a polynomial $\overline{u}$ on $\mathbb{R}^{n+k}$ such that $\overline{u}$ is constant on every parallel submanifold of M. Since the ring of W-invariant polynomials on ν_{x_0} is a polynomial ring with k generators $u_1, \cdots, u_k$ [13], $f = (\overline{u}_1, \cdots, \overline{u}_k)$ is an isoparametric polynomial map on $\mathbb{R}^{n+k}$, M is a level set of f and each level set of f is a parallel set of M.

For an infinite dimensional Hilbert space, the Laplace operator is not defined, so the definition of isoparametric map cannot be easily generalized. However the geometric analogue of Chevalley's restriction theorem is still true for the C^∞ category.

10.1. Theorem. *Let M be a rank-k isoparametric submanifold of V, W the associated affine Weyl group, and $q \in \nu_q$. Suppose $f\colon \nu_q \to \mathbb{R}$ is a smooth W-invariant function. Then f can be extended uniquely to a smooth function $\tilde{f}$ on V such that $\tilde{f}$ is constant on every parallel submanifold of M.*

Proof. Given $v \in \nu(M)_q$, we let $\tilde{v}$ denote the unique parallel normal field on M such that $\tilde{v}(q) = v$, and M_v the parallel submanifold $M_{\tilde{v}}$. Let Δ be the Weyl chamber containing q, and Δ^0 the interior of Δ. Since $\mathscr{F} = \{M_v | v \in \Delta\}$ foliates V, f has a unique and well-defined extension $\tilde{f}$ to V, i.e., $\tilde{f}(x) = f(v)$ if $x \in M_v$.

We claim that $\tilde{f}$ is smooth at nonfocal points of M. To see this we note that the map $F\colon M \times \Delta^0 \to V$, defined by $F(x, v) = x + \tilde{v}(x)$, is a diffeomorphism from $(M \times \Delta^0)$ onto an open dense subset $\mathscr{U} = F(M \times \Delta^0)$ ($\mathscr{U}$ is the set of nonfocal points of M) of V, and $F(M \times \{v\}) = M_v$. So $\tilde{f}$ is smooth on $\mathscr{U}$.

Suppose $a \in \Delta$ is a focal point. Then there are $v \in \nu(M)_q$ and a simplex $\sigma \subset \Delta$ such that $a = (q+v) \in \sigma$. We have shown that the parallel translation with respect to the normal connection conjugates the W-actions on the affine normal planes ν_x, and two points in ν_x lie in the same parallel submanifold M_v if and only if these two points lie in the same W-orbit. So $\tilde{f}|\nu_x$ is W-invariant for any $x \in M$. To prove the theorem, it suffices to prove that $\tilde{f}$ is smooth at a neighborhood of a. Although the foliation $\mathscr{F}$ does not necessarily come from a group action, we can imitate the proof of the C^∞ Chevalley restriction theorem for the group actions with sections (as in [35]) by geometric means. We proceed as follows:

(i) The map $\pi = I + \tilde{v}\colon M \to M_v$ is a fibration, so there is a local cross section s defined on an open subset U of a in M_v such that $s(a) = q$.

(ii) Let $\mathscr{M}$ be the Banach space of selfadjoint compact operators on V, and $\mathrm{O}(V)$ the Banach group of the orthogonal transformations. Let $\mathrm{O}(V)$ act on $\mathscr{M}$ by conjugation. Then all the isotropy subgroups are closed. Since

TM_x is a finite codimension closed subspace of V, we may view the shape operator A_u for $u \in \nu(M)_x$ as an element of $\mathscr{M}$. It follows from the definition of isoparametric that $A_{\tilde{v}(x)}$ and A_v lie in the same $O(V)$-orbit. From the standard theory of transformation groups the orbit map $p\colon O(V) \to O(V)\cdot A_v$ is a fibration, so there is a local cross section γ of p defined on a neighborhood of A_v such that $p(A_v) = \mathrm{id}$.

(iii) Let $I_0 = I(q,\sigma), \varsigma(x, I_0) = \bigoplus\{E_i(x) | i \in I_0\}$, and let $P_{q,x}\colon \nu_q \to \nu_x$ be defined by $P_{q,x}(q+v) = x + \tilde{v}(x)$. Then we have

$$\begin{gathered}\nu(M_v)_a = \nu(M)_q \times \varsigma(q, I_0),\\ \varsigma(s(y), I_0) = T(N_{s(y),P_{q,x}(\sigma)})_{s(y)},\\ \gamma(A_{s(y)})(\varsigma(q,I_0)) = \varsigma(s(y), I_0).\end{gathered}$$

So the map

$$\begin{aligned}\psi\colon M_v \times \nu(M)_q \times \varsigma(q, I_0) &\to \nu(M_v)\\ (y, u, w) &\to \tilde{u}(s(y)) + \gamma(A_{s(y)})(w)\end{aligned}$$

defines a vector bundle isomorphism, i.e., ψ maps $\{y\} \times \nu(M)_q \times \varsigma(q, I_0)$ isomorphically to $\nu(M_v)_y$. The restriction of the isoparametric foliation $\mathscr{F}$ on V to $y + \nu(M_v)_{s(y)}$ is the isoparametric foliation given by the slice $N_{s(y),P_{q,x}(\sigma)}$, and ψ maps leaves of $a + \nu(M_v)_a$ to $y + \nu(M_v)_{s(y)}$. Using the slice theorem 8.1, the Weyl group associated to $N_{q,\sigma}$ in $a + \nu(M_v)_a$ is W_σ, and W_σ is of rank $k_0 = k - \dim(\sigma)$.

(iv) This theorem is true if M is of finite dimension, and f is a W-invariant polynomial (Theorem C of [41]). By a theorem of Chevalley the ring of W_σ-invariant polynomials on $a + \nu(M_v)_a$ is a polynomial ring of k_0 generators, $u_1, \cdots, u_{k_0}$. Using a theorem of Schwarz [37], there exists a smooth function $\varphi\colon (a + \nu(M_v)_a) \to \mathbb{R}$ such that $\tilde{f}|(a + \nu(M_v)_a) = \varphi(u_1, \cdots, u_{k_0})$. So $\tilde{f}|(a + \nu(M_v)_a)$ is smooth. Then it follows from (iii) that $\tilde{f}$ is smooth in a neighborhood of a in V. q.e.d.

As a consequence we obtain the following analogue of the Chevalley restriction theorem:

10.2. Corollary. *Let M be an isoparametric submanifold of V, W the associated affine Weyl group, and $q \in \nu_q$. Let $C^\infty(V)^{\mathscr{F}}$ denote the set of smooth functions on V which are constant on each leaf of $\mathscr{F}$, and let $C^\infty(\nu_q)^W$ denote the set of all smooth W-invariant functions on ν_q. Let $\Phi\colon C^\infty(V)^{\mathscr{F}} \to C^\infty(\nu_q)^W$ be the restriction map, i.e., $\Phi(f) = f|\nu_q$. Then Φ is an isomorphism.*

Let M be a rank k isoparametric submanifold of V, and σ a vertex of Δ_q. Then the isotropy subgroup W_σ is a finite Weyl group, and W is the semidirect product of W_σ and a rank-k lattice Λ in ν_q. Let $C^\infty(\nu_q)^W$ denote the space of smooth W-invariant functions on ν_q. Then $\nu_q/\Lambda = T^k$ is a k-dimensional torus, and $C^\infty(T^k)^{W_\sigma} \approx C^\infty(\nu_q)^W$. Since W_σ is a finite group acting on the compact torus, there exist finitely many generators $f_1, f_2, \cdots, f_m$ of $C^\infty(\nu_q)^W$ such that $\{f_i\}$ separate orbits of W on ν_q. Therefore their extensions $\tilde{f}_i$ on V defines M. Moreover let $\tilde{f} = (\tilde{f}_1, \cdots, \tilde{f}_m)$; then $\{\tilde{f}^{-1}(c) | c \in \mathbb{R}^m\}$ is the isoparametric foliation of M. However m may be larger than k.

Appendix. Discrete Coxeter groups

We will review the definitions of proper actions and Coxeter groups, and give a characterization of Coxeter groups.

Definition. A G-action on M is called *proper* if $g_n x_n \to y$ and $x_n \to x$ in M imply that g_n has a convergent subsequence in G. If G is a discrete group, then a proper G-action is classically known as a *properly discontinuous* action.

Remark. A G-action on M is proper if and only if one of the following conditions is satisfied:

(i) the map from $G \times M$ to $M \times M$ defined by $(g, x) \to (gx, x)$ is proper,

(ii) given any compact subsets K and L of M, the set $\{g \in G | gK \cap L \neq \emptyset\}$ is compact.

Let $\mathrm{Iso}(\mathbb{R}^k)$ denote the group of isometries of $\mathbb{R}^k$, which is the semidirect product of the group $\mathrm{O}(k)$ of orthogonal transformations and the group $\mathbb{R}^k$ of translations. In particular, if $\varphi\colon \mathbb{R}^k \to \mathbb{R}^k$ is an isometry, then there exist $t_0 \in \mathbb{R}^k$ and $T \in \mathrm{O}(\mathbb{R}^k)$ such that $\varphi(x) = t_0 + T(x)$ for all $x \in \mathbb{R}^k$.

Coxeter groups can be defined either algebraically in terms of generators and relations or else geometrically [5]. We will use the geometric definition.

Definition. A subgroup W of $\mathrm{Iso}(\mathbb{R}^k)$ generated by reflections in a finite or countable set of hyperplanes $\mathscr{H} = \{l_i | i \in I\}$ is a *Coxeter group* if the following conditions are satisfied:

(i) $W(\mathscr{H}) = \mathscr{H}$, i.e., $g(l) \in \mathscr{H}$ for all $l \in \mathscr{H}$.

(ii) the induced topology of W from $\mathrm{Iso}(\mathbb{R}^k)$ is discrete,

(iii) W acts on $\mathbb{R}^k$ properly.

An infinite Coxeter group is also called an *affine Weyl group.*

It is well known [5] that if W is a Coxeter group, then $\mathscr{H}$ is locally finite (i.e., given any point x in $\mathbb{R}^k$ there exists a neighborhood U of x such that U intersects only finitely many of the l_i). The converse is also true, but we cannot find a proof in the literature, so we will give a proof here. First we make the following definitions.

Definition. Let $\mathscr{H} = \{l_i | i \in I\}$ be a family of hyperplanes in $\mathbb{R}^k$, and v_i a unit normal vector of l_i. Then the rank of $\mathscr{H}$, denoted by $\mathrm{rank}(\mathscr{H})$, is defined to be the maximal number of independent vectors in $\{v_i | i \in I\}$. The rank of the subgroup W generated by reflections in l_i is defined to be the rank of $\mathscr{H}$.

Lemma. *Let $\mathscr{H} = \{l_i | i \in I\}$ be a locally finite family of hyperplanes of rank $m < k$ in $\mathbb{R}^k$, G the subgroup of* $\mathrm{Iso}(\mathbb{R}^k)$ *generated by reflections in l_i, and v_i the unit normal vector of l_i. Suppose $v_1, \cdots, v_m$ are linearly independent. Let V denote the linear subspace spanned by $v_1, \cdots, v_m$, $p \in \bigcap\{l_i | i \leq m\}$, and $E = p + V$. Then*

(i) $g(E) \subset E$,

(ii) $\mathscr{H}^* = \{\hat{l}_i = l_i \cap E | i \in I\}$ *is a locally finite family of hyperplanes in E, and* $\mathrm{rank}(\mathscr{H}^*) = m$,

(iii) *the restriction map $\Phi\colon G \to \mathrm{Iso}(E)$ given by $\Phi(g) = g \restriction E$ is injective,*

(iv) $\Phi(G)$, *is generated by reflections in $\hat{l}_i$,*

(v) $\Phi(G)(\mathscr{H}^*) = \mathscr{H}^*$ *if* $G(\mathscr{H}) = \mathscr{H}$.

Proof. We may assume that $p = 0$, i.e., $E = V$. Since $\mathrm{rank}(\mathscr{H}) = m$ and $v_1, \cdots, v_m$ are linearly independent, we have $v_i \in V$, and $\hat{l}_i = l_i \cap E$ is a hyperplane of E for all $i \in I$. The local finiteness of $\mathscr{H}$ implies that $\{\hat{l}_i = l_i \cap E | i \in I\}$ is locally finite. Let r_i (resp. $\hat{r}_i$) denote the reflections in l_i (resp. $\hat{l}_i$) of $\mathbb{R}^k$ (resp. E). Let s_i denote the linear reflection of $\mathbb{R}^k$ in $(v_i)^\perp$. Since $l_i \cap E \neq \emptyset$, there exists $t_i \in E$ such that $r_i(x) = t_i + s_i(x)$ for all $x \in \mathbb{R}^k$. Because $v_i \in V$, we have $s_i \restriction V^\perp$ is identity. Noting that r_i is an affine transformation, we have $r_i(y+z) = r_i(y) + s_i(z)$. Given any $x \in \mathbb{R}^k$, write $x = y + z$, where $y \in V, z \in V^\perp$. Then $r_i(x) = \hat{r}_i(y) + z$, which implies that $g(x) = \Phi(g)(y) + z$, and $\hat{r}_i = r_i \restriction E$. Hence (i), (iii), (iv) and (v) follow.

Theorem. *Let $\mathscr{H} = \{l_i | i \in I\}$ be a locally finite family of hyperplanes in $\mathbb{R}^k$, and W the subgroup of* $\mathrm{Iso}(\mathbb{R}^k)$ *generated by reflections in l_i. Suppose $W(\mathscr{H}) = \mathscr{H}$. Then W is a Coxeter group.*

Proof. It follows from the above lemma that we may assume that $\mathrm{rank}(\mathscr{H}) = k$. Let v_i be the unit normal vector of l_i in $\mathbb{R}^k$. Then $\{v_{i_j} | 1 \leq j \leq k\}$ forms a basis of $\mathbb{R}^k$ if and only if $\bigcap\{l_{i_j} | 1 \leq j \leq k\}$ consists of a single point. Such a point is called a vertex of $\mathscr{H}$. Let $\mathscr{V}$ denote the set of all vertices of $\mathscr{H}$. Then it follows from the local finiteness of $\mathscr{H}$ that $\mathscr{V}$ is a discrete subset of $\mathbb{R}^k$. Since W permutes hyperplanes in $\mathscr{H}$, W permutes $\mathscr{V}$. Now suppose that $g_n \in W$ and $g_n \to \mathrm{id}$ in $\mathrm{Iso}(\mathbb{R}^k)$. Given $p \in \mathscr{V}$, $g_n(p) \to p$ is a convergent sequence in the discrete set $\mathscr{V}$. So there exists n_0 such that $g_n(p) = p$ for all $n > n_0$. The local finiteness of $\mathscr{H}$ implies that $J = \{i \in I | p \in l_i\}$ is finite. Since p is a vertex, the maximal number of independent vectors in $\{v_j | j \in J\}$ is k. Then $G = \{\varphi \in \mathrm{Iso}(\mathbb{R}^k) | \varphi(p) = p$, and for each $j \in J$, $\varphi(l_j) = l_i$ for some $i \in J\}$ is a finite subgroup. In fact if $|J| = m$, then $|G| \leq 2^m(m!)$. It can be easily seen that $g_n \in G$ for $n > n_0$. But $g_n \to \mathrm{id}$, so there exists $n_1 > n_0$ such that $g_n = \mathrm{id}$ for all $n > n_1$. This proves that W is discrete. It remains to prove that W acts on $\mathbb{R}^k$ properly. Suppose $g_n \in G$, $g_n x \to y$ and $x_n \to x$. Then there exist $t_n \in \mathbb{R}^k$ and $T_n \in \mathrm{O}(k)$ such that $g_n = t_n + T_n$. Since $\mathrm{O}(k)$ is compact, there exist a subsequence T_{n_i} and $T_0 \in \mathrm{O}(n)$ such that $T_{n_i} \to T_0$. But we have $g_{n_i} x_{n_i} = t_{n_i} + T_{n_i} x_{n_i} \to y$, $x_{n_i} \to x$, and $T_{n_i} x_{n_i} \to T_0 x$. So $t_{n_i} \to t_0 = (y - T_0 x)$, i.e., $g_{n_i} \to g_0 = t_0 + T_0$. Since $g_{n_i}(\mathscr{H}) = \mathscr{H}$ and $\mathscr{H}$ is locally finite, $g_0(\mathscr{H}) = \mathscr{H}$, i.e., $g_{n_i} \to g_0$. Because W is discrete, there exists i_0 such that $g_{n_i} = g_0$ for all $i > i_0$.

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