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COMPACTIFICATION OF COMPLETE KÄHLER MANIFOLDS OF NEGATIVE RICCI CURVATURE

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0. Introduction

The compactification of quotients of bounded symmetric domains with finite volume was obtained by Satake [19], Baily-Borel [5], Andreotti-Grauert [2] and Ash-Mumford-Rapoport-Tai [4]. Siu-Yau [21] obtained a compactification of complete Kähler manifolds with finite volume and sectional curvature pinched between two negative constants; their result may be regarded as a generalization of the compactification result for quotients with finite volume of bounded symmetric domains in the case of rank 1. In this paper, we investigate the problem of compactification of complete Kähler manifolds of negative Ricci curvature. Our main result is the following theorem.

Theorem 0.1. *Let M be a complete Kähler manifold of dimension m and let ω be the Kähler form. Assume the following.*

1. $\mathrm{Ric}(\omega) < 0$.
2. *M is very strongly $(m-2)$-pseudoconcave (cf. Definition 2.1).*
3. *The universal cover of M is Stein.*

Then M is biholomorphic to a quasiprojective manifold.

This theorem may be regarded as a generalization of the compactification results for quotients with finite volume of bounded symmetric domains of any rank (cf. §3).

Our proof depends on the weak Riemann-Roch theorem for L^2-plurigeneral (cf. §1) and the existence of Kähler-Einstein metrics (cf. §2). In §3, we shall give a purely differential geometric criterion for noncompact complete Kähler manifolds of finite volume to be quaisprojective.

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1. An L^2-Riemann-Roch inequality

1.1. Throughout this section we let (M^m, ω) denote a complete Kähler manifold with $\mathrm{Ric}(\omega) \leq -\omega$. We denote by $H^0_{(2)}\,(M, K_M^{\otimes\nu})$ the vector space of L^2 (relative to ω) holomorphic sections of the νth tensor power $K_M^{\otimes\nu}$ of the canonical bundle K_M.

Theorem 1.1. *We have the following estimate:*

$$\liminf_{\nu\to\infty} \nu^{-m} \dim H^0_{(2)}(M, K_M^{\otimes\nu}) \geq \frac{1}{m!}\int_M c_1(K_M)^m,$$

where $c_1(K_M) = -(\sqrt{-1}/2\pi)\,\mathrm{Ric}(\omega)$ is the Chern-Weil form of the canonical bundle K_M equipped with the metric induced from ω.

Remark 1.1. Theorem 1.1 remains valid even if the term on the right-hand side is infinite.

The rest of this section will be devoted to the proof of Theorem 1.1.

1.2. Definitions and notations. We will use the following definitions and notations.

1. If $\Omega \subseteq M$ is an open subset then $\mathscr{D}^{p,q}_\Omega$ will denote the space of smooth (p,q)-forms with values in $K_M^{\otimes\nu}$ and compact support in Ω. $\mathscr{D}^{p,q}_\Omega$ is a pre-Hilbert space in the usual way; we let $\mathscr{L}^{p,q}_\Omega$ denote the Hilbert space completion of $\mathscr{D}^{p,q}_\Omega$.

2. We define the Dirichlet form $Q^{p,q}_\Omega$ to be the densely defined quadratic form on $\mathscr{L}^{p,q}_\Omega$ obtained by taking the form closure (cf. [17]) of the form

$$\mathscr{D}^{p,q}_\Omega \ni f \mapsto \|\bar\partial f\|^2 + \|\vartheta f\|^2,$$

where ϑ is the formal adjoint of $\bar\partial$. In the case that Ω is a smoothly bounded relatively compact domain, the Rellich lemma implies that $Q^{p,q}_\Omega$ has discrete spectrum.

3. For the case that $\Omega = M$, there is a second way of defining $Q^{p,q}_M$, namely as the $\bar\partial$-Neumann form; this second way turns out to be equivalent to the first way due to the completeness of M (roughly speaking, the $\bar\partial$-Neumann boundary conditions get pushed to infinity). More precisely, we replace

$$\bar\partial\colon \mathscr{D}^{*,*}_M \to \mathscr{D}^{*,*}_m$$

by its graph closure to get a closed densely defined operator $\bar\partial\colon \mathscr{L}^{*,*}_M \to \mathscr{L}^{*,*}_M$ (which acts in the sense of distributions). We let $\bar\partial^*\colon \mathscr{L}^{*,*}_M \to \mathscr{L}^{*,*}_M$ be the Hilbert space adjoint of $\bar\partial$. $\bar\partial^*$ is also closed and densely defined. Then we define $Q^{p,q}_M$ as follows:

$$\mathrm{Dom}\, Q^{p,q}_M = \mathrm{Dom}\,\bar\partial \cap \mathrm{Dom}\,\bar\partial^* \cap \mathscr{L}^{p,q},$$
$$Q^{p,q}_M(f) = \|\bar\partial f\|^2 + \|\bar\partial^* f\|^2.$$

For the equivalence of this definition with the one given in 2 above, please see [10, Density Lemma, p. 89]; the basic idea is to use partitions of unity and Friedrich's mollifers to show that $\mathscr{D}_M^{p,q}$ is dense in $\operatorname{Dom} Q_M^{p,q}$ relative to the graph norm $f \mapsto \|f\|^2 + \|\bar\partial f\|^2 + \|\bar\partial^* f\|^2$. Associated to the form $Q_M^{p,q}$ is the selfadjoint densely defined operator $\Box$, the $\bar\partial$-Neumann Laplacian, defined by

$$\operatorname{Dom}\Box = \{f \in \operatorname{Dom}\bar\partial^* \cap \operatorname{Dom}\bar\partial \| \bar\partial f \in \operatorname{Dom}\bar\partial^* \text{ and } \bar\partial^* f \in \operatorname{Dom}\bar\partial\},$$
$$\Box f = \overline{\partial\partial}^* f + \bar\partial^*\,\bar\partial f$$

(compare [8, Proposition 1.3.8, p. 17]).

1.3. Denote by $N_\Omega^{p,q}(\lambda)$ the number of eigenvalues of $Q_\Omega^{p,q}$, counted with multiplicity, that are $\leq \nu\lambda$ (note the factor ν). If Ω is not relatively compact, then $Q_\Omega^{p,q}$ need not have discrete spectrum; in that case $N_\Omega^{p,q}(\lambda)$ can be defined as the dimension of a certain spectral projection. However due to Proposition 1.2 below, we need not be concerned with this extended definition.

1.4. Demailly's generalization of Weyl's formulas for the asymptotic spectrum. In the case that Ω is a smoothly bounded relatively compact domain in M, Demailly [6] has already computed $N_\Omega^{p,q}(\lambda)$ asymptotically as $\nu \to \infty$ for $\lambda \in (0, \infty)\backslash$(a countable set). We shall not need the full statement of his result. Rather, we shall be content with the following special case:

Proposition 1.1. *For $\lambda > 0$, we have*

$$\liminf_{\nu\to\infty} \frac{N_\Omega^{0,0}(\lambda)}{\nu^m} \geq \frac{1}{m!}\int_\Omega c_1(K_M)^m.$$

1.5. Absence of spectrum.

Proposition 1.2. *$Q_M^{0,0}$ has no spectrum in the interval $(0, \nu/2)$ (for $\nu \geq 2$). Hence*

$$\dim H_{(2)}^0(M, K^{\otimes\nu}) = N_M^{0,0}(0) = N_M^{0,0}\left(\tfrac{1}{3}\right)$$

for all $\nu \geq 2$.

Proof. First we look at $(0,1)$-forms. We claim that

$$Q_M^{0,1}(f) \geq (\nu - 1)\|f\|^2 \geq \frac{\nu}{2}\|f\|^2 \quad \text{for } \nu \geq 2 \tag{1}$$

for all $f \in \operatorname{Dom} Q_M^{0,1}(f)$. It suffices to verify inequality (1) for smooth compactly supported f, in which case the result is an immediate consequence of the following $\overline{\nabla}$-Bochner-Kodaira formula (cf. [20, Formula 1.3.3, p. 631]):

$$(\Box f, f) = \|\overline{\nabla} f\|^2 + (\nu - 1)(-\operatorname{Ricci}(f), f),$$

where $\mathrm{Ricci}(f) = R_{\bar\alpha}{}^{\bar\beta} f_{\bar\beta}$. Next, if $f \in \mathrm{Dom}\,\Box \cap \mathscr{L}_M^{0,0}$ we have

$$\begin{aligned}(\Box f, \Box f) &= (\bar\partial^*\psi, \bar\partial^*\psi) \quad \text{where } \psi = \bar\partial f \\ &= Q_M^{0,1}(\psi) \geq \frac{\nu}{2}\|\psi\|^2 \quad \text{by (1)} \\ &= \frac{\nu}{2}(\Box f, f).\end{aligned} \tag{2}$$

The proposition now follows.

1.6. Completion of the proof of Theorem 1.1. Let $\Omega \subset M$ be a smoothly bounded relatively compact domain. Then

$$\begin{aligned}\dim H^0_{(2)}(M, K_M^{\otimes\nu}) &= N_M^{0,0}(0) = N_M^{0,0}\left(\tfrac{1}{3}\right) \quad \text{by Proposition 1.2.} \\ &\geq N_\Omega^{0,0}\left(\tfrac{1}{3}\right) \quad \text{by the min-max principle} \\ &\geq \frac{\nu^m}{m!}\int_\Omega c_1(K_M)^m + o(\nu^m) \quad \text{by Proposition 1.1.}\end{aligned}$$

(For the min-max principle, see [17, Theorem XIII.2, p. 78, or Proposition 4, p. 270].) Therefore

$$\liminf_{\nu\to\infty} \nu^{-m} \dim H^0_{(2)}(M, K_M^{\otimes\nu}) \geq \frac{1}{m!}\int_\Omega c_1(K_M)^m. \tag{3}$$

Now let Ω expand toward all of M to get the desired result. The proof of Theorem 1.1 is now complete.

2. Compactification

In this section we shall prove Theorem 0.1.

2.1. Pseudoconcavity.

Definition 2.1. Let M be a complex manifold of dimension m. M is said to be very strongly $(m-q)$-pseudoconcave if there exists an infinite continuous exhaustion v whose complex Hessian $\sqrt{-1}\,\partial\bar\partial\nu$ is seminegative and has at least q negative eigenvalues outside of a compact subset of M in the sense of distributions. We say that v is a very strongly $(m-q)$-pseudoconcave exhaustion of M.

Let (M, ω) be the complete Kähler manifold in Theorem 0.1.

Lemma 2.1. *There exists a positive integer ν_0 such that the linear system $|\nu_0 K_M|$ embeds M as an open subset of a projective variety M^*.*

Proof (compare [3]). Using L^2-estimates on a complete Kähler manifold as in [21], we see that $\bigoplus_{\nu\geq 0} H^0(M, K_M^{\otimes\nu})$ separates points on M and forms local coordinates. Let v be a very strongly $(m-2)$-pseudoconcave exhaustion of M as in Definition 2.1. Take a sufficiently large positive number c and let $M_c = \{p \in M \mid v < c\}$. Then we may assume that $\sqrt{-1}\,\partial\bar\partial v$ has at least two negative eigenvalues on $M - M_c$. In particular there exists no divisor in M contained in

$M - M_c$. There exists a positive integer ν_1 such that the meromorphic map $\Phi\colon M \to \mathbf{P}^N$ associated with $|\nu_1 K_M|$ embeds M_c into a projective space. Then since $\operatorname{transdeg}_{\mathbf{C}} \oplus_{\nu \geq 0} H^0(M, K_M^{\otimes \nu}) \leq \dim M$ by the pseudoconcavity of M (cf. [1]) there exists a projective compactification M^p of $\Phi(M_c)$ in $\mathbf{P}^N$. We may assume that M^p is normal. Then the morphism $\Phi\colon M - Bs\Phi \to M^p$ is a biholomorphism onto its image. Let $\tilde{\Phi}\colon \widetilde{M} \to M^p$ be a resolution of the base locus and let $\widetilde{M}^p$ be the image of $\tilde{\Phi}$. By Noetherian induction, taking a positive multiple of ν_1 instead of ν_1 if necessary, we may assume that $\widetilde{M}^p$ is nonsingular. Let us consider the meromorphic map $\tilde{\Phi}^{-1}\colon \widetilde{M}^p \to M$ (this map is meromorphic because $(\Phi\,|_{M-Bs\Phi})_* v$ extends to a pluri-superharmonic exhaustion of $\widetilde{M}^p$ because v is locally constant on $Bs\Phi$). Since there exists no rational curve contained in M, $\tilde{\Phi}^{-1}$ is a morphism (cf. [9, Proposition 1, p. 113]). We note that $M^p - \widetilde{M}^p$ contains only a finite number of divisors because Φ is a biholomorphism onto its image on M_c and M_c is pseudoconcave. This implies that $Bs\Phi$ consists of only finitely many irreducible components. By Noetherian induction, we obtain the lemma. q.e.d.

We may assume that M^* is irreducible. Let us take a resolution $\mu\colon \overline{M} \to M^*$ such that the maximal divisor contained in the complement of the image of M in $\overline{M}$ has normal crossings. D exists because M is very strongly $(m-2)$-pseudoconcave. We shall identify M with its image in $\overline{M}$.

We shall fix a Hermitian metric on $\overline{M}$.

Lemma 2.2. *$\overline{M} - M$ has measure* 0.

Proof. We note that $\overline{M} - M$ is a pluripolar set by the assumption. In particular $\overline{M} - M$ has measure 0.

2.2. Construction of Kähler-Einstein metrics. Let H be an ample divisor on $\overline{M}$ such that

(a) $H + D$ is a divisor with normal crossings,

(b) $K_{\overline{M}} + D + H$ is ample,

(c) the universal covering $\widetilde{M}_H$ of $M_H^* = \overline{M} - D - H$ is biholomorphic to a bounded domain in $\mathbf{C}^m$.

The third condition is satisfied if we choose H properly, thanks to a theorem of P. A. Griffiths [12]. After performing an embedded resolution of singularities and then adding on a sufficiently ample divisor in general position, we may assume that (a) and (b) hold. By [14, Theorem 1], there exists a complete Kähler-Einstein metric ω_E^* on M_H^* such that

$$\omega_E^* = -\frac{1}{2\pi} \operatorname{Ric}(\omega_E^*).$$

Let $\widetilde{M}_H$ be the inverse image of M by the covering map $\pi\colon \widetilde{M}_H^* \to M_H^*$.

Lemma 2.3. *$\widetilde{M}_H$ is a bounded pseudoconvex domain in* $\mathbf{C}^m$.

Proof. Clearly $\widetilde{M}_H$ is a bounded domain in $\mathbf{C}^m$, since it is contained in $\widetilde{M}_H^*$. We must show that $\widetilde{M}_H$ is a domain of holomorphy. We note that the universal cover of $\widetilde{M}_H$ is Stein because the universal cover of M is Stein (by assumption) and H is ample. Then the lemma follows from the following basic fact: If a domain in $\mathbf{C}^m$ has Stein universal cover then it is a domain of holomorphy.

This fact follows easily from the characterization of domains of holomorphy (among all domains) in terms of extensions of mappings of Hartogs figures (cf. [15, p. 254, exercise 10]). q.e.d.

By [16, Main Theorem] and Lemma 2.3, we have the following lemma immediately.

Lemma 2.4. *There exists a complete Kähler-Einstein metric ω_E on M_H such that*

$$\omega_E = -\frac{1}{2\pi}\operatorname{Ric}(\omega_E).$$

2.3. Comparison of the growth of L^2-plurigenera.

Lemma 2.5. *(M_H, ω_E) has finite volume.*

Proof. Take a polydisk

$$\Delta^m = \{(z^1, \cdots, z^m) \in \mathbf{C}^m \mid |z^i| < 1, 1 \le i \le m\}$$

in M such that

$$\Delta^m \cap H = \{z^1, \cdots, z^m) \mid z^1 \cdots z^k = 0\}.$$

We consider the complete Kähler metric

$$\omega_P = \sum_{i=1}^{k} \sqrt{-1}\frac{dz^i \wedge d\bar{z}^i}{|z^i|^2(\log|z^i|^2)^2} + \sum_{i=k+1}^{m} \sqrt{-1}\frac{dz^i \wedge d\bar{z}^i}{(1-|z^i|^2)^2}$$

on $(\Delta^*)^k \times \Delta^{m-k} \subset \Delta^m$. Since ω_E is a Kähler-Einstein metric of negative Ricci curvature, applying Yau's Schwarz lemma [22] to the inclusion

$$((\Delta^*)^k \times \Delta^{m-k}, \omega_P) \hookrightarrow (M_H, \omega_E),$$

we see that there exists a positive constant C such that $\omega_E^m \le C\omega_P^m$. Then the Poincaré growth of ω_p gives that

$$H^0_{(2)}(M_H, K_{M_H}^{\otimes\nu}) \subset H^0(M, K_M^{\otimes\nu} \otimes H^{\otimes\nu-1}).$$

Since M is pseudoconcave, there exists a constant C_H such that

$$\dim H^0(M, K_M^{\otimes\nu} \otimes H^{\otimes\nu-1}) \le C_H\nu^m$$

[1, Proposition 11]. Thus by Theorem 1.1, we have

$$\int_{M_H} \omega_H^m \le m!C_H,$$

which completes the proof of the lemma.

Lemma 2.6.

$$H^0_{(2)}(M_H, K_{M_H}^{\otimes\nu}) \subseteq H^0(\overline{M}, K_{\overline{M}}^{\otimes\nu} \otimes H^{\otimes\nu-1} \otimes D^{\otimes\nu-1}).$$

Proof. Let η be an element of $H^0_{(2)}(M_H, K_{M_H}^{\otimes\nu})$. Then by the Hölder inequality, we have

$$\begin{aligned}\int_{M_H} (\eta \wedge \bar{\eta})^{\frac{1}{\nu}} &= \int_{M_H} \frac{(\eta \wedge \bar{\eta})^{\frac{1}{\nu}}}{\omega_E^m} \omega_E^m \leq \left(\int_{M_H} \frac{\eta \wedge \bar{\eta}}{(\omega_E^m)^\nu} \omega_E^m\right)^{\frac{1}{\nu}} (\mathrm{Vol}\, M_H)^{\frac{\nu-1}{\nu}} \\ &= \|\eta\|^{\frac{2}{\nu}} (\mathrm{Vol}\, M_H)^{\frac{\nu-1}{\nu}} < \infty.\end{aligned}$$

Therefore, if we can show that η extends to a meromorphic section over $\overline{M}$ of $K_{\overline{M}}^{\otimes\nu}$, it will follow from the fact that $\overline{M} - M$ has measure 0 (Lemma 2.1) together with [18, p. 243, Theorem 2.1] that

$$\eta \in H^0(\overline{M}, K_{\overline{M}}^{\otimes\nu} \otimes H^{\otimes\nu-1} \otimes D^{\otimes\nu-1}).$$

Now we show that η extends to a meromorphic section over $\overline{M}$ of $K_{\overline{M}}^{\otimes\nu}$. Note that η is meromorphic on M by [18, p. 243, Theorem 2.1]. Let ψ be any nontrivial meromorphic section over $\overline{M}$ of $K_{\overline{M}}^{\otimes\nu}$. Then η/ψ is a meromorphic function on M, and we are reduced to proving the following claim:

Claim. Every meromorphic function on M extends to a meromorphic function on $\overline{M}$ (cf. [3, Proposition 10, p. 103]).

Proof of Claim. Let $\mathscr{M}(M)$ (resp. $\mathscr{M}(\overline{M})$) denote the field of meromorphic functions on M (resp. $\overline{M}$). Restriction gives an inclusion $\mathscr{M}(\overline{M}) \hookrightarrow \mathscr{M}(M)$; it is our task to show this map is surjective. By pseudoconcavity, $\mathscr{M}(M)$ has transcendence degree m over $\mathbf{C}$, as does $\mathscr{M}(\overline{M})$ since $\overline{M}$ is projective algebraic. Hence every element of $\mathscr{M}(M)$ is algebraic over $\mathscr{M}(\overline{M})$. It is now elementary that $\mathscr{M}(M) = \mathscr{M}(\overline{M})$. This completes the proof of the Claim and the lemma.

By Yau's Schwarz lemma, we see that

$$(\omega_E^*)^m \leq \omega_E^m. \tag{4}$$

On the other hand, by construction,

$$\int_{M_H^*} (\omega_E^*)^m = (K_{\overline{M}} + D + H)^m.$$

Hence

$$\int_{M_H} \omega_E^m \geq (K_{\overline{M}} + D + H)^m, \tag{5}$$

since $\overline{M} - M$ has measure 0. We note that $K_{\overline{M}} + D + H$ is an ample divisor. Then by Lemma 2.6 and Theorem 1.1, we obtain

$$\int_{M_H} \omega_E^m \leq (K_{\overline{M}} + D + H)^m. \tag{6}$$

By (5) and (6), we obtain

$$\int_{M_H} \omega_E^m = \int_{M_H} (\omega_E^*)^m = (K_{\overline{M}} + D + H)^m,$$

since $\overline{M} - M$ is measure 0. Then by (4), we find $\omega_E^m = (\omega_E^*)^m$ on M_H. Since ω_E and ω_E^* are Kähler-Einstein metrics, $\omega_E = \omega_E^*$ on M_H. Thus the completeness of ω_E yields that $M_H = M_H^*$. Since H is an arbitrary ample divisor satisfying conditions 2.2(a), (b), (c), we conclude that $M = \overline{M} - D$. This completes the proof of Theorem 0.1.

By the same argument as above, we obtain the following theorem.

Theorem 2.1. *Let M be a quasiprojective manifold with complete Kähler metric ω such that $\mathrm{Ric}_\omega \leq -\omega$. Then M is of log-general type.*

3. Compactification of Kähler manifolds with cusps

In this section we shall give a differential geometric condition for a complete Kähler manifold of finite volume to be quasiprojective.

Let M be a noncompact complete Riemannian manifold of finite volume. We assume that the sectional curvature of M is nonpositive and greater than or equal to -1. Let $\widetilde{M}$ be the universal covering of M and let Γ be the fundamental group of M. Then Γ acts on $\widetilde{M}$ as the group of covering transformations. Let d be the distance function on $\widetilde{M}$. For a point p on $\widetilde{M}$, we set

$$\delta(p) = \inf_{g \in \Gamma - \{1\}} d(p, g(p)). \tag{7}$$

δ is invariant under Γ and hence descends to a function on M which we also denote δ. For a point p of $\widetilde{M}$ and a positive number ε, we set

(8) $\Gamma_{\varepsilon,p}$ = the subgroup of Γ generated by $\{g \in \Gamma \mid d(p, g(p)) > \varepsilon\}$,

$$M_\varepsilon = \{p \in M \mid \delta(p) \leq \varepsilon\}. \tag{9}$$

We denote by $S_1, S_2, \cdots$ all components of M_ε and let $\widetilde{S}_i$ be the inverse image of S_i by the universal covering projection. Then it is easy to see that $\Gamma_{\varepsilon,p}$, to be denoted by Γ_i, is independent of the choice of $p \in \widetilde{S}_i$. We note that if we take ε smaller than Margulis' constant, every Γ_i is quasinilpotent by Margulis' lemma. Let $\gamma\colon [0, \infty) \to \widetilde{M}$ be a ray (parametrized by arclength). We define the Busemann function G_γ associated with γ by

$$B_\gamma(p) = \lim_{t \to \infty} (d(p, \gamma(t)) - t). \tag{10}$$

A level set of a Busemann function is said to be a *horosphere*. Let $\partial\widetilde{M}$ be the differential geometric ideal boundary of $\widetilde{M}$ (cf. [7]).

Definition 3.1. We say that S_i is a cusp if there exists a unique point on $\partial\widetilde{M}$ invariant under Γ_i such that every horosphere centered at the point is invariant under Γ_i.

Definition 3.2. Let M be as above. We say that M has only cusps if there exists a positive number ε such that every S_i is a cusp.

It is well known that if $\widetilde{M}$ satisfies the visibility axiom or has negative curvature, then M has only cusps (cf. [7], [13]).

Now let us return to the case in which M is a Kähler manifold.

Definition 3.3. A Kähler manifold (M, ω) is said to have very strongly 2-negative sectional curvature if the sectional curvature of M is nonpositive and

$$\operatorname{Ric}(X, X) - R(X, Y, X, Y)$$

is negative for every pair of unit tangent vector X, Y with $X \perp Y$, where R denotes the curvature tensor of M.

For example every Kähler manifold of dimension ≥ 2 with negative curvature has very strongly 2-negative curvature, and a hermitian bounded symmetric domain without one-dimensional de Rham factors has very strongly 2-negative curvature.

The main result of this section is the following.

Theorem 3.1. *Let (M, ω) be a complete Kähler manifold of finite volume which has only cusps. Assume that M has very strongly 2-negative curvature. Then M is biholomorphic to a quasiprojective manifold.*

We note that the universal cover of M is Stein by [11, Proposition 1.17, p. 15]. Then Theorem 3.1 follows from the following proposition together with Theorem 0.1.

Proposition 3.1. *Let W be a simply connected complete Kähler manifold. Assume that W has very strongly 2-negative curvature. Then for every ray γ, B_γ is very strongly $(m-2)$-plurisubharmonic, i.e. the complex Hessian of B_γ has at least two positive eigenvalues everywhere on M in the sense of distributions.*

Proof. This proposition follows immediately from the Hessian formula for the distance function as in [21, p. 364, Proposition 1] (the formula is on p. 365).

Note added in proof. The authors sincerely thank Professor Mok for pointing out an error in an earlier version of this paper, which necessitated the addition of the third hypothesis to Theorem 0.1.

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