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A CURVATURE CHARACTERIZATION OF CERTAIN LOCALLY RANK-ONE SYMMETRIC SPACES

QUO-SHIN CHI

0. Introduction

It is an interesting question to what extent can the sectional curvatures of a Riemannian manifold M determine the metric, of which the simplest case is when all the sectional curvatures are equal; the space is then covered by a space form. Let $R(X,Y)Z$ be the curvature tensor. Define the *Jacobi operator* $K_v(\cdot), = R(\cdot, v)v$ for each vector $v \in SM$, the unit tangent bundle of M. The above statement can be rephrased in terms of K_v as: K_v has constant eigenvalue λ for all v iff M is of constant curvature λ.

The next simplest case to study is when K_v has constant eigenvalues (counting multiplicities) independent of v. We will call this case *Condition* $[O]$ from now on.

It is immediate to see that Condition $[O]$ is satisfied by spaces covered by a *two-point homogeneous space* which is a space whose isometry group is transitive on the unit sphere bundle; such nonflat spaces have long been known to be *locally rank-one symmetric*, i.e., $\nabla R = 0$, where ∇ is the connection, with all positive or all negative sectional curvatures (cf. [9], [16], [17], or [7] for a simple geometric proof). It is very natural to wonder about the converse, namely,

Conjecture. *A Riemannian manifold M is locally rank-one symmetric if Condition $[O]$ is satisfied.*

In this paper we will give a positive answer to the conjecture if $\dim M =$ an odd number, 2, 4, or $2(2k+1)$. More precisely we can show, as is done in §§1-3 and 5, that

Theorem 0. *Suppose Condition $[O]$ is satisfied. Then*

1. *If $\dim M =$ an odd number, then M has constant sectional curvature.*

2. *If $\dim M = 2, 4,$ or $2(2k+1)$, then M has either constant curvature or is covered by a standard complex projective space or its noncompact dual.*

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3. *If M is nonflat and Kählerian, then M has constant holomorphic sectional curvature if the sectional curvatures are all nonpositive or all nonnegative.*

As a corollary of this theorem we recover in §4 the well-known fact that a 4-dimensional nonflat locally harmonic manifold is locally rank-one symmetric (Theorem 3). The advantage of our proof is that it is geometric without resorting to the classification theory of symmetric spaces to see that the space is covered by $\mathbf{C}P^2$ or its dual. In addition, we show that Condition $[O]$ precisely characterizes compact rank-one symmetric spaces among compact homogeneous spaces with the same topological type as the former ones (Theorem 4).

Originally this conjecture was proposed by Osserman who was motivated by his work in [12]. Actually define, for a compact M with negative curvature, two quantities

$$\alpha(M) = \int_{SM} \operatorname{tr}(-K_v)^{1/2}\, dv = \int_{SM} \sum_i (-\lambda_i(v))^{1/2}\, dv,$$

$$\beta(M) = \sum_i \left(- \int_{SM} \lambda_i(v)\, dv \right)^{1/2},$$

where the $\lambda_i(v)$ are the eigenvalues of K_v. Then Osserman and Sarnak showed in [12] that $\alpha(M) \leq h_\Phi$, where h_Φ is the metric entropy of the geodesic flow of M, and equality holds iff M is locally rank-one symmetric. Osserman further conjectured that $h_\Phi \leq \beta(M)$, with equality iff M is locally rank-one symmetric. Note that by Schwarz inequality $\alpha(M) = \beta(M)$ precisely when Condition $[O]$ is satisfied.

We remark here that in [1] Ballmann generalized the result in [12] to compact manifolds with nonpositive curvature. He also obtained an upper bound of h_Φ with equality iff M is locally symmetric. With this and the validity of the above conjecture in symmetric spaces (easily proved using root space decomposition) it follows that $\beta(M)$ cannot in general lie between $\alpha(M)$ and Ballmann's upper bound if the sectional curvatures are only assumed to be nonpositive (but still Osserman's conjecture on the upper bound of h_Φ for rank-one case might be true).

We also remark that since v is always an eigenvector of K_v with zero eigenvalue, then whenever we mention eigenvalues of K_v we always mean those different from this trivial one.

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1. The case dim = an odd number

This case follows directly from the following theorem about the maximal number k of nontrivial k-plane distributions on certain spheres (see [14, p. 155]).

Theorem 1. *The n-sphere does not admit a continuous k-plane distribution if n is even and $1 \le k \le n-1$, or if $n \equiv 1 \pmod 4$ and $2 \le k \le n-2$.*

In order to prove Theorem 0, we need one more fact in linear algebra.

Lemma 1. *Let T be an n by n matrix-valued function on $\mathbf{R}^n$, the Euclidean n-space, such that $\operatorname{rank}(T)$ is constant throughout $\mathbf{R}^n$. Then the kernel of T is a smooth distribution of dimension $n - \operatorname{rank}(T)$ over $\mathbf{R}^n$.*

Proof. Let $k = \operatorname{rank}(T)$, and let

$$T = (X_1, X_2, \cdots, X_n),$$

where the X_i's are n by 1 matrices. Without loss of generality, we may assume $X_1, X_2, \cdots, X_k$ are linearly independent in a neighborhood of a given point p. It follows that the X_i, $k+1 \le i \le n$, are linear combinations of the first k vectors because k is the rank of T. Let

$$X_i = \sum_{\alpha=1}^{k} f_i^\alpha X_\alpha, \qquad k+1 \le i \le n.$$

Then

$$c_{\beta i} = \sum_{\alpha=1}^{k} g_{\beta\alpha} f_i^\alpha, \qquad k+1 \le i \le n,\ 1 \le \beta \le k,$$

where $c_{\beta i} = (X_\beta, X_i)$, $g_{\beta\alpha} = (X_\beta, X_\alpha)$, and (,) is the standard inner product on $\mathbf{R}^n$. Hence f_i^α are smooth functions. Now define

$$Y_i = \sum_{\alpha=1}^{k} f_i^\alpha E_\alpha + E_i, \qquad k+1 \le i \le n,$$

where the E_i are standard basis elements. One then checks easily that Y_i are smooth linearly independent vector fields spanning the kernel of T around the point p.

Proof of Theorem 0. Given p in M, let US_p be the unit sphere in M_p, the tangent space at p. For v in US_p, K_v restricted to US_p is a smooth operator. Since K_v has universally constant eigenvalues, $K_v - cI$, c being an eigenvalue, must then have constant rank on US_p; therefore the eigenspace associated with

c defines a smooth distribution over US_p by the previous lemma. Theorem 1 then implies that the eigenspace associated with c must be the whole tangent space of US_p for all unit vectors because the dimension of US_p, being one less than the dimension of M, is even. This amounts to saying that M is of constant curvature, and hence certainly is locally rank-one symmetric.

2. The case $\dim = 2(2k+1)$

Lemma 2. *Among the $4k+1$ nontrivial eigenvalues of K_v, $4k$ of them must be equal.*

Proof. The unit sphere in a tangent space has dimension $= 4k+1 \equiv 1 \pmod 4$. Now apply Lemma 1 and Theorem 1.

Denote by c the eigenvalue with multiplicity 1, and by b the one with multiplicity $4k$.

Lemma 3. *If w is an eigenvector associated with c (or b) for K_v, then v is an eigenvector associated with c (or b) for K_w.*

Proof. All sectional curvatures lie between b and c, and they attain b or c precisely when v and w are eigenvectors relative to each other. q.e.d.

For simplicity in notation, we let $m = 2k+1$, so that the manifold has dimension $2m$.

Lemma 4. *Around each point there exists a frame $(X_1, \cdots, X_{2m})$ such that $R_{ijij} = b$ or c for $i \neq j$, and $R_{ijik} = 0$ for $j \neq k$.*

Proof. Given X_1, choose X_{1+m} so that X_{1+m} is an eigenvector of K_{x_1} with eigenvalue c (for subscript notation, x_1 stands for X_1, etc.). By Lemma 3 above X_1 is a eigenvector of $K_{x_{1+m}}$ with the same eigenvalue. Now the linear subspace perpendicular to X_1 and X_{1+m} is the eigenspace of K_{x_1} and $K_{x_{1+m}}$ with eigenvalue b. Pick X_2 in this subspace and pick X_{2+m} such that X_{2+m} is the eigenvector of K_{x_2} with eigenvalue c. By Lemma 3 again X_1 and X_{1+m} belong to the eigenspace of K_{x_2} with eigenvalue b; hence it follows that X_{2+m}, in addition to X_2, belongs to the eigenspace of K_{x_1} and $K_{x_{1+m}}$ with eigenvalue b. Then choose X_3 perpendicular to $X_1, X_{1+m}, X_2, X_{2+m}$, and choose X_{3+m} such that X_{3+m} is an eigenvector of K_{x_3} with eigenvalue c. For exactly the same reason as before we know X_{3+m}, besides X_3, belongs to the eigenspace of $K_{x_1}, K_{x_{1+m}}, K_{x_2}$, and $K_{x_{2+m}}$ with eigenvalue b. Continuing in this fashion, we obtain a basis such that each X_j is an eigenvector of K_{x_i}; thus $R_{ijij} = b$ or c for $i \neq j$. Finally $R_{ijik} = (K_{x_i}(X_j), X_k) = 0$ if $j \neq k$.

Lemma 5. *Relative to the basis in the previous lemma we have $R(X_i, X_j)X_k = 0$ where $i-j$, $i-k$, $j-k \not\equiv 0 \pmod m$.*

Proof. By assumption $K_{x_i}(X_j) = bX_j$, and $K_{x_i}(X_k) = bX_k$, so that $K_{x_i}(W) = bW$, where $W = (X_j + X_k)/\sqrt{2}$. Hence $K_w(X_i) = bX_i$ by Lemma

3 above, i.e.,

$$\begin{aligned} bX_i &= R(X_i, W)W \\ &= \tfrac{1}{2}[K_{x_j}(X_i) + K_{x_k}(X_i) + R(X_i, X_j)X_k + R(X_i, X_k)X_j] \\ &= \tfrac{1}{2}[2bX_i + R(X_i, X_j)X_k + R(X_i, X_k)X_j]. \end{aligned}$$

It follows that

$$R(X_i, X_j)X_k = -R(X_i, X_k)X_j. \tag{2.1}$$

The same relation holds if we cyclicly permute i, j, k. Now the Bianchi identity says

$$\begin{aligned} 0 &= R(X_i, X_j)X_k + R(X_k, X_i)X_j + R(X_j, X_k)X_i \\ &= R(X_i, X_j)X_k - R(X_k, X_j)X_i + R(X_j, X_k)X_i \end{aligned}$$

by (2.1), so

$$\begin{aligned} R(X_i, X_j)X_k &= -2R(X_j, X_k)X_i = 2R(X_j, X_i)X_k \\ &= -2R(X_i, X_j)X_k. \end{aligned}$$

Consequently $R(X_i, X_j)X_k = 0$, proving the lemma.

The only curvature components left to be determined are $R_{i,i+m,j,j+m}$. In fact

Lemma 6. $R_{i,j,i+m,j+m} = R_{i,j+m,j,i+m} = \pm(b-c)/3$, $1 \le i \ne j \le m$.

Proof. First note that

$$\begin{aligned} R(X_i, X_j)X_{j+m} &= -R(X_i, X_{j+m})X_j, \\ R(X_j, X_i)X_{i+m} &= -R(X_j, X_{i+m})X_i, \end{aligned} \tag{2.2}$$

for the same reason as used in deriving (2.1).

Consider $W = (X_i + X_j)/\sqrt{2}$, $\overline{W} = (X_i - X_j)/\sqrt{2}$, $U = (X_{i+m} + X_{j+m})/\sqrt{2}$, and $\overline{U} = (X_{i+m} - X_{j+m})/\sqrt{2}$. Certainly $W, \overline{W}, U, \overline{U}$, and the X_k's, $k \ne i, j, i+m, j+m$, form an orthonormal basis; furthermore it is directly checked that the above X_k's and $\overline{W}$ are eigenvectors of K_w with eigenvalue b. To determine $K_w(U)$ and $K_w(\overline{U})$, one first easily deduces

$$\begin{aligned} K_w(U) = \frac{b+c}{2}U &+ [R(X_{j+m}, X_i)X_j + R(X_{i+m}, X_j)X_i]/2\sqrt{2} \\ &+ [R(X_{j+m}, X_j)X_i + R(X_{i+m}, X_i)X_j]/2\sqrt{2}. \end{aligned}$$

Then one observes by Lemmas 4 and 5 above that $R(X_{j+m}, X_i)X_j$ and $R(X_{j+m}, X_j)X_i$ are multiples of X_{i+m}, and $R(X_{i+m}, X_j)X_i$ and $R(X_{i+m}, X_i)X_j$ are multiples of X_{j+m}, i.e.,

$$\begin{aligned} R(X_{j+m}, X_i)X_j &= R_{i+m, j, j+m, i}X_{i+m} \\ &= R_{i+m, j+m, i, j}X_{i+m} \quad \text{by (2.2)}; \\ R(X_{i+m}, X_j)X_i &= R_{j+m, i, i+m, j}X_{j+m} \\ &= R_{i, j, i+m, j+m}X_{j+m} \quad \text{by (2.2)}. \end{aligned}$$

In particular,

$$R(X_{j+m}, X_i)X_j + R(X_{i+m}, X_j)X_i = R_{i,j,i+m,j+m}(X_{i+m} + X_{j+m});$$

similarly

$$\begin{aligned} R(X_{j+m}, X_j)X_i + R(X_{i+m}, X_i)X_j &= R_{i,i+m,j,j+m}(X_{i+m} + X_{j+m}) \\ &= 2R_{i,j,i+m,j+m}(X_{i+m} + X_{j+m}), \end{aligned}$$

where the last equality is obtained by the Bianchi identity and (2.2). Therefore

$$K_w(U) = \left(\frac{b+c}{2} + \frac{3}{2}R_{i,j,i+m,j+m}\right)U. \tag{2.3}$$

Similarly

$$K_w(\overline{U}) = \left(\frac{b+c}{2} - \frac{3}{2}R_{i,j,i+m,j+m}\right)\overline{U}. \tag{2.4}$$

In view of the fact that $\overline{W}$ and X_k, $k \neq i, j, i+m, j+m$ are eigenvectors with eigenvalue b, and the fact that c has multiplicity one, we see that one of the coefficients in (2.3) and (2.4) must be b, i.e.,

$$R_{i,j,i+m,j+m} = \pm(b-c)/3.$$

Finally $R_{i,j+m,j,i+m} = R_{i,j,i+m,j+m}$ by (2.2), proving the lemma.

Lemma 7. *With the moving frame above, one has*

$$\begin{aligned} &R_{ijij;h} = 0, \\ &R_{ijik;h} = 0 \quad \text{if } j,\ k \neq i+m, \\ &R_{i,j,i+m,j+m;h} = 0, \\ &R_{ijkl;h} = 0, \end{aligned}$$

where the difference between any two subscripts in the last equation $\neq 0$ $(\operatorname{mod} m)$.

Proof. Direct verification using (3.6) and Lemmas 4–6.

We need one more lemma on symmetric spaces.

Lemma 8. *Let M be a Riemannian manifold. If for every geodesic $r(t)$ the operator K_v with $v = \dot{r}(t)$ is parallel, then the entire curvature tensor is parallel along $r(t)$ and M is locally symmetric.*

Proof. See [12].

Proof of Theorem 0. Choose any $p \in M$ and a neighborhood U of p. Consider $U \setminus p$ on which one defines the unit vector X_1 to be the tangents along each geodesic emanating from p, and then choose a frame X_1, X_{1+m}, $X_2, X_{2+m}, \cdots$ according to Lemma 4. In view of Lemma 8, one needs only to show $R_{1k1l;1} = 0$, or equivalently $(\nabla_{X_1}R)(\cdot, X_1)X_1 = 0$; for then $\nabla_{X_1}[R(W, X_1)X_1] = 0$ if W is parallel along any geodesic emanating from p, i.e., K_{x_1} is parallel. In fact we only have to check $R_{1,1+m,1,l;h} = 0$ for all l by Lemma 7 above.

Observe further that we may assume in Lemma 6 that

$$R_{1,s,1+m,s+m} = R_{1,s+m,s,1+m} = (c-b)/3 \tag{2.5}$$

by changing X_s into $-X_s$ if necessary. Now (3.6) implies

$$\begin{aligned}\sum_h R_{1,1+m,1,s;h}\theta^h &= -\sum_h R_{s,s+m,s,1;h}\theta^h \\ &= (b-c)(\omega_s^{1+m} - \omega_1^{s+m}).\end{aligned} \tag{2.6}$$

Hence $R_{1,1+m,1,s;h} = -R_{s,s+m,s,1;h}$ for all h. In particular

$$R_{1,1+m,1,s;1} = -R_{s,s+m,s,1;1}. \tag{2.7}$$

Now the Bianchi identity applied to $R_{s,1,s,s+m;1}$ implies

$$R_{s,s+m,s,1;1} = R_{s,1,s,s+m;1} = -R_{s,1,1,s;s+m} - R_{s,1,s+m,1;s} = 0$$

by Lemma 7. Therefore (2.7) says $R_{1,1+m,1,s;1} = 0$, proving that the manifold is symmetric. q.e.d.

We remark here that dimension $= 2(2k+1)$ is used only to show that the operator K_v has exactly two different constant eigenvalues, one of them having multiplicity one. Thus we have in fact proved

Theorem 2. *Suppose K_v has only two distinct constant eigenvalues, one of them having multiplicity one. Then the manifold must be the complex projective space or its noncompact dual if it is simply connected.*

Proof. The space must be globally symmetric. Let c be the eigenvalue with multiplicity one, and for each unit vector X let JX be the unit eigenvector of K_x with eigenvalue c such that JX form a continuous vector field on the unit sphere bundle. Then J defines a complex structure and the space is Kählerian under J by (2.6) essentially. Now the result follows by noting that c is precisely the holomorphic sectional curvature.

3. The case dim = 4

This is the first dimension where any theorem like Theorem 1 would fail to give us any information on the possible eigenvalue distribution of K_v, due to the fact that continuous k-plane distributions for $k = 1, 2$ do exist on S^3.

Let a, b, c be the nontrivial eigenvalues for K_v. Around each point pick a smooth vector field X_1. By Lemma 1 there exist smooth vector fields X_2, X_3, X_4 such that they are the eigenvectors of K_{x_1} associated with a, b, c respectively. Denote by

$$a_{ij} = (R(X_i, X_j)X_j, X_i)$$

the sectional curvature of the plane spanned by X_i and X_j. One has

$$a_{12} = (K_{x_1}(X_2), X_2) = a,\quad a_{13} = b,\quad a_{14} = c. \tag{3.1}$$

By assumption K_{x_i}, $i = 2,3,4$, also has a, b, c, as eigenvalues; in particular $\operatorname{tr} K_{x_i} = a + b + c$. Since $\operatorname{tr} K_{x_i} = \sum_{j=1}^{4} a_{ij}$ we have

$$\begin{aligned} a_{21} + a_{23} + a_{24} &= a + b + c, \\ a_{31} + a_{32} + a_{34} &= a + b + c, \\ a_{41} + a_{42} + a_{43} &= a + b + c, \end{aligned}$$

which, together with (3.1), implies

$$\begin{aligned} &a_{12} = a,\ a_{13} = b,\ a_{14} = c, \\ &a_{23} = c,\ a_{24} = b,\ a_{34} = a. \end{aligned} \tag{3.2}$$

Now $a^2 + b^2 + c^2 = \operatorname{tr} K_v^2 = (R(X_i, v)v, R(X_i, v)v)$; one finds

$$\begin{aligned} a^2 + b^2 + c^2 = \operatorname{tr} K_{x_k}^2 &= \sum_i (R(X_i, X_k)X_k, R(X_i, X_k)X_k) \\ &= \sum_{i,s} R_{iksk}^2 = \sum_i R_{ikik}^2 + \sum_{i \neq s} R_{iksk}^2 \\ &= a^2 + b^2 + c^2 + \sum_{i \neq s} R_{iksk}, \end{aligned}$$

where the first term in the last equality follows from (3.2). Hence $R_{iksk} = 0$ if $i \neq s$; this, together with (3.2), implies

$$R_{ikjk} = a_{ik}\delta_{ij}. \tag{3.3}$$

The only R_{ijkl}'s remaining to be determined are R_{1234} and R_{1423}, and then $R_{1342} = -(R_{1234} + R_{1423})$. Set $x = R_{1234}$, $y = R_{1423}$, $R_{1342} = -x - y$. Now let $v = (X_2 + X_3)/\sqrt{2}$. The fact that

$$\begin{aligned} \operatorname{tr} K_v^2 &= a^2 + b^2 + c^2 \\ &= \sum_i \left(R\left(X_i, \frac{X_2 + X_3}{\sqrt{2}}\right) \frac{X_2 + X_3}{\sqrt{2}}, R\left(X_i, \frac{X_2 + X_3}{\sqrt{2}}\right) \frac{X_2 + X_3}{\sqrt{2}} \right) \end{aligned}$$

shows, using (3.3), that

$$(2x + y)^2 = (a - b)^2.$$

Similarly for $v = (X_1 + X_2)/\sqrt{2}$ one checks

$$(x + 2y)^2 = (b - c)^2,$$

and for $v = (X_2 + X_4)/\sqrt{2}$ one finds

$$(x-y)^2 = (c-a)^2.$$

Solving for x, y, one obtains

Case (1). a, b, c are different:

$$\left\{\begin{array}{l} x = \frac{b+c-2a}{3}, \\ y = \frac{b+a-2c}{3}, \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} x = \frac{2a-b-c}{3}, \\ y = \frac{2c-a-b}{3}. \end{array}\right. \tag{3.4}$$

Note that the second solution is the negative of the first one. Using the transformation $X_1 \to -X_1$ and $X_i \to X_i$, $i = 2, 3, 4$, one can always transform the second solution of (3.4) into the first one. Therefore

$$\begin{aligned} x = R_{1234} &= \frac{b+c-2a}{3}, \\ y = R_{1423} &= \frac{b+a-2c}{3}, \\ -x-y &= R_{1342}, \end{aligned}$$

and so

$$\begin{aligned} 2x + y &= b - a, \\ x + 2y &= b - c, \\ x - y &= c - a. \end{aligned}$$

Case (2). $a = c \neq b$: We have $x = y$, a special case of the first case. We have therefore determined all R_{ijkl}'s relative to the above chosen frame X_1, X_2, X_3, X_4. In summary

$$\begin{aligned} &a_{12} = a, \quad a_{13} = b, \quad a_{14} = c, \\ &a_{23} = c, \quad a_{24} = b, \quad a_{34} = a, \\ &R_{ikjk} = a_{ik}\delta_{ij}, \\ &R_{1234} = \frac{b+c-2a}{3} = x, \\ &R_{1423} = \frac{b+a-2c}{3} = y, \\ &R_{1342} = -x-y, \\ &2x + y = b - a, \quad x + 2y = b - c, \quad x - y = c - a. \end{aligned} \tag{3.5}$$

Now let $\theta^1, \theta^2, \theta^3, \theta^4$ be the dual forms of $X_1, \cdots, X_4$, and let ω_j^i be the connection forms. Denote by $R_{ijkl;h}$ the covariant derivatives of R_{ijkl}. One has

$$\begin{aligned} \sum_h R_{ijkl;h}\theta^h = d(R_{ijkl}) &- \sum_s R_{sjkl}\,\omega_i^s - \sum_s R_{iskl}\,\omega_j^s \\ &- \sum_s R_{ijsl}\,\omega_k^s - \sum_s R_{ijks}\,\omega_l^s. \end{aligned} \tag{3.6}$$

In particular, substituting (3.5) into (3.6) one gets

$$R_{ijkl;h} = 0 \quad \text{if } i, j, k, l \text{ are different}, \tag{3.7}$$

$$R_{ijij;h} = 0. \tag{3.8}$$

(3.6) also gives

$$-\sum_h R_{ijkj;h}\theta^h = a_{ij}\omega_k^i + a_{jk}\omega_i^k + \sum_s \omega_j^s R_{iskj} + \sum_s \omega_j^s R_{ijks}. \tag{3.9}$$

From (3.9) one checks

$$\omega_3^2(a-b) + \omega_1^4(a-b) = -\sum_h R_{2131;h}\theta^h = \sum_h R_{1242;h}\theta^h, \tag{3.10.1}$$

$$\omega_4^2(a-c) + \omega_3^1(a-c) = -\sum_h R_{1232;h}\theta^h = -\sum_h R_{2141;h}\theta^h, \tag{3.10.2}$$

$$\omega_4^3(b-c) + \omega_1^2(b-c) = -\sum_h R_{3141;h}\theta^h = \sum_h R_{1323;h}\theta^h. \tag{3.10.3}$$

We also have the second Bianchi identity

$$R_{ijkl;h} + R_{ijhk;l} + R_{ijlh;k} = 0, \tag{3.11}$$

and

$$\sum_k R_{ikjk;h} = 0, \tag{3.12}$$

since $\nabla R_{ij} = 0$ (e.g. $R_{1232;h} = -R_{1434;h}$).

Lemma 9.

$$\begin{aligned}
(\omega_3^2 + \omega_1^4)(a-b) &= \alpha\theta^1 + \beta\theta^2 + \gamma\theta^3 + \delta\theta^4, \\
(\omega_4^2 + \omega_3^1)(a-c) &= -\beta\theta^1 + \alpha\theta^2 + \delta\theta^3 - \gamma\theta^4, \\
(\omega_4^3 + \omega_1^2)(b-c) &= \gamma\theta^1 + \delta\theta^2 - \alpha\theta^3 - \beta\theta^4,
\end{aligned} \tag{3.13}$$

where $\alpha = -R_{2131;1}$, $\beta = -R_{2131;2}$, $\gamma = -R_{2131;3}$, $\delta = -R_{2131;4}$.

Proof.

$$\begin{aligned}
-R_{2141;3} &= R_{2114;3} \\
&= -R_{2131;4} \quad \text{by the Bianchi identity and (3.7)} \\
&= \delta,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
-R_{2141;4} &= -R_{1232;4} \quad \text{by (3.10.2)} \\
&= R_{1224;3} \quad \text{by Bianchi identity and (3.7)} \\
&= -R_{1242;3} = -\gamma \quad \text{by (3.10.1)},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
-R_{2141;1} &= R_{2343;1} \quad \text{by (3.12)} \\
&= -R_{2331;4} \quad \text{by Bianchi identity and (3.7)} \\
&= -R_{1332;4} \\
&= R_{1343;2} \quad \text{by Bianchi identity and (3.7)} \\
&= -\beta \quad \text{by (3.10.1) and (3.12)},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
-R_{2141;2} &= R_{1434;2} \quad \text{by (3.10.2) and (3.12)} \\
&= -R_{1442;3} \quad \text{by Bianchi identity and (3.7)} \\
&= R_{1424;3} = -R_{1323;3} \quad \text{by (3.12)} \\
&= R_{3141;3} \quad \text{by (3.10.3)} \\
&= -R_{3134;1} \quad \text{by Bianchi identity and (3.7)} \\
&= -R_{1343;1} = \alpha \quad \text{(3.10.1) and (3.12)}.
\end{aligned} \tag{3.17}$$

Thus we get the second identity of (3.13) by (3.14)–(3.17).

$$R_{2313;4} = -\beta \quad \text{by (3.16)}, \tag{3.18}$$

$$R_{2313;3} = -\alpha \quad \text{by (3.17)}, \tag{3.19}$$

$$\begin{aligned}
R_{2313;2} &= -R_{2321;3} \quad \text{by Bianchi identity and (3.7)} \\
&= -R_{3212;3} = -R_{2141;3} \quad \text{by (3.10.2)} \\
&= \delta \quad \text{by (3.14)},
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
R_{2313;1} &= -R_{3141;1} \quad \text{by (3.10.3)} \\
&= R_{3242;1} \quad \text{by (3.12)} \\
&= -R_{3221;4} \quad \text{by Bianchi identity and (3.7)} \\
&= R_{3212;4} = R_{2141;4} \quad \text{by (3.10.2)} \\
&= \gamma \quad \text{by (3.14)}.
\end{aligned} \tag{3.21}$$

Hence we get the third identity of (3.13) by (3.18)–(3.21), proving the lemma.

To prove the validity of Theorem 0, we need only to show the impossibility that a, b, c are different, then Theorem 2 implies the result. In fact, supposing they were different, by (3.13) one would get

$$\begin{aligned} d(\alpha\theta^1 + \beta\theta^2 + \gamma\theta^3 + \delta\theta^4) &= (a-b)\Omega_3^2 + (a-b)\Omega_1^4 \\ &\quad + \frac{a-b}{(a-c)(b-c)}(-\beta\theta^1 + \alpha\theta^2 + \delta\theta^3 - \gamma\theta^4) \\ &\quad \wedge (\gamma\theta^1 + \delta\theta^2 - \alpha\theta^3 - \beta\theta^4). \end{aligned} \tag{3.22}$$

Similarly, doing the same procedure for $d\omega_4^2$ and $d\omega_3^1$ yields

$$\begin{aligned} d(-\beta\theta^1 + \alpha\theta^2 + \delta\theta^3 - \gamma\theta^4) &= (a-c)\Omega_4^2 + (a-c)\Omega_3^1 \\ &\quad - \frac{a-c}{(a-b)(b-c)}(\alpha\theta^1 + \beta\theta^2 + \gamma\theta^3 + \delta\theta^4) \\ &\quad \wedge (\gamma\theta^1 + \delta\theta^2 - \alpha\theta^3 - \beta\theta^4). \end{aligned} \tag{3.23}$$

For $d\omega_4^3$ and $d\omega_1^2$ one has

$$\begin{aligned} d(\gamma\theta^1 + \delta\theta^2 - \alpha\theta^3 - \beta\theta^4) &= (b-c)\Omega_1^2 + (b-c)\Omega_4^3 \\ &\quad + \frac{b-c}{(a-b)(a-c)}(\alpha\theta^1 + \beta\theta^2 + \gamma\theta^3 + \delta\theta^4) \\ &\quad \wedge (-\beta\theta^1 + \alpha\theta^2 + \delta\theta^3 - \gamma\theta^4). \end{aligned} \tag{3.24}$$

Comparing the coefficients of $\theta^i \wedge \theta^j$ in (3.22)–(3.24) and using (3.5) give

$$\begin{aligned} A &= \alpha_{;4} - \delta_{;1} = -\frac{a-b}{(a-c)(b-c)}(\beta^2 + \gamma^2) - (a-b)(y-c), \\ B &= \beta_{;3} - \gamma_{;2} = \frac{a-b}{(a-c)(b-c)}(\alpha^2 + \delta^2) + (a-b)(y-c), \\ C &= \beta_{;3} + \delta_{;1} = \frac{a-c}{(a-b)(b-c)}(\alpha^2 + \gamma^2) + (a-c)(x+y+b), \\ D &= \alpha_{;4} + \gamma_{;2} = -\frac{a-c}{(a-b)(b-c)}(\beta^2 + \delta^2) - (a-c)(x+y+b), \\ E &= \gamma_{;2} - \delta_{;1} = -\frac{b-c}{(a-b)(a-c)}(\alpha^2 + \beta^2) - (b-c)(x-a), \\ F &= \beta_{;3} - \alpha_{;4} = \frac{b-c}{(a-b)(a-c)}(\gamma^2 + \delta^2) + (b-c)(x-a), \end{aligned}$$

where ";" appearing in subscripts again denotes covariant differentiation. Now it is easy to see $A - B = D - C = E - F$, i.e.,

$$\begin{aligned} (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)\left[\frac{a-b}{(a-c)(b-c)} - \frac{a-c}{(a-b)(b-c)}\right] \\ = 2(a-c)(x+y+b) - 2(a-b)(y-c), \end{aligned} \tag{3.25}$$

$$(\alpha^2+\beta^2+\gamma^2+\delta^2)\left[\frac{a-b}{(a-c)(b-c)}-\frac{b-c}{(a-b)(a-c)}\right] = 2(b-c)(x-a)-2(a-b)(y-c). \tag{3.26}$$

Cancelling out $\alpha^2+\beta^2+\gamma^2+\delta^2$ from (3.25) and (3.26) and introducing $s=(a+b+c)/3$, one finally gets

$$0=(a-b)^2(s-c)(s-2c)+(b-c)^2(s-a)(s-2a)+(c-a)^2(s-b)(s-2b). \tag{3.27}$$

If $s=0=(a+b+c)/3$, from (3.27) it follows that

$$0=(a-b)^2c^2+(b-c)^2a^2+(c-a)^2b^2,$$

which would certainly be absurd since a,b,c were supposed to be different; hence $s\neq 0$. The following algebraic argument settles the case $s>0$. Rescaling if necessary, one may assume $1=s=(a+b+c)/3$, and then (3.27) reads

$$0=(a-b)^2(1-c)(1-2c)+(b-c)^2(1-a)(1-2a)+(c-a)^2(1-b)(1-2b). \tag{3.28}$$

Without loss of generality one may assume $a<b<c$, so that

$$\frac{a-b}{(a-c)(b-c)}-\frac{a-c}{(a-b)(b-c)}=\frac{b+c-2a}{(a-c)(a-b)}>0;$$

therefore the left-hand side of (3.25) is nonnegative, and so is the right-hand side, i.e.,

$$(a-c)(x+y+b)\geq (a-b)(y-c)=(a-b)\frac{b+a-5c}{3}>0,$$

which implies

$$\frac{5b-a-c}{3}=x+y+b<0;$$

hence $b<\frac{1}{2}$ since $a+b+c=3$. In particular, $a<b<\frac{1}{2}$, which would imply the right-hand side of (3.28) is positive, a contradiction.

In general, we give a proof which simplifies an argument of A. Derdzinski (from personal communiation). Set

$$\varphi^1=\delta\theta^1-\gamma\theta^2+\beta\theta^3-\alpha\theta^4,$$

and let $\varphi^2,\varphi^3,\varphi^4$ be respectively the right-hand side of the third, second, and first equations in (3.13). By (3.25), $\alpha^2+\beta^2+\gamma^2+\delta^2$ is a nonzero constant so that one may assume it to be 1 by scaling. It follows then that $\varphi^i, 1\leq i\leq 4$, are orthonormal. By direct check one also has

$$\begin{aligned}\varphi^1\wedge\varphi^2-\varphi^3\wedge\varphi^4&=\theta^1\wedge\theta^2-\theta^3\wedge\theta^4,\\ \varphi^1\wedge\varphi^3-\varphi^4\wedge\varphi^2&=\theta^1\wedge\theta^3-\theta^4\wedge\theta^2,\\ \varphi^1\wedge\varphi^4-\varphi^2\wedge\varphi^3&=\theta^1\wedge\theta^4-\theta^2\wedge\theta^3.\end{aligned} \tag{3.29}$$

On the other hand, since $\operatorname{div}(\varphi^1) = B - A = C - D = F - E$, where div denotes the divergence of a vector field or a form, one obtains

$$\begin{aligned} 3\operatorname{div}(\varphi^1) &= (B - A) + (C - D) + (F - E) \\ &= \frac{(a-b)^2 + (b-c)^2 + (a-c)^2}{(a-b)(b-c)(a-c)}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\ &\quad + 2[(a-b)(s-2c) + (b-c)(s-2a) + (c-a)(s-2b)] \\ &= \frac{(a-b)^2 + (b-c)^2 + (a-c)^2}{(a-b)(b-c)(a-c)}, \end{aligned}$$

where one has used the relation $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$ and the fact that the bracketed term is zero. Hence $\operatorname{div}(\varphi^1) \neq 0$. However (3.22) is equivalent to

$$\begin{aligned} d\varphi^4 &= (a-b)\Omega_3^2 + (a-b)\Omega_1^4 + \frac{a-b}{(a-c)(b-c)}\varphi^3 \wedge \varphi^2 \\ &= (a-b)(s-2c)(\varphi^1 \wedge \varphi^4 - \varphi^2 \wedge \varphi^3) + \frac{a-b}{(a-c)(b-c)}\varphi^3 \wedge \varphi^2, \end{aligned} \tag{3.30}$$

by (3.29) and (3.5). Now let η_i, $1 \leq i \leq 4$, be the dual vector fields to φ^i, and let Λ_j^i be the corresponding connection forms. Then the first structural equation says

$$d\varphi^4 = -\Lambda_1^4(\eta_4)\varphi^4 \wedge \varphi^1 + \text{other terms not involving } \varphi^4 \wedge \varphi^1. \tag{3.31}$$

By equaling the coefficients of the term $\varphi^4 \wedge \varphi^1$ in (3.30) and (3.31), one gets $\Lambda_1^4(\eta_4) = (s-2c)(a-b)$. Similarly, $\Lambda_1^3(\eta_3) = (s-2b)(c-a)$, and $\Lambda_1^2(\eta_2) = (s-2a)(b-c)$ by (3.23) and (3.24). Therefore $\operatorname{div}(\varphi^1) = \Lambda_1^4(\eta_4) + \Lambda_1^3(\eta_3) + \Lambda_1^2(\eta_2) = 0$, contradictory to $\operatorname{div}(\varphi^1) \neq 0$.

Remark. (3.5) implies that the space is locally anti-self-dual and Einstein. (3.13) can thus also be obtained by the fact that $\operatorname{div}(W^-) = 0$, where W^- denotes the anti-self-dual part of the Weyl tensor. For details see A. Derdzinski: *Self-dual Kaehler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983) 405–433.

4. Two applications

In this section, we shall give applications of what we have shown so far, which turn out to be interesting in their own right.

Definition 1. A Riemannian manifold is said to be *locally harmonic* if every geodesic sphere has constant mean curvature.

Surprisingly, it can be shown (cf. [2]) that this geometric definition is equivalent to the well-known mean-value property of a harmonic map, namely, the average of any harmonic function over each local geodesic sphere is equal to the value of the function at the center of the sphere; and hence the name "harmonic" manifold.

We mention the following conjecture and the well-known Theorem 3.

Conjecture. *Locally harmonic manifolds must be locally rank-one symmetric.*

Theorem 3. *The conjecture is true if dimension is* 4.

The first application is that the validity of Osserman's conjecture in dim = 4 yields another proof of this theorem. To be more precise, it can be shown that for a four-dimensional harmonic manifold, K_v has globally constant eigenvalues (see [2, p. 154] for more detailed discussions).

As a second application we have

Theorem 4. *A compact homogeneous space homeomorphic to one of the compact rank-one symmetric spaces is isometric to one of them iff K_v has globally constant eigenvalues for all unit vectors v.*

Proof. By a hard classification theorem (see [4], [3], [16]) these spaces are either isometric to rank-one symmetric spaces or diffeomorphic to $\mathbf{C}P^{2k+1}$. Now our proof of the conjecture in dim $= 2(2k+1)$ settles this case.

It would be very interesting to see a geometric proof of this theorem without resorting to Lie theory.

5. The Kähler case

Proof of Theorem 0. Let M be Kählerian. Given $X \in M_p$, let Y perpendicular to X be chosen such that the plane spanned by X and Y assumes the maximal sectional curvature (minimal if the curvature of $M \leq 0$), so that Y is an eigenvector of K_x. By [3, p. 362] this plane must be holomorphic. It then follows that M has constant holomorphic sectional curvature.

6. Concluding remarks and questions

In [7] it will be shown that if a space of constant curvature satisfies the following two axioms:

1. K_v has two different constant eigenvalues (counting multiplicities) for all $v \in SM$, say b, c;

2. Let $E_c(v)$ be the span of v and the eigenspace of K_v with eigenvalue c. Then $E_c(w) = E_c(v)$ whenever $w \in E_c(v)$.

Then the space must be locally rank-one symmetric, from which follows the classification of rank-one symmetric spaces.

Note that these two conditions are natural as can be seen by the fact that for a compact rank-one symmetric space $c = 1$ and $b = 1/4$ after scaling;

furthermore $E_c(v)$ is the tangent space at the base point of v of a totally geodesic sphere of curvature c, hence the second condition follows. Consequently we would like to pose a question here.

Question. Does Condition $[O]$ imply the two axioms in this section? If the answer is yes, then Osserman's conjecture would be true.

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WASHINGTON UNIVERSITY