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DEFINITE 4-MANIFOLDS

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1. Introduction

The paucity of positive definite unimodular integral bilinear forms which are realized as the intersection form of a closed smooth 4-manifold is demonstrated by the following recent theorem of S. Donaldson:

Theorem (*Donaldson* [4]). *Let X be a smooth closed oriented 4-manifold with positive definite intersection form θ. Then θ is "standard"; i.e. over the integers $\theta \cong (1) \oplus \cdots \oplus (1)$.*

This theorem was originally proved under the assumption that X is simply connected [2], and has also been extended by M. Furuta [7] to cover X with $H_1(X; \mathbf{Z}) = 0$ by techniques similar to those used in [4]. The proofs of all these versions of the theorem rely on quite detailed ad hoc analysis and on the deep and difficult work of C. Taubes [9] (cf. [8]).

We have long felt that it would be worthwhile to give a proof of Donaldson's theorem which reduced the role played by analysis and thus be more accesible to topologists. Our work in [5] was a start in that direction. The purpose of this paper is to give a proof of Donaldson's theorem under the assumption that $H_1(X; \mathbf{Z})$ has no 2-torsion while using as analytical input only the basic work of K. Uhlenbeck [10], [11]. Our proof is in spirit similar to that of [5], using SO(3)-connections, but makes more apparent the importance of the "basepoint fibration" (see §2). By combining our techniques with Donaldson's study of orientations of moduli spaces one can presumably remove our hypothesis on $H_1(X; \mathbf{Z})$ as in [4, 4(c)]; however we do not know an elementary argument that will remove this hypothesis.

As in [5] we base our proof on a useful characterization of nonstandard integral inner product spaces. Let W be a positive definite unimodular integral inner product space and define an equivalence relation on W by declaring that $w_1 \sim w_2$ if $w_1 \equiv w_2 \pmod 2$ and $w_1^2 = w_2^2$. Note that $-w \sim w$. Set $\mu(w) = \frac{1}{2}\#(w' \in W | w' \sim w)$ and call an element $e \in W$ *minimal* if $e^2 \leq w^2$ for all $w \equiv e \pmod 2$.

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Lemma 1.1. *A positive definite unimodular integral inner product space W is nonstandard if and only if there is a minimal element $e \in W$ such that $\mu(e) = 1$, $e^2 > 1$, and $e^2 \not\equiv 0 \pmod 4$.*

Proof. For the standard form each minimal e with $e^2 > 1$ has $\mu(e)$ even (see [5]). Conversely, if W is nonstandard then we have an orthogonal direct sum decomposition $W = U \oplus V$ where U is standard, $V \neq 0$, and each nonzero $v \in V$ has $v^2 > 1$. If $v \in V$ is primitive, then since V is unimodular there is a $w \in V$ with $v \cdot w = 1$. If both v^2 and w^2 are $\equiv 0 \pmod 4$, then $(v + w^2) \equiv 2 \pmod 4$. Thus V has elements v such that $v^2 \not\equiv 0 \pmod 4$.

Let e be an element of smallest square in V such that $e^2 \not\equiv 0 \pmod 4$. For arbitrary $u + v \in U \oplus V$ we have $(e + 2u + 2v)^2 = (e + 2v)^2 + (2u)^2 \geq e^2$ since $(e + 2v)^2 \equiv e^2 \pmod 4$ and so $(e + 2v)^2 \geq e^2$. Thus e is minimal in W.

If $e^2 = (e + 2u + 2v)^2 = (e + 2v)^2 + 4u^2$, then $(e + 2v)^2 \equiv e^2 \pmod 4$; so $(e + 2v)^2 \geq e^2$ and $u^2 = 0$, so $u = 0$. Thus $e^2 = (e + 2v)^2$ and expanding we see that $v \cdot (e + v) = 0$. So $e^2 = ((e + v) + v)^2 = (e + v)^2 + v^2$, and $(e + v)^2 < e^2$ unless $v = 0$. Thus, unless $v = 0$, $(e + v)^2 \equiv 0 \pmod 4$, and so $e^2 \equiv v^2 \pmod 4$, hence $v^2 \geq e^2$. This means $e^2 = v^2$ and $(e + v)^2 = 0$, so $v = -e$. Thus $e + 2u + 2v \sim e$ if and only if $u = 0$ and $v = 0$ or $-e$. Hence $\mu(e) = 1$. q.e.d.

Now fix a smooth closed oriented 4-manifold X with no 2-torsion in $H_1(X; \mathbf{Z})$ and suppose that the intersection form of X is positive definite and nonstandard. By surgering out the free part of $H_1(X; \mathbf{Z})$ we obtain a new 4-manifold with the same intersection form, and so we may suppose that $H_1(X; \mathbf{Z}_2) = 0$. Split $H^2(X; \mathbf{Z}) = \operatorname{Fr} H^2(X; \mathbf{Z}) \oplus \operatorname{Tor} H^2(X; \mathbf{Z})$; so we may consider the intersection form of X as defined on $\operatorname{Fr} H^2(X; \mathbf{Z})$. Since the intersection form is assumed to be nonstandard, by Lemma 1.1 there is a minimal $e \in \operatorname{Fr} H^2(X; \mathbf{Z})$ such that $e^2 > 1$, $e^2 \not\equiv 0 \pmod 4$ and $\mu(e) = 1$. Let L_e be the SO(2) vector bundle over X whose Euler class is e, and consider the stabilization $L_e \oplus \varepsilon$ to an SO(3) vector bundle. In §3 we shall derive a contradiction to the existence of e by studying connections on $L_e \oplus \varepsilon$.

An SO(3) vector bundle over a 4-manifold is classified by the characteristic classes w_2 and p_1. For our bundle, $w_2(L_e \oplus \varepsilon) \equiv e \pmod 2$ and $p_1(L_e \oplus \varepsilon) = e^2$. Any other SO(3) vector bundle E with $w_2(E) = w_2(L_e \oplus \varepsilon)$ must have $p_1(E)$ differing from $p_1(L_e \oplus \varepsilon)$ by a multiple of 4. Thus if $e^2 = 4k + r$ where $1 \leq r \leq 3$, then the SO(3) vector bundles with the same w_2 as $L_e \oplus \varepsilon$ have $p_1 = 4m + r$, $m \in \mathbf{Z}$. Let E_m denote the bundle in this class with $p_1(E_m) = 4m + r$. Thus $E_k = L_e \oplus \varepsilon$.

The structure group of an SO(3) vector bundle reduces to SO(2) if there is a $v \in H^2(X; \mathbf{Z})$ such that $v \equiv w_2(E) \pmod 2$ and $v^2 = p_1(E)$. The minimality of e implies that the bundles E_m, $0 \leq m \leq k - 1$, are irreducible. Also since

$H_1(X;\mathbf{Z})$ has no 2-torsion (and $\operatorname{Fr} H_1(X;\mathbf{Z})=0$), $\operatorname{Tor} H^2(X;\mathbf{Z})=H_1(X;\mathbf{Z})$ has odd order. Now there are no reductions of E_k coming from $\operatorname{Fr} H^2(X;\mathbf{Z})$ other than $L_e \oplus \varepsilon$ because $\mu(e)=1$. All other reductions have the form $e+t$ where $0 \neq t \in \operatorname{Tor} H^2(X;\mathbf{Z})$, and no two of these are equivalent up to orientation. Thus, up to orientation, E_k has an odd number ρ of reductions to an SO(2) vector bundle.

2. The basepoint fibration

For each $m=0,\cdots,k$ let $\mathscr{A}_m$ denote the space of SO(3) connections on the vector bundle E_m over X, and let $\mathscr{G}_m$ be the group of gauge transformations of E_m. The group $\mathscr{G}_m$ acts on $\mathscr{A}_m$ with quotient the moduli space $\mathscr{B}_m$ of connections on E_m. Actually here, as in [5] and all other references, we are working with the completions of these spaces of connections and gauge transformations in appropriate Sobolev norms. As has become general practice, we shall ignore the requisite notation. Let $\mathscr{M}_m \subset \mathscr{B}_m$ be the moduli space of self-dual connections on E_m. The Atiyah-Singer index theorem measures the formal dimension of $\mathscr{M}_m$ as $2p_1(E_m)-3=8m+2r-3$. By the theorem of Freed and Uhlenbeck [6], for generic metrics on X the moduli space $\mathscr{M}_m$ is empty or a manifold of this dimension for $0 \leq m < k$, and $\mathscr{M}_k$ is a manifold of dimension $8k+2r-3$ with ρ cone singularities (corresponding to reducible connections) whose links are complex projective spaces. (It is important here that $r \neq 0$ so that none of the bundles E_m can be flat.)

Let $\mathscr{A}_k^* \subset \mathscr{A}_k$ be the subspace of irreducible connections. Using general position it is easy to see that $\pi_j(\mathscr{A}_k^*)=0$ for all j, for $\mathscr{A}_k$ is an affine space and the link in $\mathscr{A}_k^*$ of a reducible connection is a copy of S^∞. Now $\mathscr{G}_k$ acts freely on $\mathscr{A}_k^*$, so $\mathscr{A}_k^*/\mathscr{G}_k=\mathscr{B}_k^*$ is the classifying space $B\mathscr{G}_k$. By [1, Proposition 2.4] $B\mathscr{G}_k$ is homotopy equivalent to $\operatorname{Map}_k(X, B\mathrm{SO}(3))$ the space of maps which classify E_k. Hence $\mathscr{B}_k^* \cong \operatorname{Map}_k(X, B\mathrm{SO}(3))$.

If we fix a point $x_0 \in X$ we may consider those gauge transformations of E_k which act as the identity on the fiber over x_0. This is the group of based gauge transformations $\mathscr{G}_{k,0}$, which is a normal subgroup of $\mathscr{G}_k$. Let $\tilde{\mathscr{B}}_k^* = \mathscr{A}_k^*/\mathscr{G}_{k,0}$. The fibration $\mathscr{A}_k^* \to \mathscr{B}_k^*$ factors into $\mathscr{A}_k^* \to \tilde{\mathscr{B}}_k^*$ and the "basepoint fibration" $\tilde{\mathscr{B}}_k^* \to \mathscr{B}_k^*$, which is a principal SO(3)-bundle. We denote the basepoint fibration by β_k. The space $\tilde{\mathscr{B}}_k^* \cong \mathscr{A}_k^* \times_{\mathscr{G}_k} (P_k)_{x_0}$ (where P_k is the principal SO(3) bundle associated to E_k and $\mathscr{G}_k$ is viewed as the group of automorphisms of P_k). Hence an element of $\tilde{\mathscr{B}}_k^*$ may be considered as a pair $([A], f)$ consisting of a gauge equivalence class $[A]$ of connections on E_k, together with a framing f of the fiber E_{k,x_0}. When we identify $\mathscr{B}_k^*$ with $\operatorname{Map}_k(X, B\mathrm{SO}(3))$ (up to homotopy equivalence) the basepoint fibration β_k is classifed by evaluation at x_0:

$$\mathrm{eval}_{x_0}\colon \mathrm{Map}_k(X, B\mathrm{SO}(3)) \to B\mathrm{SO}(3).$$

We shall modify standard notation slightly by letting $\mathscr{M}_k^*$ denote $\mathscr{M}_k$ after removing small open cones on complex projective spaces about each of the ρ reducible connections. Let γ_k denote the restriction of the basepoint fibration β_k over $\mathscr{M}_k^*$.

If Σ is any immersed surface in X, we may restrict any of the bundles E_m to Σ. All of these restrictions to Σ are equivalent (to a bundle E_Σ, say) since these restrictions have the same w_2 and $p_1 = 0$. We consider the moduli spaces $\tilde{\mathscr{B}}_\Sigma^*$ and $\mathscr{B}_\Sigma^*$ and the basepoint fibration β_Σ.

Lemma 2.1. *Suppose that Σ is a surface in X which satisfies the property*

$(*)$ Each connection $[A] \in \mathscr{M}_k^*$ restricts to an irreducible connection in $\mathscr{B}_\Sigma^*$ under the restriction map $r_\Sigma\colon \mathscr{M}_k^* \to \mathscr{B}_\Sigma$.

Then if we choose a basepoint on Σ, $\gamma_k = r_\Sigma^*(\beta_\Sigma)$.

Proof. The map $\mathscr{M}_k^* \to \mathscr{B}_\Sigma^*$ is covered by a bundle map $\tilde{\mathscr{M}}_k^* \to \tilde{\mathscr{B}}_\Sigma^*$ taking $([A], f)$ to $([A]_\Sigma, f)$, where $[A]_\Sigma$ is restriction to Σ. This means that $\gamma_k = r_\Sigma^*(\beta_\Sigma)$.

Lemma 2.2. *There is a one point union of a finite set of loops in X such that the restriction of any $[A] \in \mathscr{M}_l^*$, $0 \le l \le k$, to this finite set of loops is irreducible.*

Proof (cf., [3, §III, (iii)]). Consider a point $([A], (x_1, \cdots, x_k))$ in $\mathscr{M}_0 \times S^k(X)$, where $S^k(X)$ denotes the symmetric product. Since $[A]$ is irreducible there is a finite set of loops such that the holonomy elements in SO(3) determined by A and these loops do not lie in any circle subgroup. In other words, $[A]$ restricts irreducibly to these loops. Deforming the loops slightly we may suppose that they all lie in $X - \{x_1, \cdots, x_k\}$. So there is a neighborhood $U \times V_1 \times \cdots \times V_k$ of $([A], (x_1, \cdots, x_k))$ in $\mathscr{M}_0 \times S^k(X)$ such that these loops lie in $X - \bigcup_{i=1}^n V_i$ and $[A'] \in U$ restricts irreducibly to these loops. It follows easily [5, Theorem 5.3] from [10], [11] that $\mathscr{M}_0$ is compact; so $\mathscr{M}_0 \times S^k(X)$ is also compact. Hence if we cover $\mathscr{M}_0 \times S^k(X)$ with sets of the form $U \times V_1 \times \cdots \times V_k$ as above, there will be a finite subcover.

Now suppose inductively that for each $0 \le k - j < k - m$ there is a finite open cover of $\mathscr{M}_{k-j} \times S^j(X)$ by open sets of the form $U \times S^j(V_1, \cdots, V_r)$, where $S^j(V_1, \cdots, V_r)$, $r \ge j$, denotes the jth symmetric product of the open subsets $V_1, \cdots, V_r$ of X, and to each such set is associated a finite set of loops in $X - \bigcup V_i$ such that each $[A] \in U$ restricts irreducibly to this set of loops.

Consider one of these open sets $U \times S^j(V_1, \cdots, V_r)$. If $([A], (x_1, \cdots, x_m)) \in \mathscr{M}_{k-m} \times S^m(X)$ with $x_1, \cdots, x_m \in \bigcup V_i$ and if $[A]$ is close enough to an

$[A'] \in U$ when restricted to $X - \bigcup V_i$, then $[A]$ must restrict irreducibly to the associated finite set of loops. Thus we get an open set $U^* \times S^m(V_1, \cdots, V_r)$ in $\mathscr{M}_{k-m} \times S^m(X)$ and for $([A], (x_1, \cdots, x_m)) \in U^* \times S^m(V_1, \cdots, V_r)$, $x_1, \cdots, x_m \in \bigcup V_i$, $[A]$ restricts irreducibly to the associated finite set of loops. Corresponding to our finite cover of $\coprod_{k-i=0}^{m-1} \mathscr{M}_{k-i} \times S^i(X)$ we obtain finitely many open sets in $\mathscr{M}_{k-m} \times S^m(X)$.

Cover the rest of $\mathscr{M}_{k-m} \times S^m(X)$ with open sets $U' \times V_1' \times \cdots \times V_m' = U' \times S^m(V_1', \cdots, V_m')$ as in the case of $\mathscr{M}_0 \times S^m(X)$ so that there is a finite set of loops in $X - \bigcup V_i'$ such that each $[A'] \in U'$ restricts irreducibly to this set of loops. Thus we obtain an open cover of $\mathscr{M}_{k-m} \times S^m(X)$ with sets $U \times S^m(V_1, \cdots, V_r)$, $r \geq m$, and we can reduce to a countable subcover $\{\mathscr{U}_i\}$ which contains all the sets $U \times S^m(V_1, \cdots, V_r)$, $r > m$. If there is no finite subcover we can find a sequence $\{([A_i], \mathbf{x}^{(i)})\}$ in $\mathscr{M}_{k-m} \times S^m(X)$ such that $([A_t], \mathbf{x}^{(t)}) \in \mathscr{U}_t - \bigcup_{i<t} \mathscr{U}_i$ for each t. Hence this sequence has no convergent subsequence. Since $S^{k-m}(X)$ is compact this means that $\{[A_i]\}$ has no convergent subsequence. Uhlenbeck's basic results [10], [11] then imply that there is a finite set of points $z_1, \cdots, z_p \in X$ and a subsequence $\{[A_{i'}]\}$ which converges in $X - \{z_1, \cdots, z_p\}$ to a connection which extends to an $[A_\infty]$ in some $\mathscr{M}_{k-j}$, $0 \leq k - j \leq k - (m + p)$. Uhlenbeck's arguments show that in the limit at each z_i an S^4 carrying an instanton of charge $\mu_i \geq 1$ is pinched off, and further $\Sigma\mu_i = j - m$ (cf. [2]). Now $\{\mathbf{x}^{(i')}\}$ also has a convergent subsequence so we may suppose that $\{\mathbf{x}^{(i')}\}$ converges to an $\mathbf{x}^{(\infty)} \in S^m(X)$. By assigning multiplicity μ_i to z_i, $i = 1, \cdots, p$, one obtains an element $\mathbf{z} \in S^{j-m}(X)$, and we have $([A_\infty], (\mathbf{x}^{(\infty)}, \mathbf{z})) \in \mathscr{M}_{k-j} \times S^j(X)$. This lies in one of the $U \times S^j(V_1, \cdots, V_r)$ in our finite cover of $\mathscr{M}_{k-j} \times S^j(X)$. Thus, for large enough i', $([A_{i'}], \mathbf{x}^{(i')}) \in U^* \times S^m(V_1, \cdots, V_r)$, in contradiction to the choice of the sequence. This completes the inductive step.

To complete the proof simply use $\mathscr{M}_k^*$ at the top level of the argument, and then take the finite collection of loops obtained.

This last lemma will be used to obtain surfaces Σ in X which satisfy condition $(*)$ of Lemma 2.1. Note that we can assume that the loops obtained in Lemma 2.2 are by general position disjoint from any given surface. Then if we connect sum this given surface with a small nullhomologous 2-dimensional surface containing the union of the loops, we obtain such a Σ. Furthermore the above proof can be used to produce disjoint copies of such one-point unions of loops (with disjoint basepoints) so that each of the unions satisfies the conclusion of Lemma 2.2. Hence if we are given d disjoint surfaces $\Sigma_1, \cdots, \Sigma_d$ in X, we may modify them to obtain disjoint surfaces $\Sigma_1', \cdots, \Sigma_d'$ so that each Σ_i' is homologous to Σ_i and so that Σ_i' satisfies condition $(*)$ of Lemma 2.1.

Proposition 2.3. *If E_k restricts nontrivially to Σ, then the* SO(3) *bundle β_Σ lifts to a $U(2)$ bundle δ_Σ.*

Proof. It suffices to show that $w_2(\beta_\Sigma) \in H^2(\mathscr{B}^*_\Sigma; \mathbf{Z}_2)$ lifts to an integral class. This is carried out in [1, §9], however we shall outline a proof here. Let $M(\Sigma) = \mathrm{Map}_E(\Sigma, B\mathrm{SO}(3))$ and $M^*(\Sigma) = \mathrm{Map}^*_E(\Sigma, B\mathrm{SO}(3))$, the based maps. The basepoint fibration $\beta_\Sigma(\tilde{\mathscr{B}}^*_\Sigma \to \mathscr{B}^*_\Sigma)$ is homotopically equivalent to the inclusion of the fiber in the fibration

$$(*) \qquad M^*(\Sigma) \to M(\Sigma) \xrightarrow{\text{eval}} B\mathrm{SO}(3).$$

The cofibration $\vee S^1 \to \Sigma \to S^2$ induces the fibration

$$(**) \qquad M^*(S^2) \to M^*(\Sigma) \to M^*(\vee S^1).$$

Now $M^*(\vee S^1) \cong \Pi\mathrm{SO}(3)$ and $M^*(S^2) \cong \Omega\mathrm{SO}(3)$; so $\pi_1(M^*(\vee S^1)) \cong \bigoplus \mathbf{Z}_2$, $\pi_1(M^*(S^2)) = 0$, and the exact sequence of $(**)$ gives $\pi_1(M^*(\Sigma)) \cong \bigoplus \mathbf{Z}_2$.

The adjoint construction gives an isomorphism

$$\pi_2(M^*(\Sigma)) \cong \mathrm{Bun}^*_\Sigma(S^2 \times \Sigma),$$

the isomorphism classes of SO(3)-bundles over $S^2 \times \Sigma$ which restrict nontrivially to $\{*\} \times \Sigma$ and trivially to $S^2 \times x_0$. Similarly $\pi_2(M(\Sigma)) \cong \mathrm{Bun}_\Sigma(S^2 \times \Sigma)$, isomorphism classes of SO(3)-bundles restricting nontrivially to $\{*\} \times \Sigma$. A $\xi \in \mathrm{Bun}_\Sigma(S^2 \times \Sigma)$ satisfies $w_2(\xi | \{*\} \times \Sigma) = 1$ and so is determined by the pair $(w_2(\xi | S^2 \times x_0), p_1(\xi)) \in \mathbf{Z}_2 \oplus \mathbf{Z}$. For $\xi \in \mathrm{Bun}^*_\Sigma(S^2 \times \Sigma)$, $w_2(\xi | S^2 \times x_0) = 0$ and p_1 gives an isomorphism $\mathrm{Bun}^*_\Sigma(S^2 \times \Sigma) \to 4\mathbf{Z} \subset \mathbf{Z}$. Also p_1 gives an isomorphism $\mathrm{Bun}_\Sigma(S^2 \times \Sigma) \to 2\mathbf{Z} \subset \mathbf{Z}$ (since $w_2(\xi | S^2 \times x_0) \equiv p_1(\xi)/2 \pmod 2$). It follows that $\pi_2(M^*(\Sigma)) \to \pi_2(M(\Sigma))$ is equivalent to multiplication by $2\colon \mathbf{Z} \to \mathbf{Z}$. Thus the exact sequence of $(*)$ shows that $\pi_1(M(\Sigma)) \cong \pi_1(M^*(\Sigma)) \cong \oplus \mathbf{Z}_2 \cong \pi$, say.

There are exact sequences:

$$\begin{array}{ccccccc}
\pi_2(M^*(\Sigma)) & \longrightarrow & H_2(M^*(\Sigma); \mathbf{Z}) & \longrightarrow & H_2(\pi; \mathbf{Z}) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle 2} & & \Big\downarrow & & \Big\downarrow{\scriptstyle =} & & \\
\pi_2(M(\Sigma)) & \longrightarrow & H_2(M(\Sigma); \mathbf{Z}) & \longrightarrow & H_2(\pi; \mathbf{Z}) & \longrightarrow & 0.
\end{array}$$

We get a splitting map $H_2(M(\Sigma); \mathbf{Z}) \to \pi_2(M(\Sigma))$ by identifying $H_2(M(\Sigma); \mathbf{Z})$ with $\Omega_2(M(\Sigma))$ and mapping $\Omega_2(M(\Sigma)) \to \mathbf{Z} = \pi(M(\Sigma))$ by sending $[f\colon N \to M(\Sigma)]$ to $p_1(\text{induced bundle over } N \times \Sigma)/2$. Using $p_1/4$ there is a consistent splitting map: $H_2(M^*(\Sigma); \mathbf{Z}) \to \pi_2(M^*(\Sigma))$ so that the diagram commutes.

Hence there are also consistent splitting maps

$$\begin{tikzcd} & H_2(M^*(\Sigma);\mathbf{Z}) \arrow[d] \\ H_2(\pi;\mathbf{Z}) \arrow[ur] \arrow[r] & H_2(M(\Sigma);\mathbf{Z}) \end{tikzcd}$$

Thus if $f\colon N \to M(\Sigma)$ represents an element of the 2-torsion subgroup $H_2(\pi;\mathbf{Z}) \subset H_2(M(\Sigma);\mathbf{Z})$ then it comes from a map $f_1\colon N \to M^*(\Sigma)$ and so $f_1^*(w_2(\beta_\Sigma)) = 0$. However $w_2(\beta_\Sigma)$ evaluates nontrivially on the generator of $\pi_2(M(\Sigma))$ (because $\pi_2(M(\Sigma)) \to \pi_2(B\mathrm{SO}(3))$ is surjective). Thus $H_2(M(\Sigma);\mathbf{Z}) \cong \mathbf{Z}\oplus$(2-torsion) and $w_2(\beta_\Sigma)$ corresponds to the map $\omega\colon \mathbf{Z}\oplus$ (2-torsion)$\to \mathbf{Z}_2$ which is trivial on the 2-torsion summand and is restriction mod 2 on the $\mathbf{Z}$-summand; so ω lifts to $\mathbf{Z}\oplus$(2-torsion)$\to \mathbf{Z}$.

Finally consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_1(M(\Sigma);\mathbf{Z}),\mathbf{Z}) \arrow[r] \arrow[d, "\cong"] & H^2(M(\Sigma);\mathbf{Z}) \arrow[r, "k"] \arrow[d, "r"] & \mathrm{Hom}(H_2(M(\Sigma);\mathbf{Z}),\mathbf{Z}) \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(H_1(M(\Sigma);\mathbf{Z}),\mathbf{Z}_2) \arrow[r] & H^2(M(\Sigma);\mathbf{Z}_2) \arrow[r, "k"] & \mathrm{Hom}(H_2(M(\Sigma);\mathbf{Z}),\mathbf{Z}_2) \arrow[r] & 0 \end{tikzcd}$$

Then $w_2(\beta_\Sigma)$ maps to ω which lifts, and a diagram chase shows that $w_2(\beta_\Sigma)$ lifts.

Let Σ be a surface satisfying the hypothesis $(*)$ of Lemma 2.1 and the hypothesis of Proposition 2.3. Let δ_Σ be the U(2) bundle obtained in Proposition 2.3, and let Λ_Σ denote the complex line bundle over $\coprod_{0\le l\le k}\mathscr{M}_l^*$ with $c_1(\Lambda_\Sigma) = r_\Sigma^*(c_1(\delta_\Sigma))$. Recall that $p_1(E_k) = 4k+r$, $1 \le r \le 3$, and $\dim\mathscr{M}_k = 8k+2r-3$ so that each of the ρ boundary components of $\mathscr{M}_k^*$ is diffeomorphic to $\mathbf{CP}^d$, $d = 4k+r-2$. Let h generate $H^2(\mathbf{CP}^d;\mathbf{Z})$.

Lemma 2.4. *On each of the ρ boundary components of $\mathscr{M}_k^*$ we have* $w_2(\Lambda_\Sigma|_{\mathbf{CP}^d}) \equiv h \pmod 2$.

Proof. Consider the basepoint fibration γ_X near a reducible connection. The slice in $\tilde{\mathscr{M}}_k$ to the SO(3) orbit of this reducible connection is a copy of $\mathbf{C}^{d+1}$ on which the isotropy group S^1 acts in the standard fashion giving the quotient $c\mathbf{CP}^d \subset \mathscr{M}_k$. This implies that $w_2(\gamma_X|_{\mathbf{CP}^d}) \equiv h \pmod 2$ (cf. also [5, p. 538]). But working mod 2, $w_2(\Lambda_\Sigma) = c_1(\Lambda_\Sigma) = r_\Sigma^*(c_1(\delta_\Sigma)) = r_\Sigma^*(w_2(\beta_\Sigma)) = w_2(\gamma_X)$ by Lemma 2.1.

3. Cutting down the moduli space

Consider a family of $d = 4k+r-2$ transversally intersecting surfaces over which E_k restricts nontrivially. As explained in the remark following

Lemma 2.2 we can modify these surfaces in their homology classes to obtain d surfaces $\Sigma_1, \cdots, \Sigma_d$ in X each satisfying the hypothesis $(*)$ of Lemma 2.1. Consider the basepoint fibrations β_1 determined by the surfaces Σ_i (where we will of course need to use different basepoints for each Σ_i) and the related complex line bundles Λ_i over $\coprod_{0 \le l \le k} \mathscr{M}_l^*$. Donaldson proves [3, Lemma 3.17] by an appeal to Sard's Theorem that for each Λ_i there is a section s_i over $\beta^*_{\Sigma_i}$ such that the pulled back section through $r^*_{\Sigma_i}$ vanishes transversally on a codimension 2 submanifold $W_i \cap \mathscr{M}_k^*$. Furthermore the sections can be chosen such that $W_{i_1} \cap \cdots \cap W_{i_r} \cap \mathscr{M}_m^*$ is cut out transversally for all $i_1, \cdots, i_r$ and $0 \le m \le k$. ($\mathscr{M}_m^* = \mathscr{M}_m$ if $0 \le m \le k$.) Let N be the 1-manifold $N = \mathscr{M}_k^* \cap W_1 \cap \cdots \cap W_d$.

Proposition 3.1. *N is compact.*

Proof. Consider a sequence of gauge equivalence classes of connections $\{[A_i]\}$ in N. If there is no convergent subsequence, then by [10], [11] there are points $X_1, \cdots, X_n \in X$, $1 \le n \le k$, and a subsequence $\{[A_{i'}]\}$ which converges to an $[A_\infty]$ over $X - \{x_1, \cdots, x_n\}$, and $[A_\infty]$ extends to an element of $\mathscr{M}_m$, $m \le k - n$ (over all of X). The n points $\{x_1, \cdots, x_n\}$ lie on at most $2n$ of the surfaces Σ_i; so at least $d - 2n$ of these surfaces, say $\Sigma_1, \cdots, \Sigma_{d-2n}$ lie in $X - \{x_i, \cdots, x_n\}$.

Now a gauge equivalence class $[A]$ lies in W_j if and only if $[A]|_{\Sigma_j}$ is in the zero set of s_j. Since each $[A_j] \in N$, the limiting connection $[A_\infty]$ lies in $W_1 \cap \cdots \cap W_{d-2n} \cap \mathscr{M}_m$. Furthermore, $m \le k - n$; so $\dim \mathscr{M}_m = 8m + 2r - 2 \le 8(k-n) + 2r - 3$. However, transversality then implies that $W_1 \cap \cdots \cap W_{d-2n} \cap \mathscr{M}_m$ is empty. This contradiction implies that N is compact.

Theorem 3.2. *Let X be a positive definite smooth closed 4-manifold with no 2-torsion in $H_1(X; \mathbf{Z})$. Then the intersection form of X is standard.*

Proof. As we have seen above we may assume that $\operatorname{Fr} H_1(X; \mathbf{Z}) = 0$, and if the form is nonstandard there is an element $e \in \operatorname{Fr} H^2(X; \mathbf{Z})$ with associated SO(2) vector bundle L_e such that $e^2 = 4k + r$, $1 \le r \le 3$. Further, the SO(3) vector bundle $E_k = L_e \oplus \varepsilon$ has an odd number ρ of reductions to SO(2), and none of the "lesser bundles" E_m, $0 \le m < k$, are reducible. So we are in the situation above and we have a compact 1-dimensional submanifold N of $\mathscr{M}_k^*$. By Lemma 2.4, the number of points modulo 2 in $N \cap \mathbf{CP}^d$ is $w_2(\Lambda_1) \cdots w_2(\Lambda_d)[\mathbf{CP}^d] = h^d[\mathbf{CP}^d] = 1$ for each boundary component $\mathbf{CP}^d$ of $\mathscr{M}_k^*$. Hence N has ρ (mod 2), i.e., an odd number, of endpoints. This evident impossibility contradicts the assumption that the intersection form of X is nonstandard.

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