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SPECTRUM OF THE LAPLACIAN ON VECTOR BUNDLES OVER $C_{2\pi}$-MANIFOLDS

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1. Introduction

Several authors studied the spectrum of an elliptic (pseudo)differential operator on a compact Riemannian manifold with emphasis on the relation to properties of its geodesics (see Chazarain [3], Colin de Verdière [4], Duistermaat and Guillemin [6], and Weinstein [18]). Under the assumption that all the geodesics are closed, more detailed properties of the spectrum of the Schrödinger operator $H = \Delta + V$ (Δ: the Laplace-Beltrami operator) with the scalar potential V were investigated by Colin de Verdière [5], Guillemin [7], Weinstein [19], and so on. On the other hand, the Schrödinger operator with a "vector" potential is regarded as the Laplacian on a vector bundle with a linear connection. In our previous papers [12], [13] we studied the spectrum of the Laplacian on the line bundle over the sphere, and clarified the relation between the asymptotic distribution of the spectrum and the holonomies of the connection. The present article considers a more general subject—the spectrum and the holonomies on the vector bundle over a $C_{2\pi}$-manifold.

Let (M, g) be a $C_{2\pi}$-manifold, that is a Riemannian manifold all of whose geodesics are closed and have a common length 2π (cf. Besse [2, Chapter 7,B]). Let E be a C^∞ complex vector bundle over M with a Hermitian structure, and let $\tilde{d}$ be a linear connection on E compatible with the Hermitian structure. With respect to a local unitary frame $\{e_1, \cdots, e_r\}$ of E ($r = \operatorname{rank} E$), we have $\tilde{d}e_\alpha = \sum \omega_\alpha^\beta e_\beta$, where $\omega = (\omega_\alpha^\beta)$ is an $r \times r$ skew-Hermitian matrix with 1-forms as its components. Associated with the connection $\tilde{d}$, let us consider a nonnegative, second order, selfadjoint, elliptic differential operator L operating on C^∞ sections of E, which is called the *Laplacian* and locally expressed with respect to the local frame $\{e_\alpha\}$ on a local coordinate neighborhood of M

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as

$$L = -\sum_{j,k} g^{jk}\nabla_j\nabla_k - 2\sum_{j,k} g^{jk}\omega_j\nabla_k - \sum_{j,k} g^{jk}(\nabla_j\omega_k + \omega_j\omega_k),$$

∇ being the Levi-Civita connection defined by g, and $\omega = \sum\omega_j\,dx_j$ (cf. [11]).

We will show that the asymptotic distribution of eigenvalues of L is described by the distribution of eigenvalues of holonomies of closed geodesics of (M, g) (§3, Theorem 3.1). In §2 we consider the so-called return operator induced from L, and calculate its principal symbol. In §3 we derive a main theorem by the symbol analysis of pseudodifferential and Fourier integral operators similarly to the arguments by Hörmander [8]. We see in §4 that the result in §3 is improved for the case of line bundles. In the final section we consider the case where the spectrum consists of clusters of eigenvalues contained in a sequence of intervals of constant width.

Throughout the symbol calculations of operators we regard the operators as acting on $C^\infty(\Theta^{1/2} \otimes E)$, $\Theta^{1/2}$ being the bundle of half-densities on M.

Remark. The outlook in our previous paper [12, §6] on the case of vector bundles (except line bundles) is somewhat incorrect. This paper (§5 in particular) plays a role of correcting it.

2. Return operator

For the Laplacian L we define the operator $P = (L + c)^{1/2}$ according to Seeley [16], where c is a positive constant. P is an elliptic, positive, selfadjoint pseudodifferential operator of order 1 with the total symbol

$$\sigma_P(x,\xi) \sim \sum_{j=0}^{\infty} p_{1-j}(x,\xi)$$

with respect to a local frame on every coordinate patch of M, where $p_{1-j}(x,\xi)$ is an $r \times r$ matrix positively homogeneous of degree $1-j$ in ξ. In particular, the principal symbol is given by

$$p(x,\xi) = p_1(x,\xi) = \Big(\sum_{j,k} g^{jk}\xi_j\xi_k\Big)^{1/2} I_r = |\xi| I_r,$$

I_r being the $r \times r$ unit matrix.

We consider the operator $U(t) = \exp(-itP)$, especially $U(2\pi)$ on the $C_{2\pi}$-manifold (M, g). $U(t)$ is a unitary operator for each $t \in \mathbf{R}$ associated with a Fourier integral distribution of the form

$$U(t;\, x, y) = (2\pi)^{-1}\int_{\mathbf{R}^n} e^{i\phi(t;\,x,\theta,y)} K(t;\, x, \theta, y)\, d\theta$$

($n = \dim M$) with respect to a local frame of E on each coordinate patch of M. Here ϕ is a nondegenerate phase function and the $r \times r$ matrix valued amplitude function K is a symbol of order 0. The function $\hat{K} = e^{i\phi}K$ is governed by the evolution equation

$$(2.1) \qquad \frac{1}{i}\partial_t \hat{K} + P\hat{K} = 0 \qquad \left(\partial_t = \frac{\partial}{\partial t}\right),$$

and accordingly the scalar function $\phi = \phi(t; x, \theta, y)$ is the solution of the eikonal equation

$$(2.2) \qquad \partial_t \phi + p(x, \phi_x) = 0, \qquad \phi(0; x, \theta, y) = (x - y) \cdot \theta,$$

($\phi_x = \partial_x \phi$). Let g_t: $T^*M \to T^*M$ be the Hamiltonian flow associated with $p(x, \xi)$, which is just the geodesic flow defined by the metric g on M. By (2.2) the Fourier integral operator $U(t)$ is associated to the homogeneous canonical relation $C_t = \{(x, \xi, y, \eta) \in (T^*M \setminus 0) \times (T^*M \setminus 0);\ (x, \xi) = g_t(y, \eta)\}$ for each $t \in \mathbf{R}$. Hence for small $|t|$ we have locally in (x, y):

$$\phi(t; x, \theta, y) = w(t; x, \theta) - y \cdot \theta,$$

$$K(t; x, \theta, y) \sim \sum_{j=0}^{\infty} K_j(t; x, \theta),$$

$K_j(t; x, \theta)$ being positively homogeneous of degree $(-j)$ in θ (cf. [9, §25.3]), and by (2.1) the K_j's satisfy the transport equations

$$(2.3) \qquad \mathscr{D}K_j = F_j(K_0, K_1, \cdots, K_{j-1}),$$

($F_0 = 0$) with the initial conditions

$$K_0(0; x, \theta) = I_r, \qquad K_j(0; x, \theta) = 0 \quad (j \geqslant 1).$$

In particular, the first transport equation is given by

$$(2.4) \qquad \begin{aligned} &\frac{1}{i}\partial_t K_0 + \frac{1}{i}\sum_{|\alpha|=1} (\partial_\xi^\alpha p)(x, w_x)\partial_x^\alpha K_0 + p_0(x, w_x)K_0 \\ &+ \frac{1}{2i}\sum_{|\alpha|=2} (\partial_\xi^\alpha p)(x, w_x)(\partial_x^\alpha w)K_0 = 0, \end{aligned}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n}$ and such like for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$.

Consider the Lagrangian submanifold

$$C_t' = \{(x, \xi, y, -\eta) \in (T^*M \setminus 0) \times (T^*M \setminus 0); (x, \xi, y, \eta) \in C_t\}.$$

For small $|t|$ we have a local diffeomorphism $T^*U \cong U \times \mathbf{R}^n \ni (x, \theta) \mapsto (x, \xi = w_x, y, -\theta) \in C_t'$, and regard $K_j(t;\ \cdot, \cdot)$ as a function of $(x, \xi) \in T^*M \setminus 0$. Then the first transport equation (2.4) is written as

$$\frac{1}{i}\frac{d}{dt}K_0 + p_0 K_0 + \frac{1}{2i}\sum_{|\alpha|=2} (\partial_\xi^\alpha p)(\partial_x^\alpha w)K_0 = 0,$$

where d/dt means the derivative along orbits of the geodesic flow g_t. We set

$$J(t;\, x,\xi) = \det[\partial(x,\theta)/\partial(x,\xi)] = \det[\partial^2 w/\partial x \partial\theta]^{-1}.$$

Then, by virtue of Liouville's formula, the transport equation can be rewritten as

$$\frac{1}{i}\frac{d}{dt}\left(|J|^{1/2}K_0\right) + \sigma_{\mathrm{sub}}(P)|J|^{1/2}K_0 = 0, \tag{2.5}$$

where $\sigma_{\mathrm{sub}}(P)$ is the subprincipal symbol of P given by

$$\sigma_{\mathrm{sub}}(P)(x,\xi) = p_0(x,\xi) - \frac{1}{2i}\sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\xi) = -\frac{i}{|\xi|}\sum_{j,k} g^{jk}\omega_j \xi_k.$$

The function $|J|^{1/2}K_0$ is just the principal symbol (with the half-density removed) of the Fourier integral operator $U(t)$, which is regarded as a map $(t;\, x,\xi) \mapsto \mathrm{End}(E_x)$ (the space of endomorphisms of the fiber of E over $x \in M$). Note that under a change of coordinates $(x,\theta) \mapsto (\tilde{x},\tilde{\theta})$ we have

$$\exp(\pi i \sigma/4)|J|^{1/2}K_0 = |\tilde{J}|^{1/2}\tilde{K}_0, \tag{2.6}$$

where $\sigma = \mathrm{sgn}[\partial^2 w/\partial\theta^2] - \mathrm{sgn}[\partial^2\tilde{w}/\partial\tilde{\theta}^2]$. (For a symmetric matrix A, $\mathrm{sgn}\, A$ denotes (the number of positive eigenvalues of A) $-$ (the number of negative eigenvalues of A).)

Dividing the interval $[0,2\pi]$ as $0 = t_0 < t_1 < \cdots < t_k = 2\pi$ such that $[t_{j-1}, t_j]$ has sufficiently small width, we consider

$$U(2\pi) = U(t_k, t_{k-1})U(t_{k-1}, t_{k-2}) \cdots U(t_1)$$

with $U(t_j, t_{j-1}) = U(t_j)U(t_{j-1})^{-1}$. Here $U(t_j, t_{j-1})$ is a Fourier integral operator associated with the canonical relation $C^{(j)} = \{(x,\xi,y,\eta);\ (x,\xi) = g_{t_j - t_{j-1}}(y,\eta)\}$ and the principal symbol $|J^{(j)}|^{1/2}K_0^{(j)}$. By virtue of the formula concerning products of Fourier integral operators we see that $U(2\pi)$ is associated to the canonical relation $C_{2\pi} = \{(x,\xi,x,\xi);\ (x,\xi) \in T^*M \setminus 0\}$ because (M,g) is a $C_{2\pi}$-manifold, and accordingly is a classical pseudodifferential operator of order 0. Let $\gamma(t)$ $(0 \leqslant t \leqslant 2\pi)$ be the closed geodesic with the initial condition $\gamma(0) = x$, $\gamma'(0) = (\theta/|\theta|)^{\#}$ ($\#$: $T^*M \to TM$ being the isomorphism induced by g), and put $x_{(j)} = \gamma(t_j)$, $\xi_{(j)}^{\#} = \gamma'(t_j)$. Then the principal symbol of $U(2\pi)$ is given by

$$K_0(2\pi;\, x,\theta) = e^{-\pi i\alpha/2}\left(\prod_{j=1}^{k}|J^{(j)}|^{1/2}\right)K_0^{(k)}K_0^{(k-1)}\cdots K_0^{(1)},$$

where α is a real constant due to the first factor in (2.6), and $|J^{(j)}|^{1/2}K_0^{(j)}$ is the principal symbol of $U(t_j, t_{j-1})$ at $(x_{(j)}, \xi_{(j)})$ with respect to the coordinates in the open set containing the arc $\{\gamma(t);\, t_{j-1} \leqslant t \leqslant t_j\}$. α is called the Maslov index corresponding to the geodesic flow g_t in T^*M (cf. [9, §21.6]).

Let us consider the holonomy of the linear connection $\tilde{d}$ on E. Let $c = c(t) = (c_1(t), \cdots, c_n(t))$ $(0 \leqslant t \leqslant T)$ be a closed curve in M, and let $\tilde{c}(t)$ be a parallel lift of $c(t)$ to E with respect to the connection $\tilde{d}$. Then there is a unitary transformation U of $E_{c(0)}$ such that $\tilde{c}(T) = U\tilde{c}(0)$. We call U the *holonomy* of $\tilde{d}$ along c, and denote it by $Q_{\tilde{d}}(c)$. U is represented by an $r \times r$ unitary matrix $U(c(t))|_{t=T}$ which is governed by the equation:

$$\frac{d}{dt} U(c(t)) + \left(\sum_{j=1}^{n} \omega_j(c(t)) c_j'(t) \right) U(c(t)) = 0,$$

$$U(c(0)) = I_r,$$

with respect to a unitary frame on a neighborhood of a point on c. Comparing this equation with (2.5), we get

$$K_0(2\pi;\, x, \theta) = e^{-\pi i \alpha/2} Q_{\tilde{d}}(\gamma(x, \theta)),$$

where $\gamma(x, \theta)$ denotes the closed geodesic $\gamma(t)$ $(0 \leqslant t \leqslant 2\pi)$ of (M, g) with $\gamma(0) = x$ and $\gamma'(0) = (\theta/|\theta|)^{\#}$.

We define $R = e^{\pi i \alpha/2} U(2\pi) = e^{\pi i \alpha/2} \exp(-2\pi P)$, which is called the *return operator* (cf. [7]).

Proposition 2.1. *R is a unitary classical pseudodifferential operator of order* 0 *whose principal symbol is given by*

$$\sigma(R)(x, \xi) = Q_{\tilde{d}}(\gamma(x, \xi)).$$

We will see that the symbol of the return operator R gives much information about the distribution of eigenvalues of L (or R).

3. Main theorem

Let $S^*M = \{(x, \xi) \in T^*M;\ |\xi| = 1\}$ be the unit cosphere bundle over M. For the $C_{2\pi}$-manifold (M, g) (over which every orbit of the geodesic flow g_t has the least period 2π), each element (x, ξ) of S^*M corresponds to a closed geodesic $\gamma = \gamma(t)$ of (M, g) by $\gamma(0) = x$, $\gamma'(0) = \xi^{\#}$. We set $Q_{\tilde{d}}(x, \xi) = Q_{\tilde{d}}(\gamma)$. Let $\sigma_1(x, \xi), \cdots, \sigma_r(x, \xi)$ be the eigenvalues of $Q_{\tilde{d}}(x, \xi)$, and set

$$\Sigma_{\tilde{d}} = \left\{ \sigma_1(x, \xi), \cdots, \sigma_r(x, \xi);\ (x, \xi) \in S^*M \right\}.$$

Notice that $\sigma_j(x, \xi)$ is continuous in (x, ξ). This fact is derived from the continuous dependence of eigenvalues of a matrix with respect to its components (cf. [10, Chapter II, §5]), and that of the solution of a linear ordinary differential equation with respect to its coefficients and initial data. Thus $\Sigma_{\tilde{d}}$ is a closed subset of the unit circle $S^1 = \{e^{2\pi i \nu};\ 0 \leqslant \nu < 1\}$. We require the following assumption:

$$(*) \qquad \Sigma_{\tilde{d}} \neq S^1.$$

Suppose $e^{2\pi i\beta} \in S^1 \setminus \Sigma_{\tilde{d}}$ $(0 \leqslant \beta < 1)$. Set

$$\lambda_k = (k + \alpha/4)^2, \qquad J_k = (\lambda_{k-1} + \beta C_{k-1}, \lambda_k + \beta C_k] \subset \mathbf{R}$$

for $k = 0, 1, 2, \cdots$, where α is the Maslov index (§2) and $C_k = \lambda_{k+1} - \lambda_k = 2k + (\alpha/2) + 1$. Let $\mu_1^{(k)}, \cdots, \mu_{N_k}^{(k)}$ be the eigenvalues of the Laplacian L associated with the connection $\tilde{d}$ which are contained in the interval J_k. We set points $\tilde{\mu}_j^{(k)}$ on S^1 as

$$\tilde{\mu}_j^{(k)} = \begin{cases} \exp\left(2\pi i \dfrac{\mu_j^{(k)} - \lambda_k}{C_{k-1}}\right) & \left(\mu_j^{(k)} < \lambda_k\right), \\ \exp\left(2\pi i \dfrac{\mu_j^{(k)} - \lambda_k}{C_k}\right) & \left(\mu_j^{(k)} \geqslant \lambda_k\right) \end{cases}$$

(see Figure 1). Our main theorem is the following.

Theorem 3.1. *Let ρ be a continuous function on S^1. Then*

$$\sum_{j=0}^{N_k} \rho\left(\tilde{\mu}_j^{(k)}\right) = c_0(\rho) k^{n-1} + o(k^{n-1}), \tag{3.1}$$

as $k \to \infty$, where $n = \dim M$ and

$$c_0(\rho) = (2\pi)^{-n} \int_{S^*M} \operatorname{Trace} \rho\left(Q_{\tilde{d}}(x, \xi)\right) dm(x, \xi)$$

$$= (2\pi)^{-n} \int_{S^*M} \left[\sum_{j=1}^{r} \rho\left(\sigma_j(x, \xi)\right) \right] dm(x, \xi),$$

dm being the measure on S^*M induced from the canonical measure on T^*M.

Roughly speaking this theorem asserts that as $k \to \infty$ the distribution of $\tilde{\mu}_j^{(k)}$ on S^1 asymptotically converges to that of the eigenvalues of the holonomies of closed geodesics.

Remark. Condition $(*)$ is satisfied by, for example, a connection "near" the flat one on the trivial bundle or "near" the connection which has a clustered spectrum (see §5).

We will prove the theorem similarly to Hörmander [8]. Let $u_j^{(k)}$ be the eigensection of L associated with the eigenvalue $\mu_j^{(k)}$. The $u_j^{(k)}$'s are also eigensections of the operator P and the return operator R. We set

$$P u_j^{(k)} = \bar{\mu}_j^{(k)} u_j^{(k)}, \qquad R u_j^{(k)} = \hat{\mu}_j^{(k)} u_j^{(k)}.$$

The eigenvalues $\hat{\mu}_j^{(k)}$ of R lie on S^1.

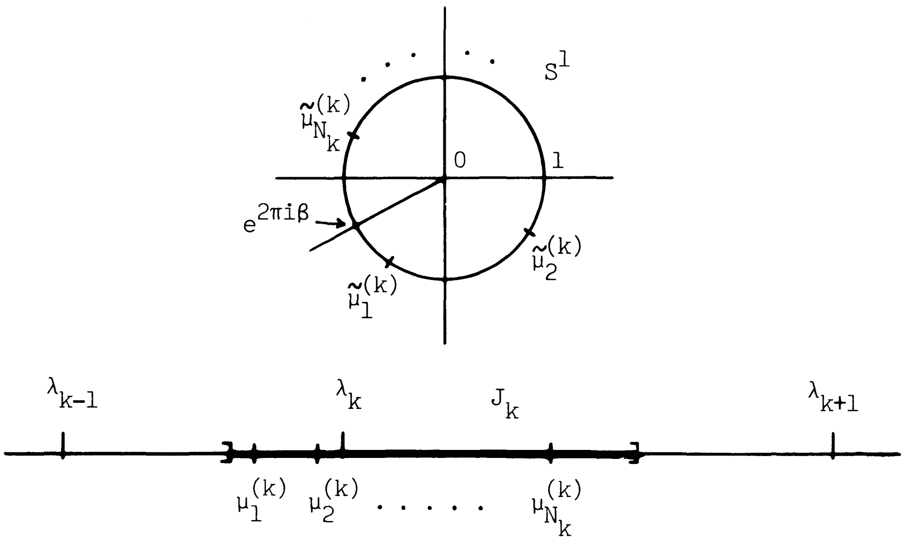

FIGURE 1

Lemma 3.2. *There is a constant M not depending on j and k such that*

$$\left|\arg \tilde{\mu}_j^{(k)} - \arg \hat{\mu}_j^{(k)}\right| \leqslant M/k,$$

where $\arg z$ *denotes the argument of* $z \in S^1 \subset \mathbf{C}$.

Proof. Let $2\pi\tilde{\nu}_j^{(k)}$ and $2\pi\hat{\nu}_j^{(k)}$ be the arguments of $\tilde{\mu}_j^{(k)}$ and $\hat{\mu}_j^{(k)}$, respectively. We have for $\tilde{\nu}_j^{(k)} < 0$,

$$\begin{aligned}
C_{k-1}\tilde{\nu}_j^{(k)} &= \mu_j^{(k)} - \lambda_k = \left(\bar{\mu}_j^{(k)}\right)^2 - c - \lambda_k \\
&= \left(\bar{\mu}_j^{(k)} - \bar{\lambda}_k\right)\left(\bar{\mu}_j^{(k)} + \bar{\lambda}_k\right) - c \qquad \left(\bar{\lambda}_k = k + (\alpha/4)\right) \\
&= \hat{\nu}_j^{(k)}\left(2\bar{\lambda}_k + \hat{\nu}_j^{(k)}\right) - c \\
&= C_{k-1}\hat{\nu}_j^{(k)} + \hat{\nu}_j^{(k)} + \left(\hat{\nu}_j^{(k)}\right)^2 - c.
\end{aligned}$$

Hence,

$$C_{k-1}\left|\tilde{\nu}_j^{(k)} - \hat{\nu}_j^{(k)}\right| = \left|\hat{\nu}_j^{(k)} + \left(\hat{\nu}_j^{(k)}\right)^2 - c\right| \leqslant M'.$$

Similarly, for $\tilde{\nu}_i^{(k)} \geqslant 0$,

$$C_k\left|\tilde{\nu}_j^{(k)} - \hat{\nu}_j^{(k)}\right| = \left|-\hat{\nu}_j^{(k)} + \left(\hat{\nu}_j^{(k)}\right)^2 - c\right| \leqslant M''$$

holds. Thus the lemma is proved.

We will study the asymptotic behavior of $\sum_{j=1}^{N_k} \rho(\hat{\mu}_j^{(k)})$.

Lemma 3.3. *Let $\rho(e^{i\phi}) = \sum_{m=-\infty}^{\infty} \hat{\rho}_m e^{im\phi}$ be the Fourier series of a C^∞ function ρ on S^1. Then*

$$\rho(R) = \sum_{m=-\infty}^{\infty} \hat{\rho}_m R^m$$

is a pseudodifferential operator of order 0 *with the principal symbol*

$$\sigma(\rho(R)) = \rho(\sigma(R)) = \sum_{m=-\infty}^{\infty} \hat{\rho}_m (\sigma(R))^m.$$

Moreover, $\rho(R)u_j^{(k)} = \rho(\hat{\mu}_j^{(k)})u_j^{(k)}$ *holds*.

Proof. See [17, p. 300], for example.

Lemma 3.4. *There exists a classical pseudodifferential operator Q of order* 0 *such that*

$$(P + Q)u_j^{(k)} = (k + \alpha/4)u_j^{(k)}$$

hold except for finitely many $u_j^{(k)}$'s.

Proof. We first show that there is a positive constant δ such that the set $\{e^{2\pi i\nu};\ \beta - \delta < \nu < \beta + \delta\}(\subset S^1)$ contains only finitely many eigenvalues of R. Consider the operator $W = e^{\pi i(1-2\beta)}R - I$. Then, by the theorem concerning the norm of pseudodifferential operators of order 0 (cf. [15], [17, p. 52]) we have

$$\inf_K \| W + K \| = \max_{\sigma \in \Sigma_{\tilde{d}}} |e^{\pi i(1-2\beta)}\sigma - 1|,$$

where the infimum is taken over all compact operators K in $L^2(E)$ (the space of L^2-sections of E), and $\|\cdot\|$ denotes the operator norm as a map of $L^2(E)$ into $L^2(E)$. By virtue of assumption $(*)$ there is a compact operator K such that $W = W' + K$ and $\|W'\| = r' < 2$. Then $\|W'u_j^{(k)}\| \leqslant r'$ and $\lim_{k\to\infty} \max_j \|Ku_j^{(k)}\| = 0$. Therefore, $\|Wu_j^{(k)}\| \leqslant r$, $r' \leqslant r < 2$, except for finitely many $u_j^{(k)}$'s, which shows the assertion. Let $\Gamma = \{\arg z = 2\pi\beta'\}$, β' being sufficiently close to β, be a ray on which there are no eigenvalues of R. Let us define Q by Dunford's integral:

$$Q = \frac{1}{2\pi i} \operatorname{Log} R = \left(\frac{1}{2\pi i}\right)^2 \int_C \operatorname{Log} z (z - R)^{-1}\, dz,$$

where C is a closed curve represented as Figure 2. Then by Seeley [16, Theorem 5] Q is a pseudodifferential operator of order 0, and

$$Qu_j^{(k)} = -\left(\bar{\mu}_j^{(k)} - \alpha/4 - k\right)u_j^{(k)}$$

for large k. Hence $(P + Q)u_j^{(k)} = \{k + (\alpha/4)\}u_j^{(k)}$. q.e.d.

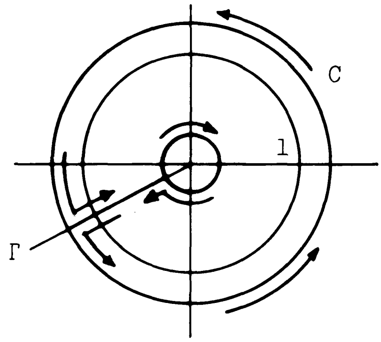

FIGURE 2

Now, set $P_0 = P + Q$. For a C^∞ function ρ we consider

$$\begin{aligned}\hat{\theta}_\rho(t) &= \operatorname{Trace}[\rho(R)\exp\{-it(P_0-(\alpha/4))\}] \\ &= \sum_{k=0}^{\infty}\sum_{j=1}^{N_k}\langle\rho(R)\exp\{-it(P_0-(\alpha/4))\}u_j^{(k)}, u_j^{(k)}\rangle \\ &= \sum_{k=0}^{\infty} a_k e^{-ikt},\end{aligned}$$

where $a_k = \sum_j \rho(\hat{\mu}_j^{(k)})$ for large k. ($\langle\cdot,\cdot\rangle$ is the natural inner product in $C^\infty(E)$.) The distribution $\hat{\theta}_\rho(t)$ is the Fourier transform of

$$\theta_\rho(\lambda) = \sum_{k=0}^{\infty} a_k\delta(\lambda - k).$$

Consider the operator $\rho(R)\exp\{-it(P_0 - (\alpha/4))\}$. The operator $\exp\{-it(P_0 - (\alpha/4))\}$ is a Fourier integral operator of order 0 for each $t \in \mathbf{R}$ (cf. §2). Following Hörmander [8], when $|t|$ is small, the distribution kernel of $\exp\{-it(P_0 - (\alpha/4))\}$ is represented as

$$(2\pi)^{-n}\int q(t;\, x, \xi, y)e^{-itp(y,\xi)+i\psi(x,\xi,y)}\,d\xi,$$

where $p(x, \xi) = |\xi|$ is the principal symbol of P_0,

$$\psi(x,\xi,y) = (x-y)\cdot\xi + O\left(|x-y|^2|\xi|\right) \quad \text{when } x \to y,$$

$$q(t;\, x,\xi,y) \sim \sum_{j=0}^{\infty} q_j(t;\, x,\xi),$$

q_j being positively homogeneous of degree $(-j)$ in ξ, and

$$q_0(0;\, x,\xi) = I_r, \qquad q_j(0;\, x,\xi) = 0 \quad (j \geqslant 1).$$

Therefore, the operator $S(t) = \rho(R)\exp\{-it(P_0 - (\alpha/4))\}$ is a Fourier integral operator of order 0, whose distribution kernel is given by

$$(2\pi)^{-n}\int s(t;\, x, \xi, y)e^{-itp(y,\xi)+i\psi(x,\xi,y)}\,d\xi,$$

for small $|t|$. If we put $\rho(R) = r(x, D_x)$,

$$s(t;\, x, \xi, y) = e^{-i\phi(t;x,\xi,y)}r(x, D_x)\big(e^{i\phi(t;x,\xi,y)}q(t;\, x, \xi, y)\big),$$

where $\phi = \psi - tp$. Hence we have

$$s(t;\, x, \xi, y) \sim \sum_{j=0}^{\infty} s_j(t;\, x, \xi),$$

s_j being positively homogeneous of degree $(-j)$ in ξ with

$$\begin{aligned} s_0(0;\, x, \xi) &= \sigma(\rho(R))(x, \xi) = \rho(\sigma(R))(x, \xi) \\ &= \rho\big(Q_{\bar{a}}(x, \xi/|\xi|)\big). \end{aligned} \tag{3.2}$$

Now we can directly apply the arguments of Hörmander [8, §4]. Set

$$\Theta_\rho(\lambda) = \int_{-\infty}^{\lambda} \theta_\rho(\nu)\,d\nu = \sum_{k\leqslant\lambda} a_k,$$

and let η be a positive and rapidly decreasing function such that the support of $\hat{\eta}$ (the Fourier transform of η) is contained in a small neighborhood of the origin and $\hat{\eta}(0) = 1$. Then we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \eta(\lambda - \mu)\Theta_\rho(\mu)\,d\mu \\ &= \int_{-\infty}^{\lambda}\left(\int_{-\infty}^{\infty} \eta(\nu - \mu)\theta_\rho(\mu)\,d\mu\right)d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\lambda}\left(\int_{-\infty}^{\infty} \hat{\eta}(t)\hat{\theta}_\rho(t)e^{it\nu}\,dt\right)d\nu \\ &= \frac{1}{(2\pi)^{n+1}}\mathrm{Trace}\int_{-\infty}^{\lambda}\left(\int \hat{\eta}(t)s(t;\, x, \xi, x)e^{it(\nu-p(x,\xi))}\,dt\,d\xi\,dx\right)d\nu \\ &\equiv \frac{1}{(2\pi)^n}\mathrm{Trace}\int_{p(x,\xi)\leqslant\lambda} s_0(0;\, x, \xi)\,d\xi\,dx \\ &= \frac{1}{n(2\pi)^n}\left(\mathrm{Trace}\int_{p(x,\xi)=1} s_0(0;\, x, \xi)\,d\xi\,dx\right)\lambda^n, \end{aligned}$$

where $f(\lambda) \equiv g(\lambda)$ means that $f(\lambda) - g(\lambda) = O(\lambda^{n-1})$ as $\lambda \to \infty$ (see [8, pp. 209–211]). Suppose $\rho \geqslant 0$, and $\Theta_\rho(\mu) \geqslant 0$ holds. From the above formula we have $\Theta_\rho(\lambda + 1) - \Theta_\rho(\lambda) \leqslant C(1 + |\lambda|)^{n-1}$, and accordingly

$$\Theta_\rho(\lambda) \equiv \int_{-\infty}^{\infty} \eta(\lambda - \mu)\Theta_\rho(\mu)\,d\mu.$$

By virtue of (3.2) we get

$$\begin{aligned} a_k &= \Theta_\rho(k) - \Theta_\rho(k-1) \\ &= \frac{1}{(2\pi)^{-n}}\left(\operatorname{Trace}\int_{S^*M} \rho(Q_{\tilde{d}}(x,\xi))\, dm(x,\xi)\right) k^{n-1} + o(k^{n-1}). \end{aligned}$$

By putting $\rho = 1$, we get the following.

Proposition 3.5. *As $k \to \infty$,*

$$N_k = (2\pi)^{-n} r \operatorname{vol}(S^*M) k^{n-1} + o(k^{n-1}).$$

Noting Lemma 3.2, we have

$$\left| \sum_{j=1}^{N_k} \rho\left(\tilde{\mu}_j^{(k)}\right) - a_k \right| \leqslant CN_k \left| \arg \tilde{\mu}_j^{(k)} - \arg \hat{\mu}_j^{(k)} \right| = O(k^{n-2}).$$

Thus Theorem 3.1 is proved for a C^∞ function ρ such that $\rho \geqslant 0$. We remark that formula (3.1) is equivalent to

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{j=1}^{N_k} \rho\left(\tilde{\mu}_j^{(k)}\right) = \frac{1}{r \operatorname{vol}(S^*M)} \int_{S^*M} \left[\sum_{j=1}^{r} \rho\left(\sigma_j(x,\xi)\right) \right] dm(x,\xi).$$

Since a nonnegative continuous function on S^1 is uniformly approximated by nonnegative C^∞ functions, we obtain the formula for a nonnegative continuous ρ. Furthermore, for any real-valued (and accordingly complex-valued) continuous ρ the theorem is proved by putting $\rho = \rho_+ - \rho_-$ with $\rho_+ = \max(\rho, 0)$ and $\rho_- = -\min(\rho, 0)$.

4. Case of line bundles

In this section we consider the case where E is a line bundle, and show that assumption $(*)$ is replaced by a weaker assumption.

We introduce the following assumption for a Hermitian line bundle E over the $C_{2\pi}$-manifold (M, g):

$(**)$ There exists a linear connection $\tilde{d}_1$ on E such that $\Sigma_{\tilde{d}_1} \neq S^1$.

Theorem 4.1. *Let ρ be a continuous function on S^1, and suppose $\rho(e^{2\pi i\beta}) = 0$. If assumption $(**)$ is satisfied, then the asymptotic expansion* (3.1) *holds for any linear connection $\tilde{d}$ on E. Here $\tilde{\mu}_j^{(k)}$ is defined as in* §3 *for the above β.*

The proof is carried out by the method of averaged operator developed by Weinstein [19]. Under assumption $(**)$ we see similarly to Lemma 3.4 that there is a pseudodifferential operator P_0 such that $\sigma(P_0)(x,\xi) = |\xi|$ and the spectrum of P_0 is contained in the set $\{\bar{\lambda}_k = k + (\alpha/4);\ k = 0, 1, 2, \cdots\}$

except for finitely many values. Let L, P, and R be the operators associated with the connection $\tilde{d}$ as in §2. Set $Q = P - P_0$, with Q a selfadjoint pseudodifferential operator of order 0. We define the so-called *averaged operator* Q_{av} as

$$Q_{\mathrm{av}} = \frac{1}{2\pi}\int_0^{2\pi} \exp(-itP_0)Q\exp(itP_0)\,dt.$$

Q_{av} is a selfadjoint pseudodifferential operator of order 0, whose principal symbol is given by

$$\sigma(Q_{\mathrm{av}}) = \frac{1}{2\pi}\int_0^{2\pi} g_t^*\sigma(Q)\,dt, \tag{4.1}$$

g_t: $T^*M \to T^*M$ being the geodesic flow (Egorov's theorem). Let $P_{\mathrm{av}} = P_0 + Q_{\mathrm{av}}$, and let $R_{\mathrm{av}} = e^{\pi i\alpha/2}\exp(-2\pi i P_{\mathrm{av}})$. Then we have

Lemma 4.2. (1) $[P_0, Q_{\mathrm{av}}] = [P_0, R_{\mathrm{av}}] = 0$.

(2) $\sigma(R_{\mathrm{av}}) = \sigma(R) = \exp\{-2\pi i\sigma(Q_{\mathrm{av}})\}$.

Proof. (1) Same as Weinstein [18, Lemma 1.1].

(2) Let us set

$$R(t) = \exp(-itP)\exp(itP_0), \tag{4.2a}$$

$$R_{\mathrm{av}}(t) = \exp(-itP_{\mathrm{av}})\exp(itP_0). \tag{4.2b}$$

These are unitary pseudodifferential operators of order 0, and $R = R(2\pi)$ and $R_{\mathrm{av}} = R_{\mathrm{av}}(2\pi)$ hold. Differentiate (4.2), and we get

$$R'(t) = i[R(t), P_0] - iQR(t), \tag{4.3a}$$

$$R'_{\mathrm{av}}(t) = i[R_{\mathrm{av}}(t), P_0] - iQ_{\mathrm{av}}R_{\mathrm{av}}(t). \tag{4.3b}$$

Let $r(t;\ x,\xi)$ and $r_{\mathrm{av}}(t;\ x,\xi)$ be the principal symbols of $R(t)$ and $R_{\mathrm{av}}(t)$, respectively. At $t = 0$, $r(0;\ x,\xi) = r_{\mathrm{av}}(0;\ x,\xi) = 1$, and we have

$$\partial_t r = -H_p r - i\sigma(Q)r, \tag{4.4a}$$

$$\partial_t r_{\mathrm{av}} = -H_p r_{\mathrm{av}} - i\sigma(Q_{\mathrm{av}})r_{\mathrm{av}}, \tag{4.4b}$$

where H_p denotes the Hamiltonian vector field associated with the function $p(x,\xi) = |\xi|$, and $H_p = dg_t/dt$ holds. Note that $\sigma(Q_{\mathrm{av}})$ is constant on each closed orbit of g_t. Integrating equations (4.4), we have

$$\sigma(R_{\mathrm{av}}) = \exp\{-2\pi i\sigma(Q_{\mathrm{av}})\} = \exp\left(-i\int_0^{2\pi} g_t^*\sigma(Q)\,dt\right) = \sigma(R).$$

Consider the one-parameter families of operators: $P(s) = P_0 + sQ$ and $P_{\mathrm{av}}(s) = P_0 + sQ_{\mathrm{av}}$ $(0 \leqslant s \leqslant 1)$. Since these families are analytic with respect to s, all the eigenvalues of $P = P(1)$ and $P_{\mathrm{av}} = P_{\mathrm{av}}(1)$ are those which are split from $\bar{\lambda}_k$ $(k = 0, 1, 2, \cdots)$ (cf. [1, Lemma 3.15]). We denote the eigenvalues of P

and P_{av} split from λ_k by $\bar{\nu}_j^{(k)}$ and $\bar{\kappa}_j^{(k)}$ ($j = 1, \cdots, m_k$), respectively. Then we have

Lemma 4.3. *There is a constant C not depending on j and k such that*

$$\left|\bar{\nu}_j^{(k)} - \bar{\kappa}_j^{(k)}\right| \leqslant C/k.$$

Proof. Consider an analytic one-parameter family of positive selfadjoint operators of order 1:

$$\bar{P}(s) = P_{\mathrm{av}} + s(Q - Q_{\mathrm{av}}) \qquad (0 \leqslant s \leqslant 1).$$

Let $\bar{\nu}_j^{(k)}(s)$ be eigenvalues of $\bar{P}(s)$ with $\bar{\nu}_j^{(k)}(0) = \bar{\kappa}_j^{(k)}$ and $\bar{\nu}_j^{(k)}(1) = \bar{\nu}_j^{(k)}$. Let $\{u_j^{(k)}(s)\}$ be the system of orthonormal eigenvalues of $\bar{P}(s)$ associated with $\bar{\nu}_j^{(k)}(s)$, that is $\bar{P}(s)u_j^{(k)}(s) = \bar{\nu}_j^{(k)}(s)u_j^{(k)}(s)$. Differentiate this equation with respect to s, and we get

$$\frac{d\bar{\nu}_j^{(k)}}{ds}(s) = \langle \bar{P}'(s)u_j^{(k)}(s), u_j^{(k)}(s)\rangle,$$

where $\bar{P}'(s) = d\bar{P}(s)/ds$. We define

$$F = \frac{i}{2\pi}\int_0^{2\pi}\left(\int_0^t \exp(-isP_0)Q\exp(isP_0)\,ds\right)dt.$$

Then we have

$$\begin{aligned}\left[\bar{P}(s), F\right] &= \left[P_0 + Q_{\mathrm{av}} + s(Q - Q_{\mathrm{av}}), F\right] \\ &= Q - Q_{\mathrm{av}} + G(s) = \bar{P}'(s) + G(s),\end{aligned}$$

where $G(s)$ is of order (-1). Hence we obtain

$$\begin{aligned}\frac{d\bar{\nu}_j^{(k)}}{ds}(s) &= \langle \bar{P}(s)Fu_j^{(k)}(s) - F\bar{P}(s)u_j^{(k)}(s) - G(s)u_j^{(k)}(s), u_j^{(k)}(s)\rangle \\ &= \langle -G(s)u_j^{(k)}(s), u_j^{(k)}(s)\rangle.\end{aligned}$$

The operator $G(s)$ defines a continuous map of $H^l(E)$ into $H^{l+1}(E)$, $H^l(E)$ being a Sobolev space with the norm $\|u\|_l = \|\bar{P}(s)^l u\|_{L^2}$. Therefore,

$$\begin{aligned}\left|\frac{d\bar{\nu}_j^{(k)}}{ds}(s)\right| &\leqslant \left\|G(s)u_j^{(k)}(s)\right\|_0\left\|u_j^{(k)}(s)\right\|_0 = \left\|G(s)u_j^{(k)}(s)\right\|_0 \\ &\leqslant C_s\left\|u_j^{(k)}(s)\right\|_{-1} = C_s\left(\bar{\nu}_j^{(k)}(s)\right)^{-1},\end{aligned}$$

C_s being a constant. By this inequality and $k - C_s' < \bar{\nu}_j^{(k)}(s) < k + C_s'$ we get the required estimation.

Proof of Theorem 4.1. Let q be a positive integer satisfying $\|Q_{\mathrm{av}}\| < q$. Then

$$\left\{\bar{\kappa}_j^{(k)}; 1 \leqslant j \leqslant m_k\right\} = \bigcup_{l=-q}^{q}\left\{\bar{\kappa}_j^{(k,l)}; 1 \leqslant j \leqslant m_{k,l}\right\},$$

where $\{\bar{\kappa}_j^{(k,l)};\ 1 \leqslant j \leqslant m_{k,l}\}$ is contained in the interval $\bar{J}_{k,l} = (\bar{\lambda}_{k+l-1} + \beta, \bar{\lambda}_{k+l} + \beta]$. Define a continuous function $\bar{\rho}_k$ on $\mathbf{R}$ by

$$\bar{\rho}_k(t) = \begin{cases} \rho(e^{-2\pi i t}) & (t \in (k-1+\beta, k+\beta]), \\ 0 & (\text{other}). \end{cases}$$

By virtue of Lemma 4.3 we see modulo $O(k^{n-2})$ that

$$\sum_{j=1}^{N_k} \rho\left(\tilde{\mu}_j^{(k)}\right) \equiv \sum_{j=1}^{N_k} \rho\left(\hat{\mu}_j^{(k)}\right) \equiv \sum_{l=-q}^{q} \sum_{j=1}^{m_{k,l}} \bar{\rho}_{-l}\left(\bar{\kappa}_j^{(k+l,-l)} - \bar{\lambda}_{k+l}\right).$$

We have the following asymptotic expansion (proved similarly to Theorem 3.1):

$$\begin{aligned} \sum_{j=1}^{m_{k,l}} \bar{\rho}_{-l}\left(\bar{\kappa}_j^{(k+l,-l)} - \bar{\lambda}_{k+l}\right) &= c_{0,l}(k+l)^{n-1} + o(k^{n-1}) \\ &= c_{0,l}k^{n-1} + o(k^{n-1}), \end{aligned}$$

where

$$c_{0,l} = (2\pi)^{-n} \int_{S^*M} \bar{\rho}_{-l}(\sigma(Q_{\mathrm{av}}))\, dm.$$

By Lemma 4.2(2) we have the asymptotic expansion (3.1) with

$$\begin{aligned} c_0(\rho) = \sum_{l=-q}^{q} c_{0,l} &= (2\pi)^{-n} \int_{S^*M} \rho(\sigma(R))\, dm \\ &= (2\pi)^{-n} \int_{S^*M} \rho(Q_{\tilde{a}}(x,\xi))\, dm(x,\xi). \end{aligned}$$

Thus the proof of Theorem 4.1 is complete.

Remarks. (i) We expect that assumption (**) is satisfied for every line bundle. Further, we conjecture that on each line bundle the holonomies of the harmonic connection are constant along every closed geodesic. This is proved for every compact symmetric space G/K of rank one because every harmonic form on it is invariant under the G-action (which is an isometric action) (cf. [14, p. 26]).

(ii) For the case of vector bundles the method of averaged operator does not go well because the symbols of operators are not necessarily commutative. Egorov's theorem (4.1) does not hold, for example.

5. Cluster theorem

For points $a^{(1)}, \cdots, a^{(s)}$ in the interval $[0,1)$, put

$$\lambda_k^{(m)} = \lambda_k + C_k a^{(m)} \qquad (m = 1, \cdots, s),$$

which are points in $[\lambda_k, \lambda_{k+1})$.

Definition. We say that the spectrum of L *makes clusters of type* $\{a^{(1)},\cdots,a^{(s)}\}$ if there is a positive constant M such that

$$I_m = \bigcup_{k=0}^{\infty} \left[\lambda_k^{(m)} - M, \lambda_k^{(m)} + M\right]$$

contains infinitely many eigenvalues of L for each m $(= 1,\cdots,s)$, and all of the eigenvalues of L are contained in $\bigcup_{m=1}^{s} I_m$.

It is well known (cf. [18]) that the spectrum associated with the flat connection on the trivial bundle over a $C_{2\pi}$-manifold makes clusters of type $\{0\}$.

Theorem 5.1. *The spectrum of L on a vector bundle over a $C_{2\pi}$-manifold makes clusters of type $\{a^{(1)},\cdots,a^{(s)}\}$ if and only if*

$$\Sigma_{\bar{a}} = \{e^{2\pi i a^{(1)}},\cdots,e^{2\pi i a^{(s)}}\}. \tag{5.1}$$

Proof. Assume (5.1). For small $\varepsilon > 0$, set $\bar{I}_{k,\varepsilon}^{(m)} = (\bar{\lambda}_k + a^{(m)} - \varepsilon,\ \bar{\lambda}_k + a^{(m)} + \varepsilon)$ $(k = 0,1,2,\cdots;\ m = 1,\cdots,s)$, and set $\tilde{I}_\varepsilon^{(m)} = e^{-\pi i\alpha/2}\exp(2\pi i \bar{I}_{k,\varepsilon}^{(m)}) \subset S^1$. Then we can see similarly as in the proof of Lemma 3.4 that there are only finitely many eigenvalues of R contained in $S^1 \setminus \bigcup_{m=1}^{s} \tilde{I}_\varepsilon^{(m)}$ for any $\varepsilon > 0$. Let $R' = \exp(-2\pi i a^{(m)})R$. There exist small $\delta' > \delta > 0$ such that no eigenvalues of R' are contained in $\{e^{2\pi i\nu};\ -\delta' < \nu < -\delta,\ \delta < \nu < \delta'\}$. We define a complex valued C^∞ function χ on S^1 such that

$$\chi(e^{2\pi i\nu}) = \begin{cases} e^{2\pi i\nu} & (\nu \in [-\delta,\delta]), \\ 1 & (\nu \in (-1/2,-\delta'] \cup [\delta',1/2]). \end{cases}$$

Consider the unitary pseudodifferential operator $\chi(R')$ (cf. Lemma 3.3). Set $\chi(R') = I + W$. Since the accumulating points of eigenvalues of W are only zero, W is compact, and accordingly of order (-1). Define

$$Q' = \frac{1}{2\pi i}\operatorname{Log}\chi(R') = \frac{1}{2\pi i}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{W^k}{k},$$

and Q' is a selfadjoint pseudodifferential operator of order (-1). Let $P^{(m)} = P - Q'$. Then eigenvalues of P and $P^{(m)}$ are different from each other only in $\bigcup_{k=0}^{\infty}\bar{I}_{k,\delta}^{(m)}$, and the eigenvalues of $P^{(m)}$ in it are $\{\bar{\lambda}_k + a^{(m)};\ k \geqslant 0\}$. Let $\{\bar{\nu}_j^{(k)};\ 1 \leqslant j \leqslant n_k\}$ be the eigenvalues of P in $\bar{I}_{k,\delta}^{(m)}$. Consider a one-parameter family of positive selfadjoint pseudodifferential operators of order 1: $P^{(m)}(s) = P^{(m)} + sQ'$ $(0 \leqslant s \leqslant 1)$, and by the arguments in the proof of Lemma 4.3 we get

$$\left|\bar{\nu}_j^{(k)} - \left(\bar{\lambda}_k + a^{(m)}\right)\right| \leqslant C/k$$

for some constant C. By noticing the relation between eigenvalues of P and $L = P^2 - c$, we get the required assertion.

Conversely, assume the spectrum of L makes clusters. Then assumption $(*)$ is satisfied by the theorem concerning the norm and symbol of a pseudodifferential operator of order 0. Let ρ_ε be a nonnegative function whose support lies in $S^1 \setminus \bigcup_{m=1}^{s} \tilde{I}_\varepsilon^{(m)}$. Then, by Theorem 3.1, $\sum_{j,k} \rho_\varepsilon(\tilde{\mu}_j^{(k)})$ is finite, hence $c_0(\rho_\varepsilon) = 0$ for any $\varepsilon > 0$. Thus $\Sigma_{\tilde{d}} \subset \{e^{2\pi i a^{(1)}}, \cdots, e^{2\pi i a^{(s)}}\}$. On the other hand, by considering ρ for any $\varepsilon > 0$ such that $\operatorname{supp} \rho \subset \tilde{I}_\varepsilon^{(m)}$, we obtain $e^{2\pi i a^{(m)}} \in \Sigma_{\tilde{d}}$. q.e.d.

Noticing the properties of holonomies $Q_{\tilde{d}}(x, \xi)$, we get the following about the type of clusters.

Proposition 5.2. *Suppose the spectrum of L on the vector bundle E makes clusters of type $A = \{a^{(1)}, \cdots, a^{(s)}\}$. Then*

(1) $s \leqslant \operatorname{rank} E$.

(2) *If $a^{(m)}$ $(\neq 0)$ belongs to A, $1 - a^{(m)}$ also belongs to A.*

(3) *If E is a line bundle, the type A of the clusters is $\{0\}$ or $\{1/2\}$. Moreover, the type is uniquely given on a fixed line bundle.*

Proof. (1) This is directly derived by the continuity of eigenvalues of $Q_{\tilde{d}}(x, \xi)$ with respect to $(x, \xi) \in S^*M$.

(2) This is obtained by the fact that $Q_{\tilde{d}}(x, -\xi) = Q_{\tilde{d}}(x, \xi)^{-1}$.

(3) The first part follows from (1) and (2). Let $\tilde{d}_1$ and $\tilde{d}_2$ be two connections on E whose connection forms are $\{\omega_1^{(j)}\}$ and $\{\omega_2^{(j)}\}$, respectively, with respect to a family of local unitary frames of E. Then $\beta = \omega_2^{(j)} - \omega_1^{(j)}$ is a globally defined (independently on j) one-form on M. Suppose spectra associated with $\tilde{d}_1$ and $\tilde{d}_2$ make clusters of type $\{0\}$ and $\{1/2\}$, respectively. Consider a one-parameter family $\tilde{d}(s)$ $(0 \leqslant s \leqslant 1)$ of connections defined by the connection forms $\{\omega_1^{(j)} + s\beta\}$. For a point x_0 on M and a 2-dimensional subspace V of $T_{x_0}^*M$ let us set $V^1 = S_{x_0}^*M \cap V$ (homeomorphic with S^1). Define a map $H: V^1 \times [0, 1] \to S^1$ by $H(x_0, \xi; s) = Q_{\tilde{d}(s)}(x_0, \xi)$. Then, (1) H is continuous, (2) $H(x_0, \xi; 0) = 1$, $H(x_0, \xi; 1) = -1$ for any $(x_0, \xi) \in V^1$, and (3) $H(x_0, -\xi; s) = H(x_0, \xi; s)^{-1}$ for any $s \in [0, 1]$. Obviously, these three conditions are contradictory. q.e.d.

Example 1 (cf. [11, §5.1], [13]). Let $\{E_m\}_{m \in \mathbf{Z}}$ be the set of equivalence classes of line bundles over S^2. On each line bundle E_m there is a unique harmonic connection $\tilde{d}_m$ whose curvature form is $\Omega_m = im\Theta/2$, Θ being the natural volume form of (S^2, g_0) (g_0 is the canonical metric). We have

$$Q_{\tilde{d}_m}(x, \xi) = (-1)^m$$

for every $(x, \xi) \in S^*S^2$. Thus, the spectrum associated with $\tilde{d}_m$ makes clusters of type $\{0\}$ if m is even, and of type $\{1/2\}$ if m is odd.

Example 2. Yang's $SU(2)$-monopole (cf. [20]). Consider the five-dimensional Euclidean space $\mathbf{R}^5 = \{(x_1, x_2, x_3, x_4, x_5)\}$. We choose coordinates $(\zeta_1, \zeta_2, \zeta_3, \theta, r)$ of $\mathbf{R}^5$ such that

$$x_i = (r\sin\theta)2\zeta_i\left(1 + |\zeta|^2\right)^{-1} \qquad (i = 1,2,3),$$

$$x_4 = (r\sin\theta)\left(1 - |\zeta|^2\right)\left(1 + |\zeta|^2\right)^{-1}, \qquad x_5 = r\cos\theta.$$

The unit sphere S^4 in $\mathbf{R}^5$ is given by $r = 1$. Set for $0 < \varepsilon < \pi/2$,

$$S^+ = \left\{(\zeta_1, \zeta_2, \zeta_3, \theta) \in S^4;\ 0 \leqslant \theta < \pi/2 + \varepsilon\right\},$$

$$S^- = \left\{(\zeta_1, \zeta_2, \zeta_3, \theta) \in S^4;\ \pi/2 - \varepsilon < \theta \leqslant \pi\right\}.$$

Let us define $su(2)$-valued 1-forms ω^+ on S^+ and ω^- on S^- by

$$\omega^+ = -\tfrac{1}{2}(1 - \cos\theta)\, dgg^{-1}, \qquad \omega^- = \tfrac{1}{2}(1 + \cos\theta) g^{-1}dg,$$

where

$$g = g(\zeta, \theta) = \frac{1}{1 + |\zeta|^2}\begin{pmatrix} 1 - |\zeta|^2 + 2i\zeta_3 & 2\zeta_2 + 2i\zeta_1 \\ -2\zeta_2 + 2i\zeta_1 & 1 - |\zeta|^2 - 2i\zeta_3 \end{pmatrix} \in SU(2).$$

Let E be the Hermitian vector bundle over S^4 of rank 2 whose transition function on $S^+ \cap S^-$ is $g(\zeta, \theta)$. Then one-forms ω^+ and ω^- induce a linear connection $\tilde{d}_1$ on E. The connection $\tilde{d}_1$ has the following properties:

(P.1) $\tilde{d}_1$ is a Yang-Mills (or harmonic) connection, that is, the functional on the space of linear connections on E,

$$\mathcal{Y}(\tilde{d}) = \int_{S^4} \|\Omega\|^2\, dv(g_0),$$

Ω being the curvature matrix of $\tilde{d}$, attains absolute minima for $\tilde{d}_1$.

(P.2) Let $f\colon E \to E$ be a diffeomorphism such that (1) f maps each fiber E_x isometrically and linearly onto one of the fiber E_y, and (2) $\bar{f}\colon (S^4, g_0) \to (S^4, g_0)$ defined by $\bar{f}(x) = y$ is an isometry. Then, the pull-back connection $f^*\tilde{d}_1$ is gauge equivalent to $\tilde{d}_1$, i.e., $f^*\tilde{d}_1 = \psi^*\tilde{d}_1$ for some gauge transformation ψ.

It follows from (P.2) that the holonomies $Q_{\tilde{d}_1}(x, \xi)$ are constant independently on $(x, \xi) \in S^*S^4$. Calculating for the equator $\theta = \pi/2$, we can see

$$Q_{\tilde{d}_1}(x, \xi) = -I_2$$

for every $(x, \xi) \in S^*S^4$. Thus the spectrum associated with $\tilde{d}_1$ makes clusters of type $\{1/2\}$.

We conclude this article by giving questions relating to the above examples.

Questions. (i) *For what kind of connections do the spectra make clusters?*

(ii) *Do the spectra make clusters for the Yang-Mills connections on the vector bundles over* (S^n, g_0), $(\mathbf{CP}^n, g_0)$, *and so on?*

For the complex line bundle over (S^n, g_0) it was shown in [12] that *the spectrum for the connection* $\tilde{d}$ *makes clusters if and only if the curvature form* Ω *of* $\tilde{d}$ *is odd, i.e.,* $\tau^*\Omega = -\Omega$ *for the antipodal map* τ *of* S^n.

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