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FOLIATIONS AND THE TOPOLOGY OF 3-MANIFOLDS. II

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Introduction

In this paper and its continuation [3] we investigate the following question: Let M be a compact oriented irreducible 3-manifold whose boundary is a torus. If N is obtained by filling ∂M along an essential curve α (i.e., N is obtained by attaching a 2-handle to ∂M along α and then capping off the resulting S^2 with a 3-cell), then does N possess a taut foliation? In this paper we consider the case when $H_2(M) \neq 0$ and in [3] we study the case when N is obtained by zero frame surgery on a knot k in S^3.

Using the existence of foliations on the filled manifolds we obtain a number of topological corollaries.

We now state (for reasons of clarity) a slightly less general version of the main result (Theorems 1.7, 1.8) of this paper.

Theorem. *Let M be an atoroidal Haken 3-manifold whose boundary is a torus and $H_2(M) \neq 0$. Let S be any Thurston norm minimizing surface representing a class of $H_2(M)$. Then with at most 1-exception (up to isotopy) the manifold N obtained by filling M along an essential simple closed curve in ∂M possesses a taut finite depth foliation $\mathscr{F}$ such that S is a leaf of $\mathscr{F}$ and the core of the filling is transverse to $\mathscr{F}$.*

Combining our main result with the work of Alexander, Reeb, Novikov, and Thurston (see [2, 2.5 and 2.8]) and some 3-dimensional topology we obtain the following results.

Corollary 2.14. *Let M be a connected sum of $M_1, \cdots, M_r$ where each M_i is either an oriented torus or sphere bundle over S^1. If k is a knot in M which does not lie in a 3-cell, then k is determined by its complement.*

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Corollary (*see* 2.4 *and* 2.7). *Let M be a Haken atoroidal* 3-*manifold such that ∂M is a torus. Let S be any closed Thurston norm minimizing surface. With at most one exception (up to isotopy) the following holds. If N is obtained by filling M along an essential curve $\alpha \subset \partial M$, then*

(1) *S is norm minimizing in N,*

(2) *S is incompressible in N,*

(3) *the core of the filling is of infinite order in $\pi_1(N)$, and*

(4) *N is irreducible.*

Remarks. By Thurston, any homology class can be represented by a norm minimizing surface.

Closed norm minimizing surfaces in hyperbolic 3-manifolds satisfy the hypotheses of the previous corollary.

Analogous versions exist for manifolds with more boundary components and manifolds which have essential tori but are atoroidal in a weaker sense.

Apply this corollary to the 3-manifold $M - \mathring{N}(k)$ where k is a homotopically trivial knot to obtain:

Corollary 2.9. *Let M be a compact* 3-*manifold such that $H_2(M) \neq 0$. If k is a knot in M such that k is homotopically trivial and $M - \mathring{N}(k)$ is atoroidal and irreducible, then each nontrivial surgery on k yields an irreducible* 3-*manifold N such that k (viewed in N) is of infinite order in $\pi_1(N)$.*

Applying Corollary 2.4 to homologically trivial knots in solid tori we obtain the following results.

Corollary 2.5. *Let k be a knot in $D^2 \times S^1$ of winding number* 0 (*i.e.*, $\langle k, D^2 \times \mathrm{pt.}\rangle = 0$) *such that k does not lie in a* 3-*cell in $D^2 \times S^1$. If M is obtained by nontrivial surgery on k, then $M \neq D^2 \times S^1$. In particular ∂M is incompressible.*

Corollary 2.6. *Let $f_1\colon W \to S^3$, $f_2\colon W \to S^2 \times S^1$ be embeddings of $W = D^2 \times S^1$ such that, for $i = 1, 2$, $f_i(W)$ is a standardly embedded solid torus* (*i.e.*, $f_2(W) = N(\mathrm{pt.} \times S^1)$). *If k is a nontrivial knot in W, then $f_i(k)$ is a nontrivial knot for some i, i.e., it does not bound a* 2-*cell.*

Corollaries 2.5 and 2.6 respectively give positive solutions to three old problems of J. Martin [9, problems 1.18A, B, C]. Problem 1.18B, an interesting special case of Martin's first problem, has been independently solved by Bleiler and Scharlemann [1] using a very complicated combinatorial argument.

The paper is organized as follows. We assume that the reader is familiar with the results and terminology of [2]. In §0 we introduce new terminology and prove some basic lemmas which were implicitly proven in [2, §3]. In §1 we prove Theorem 1.7 and a generalization. In §2 we give topological consequences of the foliations results of §1.

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0. Preliminaries, corrections, and amplifications

The notation and terminology in this paper will follow that of [2].

Definitions 0.1. Let (M, γ) be a sutured manifold. A *product annulus* in the sutured manifold (M, γ) is an annulus A properly embedded in M such that $\partial A \subset R(\gamma)$, $\partial A \cap R_+(\gamma) \neq \varnothing$, and $\partial A \cap R_-(\gamma) \neq \varnothing$. A *product disc* is a disc D properly embedded in M such that $|\partial D \cap \gamma| = 2$ and $\partial D \cap \gamma$ consists of essential arcs in γ. Product discs and annuli detect where a sutured manifold is locally a product. (M, γ) is a *product sutured manifold* if $M = R \times I$, $\gamma = \partial R \times I$, $R_+(\gamma) = R \times 1$, and $R_-(\gamma) = R \times 0$.

Definitions 0.2. A sequence

$$(M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

of sutured manifold decompositions is said to be *groomed* if each (M_i, γ_i) is taut, no subset of toral components of $S_i \cup R(\gamma_{i-1})$ is homologically trivial in $H_2(M_{i-1})$ and for each component V of $R(\gamma_{i-1})$, either $S_i \cap V$ is a union of parallel, coherently oriented, nonseparating closed curves or $S_i \cap V$ is a union of arcs such that for each component δ of ∂V, $|\delta \cap \partial S_i| = |\langle \delta, \partial S_i \rangle|$. A groomed sutured manifold sequence is *well groomed* if it has the additional property that for each component V of $R(\gamma_{i-1})$ which intersects S_i in arcs, $V \cap S_i$ is a union of parallel arcs. A properly embedded surface S in (M, γ) is said to be (*well*) *groomed* if the decomposition $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ is (well) groomed.

If N is a codimension-0 submanifold of M, then define δN, the *frontier* of N, to be $\partial N \cap (\overline{M - N})$.

Correction 0.3. Correct [2, Definition 3.1] to further require that "if $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ is a sutured manifold decomposition, then no component of ∂S bounds a disc in $R(\gamma)$ and no component of S is a disc D with $\partial D \subset R(\gamma)$." Without these additional trivial properties, as pointed out by S. Miyoshi and M. Scharlemann many of the results of [2] (e.g., 3.5, 3.12, and 5.1) are not "precisely stated." With the latter excluded (which among other things outlaws Reeb components) some of the results of [2] (e.g., 3.5 and 5.1) can be stated more clearly. All the decompositions in [2] have these additional properties. Here are examples to illustrate what goes wrong.

EXAMPLE 0.3. (A) Pick a sutured manifold (M, γ) which is taut such that the sutured manifold (M_1, γ_1) obtained by attaching a 2-handle to an annular component of γ is not taut. (E.g., $M_1 = B^3$ and $\gamma_1 \subset \partial B^3$ consists of three concentric annuli. $M = M_1 - \mathring{N}(k)$ where k is an arc with endpoints in the disc components of $R(\gamma_1)$.) The taut $(M, \gamma) \cup (D^2 \times I, (\partial D^2) \times I)$ is obtained by decomposing (M_1, γ_1) along a product annulus. This contradicts both Lemma 3.5 and Lemma 3.12 of [2].

EXAMPLE 0.3. (B) Let (M_1, γ_1) be a taut sutured manifold with a planar component $P \subset R_+(\gamma_1)$ (e.g., $((S^1 \times I) \times I, \partial(S^1 \times I) \times I)$). Let (M, γ) be the sutured manifold obtained by gluing M_1 to $D^2 \times S^1$ by identifying P with a subset of $\partial D^2 \times S^1$ which is disjoint from a meridinal disc D. Let $\gamma = \gamma_1 - \partial P$. The taut $(M_1, \gamma_1) \cup (D^2 \times I, \partial(D^2) \times I)$ is obtained by decomposing the nontaut (M, γ) along $P \cup D$ again contradicting [2, Lemma 3.5].

Lemma 0.4 (*corrected Lemma* 3.5 *of* [2]). *Let* $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ *be a sutured manifold decomposition. If* (M', γ') *is taut, then* (M, γ) *is taut.*

Proof. Recall [2, 3.1] that $M' = M - \mathring{N}(S)$, so topologically M is obtained by gluing a $S_+ \subset R_+(\gamma')$ to a $S_- \subset R_-(\gamma')$. Also $\gamma = \gamma'$ modified along ∂S.

It suffices to assume that M and M' are irreducible and M is connected.

Case 1. $R(\gamma)$ *is compressible*. Let E be a nontrivial compressing disc of $R(\gamma)$ such that $|E \cap S| + |E \cap \partial S|$ is minimal. There exists (after possibly reversing the transverse orientation on (M, γ)) a disc component E_1 of $E \cap M'$ such that $\partial E_1 \subset R_+(\gamma')$. Let E' be the disc in $R_+(\gamma)$ which ∂E_1 bounds. Note that possibly $\partial E' \subset S_+$. By minimality some component of $\partial S_+ \cap E'$ is a circle. Therefore either some component δ of ∂S bounds a disc in $R_+(\gamma)$ or some disc component D of S satisfies $\partial D \subset R(\gamma)$.

Case 2. $R(\gamma)$ *is incompressible*. If (M, γ) is not taut, then (after possibly reversing the transverse orientation on (M, γ)) there exists an incompressible surface T such that $\partial T = s(\gamma)$ and $[T] = [R_+(\gamma)] \in H_2(M, \gamma)$, and $\chi(T) > \chi(R_+(\gamma))$. By deleting components of T if necessary we can assume that no component of T bounds a submanifold of M. Assume that such a T has been chosen so that $|T \cap S|$ is minimal. By minimality no component of $S - T$ is a disc E with $\partial E \subset T$. Conversely no component of $T - S$ is a disc E with $\partial E \subset S$ for, as in Case 1, such a disc implies that either (M', γ') is not taut or some component δ of ∂S bounds a disc in $R(\gamma)$. Let T' be the surface obtained by doing oriented cut and paste with T and S. Since $T \cup R_+(\gamma) \cup$ (half of γ) bounds in M, T' can be isotoped slightly so that $T' \cap S = \varnothing$. T' is isotopic to a surface (also called T') in M' such that $\partial T' = s(\gamma')$ and

$[T'] = [R_+(\gamma')] \in H_2(M', \gamma')$. By construction no component of T' is a 2-sphere so $x(T') = -\chi(T') + |\text{Disc components of } T'|$. Also $\chi(T') = \chi(T) + \chi(S)$, $\chi(R_+(\gamma')) = \chi(R_+(\gamma)) + \chi(S) = -x(R_+(\gamma') + |\text{Disc components of } R_+(\gamma')|)$. It follows that either $x(T') < x(R_+(\gamma'))$ or $|\text{Disc components of } T'| > |\text{Disc components of } R_+(\gamma')|$. In either case (M', γ') is not taut. q.e.d.

The following three lemmas were implicitly proven in §3 of [2].

Lemma 0.5. *Let* $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ *be a sutured manifold decomposition such that* (M', γ') *is taut. Let* R *be a subsurface of* $R(\gamma)$ *(i.e.,* R *inherits an orientation from* $R(\gamma)$*). Isotope* R *slightly so that it is properly embedded in* M *and* $\partial R \cap \gamma$ *consists of essential curves. Let* S_1 *be the surface obtained by doing oriented cut and paste to* S *and* R*. Then the sutured manifold* (M_1, γ_1) *obtained by decomposing* (M, γ) *along* S_1 *is taut.*

Proof. There exists a set D of product discs and annuli in (M_1, γ_1) such that the sutured manifold obtained by decomposing (M_1, γ_1) along D is a union of (M', γ') and a product sutured manifold. The result now follows from [2, Lemma 3.5].

Lemma 0.6. *Let* $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ *be a sutured manifold decomposition such that* (M', γ') *is taut and for each component* δ *of* $\partial R(\gamma)$, $|\delta \cap \partial S| = |\langle \delta, \partial S \rangle|$. *If* λ *is a set of pairwise disjoint oriented simple essential curves in* $R(\gamma)$ *such that for each component* δ *of* $\partial R(\gamma)$, $|\lambda \cap \delta| = |\langle \lambda, \delta \rangle|$ *and* $[\lambda] = [\partial S \cap R(\gamma)] \in H_1(R(\gamma), \partial R(\gamma))$, *then there exists a sutured manifold decomposition* $(M, \gamma) \overset{T}{\rightsquigarrow} (M_1, \gamma_1)$ *such that* (M_1, γ_1) *is taut,* $\partial T \cap R(\gamma) = \lambda$, *and* $[T] = [S] \in H_2(M, \partial M)$. *In fact,* T *can be chosen such that* $T \cap (M - \mathring{N}(\partial M)) = S \cap (M - \mathring{N}(\partial M))$.

Proof. (Compare 3.9, 3.10 in [2].) Find a sequence $\partial S \cap R(\gamma) = \lambda_0, \lambda_1, \cdots, \lambda_p = \lambda$ such that $\partial W_i = \lambda_{i+1} \cup (-\lambda_i)$ for some compact subsurface W_i of $R(\gamma)$ (i.e., W_i inherits orientation from $R(\gamma)$), where $-\lambda_i$ denotes λ_i oppositely oriented. Each λ_i should have the property that for each component δ of $\partial R(\gamma)$, $|\lambda_i \cap \delta| = |\langle \lambda_i, \delta \rangle|$. Let $T_0 = S$, let $T_{i+1} = T_i \cup W_i$ isotoped slightly such that $\partial T_{i+1} \cap R(\gamma) = \lambda_{i+1}$, and let $T = T_p$. By Lemma 0.5, (M_1, γ_1) is taut.

Lemma 0.7. *Let* (M, γ) *be a taut sutured manifold. If* $0 \neq y \in H_2(M, \partial M)$, *then there exists a properly embedded groomed surface* S *such that* $[S] = y \in H_2(M, \partial M)$.

Proof. Let N be the manifold obtained by doubling M along $R(\gamma)$. Let $z \in H_2(N, \partial N)$ be the class obtained by doubling y. Now apply Lemma 3.11 of [2] to obtain the norm minimizing surface $T \subset N$ where $z = [T] \in H_2(N, \partial N)$. It follows from the first paragraph of [2, p. 457] that if δ is a component of $\partial R(\gamma)$, then $|\delta \cap T| = |\langle \delta, T \rangle|$.

Let S' be a union of components of $T \cap M$ such that S' is nonseparating and $[S', \partial S'] = [T, \partial T] \cap M = y \in H_2(M, \partial M)$. By the proof of Lemma 3.13 of [2] we conclude that the decomposition $(M, \gamma) \overset{S'}{\rightsquigarrow} (M', \gamma')$ yields a taut sutured manifold. If V is a component of $R(\gamma)$ such that $\partial V \cap S' \neq \varnothing$, then $V \cap S'$ is homologous in $H_1(V, \partial V)$ to a set of arcs λ such that $|\delta \cap \lambda| = |\langle \delta, \lambda \rangle|$ for each component δ of ∂V. By Lemma 3.9 of [2] if $\partial V \cap S' = \varnothing$, then $V \cap S'$ is homologous to a set λ of parallel coherently oriented closed curves. By Lemma 0.6 we can modify S' near ∂M to find the desired S.

1. Foliating certain knot spaces

Definition 1.1. Let M be a 3-manifold such that ∂M contains a torus T. N is said to be obtained by *filling* (or *Dehn filling*) M along an essential simple closed curve α in T if N is obtained by first attaching a 2-handle to M along α and then capping off the resulting 2-sphere with a 3-cell. N will often be denoted $M(\alpha)$.

$M(\alpha)$ is obtained by attaching a solid torus called the *filling* to M, and $M = M(\alpha) - \mathring{N}(k)$ where k is the *core* of the filling. If k is a knot in M, then one can view a Dehn surgery on k as a filling on $M - \mathring{N}(k)$. Therefore the objects core and filling are "well defined" in the surgered manifold.

Remarks 1.2. Filling M along isotopic simple closed curves yields homeomorphic manifolds.

Definition 1.3. An I(njective)-*cobordism* between closed connected oriented surfaces T_0 and T_1 is a compact oriented 3-manifold V such that $\partial V = T_0 \cup T_1$ and for $i = 0, 1$ the induced maps j_i: $H_1(T_i) \to H_1(V)$ are injective.

Recall the following old and well-known result which follows from the fact that the alternating sum of the ranks of the terms of the long exact homology sequence is zero and from the fact that Poincaré duality implies that

$$\operatorname{rank}(H_1(M)) = \operatorname{rank}(H^2(M, \partial M)) = \operatorname{rank}(H_2(M, \partial M)),$$
$$\operatorname{rank}(H_1(M, \partial M)) = \operatorname{rank}(H^2(M)) = \operatorname{rank}(H_2(M)).$$

Lemma 1.4. *If M is a compact oriented 3-manifold, then*

$$\operatorname{rank}(\operatorname{image} \partial\colon H_2(M, \partial M) \to H_1(\partial M)) = \operatorname{rank}(\ker i\colon H_1(\partial M) \to H_1(M)) = \tfrac{1}{2}\operatorname{rank}(H_1(\partial M)),$$

where ∂ and i are the boundary and inclusion maps of the long exact homology sequence.

Lemma 1.5. *If V is an I-cobordism between surfaces T_0 and T_1, then*

(a) *there exists a natural isomorphism φ: $H_1(T_0; \mathbb{Q}) \to H_1(T_1; \mathbb{Q})$ defined by $j_1(\varphi(x)) = j_0(x)$ where j_i: $H_1(T_i; \mathbb{Q}) \to H_1(V; \mathbb{Q})$ for $i = 1, 0$ is the natural inclusion map.*

(b) genus T_1 = genus T_0.

(c) *If T_1 and T_0 are tori, then there exists a bijection t: {isotopy classes of oriented essential simple closed curves in T_1} $\to$ {isotopy classes of oriented essential simple closed curves in T_0} defined by the equation*

$$j_1[t(\alpha)] = \lambda j_0[\alpha] \quad \textit{for some } \lambda \in \mathbb{Q},\ \lambda > 0.$$

Finally if V is a cobordism between surfaces T_0 and T_1 of the same genus and j_0 is injective, then V is an I-cobordism.

Proof. (a), (b), and (c) follow immediately from Lemma 1.4 and the fact that for a torus T there is a bijection between projective classes of $H_1(T)$ and oriented essential simple closed curves.

To complete the proof note that Lemma 1.4 implies that $j_1(H_1(T_1; \mathbb{Q})) \subset j_0(H_1(T_0; \mathbb{Q}))$. Therefore if $a \in H_1(T_1)$, there exists a properly embedded surface R such that $[\partial R \cap T_1] = ra$ for some nonzero integer r. If $b \in \ker j_1$, then $0 = \langle b, R\rangle = \langle b, a\rangle$. Since this is true for every a, $b = 0$.

Definition 1.6. Let M be a compact oriented 3-manifold, S a properly embedded oriented surface in M, and P a toral component of ∂M such that $P \cap S = \varnothing$. M is *S_P-atoroidal* if boundary parallel tori are the only surfaces which are I-cobordant to P by cobordisms contained in $M - S$. If k is a knot in M disjoint from S and $M - \mathring{N}(k)$ is S_P-atoroidal, where $P = \partial N(k)$, then we say that M is *S_k-atoroidal*. If the boundary component P or knot k is understood, then we say that M is S-atoroidal.

Theorem 1.7. *Let M be a Haken 3-manifold whose boundary is a torus and $H_2(M) \neq 0$. Let S be any Thurston norm minimizing surface representing a nontrivial class $y \in H_2(M)$. If M is S-atoroidal, then with at most one exception (up to isotopy) the manifold N obtained by filling M along an essential simple closed curve in ∂M possesses taut foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ such that*

(1) *S is a compact leaf of both $\mathscr{F}_0$ and $\mathscr{F}_1$,*

(2) *$\mathscr{F}_0$ is of finite depth,*

(3) *$\mathscr{F}_1$ is C^∞ except possibly along the toral components of S, and*

(4) *the core C of the filling is transverse to $\mathscr{F}_0$ and $\mathscr{F}_1$ and C is of infinite order in $\pi_1(N)$.*

Proof. Step 1. There exists a sequence

$$(M, \partial M) = (M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

of sutured manifold decompositions with the following properties:

(1) Each (M_i, γ_i) is taut and each separating component of S_{i+1} is a product disc.

(2) Some component of γ_n is the torus ∂M.

(3) (M_n, γ_n) is a union of a product sutured manifold and a sutured manifold $(H, \delta) = T^2 \times I$ where $\partial M = T^2 \times 0$, $\delta \cap (T^2 \times 1) \neq \varnothing$.

Proof of Step 1. Suppose that the sequence

$$(M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_k, \gamma_k)$$

has been constructed and satisfies (1) and (2) of Step 1. Let S_{k+1} be a maximal set of nonboundary parallel, pairwise disjoint, pairwise nonparallel properly embedded product discs in (M_k, γ_k). Extend our sequence by $(M_k, \gamma_k) \overset{S_{k+1}}{\rightsquigarrow} (M_{k+1}, \gamma_{k+1})$ to obtain a new one which still satisfies (1) and (2) and furthermore (M_{k+1}, γ_{k+1}) possesses no nonboundary parallel product discs.

Let (H, δ) be the component of (M_{k+1}, γ_{k+1}) containing ∂M. ∂H contains no 2-spheres since (H, δ) is taut. If the natural map $H_1(\partial H - \partial M) \to H_1(H)$ is injective, then Lemmas 1.4 and 1.5 imply that H is an I-cobordism and $\partial H - \partial M$ is a torus Q. Since M is S-atoroidal, $H = \partial M \times I$. Apply (1) to the integer $i \leqslant k$ for which H first appears as a component of M_{i+1} to conclude that $Q \cap \gamma_{k+1} \neq \varnothing$. Now use Theorem 4.2 of [2] to find a sutured manifold hierarchy of $(M_{k+1}, \gamma_{k+1}) - (H, \delta)$ and use this hierarchy to extend our sequence to one satisfying Step 1.

On the other hand if $H_1(\partial H - \partial M) \to H_1(H)$ is not injective, then there exists a nontrivial $y \in H_2(H, \partial H)$ such that $[y \cap \partial M] = 0$. By Lemma 0.4 there exists a nonseparating surface S_{k+2} such that $y = [S_{k+2}] = H_2(H, \partial H)$ and such that the decomposition $(M_{k+1}, \gamma_{k+1}) \overset{S_{k+2}}{\rightsquigarrow} (M_{k+2}, \gamma_{k+2})$ extends our sequence and still satisfies (1) and (2). Note that $\partial S_{k+2} \cap \partial M = \varnothing$, since $[y \cap \partial M] = 0$. By §4 of [2] we obtain the sutured manifold complexity inequalities $C(M_i, \gamma_i) \leqslant C(M_j, \gamma_j)$ if $i > j$ and $C(M_{k+2}, \gamma_{k+2}) < C(M_{k+1}, \gamma_{k+1})$. The preceding paragraph and the fact that sutured manifold complexity takes values in a well-ordered set imply that our sequence will satisfy the conclusions of Step 1 after a finite number of extensions.

Step 2. With at most one exception (up to isotopy) the manifold N obtained by filling M along an essential simple closed curve in ∂M possesses a sutured manifold hierarchy whose first term is obtained by decomposing along S.

Proof of Step 2. For each i attach a solid torus to each $\partial M \subset \partial M_i$ to obtain the sequence

$$(N, \varnothing) = (N_0, \delta_0) \overset{S_1}{\rightsquigarrow} (N_1, \delta_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (N_n, \delta_n).$$

By Step 2, the component H of N_n containing ∂M satisfies $H = D^2 \times S^1$ and $s(\gamma_n) \cap H$ is a union of $0 \neq 2r$ parallel essential simple closed curves in $\partial D^2 \times S^1$. Let λ be one such suture. If N was not obtained from the filling, corresponding to the case where λ bounds a disc in H, then decomposing (N_n, δ_n) along a $D^2 \times$ pt. extends our sequence to the desired sutured manifold hierarchy.

Proof of Theorem 1.7. Apply Theorem 5.1 of [2] to the sutured manifold hierarchy of Step 2 to obtain the desired foliations. By construction S is a leaf of these foliations and the core C of the filling is transverse to the foliation. By Novikov [10], C is homotopically of infinite order. q.e.d.

There is a natural generalization of Theorem 1.7 in the case that M has more than one boundary component. We state this generalization in terms of sutured manifolds. The proof is verbatim the proof of Theorem 1.7.

Theorem 1.8. *Let (M, γ) be a taut connected sutured manifold, P be a toral component of γ, and suppose that $H_2(M, \partial M - P) \neq 0$. Let S be a properly embedded nonseparating surface in M such that $\partial S \cap P = \varnothing$ and the decomposition $(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$ yields a taut sutured manifold. If M is S_P-atoroidal, then with at most one exception (up to isotopy) the sutured manifold $(N, \gamma - P) = (N, \delta)$ obtained by filling M along an essential simple closed curve in P possesses foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ such that*:

(1) *$\mathscr{F}_0$ and $\mathscr{F}_1$ are tangent to $R(\delta)$.*

(2) *For $i = 1, 2$, $\mathscr{F}_i$ is transverse to δ and $\mathscr{F}_i \mid \delta$ has no Reeb components.*

(3) *Every leaf of $\mathscr{F}_0$ and $\mathscr{F}_1$ nontrivially intersects a transverse closed curve or transverse arc with endpoints in $R(\gamma)$ unless $\partial N = R_+(\delta)$ or $R_-(\delta)$ in which case this holds only for interior leaves.*

(4) *If $\partial S \subset \gamma$, then S is a leaf of $\mathscr{F}_0$ and $\mathscr{F}_1$.*

(5) *$\mathscr{F}_0$ is of finite depth.*

(6) *$\mathscr{F}_1$ is C^∞ except possibly along toral components of $R(\gamma) \cup S$.*

(7) *The core C of the filling is transverse to $\mathscr{F}_0$ and $\mathscr{F}_1$ and C is of infinite order in $\pi_1(N)$.*

2. Applications

Definitions 2.1. Two unoriented knots k_1 and k_2 are *equivalent* in the compact oriented 3-manifold M if there exists an orientation preserving homeomorphism $f: (M, k_1) \to (M, k_2)$. A knot k is *determined by its complement* in the oriented 3-manifold M if for any knot $k' \subset M$, k is equivalent to k' if and only if there exists an orientation preserving homeomorphism between $M - k$ and $M - k'$.

Remarks 2.2. There exists an orientation preserving homeomorphism between $M - \mathring{N}(k)$ and $M - \mathring{N}(k')$ if and only if there exists an orientation preserving homeomorphism between $M - k$ and $M - k'$.

If every nontrivial Dehn surgery on k yields a manifold not homeomorphic (by an orientation preserving homeomorphism) to M, then k is determined by its complement. If $M - \mathring{N}(k)$ is irreducible, then it is well known that the converse is almost true. If a nontrivial surgery on a knot determined by its complement yields M, then there exists a homeomorphism f: $(M - \mathring{N}(k), \partial N(k)) \to (M - \mathring{N}(k), \partial N(k))$ such that $f(m) \neq \pm m$ where m is the meridian of k. Since f is also orientation preserving and $f(\lambda) = \pm\lambda$, where λ is the longitude, f is of infinite order. It follows by the Jaco-Shalen, Johannson theory [7], [8] that either λ bounds a disc D in $M - \mathring{N}(k)$ or two copies of λ bound an annulus A such that $0 \neq [\partial A] \in H_1(\partial N(k))$. (If $M - \mathring{N}(k) \neq D^2 \times S^1$, then by [8, §27] f^n is isotopic to a map which is a product of "Dehn twists" along embedded annuli and tori. A Dehn twist, as defined in [8], along a surface $T \subset M$ is a homeomorphism of M which is the identity on $M - \mathring{N}(T)$. Only twists along annuli A such that $0 \neq [\partial A] \in H_1(\partial N(k))$ will affect $f \mid \partial N(k)$, thus $\partial[A] = 2[\lambda]$.) Conversely if such a D or A exists, then there exist an infinite number of distinct surgeries on k which yield M.

Corollary 2.3. *If k is a knot in $S^2 \times S^1$ which is not contained in a 3-cell, then k is determined by its complement.*

Proof. If k is homologically nontrivial, then the trivial surgery is the unique surgery on k which yields a homology $S^2 \times S^1$.

If k is homologically trivial, let S be a closed connected nonseparating orientable surface of smallest genus. By hypothesis, genus $S > 0$. Since S is compressible in $S^2 \times S^1$ it will follow from Corollary 2.4 that S will remain a minimal genus (therefore incompressible) nonseparating surface in any manifold obtained by nontrivial surgery on k. Since $S^2 \times S^1$ contains no incompressible surfaces of positive genus the result follows.

Corollary 2.4. *Let M be a Haken manifold whose boundary is a nonempty union of tori. Let S be a Thurston norm minimizing surface representing an element of $H_2(M, \partial M)$ and let P be a component of ∂M such that $P \cap S = \varnothing$. Then with at most one exception (up to isotopy) S remains norm minimizing in each manifold $M(\alpha)$ obtained by filling M along an essential simple closed curve α in P. In particular S remains incompressible in all but at most one manifold obtained by filling P.*

If M is also S_P-atoroidal then $M(\alpha)$ is also irreducible.

Proof. Haken (see [6]) has shown that for any compact 3-manifold there is an integer n such that any n pairwise disjoint incompressible surfaces have two distinct ones parallel. It follows that in M there is an I-cobordism V (possibly a boundary collar) from P to an incompressible torus T such that $V \cap S = \varnothing$ and $N = M - \mathring{V}$ is S_T-atoroidal. Note that S is norm minimizing in N.

Let $M(\alpha)$ and $V(\alpha)$ (resp. $N(\delta)$) denote the manifolds obtained by filling M and V (resp. N) along an essential simple closed curve α (resp. δ) on P (resp. T). Let $\beta \subset T$ be either the one exceptional simple closed curve which arose by applying Theorem 1.8 (or equivalently Theorem 1.7 if $\partial M = P$) to N (where T is the distinguished boundary component and $\gamma = \partial N$) or $\varnothing$ if no exceptional curve arose. Therefore if $\delta \neq \beta$, then by Thurston (see [2, 2.5]) S is norm minimizing in $N(\delta)$. By Novikov and Alexander (see [2, 2.8]) $N(\delta)$ is irreducible. Let α be a simple closed curve in P such that $t(\alpha) \neq \beta$ (t as defined in Lemma 1.5).

If T is incompressible in $M(\alpha)$, then S remains norm minimizing in $M(\alpha)$. To see this, let R be an incompressible surface such that $[R] = [S] \in H_2(M(\alpha), \partial M(\alpha))$. $[R \cap T] = [S \cap T] = 0 \in H_1(T)$, so one can obtain a surface H by attaching annuli to $R - \mathring{N}(T)$ and deleting $R \cap V(\alpha)$ which satisfies $[H] = [R] \in H_2(M(\alpha), \partial M(\alpha))$, $\chi(H) \geqslant \chi(R)$, and $H \cap V(\alpha) = \varnothing$. Since S is norm minimizing, $\chi(S) \geqslant \chi(H) \geqslant \chi(R)$.

If T is compressible in $M(\alpha)$, then by Lemma 1.5 the compression must be along the curve $t(\alpha) \subset T$. Therefore $t(\alpha)$ bounds a disc in $V(\alpha)$, $V(\alpha) = D^2 \times S^1 \# W(\alpha)$, and $M(\alpha) = N(t(\alpha)) \# W(\alpha)$ for some closed 3-manifold $W(\alpha)$. Since S remains norm minimizing in $N(t(\alpha))$, S remains norm minimizing in $M(\alpha)$.

Since $t(\alpha) = \beta$ for at most one α the result follows.

Corollary 2.5. *Let k be a knot in $D^2 \times S^1$ of winding number* 0 (*i.e.*, $\langle k, D^2 \times \text{pt.}\rangle = 0$) *such that k is not contained in a* 3*-cell in $D^2 \times S^1$. If M is obtained by nontrivial surgery on k, then $M \neq D^2 \times S^1$. In particular ∂M is incompressible.*

Proof. Let S be a Thurston norm minimizing surface in $D^2 \times S^1 - \mathring{N}(k)$ such that ∂S is a meridian of $\partial D^2 \times S^1$. S is not a disc since k does not lie in a 3-cell. By Corollary 2.4 S remains norm minimizing, hence incompressible in M. ∂M is incompressible because any compressing disc of ∂M would, by Lemma 1.4, have boundary isotopic to ∂S contradicting the incompressibility of S.

Corollary 2.6. *Let f_1: $W \to S^3$, f_2: $W \to S^2 \times S^1$ be embeddings of $W = D^2 \times S^1$ such that, for $i = 1, 2$, $f_i(W)$ is a standardly embedded solid torus* [*i.e.*, $f_2(W) = N(\text{pt.} \times S^1)$]. *If k is a nontrivial knot in W, then $f_i(k)$ is a nontrivial knot for some i, i.e., it does not bound a* 2*-cell.*

Proof. We need only consider the case that k is homologically trivial in W, for otherwise $f_2(k)$ is homotopically nontrivial. If k lies in a 3-cell but is nontrivial, then $f_1(k)$ is nontrivial in S^3. Therefore we can assume that k is homologically trivial and that k does not lie in a 3-cell. Let $S \subset W - \mathring{N}(k)$ be a minimal genus surface with $\partial S = k$. By Corollary 2.4, for all but at most one filling on ∂W, S remains minimal genus. Therefore for some i, $f_i(S)$ is a minimal genus surface for $f_i(k)$.

Remark. Corollaries 2.5 and 2.6 give positive solutions to three old problems of J. Martin (see [9], problems 1.18A, B, C). Problem 1.18B, an important special case of 1.18A, was independently proven by Scharlemann and Bleiler [1]. I would like to thank B. Wajnryb for bringing Martin's third problem to my attention.

The problem of when nontrivial surgery on a knot $k \subset D^2 \times S^1$ yields $D^2 \times S^1$ is further investigated in [4].

Corollary 2.7. *Let M be a Haken 3-manifold such that ∂M contains a toral component P. If $H_2(M, \partial M - P) \neq 0$, then with at most one exception (up to isotopy) the following holds. If $M(\alpha)$ is obtained by filling M along the essential simple closed curve $\alpha \subset P$, then the core of the filling is of infinite order in $\pi_1(N)$.*

Proof. Let C denote the core of the filling.

Case 1. ∂M is a union of tori and M is S_P-atoroidal for S a norm minimizing surface representing an element of $H_2(M, \partial M)$ such that $\partial S \cap P = \varnothing$.

Proof. Apply Theorem 1.8.

Case 2. ∂M is a union of tori.

Proof. Let S be a norm minimizing surface representing an element of $H_2(M, \partial M)$ such that $\partial S \cap P = \varnothing$. Let T, V, N, and β be as in the proof of Corollary 2.4. Let α be an essential simple closed curve in P such that $t(\alpha) \neq \beta$. C is of infinite order in $H_1(V(\alpha))$, hence in $\pi_1(V(\alpha))$, since the natural map $H_1(P) \to H_1(V)$ is injective and C generates $H_1(P)/[\alpha] = \mathbb{Z}$. If T is incompressible, then C is of infinite order in $\pi_1(M(\alpha))$. If T is compressible, then as in Corollary 2.4, $V(\alpha) = (D^2 \times S^1) \# W(\alpha)$ for $W(\alpha)$ a closed 3-manifold. Note that each $\partial D^2 \times \text{pt.}$ is isotopic to $t(\alpha)$. C is homotopic in $\pi_1(V(\alpha)) = \mathbb{Z} * \pi_1(W)$ to a word which projects nontrivially into $\mathbb{Z}$, since it is homologous to an element of $H_1(T)$ outside the subspace spanned by $[t(\alpha)]$. Therefore C projects in $\pi_1(M(\alpha)) = \pi_1(N(t(\alpha))) * \pi_1(W(\alpha))$ to an element of $\pi_1(N(\alpha))$ demonstrated to have infinite order in Case 1. Since $t(\alpha) = \beta$ for at most one α, Case 2 follows.

Case 3. General case.

Proof. If ∂M is incompressible, then $D(M) = \{M \text{ doubled along } M - P\}$ satisfies the hypothesis of Case 2. Since ∂M is incompressible, the result for $D(M)$ implies the result for M.

If ∂M is compressible, then let H be the manifold obtained by splitting M open along a maximal set of compressing discs to obtain a 3-manifold with incompressible boundary. The component H containing P satisfies the hypotheses of the corollary and $\pi_1(H(\alpha))$ is a free factor in $\pi_1(M(\alpha))$. By the previous paragraph the conclusions of the corollary hold for H and therefore M.

Corollary 2.8. *Let M be a compact oriented manifold whose interior supports a hyperbolic structure with finite volume. Suppose that there exist P a component of ∂M and S a Thurston norm minimizing surface representing an element of $H_2(M, \partial M)$ such that $S \cap P = \varnothing$. With at most one exception (up to isotopy) the following holds. If α is an essential simple closed curve in P, then the manifold obtained by filling M along α possesses a taut foliation $\mathscr{F}$ such that $\mathscr{F}$ is transverse to ∂N and $\mathscr{F} \mid \partial N$ has no 2-dimensional Reeb components. Furthermore N is irreducible, S is norm minimizing in N, and the core of the filling is of infinite order in $\pi_1(N)$.*

Proof. By Thurston [12] M is irreducible and atoroidal. The result follows from Theorem 1.8 and the results of Novikov and Thurston (see [2, 2.5 and 2.8]).

Corollary 2.9. *Let M be a compact 3-manifold such that $H_2(M) \neq 0$. Let k be a knot in M such that k is homotopically trivial and $M - \mathring{N}(k)$ is atoroidal and irreducible. Then each nontrivial Dehn surgery on k yields an irreducible 3-manifold N and the core of the filling is of infinite order in $\pi_1(N)$.*

Proof. k is homotopically trivial so $H_2(M - \mathring{N}(k)) \neq 0$. In particular there exists a closed Thurston norm minimizing surface $S \neq S^2$. Apply Theorem 1.7 to $M - \mathring{N}(k)$ to conclude that for all but possibly one surgery on k, the surgered manifold possesses a taut foliation $\mathscr{F}$ such that S is a leaf and the core of the filling is transverse to $\mathscr{F}$. Since k is homotopically trivial in M the exceptional surgery must be the trivial one. The result now follows from Novikov's and Thurston's theorems.

Corollary 2.10. *If M is a closed reducible 3-manifold and k is a knot in M such that $M - \mathring{N}(k)$ is irreducible and S-atoroidal for some S representing a norm minimizing element of $H_2(M - \mathring{N}(k))$, then nontrivial surgery on k yields an irreducible 3-manifold.*

Remark 2.11. Corollaries 2.9 and 2.10 give a tiny bit of supporting evidence for the Poincaré conjecture. They show that all surgeries on a large class of knots yield manifolds which contain no fake 3-cells.

Theorem 2.12. *Let M be a torus bundle over S^1. If k is a knot in M which does not lie in a 3-cell, then k is determined by its complement.*

Idea of Proof. M determines the homology group $H_1(M)$ and a set $\mathscr{A}$ of matrices in $\mathrm{SL}(2, \mathbb{Z})$. Incompressible surfaces in M satisfy the geometric properties of Lemma 2.13. We will show that any nontrivial surgery on k yields a manifold not homeomorphic to M (by an orientation preserving homeomorphism) by showing that one of these properties or invariants is changed. By Remark 2.2 the result will follow.

Lemma 2.13. *Let M be an oriented torus bundle over S^1. M determines a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ with the property that the monodromy of any torus fibration of M is either of the form BAB^{-1} for $B \in \mathrm{SL}(2,\mathbb{Z})$ or $CA^{-1}C^{-1}$ for $C \in [\mathrm{GL}(2,\mathbb{Z}) - \mathrm{SL}(2,\mathbb{Z})]$.*

Let S be an incompressible surface in M and let $\mathscr{F}$ denote a torus fibration. Then $\chi(S) = 0$, and S is isotopic to either a leaf of $\mathscr{F}$ or to a surface transverse to $\mathscr{F}$. If S is a nonseparating torus, then S is the fiber of a torus fibration of M. If S is nonorientable, then $|S \cap F| = 1$ for each leaf F of $\mathscr{F}$. If S is orientable and separating, then $S = \partial N(K)$ for some Klein bottle K.

Proof. Let F be a fiber of $\mathscr{F}$. A choice of basis for $H_1(F)$ determines a unique monodromy map $A \in \mathrm{SL}(2,\mathbb{Z})$ as follows. Let $\mathbf{X}$ be any vector field in M transverse to $\mathscr{F}$ such that $\mathbf{X}$ and the basis of F induce the given orientation on M. A is the linear homeomorphism isotopic to the homeomorphism of F obtained by flowing F to F along $\mathbf{X}$. Two choices of $\mathbf{X}$ yield homotopic maps, hence the same A. If one reverses the order of basis elements of $H_1(F)$ and uses $-\mathbf{X}$ (to maintain the same orientation of M) one obtains the monodromy map

$$A' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Two bases of F which differ by an element $B \in \mathrm{SL}(2,\mathbb{Z})$ yield monodromies which are conjugate by B. We have now shown that for a fixed fiber F of M the set of monodromies is that given in the first paragraph of the lemma.

Let $\tilde{M} = T^2 \times \mathbb{R}$ be the infinite cyclic cover corresponding to this fibration and $\tilde{S}$ a lift of S. Since $\pi_1(\tilde{M})$ is abelian so is $\pi_1(\tilde{S})$. Therefore $\chi(S) = 0$. By [11], S can be isotoped to be either transverse to $\mathscr{F}$ or be a leaf of $\mathscr{F}$. If F is a leaf of $\mathscr{F}$, then $F - \mathring{N}(S)$ is a union of annuli; hence, $M - \mathring{N}(S)$ is an annulus bundle over S^1. Therefore if S is a nonseparating torus, $M - \mathring{N}(S) = S \times I$, so S is a fiber of a fibration. If S is oriented and separating, then $S = \partial N(K)$ for some Klein bottle K. It remains to show $|F \cap S| = 1$ if S is a Klein bottle.

If M contains a Klein bottle K, then by choosing a basis for $H_1(F)$ so that the first generator g is a component of $F \cap K$ we see that $Ag = -g$ so that the monodromy is $\left[\begin{smallmatrix} -1 & r \\ 0 & -1 \end{smallmatrix}\right]$. To show that $|F \cap K| = 1$ it suffices to assume that $r = 0$, for if a K existed (in a bundle with $r \neq 0$) with $|F \cap K| \neq 1$, then by doing a Dehn surgery along a component of $F \cap K$ one would have constructed an example in the bundle with $r = 0$. If $r = 0$ there exist Klein bottles

K_1, K_2 in M such that $K_1 \cap F$ is isotopic to g and $K_2 \cap F$ is isotopic to the second generator. Observe that $|F \cap K| = |F \cap K \cap K_2|$ is odd since K is nonorientable. Since there are only four isotopy classes of essential simple closed curves in a Klein bottle (hence, in K_2 and K), the only possibility is that in both K and K_2, $K \cap K_2$ is the core curve of a Möbius band and $|F \cap K| = 1$.

We now complete the proof of the first assertion of the lemma. Let F' be the fiber of another fibration. By choosing a basis for $H_1(F)$ so that the first generator g is a component of $F \cap F'$, we see that $Ag = g$ so that the monodromy is $[\begin{smallmatrix}1 & r\\ 0 & 1\end{smallmatrix}]$. Therefore M is explicitly constructed by doing a multiple Dehn twist to an essential simple closed curve C in a fiber of the 3-torus T^3. If $M \neq T^3$, then fibers of M come from geodesic tori in T^3 which are disjoint from C. By following such tori as they sweep out M, we observe that the monodromy of any fibration of M can be taken to be A.

Proof of Theorem 2.12. *Case* 1. $0 \neq [k] \in H_1(M, \mathbf{Q})$.

Proof. Only the trivial surgery on k yields a manifold with the same homology groups as M.

Case 2. $k \cap F \neq \varnothing$ for every nonseparating torus in M.

Proof. Assume that k is homologically trivial in $H_1(M, \mathbf{Q})$. Let S be a Thurston norm minimizing surface in $M - \mathring{N}(k)$ homologous in $H_2(M)$ to $[F]$ where F is a torus fiber. By the hypothesis of Case 2, $\chi(S) < 0$. By Corollary 2.4, if N is obtained by nontrivial surgery on k, then S is norm minimizing hence incompressible in N. By Lemma 2.13, N is not a torus bundle over S^1. q.e.d.

By Lemma 2.13 and Case 2 we can assume that there exists a fiber F which is disjoint from k. Let $Q = M - \mathring{N}(F) = T^2 \times I$ and $Q' = Q - \mathring{N}(k)$.

Case 3. There exists no annulus A of the form $\gamma \times I \subset Q$ such that γ is an essential simple closed curve in T^2 and $A \cap k = \varnothing$.

Proof. Glue the components of ∂Q together to obtain T^3. Let S be a norm minimizing surface representing a class in $H_2(T^3 - \mathring{N}(k))$ distinct from F. By the hypothesis of Case 3, $\chi(S) < 0$. As in Case 2 we conclude that nontrivial surgery on $K \subset T^3$ does not yield a T^2 bundle over S^1. This implies that N surgered along k does not yield $T^2 \times I$, so M nontrivially surgered along k does not yield a T^2 bundle over S^1. q.e.d.

Let $W = Q - \mathring{N}(A) = D^2 \times S^1$, where A is an annulus in Q' of the form $\gamma \times I$ where γ is an essential simple closed curve in F.

Case 4. k is homologically trivial in W.

Proof. If a nontrivial surgery on k yields M, then the corresponding surgery on N must yield a $T^2 \times I$. Since $T^2 \times I$ is irreducible and every essential torus is boundary parallel, the corresponding surgery on W must yield a $D^2 \times S^1$. By Corollary 2.5, ∂W becomes incompressible after nontrivial surgery along k.

Case 5. k is not isotopic to the core of W and the result holds for knots isotopic to the core of W.

Proof. By Lemma 1.5 if distinct surgeries on $k \subset W$ yield $D^2 \times S^1$, then the corresponding curves on ∂W which bound discs in the surgered W are distinct. Therefore if distinct surgeries on k yield M, then distinct surgeries on C, the core of W, yield M.

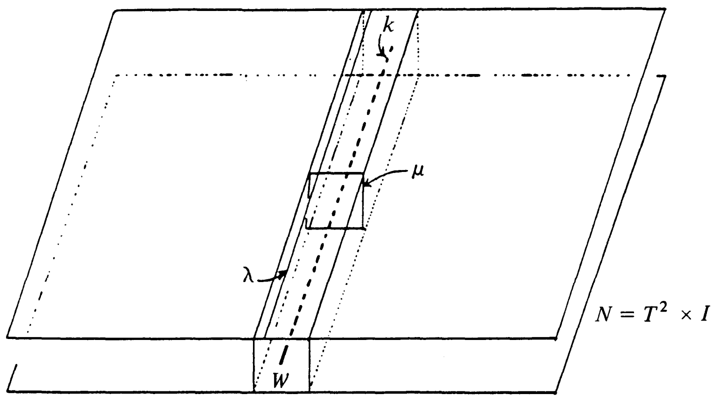

FIGURE 2.1

Case 6. k is contained in a fiber F of a fibration $\mathscr{F}$ over S^1.

Proof. First note that this case is equivalent to the case that k is isotopic to the core of W, since W can be viewed as $N(C)$, where C is a simple closed curve in a fiber. Give ∂W a natural framing (Figure 2.1) as follows. Let μ be an essential simple closed curve in ∂W which bounds a disc in W and let λ be an essential simple closed curve in $\partial N \cap \partial W$. Let $N(m/p)$, $W(m/p)$, and $M(m/p)$ denote the manifolds N, W, and M surgered along k so that the meridian of the filling glues to a curve homologous to $m\mu + p\lambda$. By changing the sign of p, if necessary, we can assume that $m \geqslant 0$.

We now show that if $m \neq 1$, then $N(m/p) \neq T^2 \times I$, hence $M(m/p) \neq M$. $N(m/p)$ is obtained by attaching a thickened annulus to a solid torus (Figure 2.2(a)), where a meridian of the solid torus intersects a boundary component of the annulus m times. Therefore $N(m/p)$ can be viewed as a mapping cylinder of a thickened bouquet of m circles, hence $\pi_1(N(m/p))$ contains a free group on m generators. Therefore $N(m/p) = T^2 \times I$ implies that $m = 1$. One could argue geometrically as follows. Decompose the sutured manifold (H, γ) (where $H = N(m/p)$ and $R_+(\gamma)$ and $R_-(\gamma)$ each consist of a boundary torus) along a product annulus A to obtain the sutured manifold (H', γ') (Figure 2.2(b)), where $H' = D^2 \times S^1$ and each suture of γ' is homotopic to m times a generator of $\pi_1(D^2 \times S^1)$. Since decomposing along product annuli preserves the quality of being a product and (H', γ') is a product if and only if $m = 1$, our assertion follows.

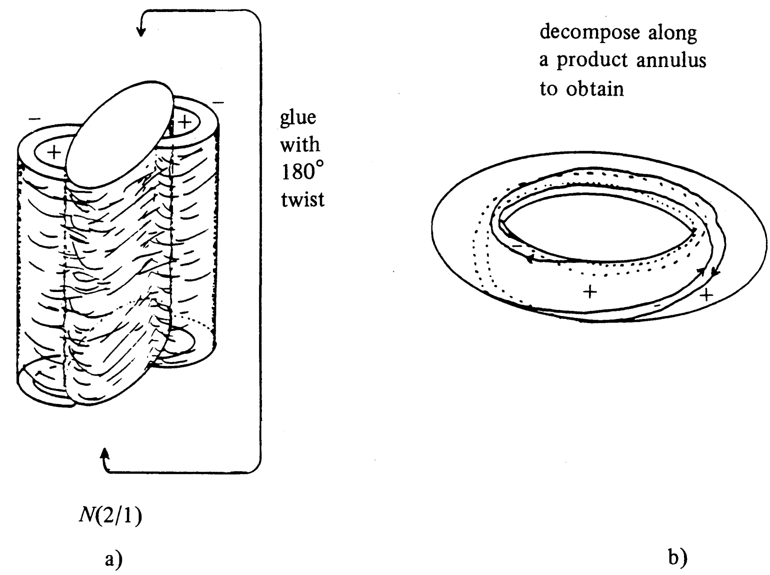

FIGURE 2.2

$M(1/p)$ is a torus bundle over S^1 whose monodromy is given by the matrix $A\left[\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right] = D$, where $A \in \mathrm{SL}(2, \mathbb{Z})$ is the monodromy of the fibration $\mathscr{F}$. We now show that if $M(1/p)$ is homeomorphic to M, by an orientation preserving homeomorphism, then $p = 0$. By Lemma 2.13 such a homeomorphism exists if and only if D is conjugate to $A^{\pm 1}$ by a $B \in \mathrm{GL}(2, \mathbb{Z})$ where $\det(B) = \pm 1$. Since trace is an invariant of conjugacy the existence of such a B implies that A is upper triangular. An easy calculation now yields $p = 0$.

Corollary 2.14. *Let M be a connected sum of $M_1, \cdots, M_r$ where each M_i is either an oriented torus or sphere bundle over S^1. If k is a knot in M which does not lie in a* 3*-cell, then k is determined by its complement.*

Proof. By the Milnor-Kneser prime decomposition theorem [5] it suffices to consider the case that $M - k$ is irreducible. If $r = 1$, then the result follows by Corollary 2.3 and Theorem 2.12. If $r \geqslant 2$, then $H_2(M - k) \neq 0$, so a norm minimizing surface S disjoint from k exists. If M is S-atoroidal, then apply Theorem 1.7 to conclude that nontrivial surgery on k yields an irreducible

3-manifold. Otherwise apply the argument of the proof of Corollary 2.4 to conclude that nontrivial surgery on k yields $N\#W$, where $H_2(W) = 0$ and N is irreducible.

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