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DUPIN HYPERSURFACES, GROUP ACTIONS AND THE DOUBLE MAPPING CYLINDER

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Introduction

A basic question in Riemannian geometry asks how geometric properties of a manifold are reflected in its topology. Here we shall consider the case of closed *Dupin hypersurfaces* in the euclidian sphere S^{n+1}, $n \geqslant 1$. Recall [18] that these are closed submanifolds E^n for which, in particular, the number of eigenvalues (principal curvatures) $\lambda_i(x)$ of the second fundamental form is independent of $x \in E^n$. In this case [4] the eigenspaces of $\lambda_i(x)$ define a foliation of E, and E is Dupin if λ_i is constant on each leaf. In the special case that the λ_i are constant on E, E is called an *isoparametric hypersurface*.

An analogous question in transformation groups asks what topological restrictions are forced on a closed manifold, M, by the existence of a "large" effective action of a compact Lie group, G. The simplest case is that of transitive actions, $M = G/H$. We shall consider the next simplest case—when the principal orbits G/H have codimension one; these are called *cohomogeneity one actions*. We shall confine ourselves to the case of *strict* cohomogeneity one actions; i.e. excluding the fairly trivial case when all orbits have the same dimension.

An important class of examples of (strict) cohomogeneity one actions is that of linear actions on S^{n+1}, $n \geqslant 1$. Since the principal orbits of these actions are precisely the *homogeneous* isoparametric hypersurfaces, they may be thought of as "linear models" or "test spaces" in the sense of Hsiang for both general Dupin hypersurfaces and general cohomogeneity one actions. These linear actions, moreover, have been classified by Hsiang and Lawson [13].

Now let j: $E^n \subset M^{n+1}$ be either a closed Dupin hypersurface ($M = S^{n+1}$) or the principal orbit of a strict cohomogeneity one action, and let F denote a

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path component of the homotopy fiber of the inclusion $j\colon E \to M$. In the case of our linear models, $E = G/H$ is contractible in $M = S^{n+1}$ and so $F \cong G/H \times \Omega S^{n+1}$. Thus the list of possible homotopy fibers for the linear models can be read off from [13].

The general situation is described in our

Theorem A. *Suppose $E \subset M^{n+1}$ $(n \geqslant 1)$ is either a closed Dupin hypersurface or the principal orbit of a strict cohomogeneity one action $(M, G$ closed$)$. Then F is a nilpotent space, and there is a linear model whose homotopy fiber has the same fundamental group, integral homology, and rational homotopy type as F.*

Call a space X *rationally Ω-elliptic* if the total rational homotopy, $\pi_*(\Omega X, *) \otimes \mathbb{Q}$, of the loop space is finite dimensional. A classic theorem of Serre [19] asserts that Lie groups are rationally Ω-elliptic; hence so are homogeneous spaces. Since the homotopy fibers for our linear models have the form $G/H \times \Omega S^{n+1}$ they, too, are rationally Ω-elliptic. Thus Theorem A gives

Theorem B. *Let E be a closed Dupin hypersurface in S^{n+1} and let M^{n+1} admit a strict cohomogeneity one action $(M, G$ closed$)$. Then E and M are rationally Ω-elliptic.*

Remarks. 1. Theorem B is new even for isoparametric hypersurfaces.

2. Since E and M are rationally Ω-elliptic it follows from [9, Corollary 2.3] and [6] that

$$\pi_i(E) \otimes \mathbb{Q} = 0, \quad i \geqslant 2n, \quad \text{and} \qquad \pi_i(M) \otimes \mathbb{Q} = 0, \quad i \geqslant 2n+2.$$

3. If M admits a nonstrict cohomogeneity one action, then some covering space of M has the homotopy type of a principal orbit. Thus M is still rationally Ω-elliptic.

4. If M admits a transitive action it is a homogeneous space and so rationally Ω-elliptic. On the other hand, $(S^k \times S^l)\#(S^k \times S^l)$, $k \geqslant l \geqslant 2$, is not rationally Ω-elliptic [10, Theorem 5.4] but does admit a cohomogeneity two $\mathrm{SO}(k) \times \mathrm{SO}(l)$ action.

Our topological results can be applied to yield new geometric information about closed Dupin hypersurfaces $E \subset S^{n+1}$. In fact Thorbergson obtains explicit formulae [23], [15] connecting the homology $H_*(E; \mathbb{Z}_2)$ with the multiplicities of the principal curvatures, λ_i. In particular he shows that the number, g, of principal curvature functions satisfies $2g = \dim H_*(E; \mathbb{Z}_2)$ and that as in the case [17] of isoparametric hypersurfaces, $g = 1, 2, 3, 4$ or 6.

On the other hand, in this case $F \cong E \times \Omega S^{n+1}$ and so we are able to compute $H_*(E; \mathbb{Z})$ directly (cf. §2). We recover the fact that $g = 1, 2, 3, 4$ or 6 independently and then, with the aid of Thorbergson's formulas, establish

Theorem C. *Let E be a closed Dupin hypersurface. Then*:

(i) *E is nilpotent.*

(ii) *There are two integers k, l (possibly equal) such that each principal curvature has multiplicity k or l.*

(iii) *The integral homology of E determines k, l, and the number, g, of principal curvatures. Conversely, g, k, and l determine the fundamental group, integral homology, and rational homotopy type of E.*

(iv) *The integers g, k, l satisfy the following restrictions.*

(a) *If $k \neq l$, then $g = 2$ or 4 and k and l are each the multiplicity of $g/2$ principal curvatures.*

(b) *If $g = 3$, then $k = 1, 2, 4$ or 8.*

(c) *If $g = 4$ and $k = l$, then $k = 1$ or 2. If $g = 4$ and $k > l \geqslant 2$, then $k + l$ is odd.*

(d) *If $g = 6$, then $k = 1$ or 2.*

Remarks. 1. In §2 we list (Table 2.2) all the possibilities for $\pi_1(E)$, $H_*(E; \mathbb{Z})$, and the $\mathbb{Q}$-homotopy type of E in terms of k, l, and g.

2. It is immediate from Theorem C that $n = \frac{1}{2}(k + l)g$. In particular

$$g = 1 \Leftrightarrow k + l > n; \quad g = 2 \Leftrightarrow k + l = n; \quad g = 3, 4 \text{ or } 6 \Leftrightarrow k + l < n.$$

3. Except when $g = 4$ the above restrictions in (ii) and (iv) include all known restrictions (cf. Münzner [17] and Abresch [1]) for isoparametric hypersurfaces. Theorems A and C thus provide further evidence for the conjecture that a closed Dupin hypersurface is Lie equivalent (cf. [18]) to an isoparametric hypersurface. (This is classical for $g = 1$ and has been proved by Cecil and Ryan [4] for $g = 2$ and by Miyaoka [15] for $g = 3$.)

The feature common to the inclusion $j\colon E^n \to M^{n+1}$ of a Dupin hypersurface or the principal orbit of a strict cohomogeneity one action is that in both cases there is a decomposition

$$M = DB_0 \cup_E DB_1$$

of M as the union of two linear disc bundles $DB_0 \to B_0$, $DB_1 \to B_1$ with common boundary E. In the case of group actions this is due to Mostert [16], and in the case of isoparametric hypersurfaces to Münzner [17]; cf. also [11]. Thorbergson [23] obtains a decomposition (as above) of S^{n+1} into (non-smooth) ball bundles for general closed Dupin hypersurfaces and we extend his argument in §2 to get the linear disc bundle decomposition in this case.

Our analysis of the homotopy fiber, however, is chiefly carried out in a significantly wider context. We consider continuous maps $\phi_i\colon E \to B_i$, $i = 0, 1$, between general topological spaces such that over each path component of each B_i the homotopy fiber of ϕ_i has the weak homotopy type of a sphere (possibly of differing dimensions). Then M, above, generalizes to the *double mapping cylinder*

$$DE = B_0 \cup_{\phi_0} (E \times I) \cup_{\phi_1} B_1$$

and j becomes the inclusion $x \to (x, \frac{1}{2})$ of E into DE.

In spite of this increase in generality, we still reach almost the same conclusion as in Theorem A. In fact, let F be a path component of the homotopy fiber of j. Then we prove

Theorem D. *With the hypotheses above, F is a nilpotent, rationally Ω-elliptic space. Moreover, if E has finite Lusternik-Schnirelmann category then F has the fundamental group, integral homology, and rational homotopy type of*

(i) *a point or a sphere, or*

(ii) *the homotopy fiber of a linear model, as in Theorem* A, *or*

(iii) *one of the two "exceptional spaces" $A_4(4) \times \Omega S^{17}$ and $A_6(4) \times \Omega S^{25}$ defined in* §1.

Remarks. 1. Case (i) does not occur unless either all homotopy fibers of the ϕ_i's are weakly S^0 or E has infinitely many components.

2. We do not know if the exceptional spaces in (iii) actually occur.

The body of the paper is organized as follows:

1. Classification theorems.
2. Dupin hypersurfaces.
3. Fundamental group and homology.
4. Rational homotopy theory.
5. Rational classification.
6. Integral restrictions.

In §1 we state more precise classification theorems (1.3, 1.8), which immediately imply Theorems A, B, and D above. In §2 we apply the classification theorems to prove Theorem C. After an elementary topological reduction (Proposition 1.2) we calculate, in §3, the fundamental group and integral homology of a homotopy fiber, F, and prove that F is nilpotent (Theorem 1.3). The classification by rational homotopy type (Theorem 1.8) is carried out in §§4, 5, and 6. A more detailed plan for the proof of this result is given after its statement in §1.

1. Classification theorems

Consider continuous maps ϕ_i: $E \to B_i$, $i = 0, 1$, as in the introduction. Thus over each path component of each B_i the homotopy fiber of ϕ_i has the weak homotopy type of a sphere. By abuse of language we shall refer to these simply as the *fiber spheres*, S^k of ϕ_i.

Denote by DE the double mapping cylinder of (ϕ_0, ϕ_1) and by F a path component of a homotopy fiber of the inclusion j: $E \to DE$. We call F a *cylinder fiber* for DE.

In §3 we reduce the analysis of cylinder fibers to the important special case

$$\text{(1.1)} \qquad \begin{array}{l} E,\ B_0,\ B_1 \text{ are connected CW complexes;} \\ DE \text{ is 1-connected;} \\ \text{The fiber spheres } S^k,\ S^l \text{ of } \phi_0,\ \phi_1 \text{ satisfy } k,\ l \geqslant 1. \end{array}$$

We may, in any case, suppose DE path connected.

Our reduction is then contained in

Proposition 1.2. *Let F be a cylinder fiber for the double mapping cylinder DE of* (ϕ_0, ϕ_1). *If DE is path connected, then F has the weak homotopy type of*

(i) *a point, or*

(ii) *a sphere* S^k, $k \geqslant 1$, *or*

(iii) *the cylinder fiber* $\bar{F}$ *of a double mapping cylinder* $D\bar{E}$ *of a pair* $(\bar{\phi}_0, \bar{\phi}_1)$ *that satisfies* (1.1).

Remarks. 1. If there are more than two fiber spheres of positive dimension, then DE is not path connected.

2. Case (i) above occurs precisely when all fiber spheres are S^0.

3. Case (ii) occurs if and only if exactly one fiber sphere has positive dimension, k, and E has infinitely many path components.

4. If E has finite Lusternik-Schnirelmann category so does $\bar{E}$ in (iii).

Now suppose (1.1) is satisfied and $l = 1$. Then $\phi_0(S^l)$ defines an element in $\pi_1(B_0)$, which in turn acts on the homology $H_k(S^k, \mathbb{Z})$ of the fiber sphere S^k of ϕ_0. We say that ϕ_0 *is twisted* if this action is nontrivial, i.e. acts by -1. Likewise ϕ_1 *is twisted* if $k = 1$ and $\phi_1(S^k)$ acts by -1 on the homology $H_l(S^l; \mathbb{Z})$ of the fiber of ϕ_1.

The fundamental group and integral homology of a cylinder fiber are determined by k, l, and twists:

Theorem 1.3. *Let F be a cylinder fiber for a double mapping cylinder DE of* (ϕ_0, ϕ_1) *satisfying* (1.1). *Then E and F are nilpotent and F is rationally* Ω*-elliptic. Moreover* $\pi_1(F)$ *and* $H_*(F; \mathbb{Z})$ *are as given in Tables* 1.4 *and* 1.5.

In Table 1.4, Q denotes the order 8 subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of the unit quaternions $S^3 \subset \mathbb{H}$. It is generated by two elements a, b with the relations $aba^{-1} = a^{-1}ba = b^{-1}$ and $bab^{-1} = b^{-1}ab = a^{-1}$.

Remarks. 1. In Table 1.5 and henceforth we use the convention $H_i(F; \mathbb{Z}) = 0$ unless i appears explicitly in the table.

2. The nilpotence is proved in Proposition 3.5 and the ellipticity in §6. Tables 1.4 and 1.5 are established in §3.

TABLE 1.4. $\pi_1(F)$

(k,l)	$k,l > 1$	$k > l = 1$	$k = l = 1$ no ϕ_i twisted	$k = l = 1$ one ϕ_i twisted	$k = l = 1$ both ϕ_i twisted
$\pi_1(F)$	$\{1\}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	Q

TABLE 1.5. $H_*(F; \mathbb{Z})$

(k,l)	$H_i(F; \mathbb{Z})$	
$k \neq l$ no twists	$\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}$	$i = 0$ or $i \equiv k, l \bmod(k+l)$ $i > 0$ and $i \equiv 0 \bmod(k+l)$
$k = l$ no twists	$\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}$	$i = 0$ $i > 0$ and $i \equiv 0 \bmod(k)$
$k > l = 1$ ϕ_0 twisted	$\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}$ $\mathbb{Z}_2$	$i = 0$ or $i \equiv \pm 1 \bmod(2k+2)$ $i > 0$ and $i \equiv 0 \bmod(2k+2)$ $i \equiv k, k+1 \bmod(2k+2)$
$k = l = 1$ ϕ_0 twisted ϕ_1 not twisted	$\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}_2$ $\mathbb{Z}_2$ $\mathbb{Z} \oplus \mathbb{Z}$	$i = 0$ or $i \equiv 3 \bmod(4)$ $i \equiv 1 \bmod(4)$ $i \equiv 2 \bmod(4)$ $i > 0$ and $i \equiv 0 \bmod(4)$
$k = l = 1$ ϕ_0, ϕ_1 both twisted	$\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$i = 0$ $i > 0$ and $i \equiv 0 \bmod(3)$ $i \equiv 1 \bmod(3)$

Corollary 1.6. *Under the hypotheses* (1.1) *the* mod 2 *Poincaré series* $P(t) = \Sigma_p \dim H_p(F; \mathbb{Z}_2)t^p$ *for* F *is given by*

$$P_F(t) = \frac{(1+t^k)(1+t^l)}{1-t^{k+l}}.$$

Let $K = k$ (resp. $2k+1$) if ϕ_0 is untwisted (resp. twisted), and let $L = l$ (resp. $2l+1$) if ϕ_1 is untwisted (resp. twisted). Then we have

Corollary 1.7. *Under the hypotheses* (1.1) *the rational Poincaré series* $P(t) = \Sigma_p \dim H_p(F; \mathbb{Q})t^p$ *for* F *is given by*

$$P_F(t) = \frac{(1+t^K)(1+t^L)}{1-t^{K+L}}.$$

As with homology and fundamental group, we can describe the rational homotopy type of the cylinder fiber in terms of k, l, and twists:

Theorem 1.8. *Suppose E has finite Lusternik-Schnirelmann category and* (1.1) *holds. The possibilities for the rational homotopy type of the cylinder fiber, F, are then as given in Table* 1.9.

Moreover, the exceptional cases $A_4(4) \times \Omega S^{17}$, $A_6(4) \times \Omega S^{25}$ do not occur either in the case that $DE = S^{n+1}$ and the ϕ_i are normal sphere bundles for the (smooth manifolds) B_i, or in the case of cohomogeneity one actions.

Here the spaces $A_m(k)$ (k even, $m = 1,2,3,4$ or 6) are the unique (up to rational homotopy type) 1-connected spaces whose cohomology algebra $H^*(A_m(k);\ \mathbb{Q})$ is generated by two elements x, y of degree k subject to the relations

$$x^m = x^2 + y^2 = 0 \quad \text{if } m = 1,2 \text{ or } 4,$$
$$x^m = x^2 + 3y^2 = 0 \quad \text{if } m = 3 \text{ or } 6.$$

Remarks. 1. The right-hand column of Table 1.9 exhibits "linear models" as explained in the introduction. The notation is taken from [13, Theorem 5, Table II].

TABLE 1.9

(k, l) and twists	$\mathbb{Q}$ homotopy type of F	Group; representation
$k = l = 1$ ϕ_0, ϕ_1 both twisted	$\begin{cases} [SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)] \times \Omega S^4 \\ [SO(4)/(\mathbb{Z}_2 \times \mathbb{Z}_2)] \times \Omega S^7 \end{cases}$	$SO(3);\ S^2\rho_3 - \theta$ $SO(4)$; see Remark 5
$k = l = 1$ ϕ_0 twisted, not ϕ_1	$[(SO(2) \times SO(3))/\mathbb{Z}_2] \times \Omega S^5$	$SO(2) \times SO(3);\ \rho_2 \otimes_{\mathbf{R}} \rho_3$
$k = l = 1$ ϕ_0, ϕ_1 not twisted	$\begin{cases} S^1 \times S^1 \times \Omega S^3 \\ S^1 \times \Omega S^2 \end{cases}$	$SO(2) \times SO(2);\ \rho_2 + \rho_2$ $SO(2);\ \rho_2 + \theta$
$k > l = 1$, k odd ϕ_0 twisted	$S^1 \times S^{2k+1} \times \Omega S^{2k+3}$	$SO(2) \times SO(k+2);\ \rho_2 \otimes_{\mathbf{R}} \rho_{k+2}$
$k > l = 1$ ϕ_0 not twisted	$\begin{cases} S^1 \times S^k \times \Omega S^{k+2} \\ S^1 \times S^k \times S^{k+1} \times \Omega S^{2k+3} \\ (k \text{ even}) \end{cases}$	$SO(2) \times SO(k+1);\ \rho_2 + \rho_{k+1}$ $SO(2) \times SO(k+2);\ \rho_2 \otimes_{\mathbf{R}} \rho_{k+2}$
$k > l \geqslant 2$	$S^k \times S^l \times \Omega S^{k+l+1}$	$SO(k+1) \times SO(l+1);\ \rho_{k+1} + \rho_{l+1}$
$k = l$ odd	$\begin{cases} S^k \times S^k \times \Omega S^{2k+1} \\ S^k \times \Omega S^{k+1} \end{cases}$	$SO(k+1) \times SO(k+1);\ \rho_{k+1} + \rho_{k+1}$ $SO(k+1);\ \rho_{k+1} + \theta$
$k = l$ even	$S^k \times S^k \times \Omega S^{2k+1}$ $S^k \times \Omega S^{k+1}$	$SO(k+1) \times SO(k+1);\ \rho_{k+1} + \rho_{k+1}$ $SO(k+1);\ \rho_{k+1} + \theta$
$k = l = 2$	$SU(3)/T^2 \times \Omega S^7$ $Sp(2)/T^2 \times \Omega S^9$ $G_2/T^2 \times \Omega S^{13}$	$SU(3)$; Ad $Sp(2)$; Ad G_2; Ad
$k = l = 4$	$Sp(3)/Sp(1)^3 \times \Omega S^{13}$ $A_4(4) \times \Omega S^{17}$ $A_6(4) \times \Omega S^{25}$	$Sp(3);\ \Lambda^2\nu_3 - \theta$
$k = l = 8$	$F_4/Spin(8) \times \Omega S^{25}$	$F_4;\ \phi_4$

2. In the middle column spaces within a single parenthesis have the same rational homotopy type; e.g., $[\mathrm{SO}(3)/\mathbb{Z}_2 \times \mathbb{Z}_2] \times \Omega S^4 \simeq_{\mathbf{Q}} [\mathrm{SO}(4)/\mathbb{Z}_2 \times \mathbb{Z}_2] \times \Omega S^7$. Such "duplications," and others we have not indicated, arise from rational homotopy equivalences such as $\Omega S^{2k} \simeq_{\mathbf{Q}} S^{2k-1} \times \Omega S^{4k-1}$ or $\mathrm{SO}(4) \simeq_{\mathbf{Q}} S^3 \times S^3$.

3. The rational homotopy type of F is either of the form $S^k \times S^l \times \Omega S^{k+l+1}$ or of the form $A_m(k) \times S^{mk+1}$ with $m = 1, 2, 3, 4$ or 6 and k even. When $m = 1$ or 2 we have

$$A_1(k) \simeq_{\mathbf{Q}} S^k; \qquad A_2(k) = S^k \times S^k$$

and these spaces occur for each even k. When $m = 3, 4$ or 6 then k must be $2, 4$ or 8; in particular

$$\mathrm{SU}(3)/T^2 \simeq_{\mathbf{Q}} A_3(2); \quad \mathrm{Sp}(2)/T^2 \simeq_{\mathbf{Q}} A_4(2); \quad G_2/T^2 \simeq_{\mathbf{Q}} A_6(2);$$
$$\mathrm{Sp}(3)/\mathrm{Sp}(1)^3 \simeq_{\mathbf{Q}} A_3(4); \qquad F_4/\mathrm{Spin}\,8 \simeq_{\mathbf{Q}} A_3(8).$$

4. The right-hand column is not a complete list of linear cohomogeneity one actions (cf. [13]), but for any such action there is in Table 1.9 an action on the same sphere whose cylinder fiber has the same integral homology, fundamental group, and rational homotopy type.

5. The example SO(4) in the first row of 1.9 was omitted in [13]. The representation in question is the adjoint action of SO(4) on $\mathfrak{g}_2/\mathfrak{so}(4)$.

6. If in (1.1) we assume only that the homotopy fiber of ϕ_1 is either S^1 or 1-connected and of the *rational* homotopy type of a sphere, then all the spaces $A_m(k) \times \Omega S^{mk+1}$ ($m = 1, 2, 3, 4, 6$, k even) can occur as the rational homotopy type of the cylinder fiber, and these are the *only* additional rational homotopy types. This is essentially the content of Theorem 5.1, which is the heart of the proof of 1.8, and we shall not elaborate further.

The proof of Theorem 1.8 is carried out mostly in the framework of rational homotopy theory:

In §4 we pass from topology to commutative graded differential algebras (over $\mathbf{Q}$) via Sullivan's theory of minimal models. In §5 we state and prove an algebraic classification, Theorem 5.1 (referred to above), in the context of minimal models. This turns out directly to imply most of Theorem 1.8. In particular, it is here that the rational homotopy types $A_m(k)$ appear and where also the reason that m is limited (to $1, 2, 3, 4$ or 6) becomes clear: we need $\tan^2 \pi/m$ to be rational.

In §6 we return to topology and obtain the Ω-ellipticity of F directly (cf. 1.3). In the case $F \simeq A_m(k) \times \Omega S^{mk+1}$ we then use Atiyah and Adam's results on Hopf invariant one maps [2] to obtain the restrictions $k = 2, 4$ or 8 when

$m = 3, 4$ or 6 and $k = 2$ or 4 when $m = 4$ or 6. This completes the proof of Theorem 1.8 except for ruling out the two exceptional cylinder fibers in the case where the B_i are submanifolds of S^{n+1} and the case of cohomogeneity one actions.

This is accomplished in the first case via characteristic classes and the Hirzebruch signature theorem. In the second we use the Borel-de Siebenthal [3] classification of maximal subgroups of compact Lie groups.

2. Dupin hypersurfaces

This section is devoted to a proof of Theorem C modulo Theorems 1.3 and 1.8.

Let $E^n \subset S^{n+1}$ be a closed connected Dupin hypersurface with principal curvature functions $\lambda_1 < \lambda_2 < \cdots < \lambda_g$. Denote by $(m_1, m_2, \cdots, m_g)$ the corresponding set of multiplicities.

Let B_0 (resp. B_1) be the set of focal points corresponding to the smallest (resp. largest) λ_i. Then [4] B_0 (resp. B_1) is a submanifold of S^{n+1} of codimension $m_1 + 1$ (resp. $m_g + 1$). Moreover, the focal map ϕ_0: $E \to B_0$ (resp. ϕ_1: $E \to B_1$) is a submersion whose fibers are the leaves of the foliation defined by λ_1 (resp. λ_g). These leaves are (umbilic) m_1- (resp. m_g-) spheres in S^{n+1}.

We claim that S^{n+1} is the double mapping cylinder DE of the focal maps (ϕ_0, ϕ_1) described above. This follows directly from [23]. In fact, for each $p \in B_0$ the "fiber" $\phi_0^{-1}(p)$ is exactly the set of points in E at minimal distance $(= \cot^{-1}\lambda_1(\phi_0^{-1}(p)))$ to p. In particular $B_0 \cup B_1$ is the cut locus of E, and so $S^{n+1} = D_0 \cup D_1$, where D_0 (resp. D_1) is the "cone"-bundle over B_0 (resp. B_1) whose fiber at $p \in B_0$ (resp. $q \in B_1$) is the geodesic cone in S^{n+1} with vertex at p and base $\phi_0^{-1}(p)$ (resp. q and $\phi_0^{-1}(q)$).

Next we identify ϕ: $E \to B_i$ ($i = 0$ or 1) with the normal sphere bundle of B_i. For $p \in B_i$ let $d_p = d(p, E)$ and denote by C_p the set of vectors in $T_p(S^{n+1})$ of length d_p, tangent to minimal geodesics from p to E. A simple first variation argument shows that for $X \in T_p(B_i)$, $\langle h, X\rangle$ is constant for all $h \in C_p$. Hence there is a unique $Y_p \in T_p(B_i)$ such that $h - Y_p \in T_p^{\perp}(B_i)$, $h \in C_p$. The map $\exp h \mapsto (h - Y_p)/|h - Y_p|$ is the desired identification.

We may now apply Theorems 1.3 and 1.8. In particular, E is nilpotent ((i) of Theorem C). Moreover, setting $\{k, l\} = \{m_1, m_g\}$ and observing that $F \simeq E \times \Omega S^{n+1}$ we see that $\pi_1(E)$ is determined by k, l, and twists as described in Table 1.4.

Observe as well that F has the rational homotopy type of one of the spaces in column 2 of Table 1.9, excluding the two exceptional cases. A case by case check of all the possibilities then shows that $2n = r(k + l)$ where $r = 1, 2, 3, 4$ or 6; moreover if $k \neq l$ then $r = 2$ or 4.

On the other hand from Corollary 1.6 we deduce that the mod 2 Poincaré polynomial for E is $(1 + t^k)(1 + t^l)(1 - t^{k+1})^{-1}(1 - t^n)$. Evaluating at $t = 1$ gives $\dim H_*(E; \mathbb{Z}_2) = 4n/k + l$. But the Morse theory argument in [23] ([15]) shows that the $2g$ integers $0, \Sigma_{i=1}^{s} m_i$ $(1 \leqslant s \leqslant g - 1)$, $\Sigma_{i=s}^{g} m_i$ $(2 \leqslant s \leqslant g)$, n are the degrees of a basis of $H_*(E; \mathbb{Z}_2)$. It follows that $\dim H_*(E; \mathbb{Z}_2) = 2g$; whence $2n = g(k + l)$. Hence $g = r = 1, 2, 3, 4$ or 6 and if $k \neq l$ then $g = 2$ or 4.

Consider the sequence $(m_1, \cdots, m_g)$. If $k \neq l$ and $g = 2$ it is just (k, l). If $k \neq l$ and $g = 4$ then $n = 2(k + l)$ and the Poincaré polynomial for E is $1 + t^k + t^l + 2t^{k+l} + t^{2k+l} + t^{2k+2l} + t^{2k+2l}$. Comparing with the degrees predicted by the Morse theory we find $(m_1, \cdots, m_g) = (k, l, k, l)$. Finally, if $k = l$ then $H_i(E; \mathbb{Z}_2) = 0$ unless $i \equiv 0 \pmod k$ and it follows by induction on s that k divides each m_i. Since $n = \Sigma_{i=1}^{g} m_i = gk$ in this case we have $m_i = k$ for all i. This proves Theorem C(ii).

For (iii) note that $H_*(E; \mathbb{Z})$ determines n and hence $H_*(F; \mathbb{Z})$, and hence (via Table 1.5) k and l, and hence $g = 2n/(k + l)$. Conversely g, k, and l determine n and hence $\pi_1(E)$ and $H_*(E; \mathbb{Z})$ from Tables 1.4 and 1.5. But $\pi_1(E)$ determines twists and k, l, n and twists determine the rational homotopy type of F (whence that of E) via Table 1.9.

Theorem C(iv)(a) is already proved and (b), (c), (d) follow directly from Table 1.9. In the same way one verifies Table 2.1.

3. Fundamental group and homology

In this section we prove Proposition 1.2, show that the cylinder fibers, F, are nilpotent, and establish the classification of $\pi_1(F)$ and $H_*(F; \mathbb{Z})$ (cf. Tables 1.4, 1.5). Recall we consider maps ϕ_i: $E \to B_i$ as described at the start of §1, whose homotopy fibers are all spheres.

From now on we shall also assume that

(3.1) $\quad DE$ is connected and E, B_0, B_1 are CW-complexes.

Indeed, if necessary, we simply replace E, B_0, B_1 by the CW-complexes of their singular simplices. This has no effect on the weak homotopy type of fiber spheres and of cylinder fibers, and the L.-S. (Lusternik-Schnirelmann) category of E is not increased under this process.

TABLE 2.1. Dupin hypersurfaces

g, k, l	$\pi_1(E)$	$H_i(E; \mathbb{Z})$	$\mathbb{Q}$ homotopy type of E
$g = 1$ $k = l = 1$	$\mathbb{Z}$	$\mathbb{Z}$, $i = 0, 1$	S^1
$g = 1$ $k = l > 1$	$\{1\}$	$\mathbb{Z}$, $i = 0, k$	S^k
$g = 2$ $k = l = 1$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$ $i = 0, 2$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = 1$	$S^1 \times S^1$
$g = 2$ $k = l > 1$	$\{1\}$	$\mathbb{Z}$, $i = 0, 2k$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = k$	$S^k \times S^k$
$g = 2$ $k > l = 1$	$\mathbb{Z}$	$\mathbb{Z}$, $i = 0, 1, k, k+1$	$S^k \times S^1$
$g = 2$ $k > l \geqslant 2$	$\{1\}$	$\mathbb{Z}$, $i = 0, l, k, k+l$	$S^k \times S^l$
$g = 3$ $k = l = 1$	Q	$\mathbb{Z}$, $i = 0, 3$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $i = 1$	$SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \mathbb{Q} S^3$
$g = 3$ $k = l = 2$	$\{1\}$	$\mathbb{Z}$, $i = 0, 6$ $\mathbb{Z} \oplus \mathbb{Z}$ $i = 2, 4$	$SU(3)/T^2 \simeq_{\mathbb{Q}} A_3(2)$
$g = 3$ $k = l = 4$	$\{1\}$	$\mathbb{Z}$, $i = 0, 12$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = 4, 8$	$Sp(3)/Sp(1)^2 \simeq_{\mathbb{Q}} A_3(4)$
$g = 3$ $k = l = 8$	$\{1\}$	$\mathbb{Z}$, $i = 0, 24$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = 8, 16$	$F_4/Spin(8) \simeq_{\mathbb{Q}} A_3(8)$
$g = 4$ $k = l = 1$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$, $i = 0, 3, 4$ $\mathbb{Z} \oplus \mathbb{Z}_2$, $i = 1$ $\mathbb{Z}_2$, $i = 2$	$[SO(2) \times SO(3)]/\mathbb{Z}_2 \simeq_{\mathbb{Z}} S^1 \times S^3$
$g = 4$ $k > l = 1$ k odd	$\mathbb{Z}$	$\mathbb{Z}$, $i = 0, 1, 2k+1, 2k+2$ $\mathbb{Z}_2$ $i = k, k+1$	$SO(2) \times SO(k+2)/\mathbb{Z}_2 \times SO(k)$ $\simeq_{\mathbb{Q}} S^1 \times S^{2k+1}$
$g = 4$ $k > l = 1$ k even	$\mathbb{Z}$	$\mathbb{Z}$, $i = 0, 1, k, k+2, 2k+1, 2k+2$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = k+1$	$SO(2) \times SO(k+2)/\mathbb{Z}_2 \times SO(k)$ $\simeq_{\mathbb{Q}} S^1 \times S^k \times S^{k+1}$
$g = 4$ $k > l \geqslant 2$ $k + l$ odd	$\{1\}$	$\mathbb{Z}$, $i = 0, l, k, k+2l, 2k+l, 2k+2l$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = k+l$	$S^k \times S^1 \times S^{k+1}$
$g = 4$ $k = l = 2$	$\{1\}$	$\mathbb{Z}$, $i = 0, 8$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = 2, 4, 6$	$Sp(2)/T^2 \simeq_{\mathbb{Q}} A_4(2)$
$g = 6$ $k = l = 1$	Q	$\mathbb{Z}$, $i = 0, t$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $i = 1, 4$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = 3$	$SO(4)/\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq_{\mathbb{Q}} S^3 \times S^3$
$g = 6$ $k = l = 2$	$\{1\}$	$\mathbb{Z}$, $i = 0, 12$ $\mathbb{Z} \oplus \mathbb{Z}$, $i = 2, 4, 6, 8, 10$	$G_2/T_2 \simeq_{\mathbb{Q}} A_6(2)$

We want to reduce further to the case where DE is simply connected and E, B_0, B_1 are connected.

Here and later on, the simple behavior of double mapping cylinders under pull-back plays an important role:

Let ρ: $X \to DE$ be a fibration. View $DE = D_0 \cup_E D_1$ as the union of the mapping cylinders $D_0 = B_0 \cup_{\phi_0}(E \times [0, \frac{1}{2}])$, $D_1 = (E \times [\frac{1}{2}, 1]) \cup_{\phi_1} B_1$, and set $X_i = \rho^{-1}(D_i)$, $i = 0, 1$, and $X_E = \rho^{-1}(E \times \{\frac{1}{2}\})$. Let ψ_i: $X_E \to X_i$ denote the inclusions and observe that up to homotopy we can identify ϕ_i: $E \to B_i$ with the inclusions $E \to D_i$, $i = 0, 1$. It is now straightforward to prove

Lemma 3.2. *Let ρ: $X \to DE$ and ψ_i: $X_E \to X_i$ be as above. Then up to weak homotopy type we have*:

(i) *The homotopy fibers of ψ_i are the same as those of ϕ_i, $i = 0, 1$.*

(ii) *$X = DX_E$ is the double mapping cylinder of (ψ_0, ψ_1).*

(iii) *The homotopy fiber of the inclusion $X_E \to X$ is the same as that of the inclusion $E \to DE$.*

Our first application of this is to the universal covering ρ: $\widetilde{DE} \to DE$.

Proposition 3.3. *The universal covering space $\widetilde{DE}$ of DE has the homotopy type of one of the following types of spaces*:

(i) *Double mapping cylinder $D\overline{E}$ for maps $\overline{\phi}_0$, $\overline{\phi}_1$: $\overline{E} \to \overline{B}_0$, $\overline{B}_1$ with $\overline{E}$ connected*;

(ii) *Open mapping cylinder for $\overline{\phi}$: $\overline{E} \to \overline{B}$ with $\overline{E}$ connected*;

(iii) *$\overline{E} \times \mathbb{R}$ with $\overline{E}$ connected.*

The homotopy fiber of the $\overline{\phi}_i$'s in (i) *and $\overline{\phi}$ in* (ii) *have the weak homotopy type of a sphere of positive dimension. Moreover, in all cases each cylinder fiber F of DE is the homotopy fiber of the natural inclusion of $\overline{E}$.*

Remarks 3.4. 1. Proposition 3.3 follows easily from 3.2 applied to ρ: $\widetilde{DE} \to DE$ by considering the path components of $\rho^{-1}(E)$, $\rho^{-1}(B_0)$, and $\rho^{-1}(B_1)$, and Proposition 1.2 is immediate from 3.3.

2. Case (iii) occurs exactly when all fiber spheres of ϕ_0, ϕ_1 are S^0. Case (ii) occurs when exactly one fiber sphere has positive dimension and $\rho^{-1}(E)$ has infinitely many components. Since DE is connected, at most two fiber spheres can have positive dimension.

Since Proposition 1.2 is now established we will assume (1.1) throughout the remaining part of the paper. Then, since DE is 1-connected, we have in particular that the homotopy fiber of the inclusion $E \to DE$ is connected and we denote it by F.

Proposition 3.5. *Let F be the cylinder fiber of a DE satisfying* (1.1). *Then E and F are nilpotent spaces. Moreover ϕ_i: $E \to B_i$ is twisted* (*cf.* §1) *if and only if it is not* (*up to homotopy*) *an orientable fibration* (*$\mathbb{Z}$-coefficients*).

Proof. Consider the diagram:

$$\begin{tikzcd}
S^l \simeq F_1 \arrow[dr, "\gamma_1"'] & & F_2 \simeq S^k \arrow[dl, "\gamma_0"] \\
& E \arrow[dl, "\phi_0"'] \arrow[dr, "\phi_1"] & \\
B_0 & & B_1
\end{tikzcd} \tag{3.6}$$

If $k = 1$ then γ_0 represents an element $a_0 \in \pi_1(E)$. If $k > 1$ put $a_0 = 1 \in \pi_1(E)$. Similarly let $a_1 \in \pi_1(E)$ be represented by γ_1, or be 1, according as $l = 1$, or $l > 1$.

Now set $b_1 = (\phi_1)_* a_0 \in \pi_1(B_1)$ and $b_0 = (\phi_0)_* a_1 \in \pi_1(B_0)$. Since the cyclic subgroups (a_0), $(a_1) \subset \pi_1(E)$ are normal, $(a_0) \cdot (a_1)$ is the normal subgroup they generate. Since $(\phi_0)_*$, $(\phi_1)_*$ are surjective on π_1 it follows that (b_0), (b_1) are normal subgroups too, and $(\phi_0)_*$, $(\phi_1)_*$ induce isomorphisms

$$\pi_1(E)/(a_0) \cdot (a_1) \underset{\cong}{\to} \pi_1(B_0)/(b_0), \pi_1(B_1)/(b_1).$$

By Van Kampen's theorem this group can be identified with a quotient of $\pi_1(DE)$; hence it is trivial and

$$\pi_1(E) = (a_0) \cdot (a_1), \quad \pi_1(B_i) = (b_i), \qquad i = 0, 1. \tag{3.7}$$

In particular the commutator subgroup of $\pi_1(E)$ is contained in $(a_0) \cap (a_1)$ and is central. Thus $\pi_1(E)$ is nilpotent. If $k = 1$ then $(\phi_0)_*\colon \pi_i(E) \to \pi_i(B_0)$ is injective for $i \geqslant 2$ and maps a_0 to 1. Thus a_0 (and also a_1) acts trivially on $\pi_i(E)$, $i \geqslant 2$. Hence E is nilpotent and, since DE is simply connected, F is nilpotent as well.

The last assertion in 3.5 follows from the observation above that b_i generates $\pi_1(B_1)$. q.e.d.

Our next goal is to establish Table 1.5 for the integral homology $H_*(F; \mathbb{Z})$ of the cylinder fiber for a DE satisfying (1.1).

Proof of 1.5. By applying Lemma 3.2 with X contractible we may reduce to the case that $DE \simeq \{\text{pt}\}$ and $F \simeq E$. The Mayer-Vietoris sequence for the double cylinder then reduces to isomorphisms

$$H_i(E; G) \overset{\cong}{\to} H_i(B_0; G) \oplus H_i(B_1; G), \qquad i \geqslant 1, \tag{3.8}$$

for any abelian group G.

Now $G = H_k(S^k; G)$ is a $\pi_1(B_0)$-module which is trivial unless $l = 1$ and ϕ_0 is twisted; in this case the generator b_0 of $\pi_1(B_0)$ acts by -1 (cf. proof of 3.5). We denote by $H^\varepsilon_*(B_0; G)$ the homology of B_0 with coefficients in this module, and define $H^\varepsilon_*(B_1; G)$ in the same way.

Then combining (3.8) with the Serre spectral sequence for the fibrations ϕ_i: $E \to B_i$ we obtain isomorphism

(3.9) $H^{\varepsilon}_{i-k}(B_0; G) \xrightarrow{\cong} H_i(B_1; G)$; $H^{\varepsilon}_{i-l}(B_1; G) \xrightarrow{\cong} H_i(B_0; G)$, $\quad i \geqslant 1$.

Thus if neither ϕ_i is twisted we have $H^{\varepsilon}_*(B_i; \mathbb{Z}) = H_*(B_i; \mathbb{Z})$ and 1.5 follows (with $G = \mathbb{Z}$) in these cases via an obvious induction.

Suppose now that ϕ_0 is twisted and so $l = 1$. Clearly $H^{\varepsilon}_*(B_i, \mathbb{Z}_2) = H_*(B_i; \mathbb{Z}_2)$ in any case, so it follows from (3.8) and (3.9) that the Poincaré series for the $\mathbb{Z}_2$-homology of B_0, B_1 and E (cf. Corollary 1.6) are given respectively by

(3.10) $$\frac{1+t}{1-t^{k+1}}, \quad \frac{1+t^k}{1-t^{k+1}}, \quad \frac{(1+t)(1+t^k)}{1-t^{k+1}}.$$

Next recall that $b_0 \in \pi_1(B_0)$ is given as $S^1 \xrightarrow[\gamma_1]{} E \xrightarrow[\phi_0]{} B_0$ and let $\tilde{E} \xrightarrow{\tilde{\phi}} \tilde{B}_0$ be the double cover of ϕ_0 corresponding to $2 \cdot \pi_1(B_0)$. (This is the proper subgroup of $\pi_1(B_0)$ acting trivially on $H_k(S^k; \mathbb{Z})$.) The double cover $\tilde{B}_0 \to B_0$ leads to the standard row- and column-exact commutative diagram of chain complexes

(3.11)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & C_*(B_0) \arrow[d, "\lambda"] \arrow[dr, ".2"] & & \\
0 \arrow[r] & C^{\varepsilon}_*(B_0) \arrow[r, "\lambda^{\varepsilon}"] \arrow[dr, ".2"'] & C_*(\tilde{B}_0) \arrow[r, "\rho"] \arrow[d, "\rho^{\varepsilon}"] & C_*(B_0) \arrow[r] & 0 \\
 & & C^{\varepsilon}_*(B_0) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

in which $C_*(\ )$ denotes singular chains and $H(C_*^{\varepsilon}) = H_*^{\varepsilon}(\ ;\ \mathbb{Z})$. Similar considerations apply to ρ_E: $\tilde{E} \to E$.

In particular the cokernels of $(\rho_E)_*$, ρ_*, and ρ^{ε}_* consist of 2-torsion. Moreover, because E is nilpotent (cf. 3.5) it follows that $\ker(\rho_E)_* \subset H_i(\tilde{E}; \mathbb{Z})$ is a 2^{r_i}-torsion group for some r_i.

The double cover we are considering leads to a map of Gysin sequences in which we denote the connecting homomorphisms by ∂_*. This map, combined with (3.8), (3.11), and our observations above, yields

Lemma 3.12. *For all i, the kernel and cokernel of*

$$\rho_*\partial_*\lambda^{\varepsilon}_*: H^{\varepsilon}_i(B_0; \mathbb{Z}) \to H_{i-k-1}(B_0; \mathbb{Z})$$

are 2^{s_i}-torsion groups.

Suppose now that ϕ_1 is *not twisted*, so that $H^{\varepsilon}(B_1; \mathbb{Z}) = H(B_1; \mathbb{Z})$. Then (3.9) and (3.12) imply that modulo 2^{n_i}-torsion groups $H_i(B_0; \mathbb{Z})$ is $\mathbb{Z}$ if $i \equiv 0, 1 \bmod (2k+2)$ and zero otherwise. On the other hand (3.10) gives $H_*(B_0; \mathbb{Z}_2)$ and from this and the universal coefficient theorem we compute

$$(3.13) \quad H_i(B_0; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \equiv 0, 1 \bmod (2k+2), \\ \mathbb{Z}/2^{n_i}, & i \equiv k+1 \bmod (2k+2), \text{ some } n_i \geqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (3.9) with $m_i = n_{i+k+1}$ that

$$(3.14) \quad H_i^{\varepsilon}(B_0; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2^{m_i}, & i \equiv 0 \bmod (2k+2), \\ \mathbb{Z}, & i \equiv k+1, k+2 \bmod (2k+2), \\ 0, & \text{otherwise.} \end{cases}$$

Now, applying (3.11) we obtain

$$(3.15) \quad H_i(\tilde{B}_0; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \equiv 1 \bmod (k+1), \\ \mathbb{Z} \oplus \mathbb{Z}/2^{n_i - 1}, & i \equiv k+1 \bmod (2k+2), \\ \mathbb{Z} \oplus \mathbb{Z}/2^{m_i - 1}, & i \equiv 0 \bmod (2k+2), \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, suppose $k = l = 1$ and *both fibrations are twisted*. In this case (3.12) applies to both fibrations and the same argument as above then shows that for $j = 0, 1$ and for integers $n_{i,j} \geqslant 1$

$$(3.16) \quad H_i(B_j; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \equiv 0 \bmod (3), \\ \mathbb{Z}/2^{n_{i,j}}, & i \equiv 1 \bmod (3), \\ 0, & i \equiv 2 \bmod (3). \end{cases}$$

Again consider the double cover $\tilde{\phi}_0\colon \tilde{E} \to \tilde{B}_0$ of ϕ_0. As above we may use (3.9) and (3.16) to compute $H^{\varepsilon}_*(B_i, \mathbb{Z})$ and combine this with (3.11) to get

$$(3.17) \quad H_i(\tilde{B}_0; \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2^{n_{i+1,1} - 1}, & i \equiv 0 \bmod (3), \\ \mathbb{Z}/2^{n_{i,0} - 1}, & i \equiv 1 \bmod (3), \\ \mathbb{Z}, & i \equiv 2 \bmod (3). \end{cases}$$

In both cases the fact that $\tilde{E} \to E \to B_1$ is a $\mathbb{Z}_2$-oriented S^1-fibration implies, in view of (3.10), that $\dim H_i(\tilde{E}; \mathbb{Z}_2) \leqslant 2$ for all i. On the other hand the $\mathbb{Z}_2$-oriented S^1-fibration $\tilde{E} \to \tilde{B}_0$ pulls back from $E \to B_0$; hence its $\mathbb{Z}_2$ Serre spectral sequence collapses. A calculation from (3.15) (resp. 3.17) now shows that $H_*(\tilde{B}_0; \mathbb{Z})$ is torsion free. Thus $n_i = m_i = 1$ (resp. $n_{i,0} = n_{i,1} = 1$). Substitution in (3.8) establishes the remaining cases of Table 1.5.

Proof of 1.4. As above we assume $F = E$, $DE \simeq \{\mathrm{pt}\}$. Recall from the proof of (3.5) that $\pi_1(E) = (a_0) \cdot (a_1)$. If $(k, l) \neq (1,1)$, or if there is at most one twist, it follows that $\pi_1(E)$ is abelian and we apply Table 1.5.

Finally, suppose $k = l = 1$ and both ϕ_0 and ϕ_1 are twisted. Then $a_0 a_1 a_0^{-1} = a_0^{-1} a_1 a_0 = a_1^{-1}$ and $a_1 a_0 a_1^{-1} = a_1^{-1} a_0 a_1 = a_0^{-1}$ and thus Q maps onto $\pi_1(E)$. It remains to show that the order $|\pi_1(E)|$ is at least 8. Consider the fibration $\tilde{E} \to B_1$ discussed at the end of the proof of 1.5 above. It is twisted, and by (3.16) and (3.9) the E_2-term of its spectral sequence ($\mathbb{Z}$-coefficients) satisfies $E^2_{1,0} = E^2_{0,1} = \mathbb{Z}_2$, and $E^2_{2,0} = 0$. Hence $H_1(\tilde{E}; \mathbb{Z})$ is a group of order 4. Thus $|\pi_1(\tilde{E})| \geqslant 4$ and so $|\pi_1(E)| \geqslant 8$.

4. Rational homotopy theory

In this section we begin the proof of Theorem 1.8 by passing from topology to commutative graded differential algebras via Sullivan's theory of minimal models.

The reader is referred to [20] and [8] for details of this theory. Here we recall briefly some of the basic definitions and results. All vector spaces and algebras considered in this section are defined over $\mathbb{Q}$.

For a graded vector space $X = \sum_{k \geqslant 0} X^k$ we define

$$\wedge X = \text{exterior algebra}\,(X^{\mathrm{odd}}) \otimes \text{symmetric algebra}\,(X^{\mathrm{even}}).$$

Then $\wedge X$ is augmented by the ideal $\wedge^+ X$ generated by X.

Denote by $\mathscr{A}$ the category of augmented commutative graded differential algebras, $A = \sum_{k \geqslant 0} A^k$, satisfying $H^0(A) = \mathbb{Q}$. Its objects and morphisms will be called $\mathscr{A}$-DGA's and $\mathscr{A}$-morphisms.

If $A \to A \otimes \wedge X$ is an $\mathscr{A}$-morphism and X admits a well-ordered basis $\{x_\alpha\}$ for which the differential d satisfies $dx_\alpha \in A \otimes \wedge(X_{<\alpha})$, then $A \otimes \wedge X$ is a *KS extension* of A. When $A = \mathbb{Q}$ it has the form $(\wedge X, d)$ and is called simply a *KS complex*. A DGA-morphism inducing a cohomology isomorphism is called a *quism* and is written $\xrightarrow{\simeq}$.

Any $\mathscr{A}$-morphism ϕ: $A \to B$ embeds in a commutative diagram of $\mathscr{A}$-morphisms,

$$\begin{tikzcd} & B \\ A \arrow[ur, "\phi"] \arrow[r] & A \otimes \wedge X, \arrow[u, "\simeq"'] \end{tikzcd}$$

$A \otimes \wedge X$ is a KS extension, and $A \otimes \wedge X$ is called a *Sullivan model for* ϕ. The DGA $(\wedge X, \bar{d}) = \mathbb{Q} \otimes_A (A \otimes \wedge X)$ is called a *Sullivan fiber for* ϕ. If $\operatorname{Im} \bar{d} \subset \wedge^+ X \cdot \wedge^+ X$, the model is called *minimal*; minimal models exist and are unique up to isomorphism. If $A = \mathbb{Q}$, then $\wedge X$ is a (*minimal*) *Sullivan model for* B.

Finally, $\mathscr{A}$ admits a homotopy theory as described in [20] or [8]. If ϕ: $\wedge X \to A$ is an $\mathscr{A}$-morphism from a KS complex and if ψ: $B \to A$ is an $\mathscr{A}$-quism, then there is a unique homotopy class of $\mathscr{A}$-morphisms χ: $\wedge X \to B$ such that $\psi\chi \sim \phi$.

Let $\mathscr{G}$ be one of the directed graphs $\cdot$, $\cdot \to \cdot$ or $\cdot \to \cdot \leftarrow \cdot$ regarded as a category. Two functors F, G from $\mathscr{G}$ to $\mathscr{A}$ are connected by an *elementary equivalence* if there are $\mathscr{A}$-quisms $F(\cdot) \to G(\cdot)$ for each vertex which make the obvious diagram commute. The equivalence relation generated by elementary equivalences is called a *c-equivalence*. Two objects in $\mathscr{A}$ are c-equivalent ($\mathscr{G} = \cdot$) precisely when their minimal Sullivan models are isomorphic. (We say they have the *same homotopy type*.) Homotopic $\mathscr{A}$-morphisms are c-equivalent ($\mathscr{G} = \cdot \to \cdot$).

The passage from topology to algebra is via the contravariant Sullivan-de Rham functor A_{PL} that associates to a pointed path connected space, S, the $\mathscr{A}$-DGA of rational PL-forms on the singular simplices of S. A *Sullivan model for a space* S (resp., a *Sullivan model for a continuous map* ϕ, *a Sullivan fiber for* ϕ) is a Sullivan model for $A_{\mathrm{PL}}(S)$ (resp., a Sullivan model for $A_{\mathrm{PL}}(\phi)$, a Sullivan fiber for $A_{\mathrm{PL}}(\phi)$).

On the other hand if the Sullivan model of a space F occurs as the Sullivan fiber of an $\mathscr{A}$-morphism (or continuous map) we often abuse language and refer simply to F as the Sullivan fiber of the morphisms or map. This is justified by the following result [8] (proved in the simply connected case by Grivel [7]):

Theorem 4.1. *Suppose* ϕ: $E \to B$ *is a continuous map between path connected spaces with path connected homotopy fiber* F *and Sullivan fiber* $(\wedge X, d)$. *If* $\dim H^i(F; \mathbb{Q}) < \infty$ *for all* i, *and* $\pi_1(B)$ *acts nilpotently in each* $H^i(F; \mathbb{Q})$, *then* $(\wedge X, d)$ *is a Sullivan model for* F.

We can now describe the analogue in $\mathscr{A}$ of the double mapping cylinder DE of ϕ_0, ϕ_1: $E \to B_0$, B_1. Observe that the contravariance of A_{PL} causes all the arrows to be reversed.

Consider a pair of $\mathscr{A}$-morphisms

$$A_0 \xrightarrow{\phi_0} A \xleftarrow{\phi_1} A_1$$

and proceed as follows: Extend ϕ_0 to an $\mathscr{A}$-morphism Φ_0: $A_0 \otimes C \to A$ with C acyclic and so that $A = \operatorname{Im}\Phi_0 + \operatorname{Im}\phi_1$. Define an $\mathscr{A}$-DGA, DA, by

$$DA = \{(x, y) \in (A_0 \otimes C) \oplus A_1 \mid \Phi_0 x = \phi_1 y\}.$$

There is a short exact sequence of differential spaces

$$(4.2) \qquad 0 \to DA \xrightarrow{\lambda} (A_0 \otimes C) \oplus A_1 \xrightarrow{\Phi_0 - \phi_1} A \to 0$$

and an $\mathscr{A}$-morphism $\varepsilon = \Phi_0 \circ \lambda = \phi_1 \circ \lambda$:

$$(4.3) \qquad \varepsilon\colon DA \to A.$$

Definition 4.4. DA is called a *double cylinder* for (ϕ_0, ϕ_1) and a Sullivan fiber for ε is called a *Sullivan cylinder fiber*. The long exact cohomology sequence determined by (4.2) is called the *long exact sequence* of the double cylinder.

Standard arguments give the next three lemmas.

Lemma 4.5. *With the terminology above*:

(i) *Sullivan fibers of c-equivalent morphisms have the same homotopy type.*

(ii) *The c-equivalence class of* ε: $DA \to A$ *does not depend on the choice of* C *or of the extension* Φ_0, *and depends only on the c-equivalence class of* $A_0 \to A \leftarrow A_1$.

(iii) *In particular, the isomorphism class of the minimal Sullivan cylinder fiber depends only on the c-equivalence class of* $A_0 \to A \leftarrow A_1$

(iv) *Suppose* ε_A, ε, *and* ε_F *are the double cylinders for* (ψ_0, ψ_1), (ϕ_0, ϕ_1), *and* (α_0, α_1), *where*

$$\begin{tikzcd}
A_0 \arrow[r, "\psi_0"] \arrow[d] & A \arrow[d] & A_1 \arrow[l, "\psi_1"'] \arrow[d] \\
A_0 \otimes \wedge X_0 \arrow[r, "\phi_0"] \arrow[d] & A \otimes \wedge X \arrow[d] & A_1 \otimes \wedge X_1 \arrow[l, "\phi_1"'] \arrow[d] \\
\wedge X_0 \arrow[r, "\alpha_0"'] & \wedge X & \wedge X_1 \arrow[l, "\alpha_1"]
\end{tikzcd}$$

is a commutative $\mathscr{A}$*-diagram in which the vertical arrows are KS extensions. Then* ε *and* ε_F (*resp.*, ε *and* ε_A) *have isomorphic minimal Sullivan fibers provided* ψ_0 *and* ψ_1 *are quisms* (*resp.* α_0 *and* α_1 *are quisms and* $H^1(A) = H^1(A_i) = 0$, $i = 0, 1$).

Lemma 4.6. *Suppose* ϕ_0: $A_0 \to A$ *is an* $\mathscr{A}$*-morphism with* S^k (*k odd*) *as Sullivan fiber, and suppose* m: $\wedge X \xrightarrow{\simeq} A$ *is a Sullivan model. There is then a homotopy commutative* $\mathscr{A}$*-diagram*

$$\begin{tikzcd}
\wedge y \otimes \wedge X \arrow[r, "\simeq"] \arrow[d, "\mathbb{Q} \otimes_{\wedge y}"'] & A_0 \arrow[d, "\phi_0"] \\
\wedge X \arrow[r, "m"'] & A
\end{tikzcd}$$

in which $\deg y = k + 1$ *and* $\wedge y \to \wedge y \otimes \wedge X$ *is a KS extension.*

Lemma 4.7. *Suppose ϕ_i: $E \to B_i$, $i = 0, 1$, are maps between connected spaces and j: $E \to DE$ is the inclusion of E in the double cylinder of the ϕ_i's. Then $A_{\mathrm{PL}}(j)$ is c-equivalent to the "inclusion" ε of the double cylinder for $(A_{\mathrm{PL}}(\phi_0), A_{\mathrm{PL}}(\phi_1))$.*

In particular, a cylinder fiber for these two $\mathscr{A}$-morphisms is a Sullivan fiber for j.

In order to apply these techniques in the setting (1.1) we need to compute the Sullivan fibers of the ϕ_i. The answer is contained (cf. also Corollary 1.7) in the following proposition (in which ϕ_1, l may be replaced by ϕ_0, k).

Proposition 4.8. *Assume* (1.1) *holds. If ϕ_1 is twisted, then l is odd. The Sullivan fiber of ϕ_1 is S^l if ϕ_1 is untwisted and S^{2l+1} if ϕ_1 is twisted.*

Proof. The assertion in the untwisted case follows from 4.1.

Assume $k = 1$ and ϕ_1 is twisted. The inclusion of the fiber γ_1: $F_1 \to E$ of ϕ_1 defines an element $a \in \pi_l(E)$ $(a = a_1$ if $l = 1)$. The action of π_1 on π_l satisfies $a_0 \cdot a = -a$ $(a_0 \cdot a = a^{-1}$ if $l = 1)$. From the nilpotency of E it follows that $2^n a = 0$ $(a^{2^n} = 1$ if $l = 1)$.

Consider the double cover $\tilde{B}_1 \to B_1$ such that E pulls back to an orientable fibration $\tilde{E} \to \tilde{B}_1$, with fiber F_1. The generator of $\pi_l(F_1)$ determines $\tilde{a} \in \pi_l(\tilde{E})$ covering a; hence $2^n\tilde{a} = 0$ and $H_l(F_1; \mathbb{Q})$ vanishes in $H_l(\tilde{E}; \mathbb{Q})$. An elementary cohomology spectral sequence argument for $\tilde{E} \to \tilde{B}_1$ now shows that l is odd.

Next apply (4.1) to obtain a Sullivan model for $\tilde{E} \to \tilde{B}_1$ of the form $\wedge Y \to \wedge Y \otimes \wedge x$ with $\wedge Y$ the minimal model for $\tilde{B}_1$ and $\deg x = l$. Then x is dual to $\tilde{a}$. Since $\tilde{a}$ vanishes in $\pi_l(\tilde{E}) \otimes \mathbb{Q}$ it follows that (for appropriate choice of $Y \subset \wedge Y$) that $0 \neq dx = y \in Y$. Moreover the involutions of $\tilde{E}$ and $\tilde{B}_1$ determine an involution ω of $\wedge Y \otimes \wedge x$ which may be taken to preserve Y and to map x to $-x$.

Because E is nilpotent,

$$(\wedge Y \otimes \wedge x)^{\omega = \mathrm{id}} \underset{\psi}{\to} \wedge Y \otimes \wedge x,$$

which represents $\tilde{E} \to E$, is a quism. Since $dy = 0$, $\wedge(Y/y)$ is a minimal KS complex and because ψ is a quism so is $\wedge(Y/y)^{\omega = \mathrm{id}} \to \wedge(Y/y)$. This forces ω to act by the identity in Y/y.

Finally, since $(\wedge Y)^{\omega = \mathrm{id}} \to (\wedge Y \otimes \wedge x)^{\omega = \mathrm{id}}$ is c-equivalent to $A_{\mathrm{PL}}(B_1) \to A_{\mathrm{PL}}(E)$, its Sullivan fiber is the Sullivan fiber of ϕ_1. Our remarks above show that $(\wedge Y)^{\omega = \mathrm{id}} = \wedge(y^2) \otimes \wedge(Y/y)$ and $(\wedge Y \otimes \wedge x)^{\omega = \mathrm{id}} \simeq \wedge(Y/y)$; hence this fiber is S^{2l+1}.

5. Rational classification

This section is devoted to the proof of Theorem 5.1 below which is the main step in the proof of Theorem 1.8.

Let ϕ_i: $A_i \to A$, $i = 0,1$, be $\mathscr{A}$-morphisms with Sullivan fibers S^k, S^l $(k, l \geqslant 1)$ and let ε: $DA \to A$ be a double cylinder (cf. 4.4) for (ϕ_0, ϕ_1).

Theorem 5.1. *The Sullivan cylinder fiber*, $(\Lambda W, d)$, *for* (ϕ_0, ϕ_1) *has at most three odd generators and at most two even generators* (*i.e.*, $\dim W^{\text{odd}} \leqslant 3$, $\dim W^{\text{even}} \leqslant 2$).

Moreover, unless $k = l$ *and is even the cylinder fiber is* $S^k \times S^l \times \Omega S^{k+l+1}$. *If* $k = l$ *and is even, and every class in* $H^+(A)$ *is nilpotent, then the cylinder fiber is one of*

$$A_m(k) \times \Omega S^{mk+1}, \quad m = 1,2,3,4 \text{ or } 6,$$

and all these possibilities can be realized.

For the proof of 5.1 we distinguish three cases: k, l odd; k odd, l even; k, l even.

5.2. *The case* k *and* l *are odd*. Here we use Lemmas 4.5 and 4.6 to replace ϕ_0, ϕ_1 by

$$\Lambda y_{k+1} \otimes \Lambda X \to \Lambda X \leftarrow \Lambda z_{l+1} \otimes \Lambda X.$$

By Lemma 4.5(iv) the Sullivan cylinder fiber is unchanged if we pass to $\Lambda y_{k+1} \to \mathbb{Q} \leftarrow \Lambda z_{l+1}$. Thus it is $S^k \times S^l \times \Omega S^{k+l+1}$, and (5.1) is proved in this case.

5.3. *The case* k *is odd and* l *is even*. Choose a model $\Lambda Z \xrightarrow{\simeq} A_1$ for A_1. Since the Sullivan fiber of ϕ_1 is S^l (l even) this extends to a commutative $\mathscr{A}$-diagram,

$$\begin{tikzcd} \Lambda Z \arrow[r] \arrow[d, "\simeq"'] & \Lambda Z \otimes \Lambda(u,x) \arrow[d, "\simeq"] \\ A_1 \arrow[r, "\phi_1"'] & A \end{tikzcd}$$

in which $\deg u = l$, $du = 0$, and $dx = u^2 - \Phi$, Φ a cocycle in ΛZ. Apply Lemma 4.6 to the right-hand quism and to ϕ_0. Because of Lemma 4.5 we may reduce in this way to the case that $A_0 \xrightarrow{\phi_0} A \xleftarrow{\phi_1} A_1$ has the form

$$\Lambda y \otimes [\Lambda Z \otimes \Lambda(u,x)] \xrightarrow{\rho} \Lambda Z \otimes \Lambda(u,x) \xleftarrow{i} \Lambda Z, \tag{5.4}$$

where $\rho = \mathbb{Q} \otimes_{\Lambda y}$, and $\deg y = k + 1$.

This is c-equivalent to, and hence has the same Sullivan cylinder fiber as,

$$\begin{aligned}(5.5)\qquad \Lambda y \otimes [\Lambda Z \otimes \Lambda(u,x)] \to \Lambda y \otimes [\Lambda Z \otimes \Lambda(u,x)] \otimes \Lambda(v) \\ \leftarrow \Lambda y \otimes \Lambda Z \otimes \Lambda(v),\end{aligned}$$

where $dv = y$. By Lemma 4.5(iv) the Sullivan cylinder fiber is unaffected if we apply first $\mathbb{Q} \otimes_{\Lambda y}$ and then $\mathbb{Q} \otimes_{\Lambda Z}$ to (5.5). Reversing the argument we see we may suppose $Z = 0$ in (5.4). A simple computation then identifies the cylinder fiber as $S^k \times S^l \times \Omega S^{k+l+1}$, and (5.1) is proved in this case.

5.6. *The case $k \leqslant l$, both even.* By Lemma 4.5(i–iii) we may suppose $A = \Lambda X$, $A_i = \Lambda X_i$ are minimal KS complexes. As in (5.3), the fact that ϕ_0, ϕ_1 have S^k, S^l as Sullivan fibers translates to commutative $\mathscr{A}$-diagrams

$$\begin{tikzcd} \Lambda X_i \arrow[r, "\phi_i"] \arrow[dr] & \Lambda X \\ & \Lambda X_i \otimes \Lambda(a_i, u_i), \arrow[u, "\psi_i"', "\simeq"] \end{tikzcd}$$

where $\deg a_0 = k$, $\deg a_1 = l$, $da_0 = da_1 = 0$, and $du_i = a_i^2 - \Phi_i$, Φ_i a cocycle in ΛX_i.

Because ΛX, ΛX_i are minimal, ϕ_0 is an isomorphism in degrees $< k$ and injective in degree k, while ϕ_1 is an isomorphism in degrees $< l$ and injective in degree l. Both ψ_0 and ψ_1, moreover, are surjective.

We now divide into two subcases:

Suppose $k < l$. Then $\psi_0(a_0) = \phi_1(\Psi)$, Ψ a cocycle in ΛX_1, while $\psi_1(a_1) = \psi_0(\Omega + \Omega' \otimes a_0) + d\Omega''$ for cocycles Ω, $\Omega' \in \Lambda X_0$. We can modify ψ_1 so that $\psi_1(a_1) = \psi_0(\Omega + \Omega' \otimes a_0) = \phi_0(\Omega) + \phi_0(\Omega') \cdot \psi_0(a_0)$. Moreover $\phi_0(\Omega') = \phi_1(\Omega_1)$ for some cocycle $\Omega_1 \in \Lambda X_1$.

Now consider the $\mathscr{A}$-diagram $B_0 \xrightarrow{\sigma_0} B \xrightarrow{\sigma_1} B_1$ given by

$$(5.7)\qquad \Lambda(c_0, b_0, w) \xrightarrow{\sigma_0} \Lambda(c_0, b_0, c_1, b_1, w, v_0, v_1) \xleftarrow{\sigma_1} \Lambda(c_1, b_1, w),$$

in which $\deg w = l - k$, $\deg c_0 = l$, $\deg b_0 = 2k$, $\deg c_1 = k$, $\deg b_1 = 2l$, and all these generators are cocycles, while $dv_0 = c_1^2 - b_0$, $dv_1 = (c_0 + wc_1)^2 - b_1$. Map $c_0 \to \Omega$, $b_0 \to \Phi_0$, $c_1 \to \Psi$, $b_1 \to \Phi_1$. Map $w \to \Omega'$ on the left and $w \to \Omega_1$ on the right, and $v_0 \to \psi_0(u_0)$, $v_1 \to \psi_1(u_1)$. This defines an $\mathscr{A}$-morphism from (5.7) to $A_0 \underset{\phi_0}{\to} A \underset{\phi_1}{\leftarrow} A_1$. Moreover, the Sullivan fiber of σ_i is mapped isomorphically to that of ϕ_i. Hence the Sullivan fiber of $B_i \to A_i$ is mapped isomorphically to that of $B \to A$, and it follows from Lemma 4.5(iv) that (5.7) and $A_0 \to A \leftarrow A_1$ have the same cylinder fibers.

Now if we put $w = 0$ in (5.7), then the cylinder fiber is unaffected. The projection $\Lambda(c_0, b_0, c_1, b_1, v_0, v_1) \to \Lambda(c_0, c_1)$ sending v_i, dv_i to zero is a quism. We are thus reduced to computing a Sullivan cylinder fiber for

$$(5.8) \qquad \Lambda(c_0, b_0) \to \Lambda(c_0, c_1) \leftarrow \Lambda(c_1, b_1),$$

where $b_0 \to c_1^2$ and $b_1 \to c_0^2$. This is again $S^k \times S^l \times \Omega S^{k+l+1}$, which proves (5.1) in this case.

Suppose $k = l$. If the cohomology classes in $H(\Lambda X)$ represented by $\psi_0(a_0)$ and $\psi_1(a_1)$ are linearly dependent (they are each necessarily nonzero!) we can choose a_0, a_1, ψ_0, and ψ_1 so that $\psi_0(a_0) = \psi_1(a_1)$. In this case a variation of the argument given above reduces us to the case that $A_0 \underset{\phi_0}{\to} A \underset{\phi_1}{\leftarrow} A_1$ has the form

$$(5.9) \qquad \Lambda(b_0) \to \Lambda(c) \leftarrow \Lambda(b_1),$$

with $\deg c = k$ and b_0, b_1 both mapping to c^2. In this case the Sullivan cylinder fiber is $S^k \times \Omega S^{k+1} = A_1(k) \times \Omega S^{k+1}$.

If on the other hand, the classes represented by $\psi_0(a_0)$ and $\psi_1(a_1)$ are linearly independent, then the above arguments reduce us to the case that $A_0 \underset{\phi_0}{\to} A \underset{\phi_1}{\leftarrow} A_1$ has the form

$$\Lambda(c_0, b_0) \to \Lambda(a_0, a_1) \leftarrow \Lambda(c_1, b_1),$$

with a_i, c_i of degree k, $\phi_i(b_i) = a_i^2$, and each of the pairs $\{a_0, a_1\}$, $\{\phi_0(c_0), a_0\}$, and $\{\phi_1(c_1), a_1\}$ form a basis for X^k.

Now put $x = \phi_0(c_0)$ and $y = a_0$. Then ϕ_0, ϕ_1 are the inclusions

$$(5.10) \qquad \Lambda(x, y^2) \to \Lambda(x, y) \leftarrow \Lambda\big(\lambda x + \mu y, (\lambda' x + \mu' y)^2\big),$$

where λ, μ, λ', $\mu' \in \mathbb{Q}$ satisfy $\lambda\mu' - \lambda' u \neq 0$ and $\lambda' \neq 0$. This leads to the final subdivision into cases:

(i) $\lambda = 0$, $\mu' \neq 0$: Here $\operatorname{Im}\phi_0 + \operatorname{Im}\phi_1 = A$, $\operatorname{Im}\phi_0 \cap \operatorname{Im}\phi_1 = \Lambda(y^2)$, and our desired cylinder fiber is the Sullivan fiber of $\Lambda(y^2) \to \Lambda(x, y)$. This is $S^k \times \Omega S^{k+1}$.

(ii) $\lambda \neq 0$, $\mu' = 0$: Here $\operatorname{Im}\phi_0 + \operatorname{Im}\phi_1 = A$ and $\operatorname{Im}\phi_0 \cap \operatorname{Im}\phi_1 = \Lambda(x^2)$. The cylinder fiber is $S^k \times \Omega S^{k+1}$.

(iii) $\mu = 0$: Here $x \in A_0$, A, A_1. Put $x = 0$ (without affecting the Sullivan cylinder fiber) and deduce from (5.9) that the cylinder fiber is $S^k \times \Omega S^{k+1}$.

(iv) $\lambda = \mu' = 0$: This is identical with (5.8), except that $k = l$, and the Sullivan cylinder fiber is $S^k \times S^k \times \Omega^{2k+1} = A_2(k) \times \Omega S^{2k+1}$.

(v) λ, μ, λ', μ' *are all nonzero*: Put $\alpha = \mu/\lambda$ and $\beta = \lambda'/\mu'$. Then (5.10) is equivalent to

$$(5.11) \qquad \Lambda(x, y^2) \to \Lambda(x, y) \leftarrow \Lambda\big(x + \alpha y, (\beta x + y)^2\big),$$

where $\alpha \neq 0$, $\beta \neq 0$, and $\alpha\beta \neq 1$.

In the remaining part of this section we determine the possible Sullivan cylinder fibers for the case (5.11) above. The main step is to find $\operatorname{Im}\phi_0 \cap \operatorname{Im}\phi_1$ and a complement for $\operatorname{Im}\phi_0 + \operatorname{Im}\phi_1$.

The computations are noticably different from the previous ones.

We begin by extending the coefficient field from $\mathbb{Q}$ to $\mathbb{C}$ and by choosing complex numbers θ, $\sigma \in \mathbb{C}$ so that

$$\tan^2\theta = -\alpha\beta \text{ and } \sigma\tan\theta = \alpha.$$

Put $\bar{x} = x$ and $\bar{y} = \sigma y$.

Now embed $\Lambda(\bar{x}, \bar{y})$ into the algebra of holomorphic functions in $\mathbb{C}^2$ by mapping $\bar{x} \to u\cos v$ and $\bar{y} \to u\sin v$, u, $v \in \mathbb{C}$. A basis (as a complex vector space) for the image is given by

$$(5.12)\qquad \begin{aligned} &u^{2p+q}\cos qv, \quad p \geqslant 0,\, q \geqslant 0,\\ &u^{2p+q}\sin qv, \quad p \geqslant 0,\, q \geqslant 1. \end{aligned}$$

The complexification of $\Lambda(x, y^2)$ is $\Lambda(\bar{x}, \bar{y}^2)$ and its image has a basis

$$(5.13)\qquad u^{2p+q}\cos qv, \qquad p \geqslant 0,\, q \geqslant 0.$$

The complexification of $\Lambda(x + \alpha y, (\beta x + y)^2)$ is $\Lambda(\cos\theta\bar{x} + \sin\theta\bar{y}, (-\sin\theta\bar{x} + \cos\theta\bar{y})^2)$ and a basis for its image is

$$(5.14)\qquad \cos q\theta\left(u^{2p+q}\cos qv\right) + \sin q\theta\left(u^{2p+q}\sin qv\right), \qquad p, q \geqslant 0.$$

The intersection of the spaces spanned by (5.13) and (5.14) has for a basis u^{2p}, $p \geqslant 0$, if $\theta \notin \mathbb{Q}\cdot\pi$ or $u^{2p+qm}\cos qmv$, p, $q \geqslant 0$, if $\theta = r\pi/m$, r, $m \in \mathbb{Z}$, $(r, m) = 1$.

A complement of the span of (5.13) and (5.14) in the span of (5.12) has for a basis ϕ if $\theta \notin \mathbb{Q}\cdot\pi$ or $u^{2p+qm}\sin qmv$, $p \geqslant 0$, $q \geqslant 1$, if $\theta = r\pi/m$ as above.

Using these relations we translate back to $\Lambda(x, y)$. Put $\varepsilon(m) = 0$ or 1 according as m is even or odd. Let f_m, $g_m \in \mathbb{Q}[s, t]$ be the homogeneous polynomials for which

$$u^m\cos mv = \left(u\cos v\right)^{\varepsilon(m)} f_m\left(u^2, u^2\cos^2 v\right),$$

$$u^m\sin mv = \left(u\sin v\right)\left(u\cos v\right)^{1-\varepsilon(m)} g_m\left(u^2, u^2\cos^2 v\right).$$

Thus with a_m, $b_m \in \Lambda(x, y)$ defined by

$$a_m = x^{\varepsilon(m)} f_m\left(\beta x^2 - \alpha y^2, \beta x^2\right),$$

$$b_m = yx^{1-\varepsilon(m)} g_m\left(\beta x^2 - \alpha y^2, \beta x^2\right)$$

we have obtained the following.

Proposition 5.15. *If θ is not a rational multiple of π then*

$$\Lambda(\beta x^2 - \alpha y^2) = \operatorname{Im}\phi_0 \cap \operatorname{Im}\phi_1; \qquad \Lambda(x, y) = \operatorname{Im}\phi_0 + \operatorname{Im}\phi_1.$$

If $\theta = r\pi/m$ (r, m relatively prime integers) then

$$\Lambda(\beta x^2 - \alpha y^2, a_m) = \operatorname{Im}\phi_0 \cap \operatorname{Im}\phi_1,$$

$$\Lambda(x, y) = (\operatorname{Im}\phi_0 + \operatorname{Im}\phi_1) \oplus b_m \cdot \Lambda(\beta x^2 - \alpha y^2, a_m).$$

Choose now an acyclic Sullivan model, C, and extend ϕ_0 to Φ_0: $A_0 \otimes C \to A$ so that $\operatorname{Im}\Phi_0 + \operatorname{Im}\phi_1 = A$. Recall the short exact sequence

$$0 \to DA \to (A_0 \otimes C) \oplus A_1 \xrightarrow{\Phi_0 - \phi_1} A \to 0. \tag{5.16}$$

Since A_0, A_1, A are concentrated in even degrees we can easily combine (5.16) and Proposition 5.15 to calculate $H(DA)$. Suppose first $\theta = \pi r/m$ with r, m relatively prime integers. Let $\bar{b}_m \in H^{km+1}(DA)$ be the image of b_m under the connecting homomorphism of (5.16). Then $H(DA) \cong \Lambda(\beta x^2 - \alpha y^2, a_m, \bar{b}_m)$ and ε: $DA \to A$ is c-equivalent to the morphism

$$(\Lambda(\beta x^2 - \alpha y^2, a_m, \bar{b}_m), 0) \to (\Lambda(x, y), 0).$$

The Sullivan fiber of this morphism has the form $(\Lambda(x, y, w, z), d) \otimes (\Lambda c, 0)$ with $\deg c = km$ and

$$dw = \beta x^2 - \alpha y^2, \qquad dz = a_m.$$

Now $f_m(s, t) = \lambda_m t^{(m - \varepsilon(m))/2} + s h_m(s, t)$ for some $0 \neq \lambda_m \in \mathbb{Q}$. It follows from the definition of a_m that $dz = \lambda'_m x^m + \Phi\, dw$, with $0 \neq \lambda'_m \in \mathbb{Q}$ and $\Phi \in \Lambda(x, y)$. On the other hand we have $d(\beta w) = (\beta x)^2 + \tan^2\theta y^2$.

Now a suitable change of basis reduces the Sullivan fiber to the form $(\Lambda(x, y, z, w), d) \otimes (\Lambda c, 0)$ with

$$dw = x^2 + \tan^2\theta y^2, \qquad dz = x^m.$$

Since $\tan^2\theta = -\alpha\beta$ is rational and nonzero and since $\theta = \pi r/m$ we must have $m = 3, 4$ or 6. For each value of m there is a unique possibility for $\tan^2\theta$, namely 3, 1, and 1/3. The respective Sullivan fibers are thus $A_m(k) \times \Omega S^{mk+1}$, $m = 3, 4$ or 6.

It remains to consider the case $\theta \notin \mathbb{Q} \cdot \pi$.

By Proposition 5.15 we may then in (5.16) take $C = \mathbb{Q}$. Thus $DA \xrightarrow{\varepsilon} A$ can be identified with the inclusion $\operatorname{Im}\phi_0 \cap \operatorname{Im}\phi_1 \to \Lambda(x, y)$. It follows (again by (5.15)) that in this case the Sullivan fiber is given by

$$(\Lambda(x, y, z), d), \qquad dz = \beta x^2 - \alpha y^2. \tag{5.17}$$

To complete the proof of Theorem 5.1 we now rule out (5.17) as a cylinder fiber when the elements of $H^+(A)$ are nilpotent.

Indeed if (5.17) is the cylinder fiber there is a quism

$$\psi\colon (DA \otimes \Lambda(x, y, z), \delta) \to (A, d)$$

in which δx, $\delta y \in DA$ and $\delta z = \beta x^2 - \Phi x - \alpha y^2 - \Psi y + \Omega$, with Φ, Ψ, $\Omega \in DA$. By modifying x and y (replace x by $x + \Phi/2\beta$) we can arrange that $\Phi = \Psi = 0$. From $\delta^2 z = 0$ we deduce $\delta x = 0$ and so x represents a nonnilpotent class in $H(A)$.

6. Integral restriction

In this last section we complete the proofs of 1.3 and 1.8. In both cases our point of departure is Theorem 5.1.

Completion of 1.3. Having calculated $\pi_1(F)$, $H_*(F; \mathbb{Z})$, and established the nilpotence of F in §3, we have only to show that F is rationally Ω-elliptic. By 1.2 we may suppose that (1.1) holds.

Since DE is simply connected and $H^*(F; \mathbb{Q})$ is finite dimensional in each degree, Theorem 4.1 asserts that a Sullivan fiber for $j\colon E \to DE$ is a Sullivan model for F.

On the other hand, by Lemma 4.7, a Sullivan cylinder fiber for $(A_{\mathrm{PL}}(\phi_0), A_{\mathrm{PL}}(\phi_1))$ is a Sullivan fiber for j. By Proposition 4.8 each $A_{\mathrm{PL}}(\phi_i)$ has a sphere as Sullivan fiber and so by Theorem 5.1 the Sullivan cylinder fiber ($=$ Sullivan model for F) has at most five generators. But these are dual to a basis of $\pi_*(F) \otimes \mathbb{Q}$ and so F is Ω-elliptic.

Corollary 6.1. *DE is Ω-elliptic if and only if E is.*

We are now ready for the

Completion of 1.8. Since by assumption E has finite L.-S. category each class in $H^+(E; \mathbb{Q})$ is nilpotent. Identify a minimal model for F with a Sullivan cylinder fiber for $(A_{\mathrm{PL}}(\phi_0), A_{\mathrm{PL}}(\phi_1))$, as above. Then Table 1.9 is an immediate consequence of 5.1 except when

(6.2) $\quad k = l$ even and $F \simeq_{\mathbb{Q}} A_m(k) \times \Omega^{mk+1}$, $m = 3, 4$ or 6.

In these cases, however, Table 1.9 follows directly from Lemmas 6.3 and 6.4 below.

Lemma 6.3. *Suppose $\phi_i\colon E \to B_i$ satisfy* (1.1) *and* (6.2). *If E has finite L.-S. category then the connecting homomorphism*

$$\partial\colon \pi_{mk+1}(DE) \otimes \mathbb{Q} \to \pi_{mk}(F) \otimes \mathbb{Q}$$

is nonzero.

Proof. The minimal model for F has the form $\wedge X = \wedge(x, y, c, u, v)$ with x, y, c cocycles of degrees k, k, mk and $du = x^m$, $dv = x^2 + y^2$ (resp. $x^2 + 3y^2$) if $m = 4$ (resp. $m = 3$ or 6). Theorem 4.1 asserts that the sequence $F \to E \to DE$ is modelled by $\wedge X \leftarrow \wedge Y \otimes \wedge X \leftarrow \wedge Y$; in particular we get a sequence of surjections

$$(\wedge Y \otimes \wedge X, D) \to (\wedge X, d) \to (\wedge c, 0)$$

Since E has finite L.-S. category it follows [5, §6] that in $\wedge Y \otimes \wedge X$, Dc has a nontrivial component, $a \in Y$. The duality between model generators and $\pi_* \otimes \mathbb{Q}$ identifies the map $c \mapsto a$ as the dual of ∂ (cf. [22]).

Lemma 6.4. *Suppose ϕ_i: $E \to B_i$ satisfy* (1.1) *and* (6.2), *and that*

$$\partial\colon \pi_{mk+1}(DE) \otimes \mathbb{Q} \to \pi_{mk}(F) \otimes \mathbb{Q}$$

is nonzero. Then $k = 2, 4$ or 8 and, if $m = 4$ or 6 then $k \neq 8$.

Proof. Since $\dim \pi_{mk}(F) \otimes \mathbb{Q} = 1$ we may choose a map α $S^{mk+1} \to DE$ so that the composite

$$\pi_{mk+1}(S^{mk+1}) \otimes \mathbb{Q} \to \pi_{mk+1}(DE) \otimes \mathbb{Q} \xrightarrow{\partial} \pi_{mk}(F) \otimes \mathbb{Q}$$

is an isomorphism. Convert α to a fibration and apply (3.2) to replace DE by a space of the homotopy type of S^{mk+1}. Thus in addition to the hypotheses of 6.4 we may assume

$$(6.5) \qquad DE \simeq S^{mk+1} \quad \text{and} \quad \partial\colon \pi_{mk+1}(DE) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_{mk}(F) \otimes \mathbb{Q}.$$

In particular, d_{mk+1} is the only nontrivial differential in the Serre cohomology spectral sequence for $E \to DE$. Since, moreover, $H^*(F; \mathbb{Q}) = H(A_m(k)) \otimes H^*(\Omega S^{mk+1}; \mathbb{Q})$ it is easy to compute d_{mk+1} ($\mathbb{Q}$-coefficients) using (6.5) and to deduce that $H^*(j)$: $H^*(E; \mathbb{Q}) \to H^*(F; \mathbb{Q})$ is in fact an isomorphism

$$(6.6) \qquad H^*(E; \mathbb{Q}) \xrightarrow{\cong} H(A_m(k)).$$

On the other hand, using 1.5 we obtain from the same spectral sequence with $\mathbb{Z}$-coefficients that for $j \leqslant mk$,

$$(6.7) \qquad H^j(E; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, mk, \\ \mathbb{Z} \oplus \mathbb{Z}, & j = n \cdot k, 0 < n < m, \\ 0, & \text{otherwise.} \end{cases}$$

The Mayer-Vietoris sequence for DE implies that

$$(6.8) \qquad H^j(B_0; \mathbb{Z}) \oplus H^j(B_1; \mathbb{Z}) \xrightarrow{\cong} H^j(E; \mathbb{Z}), \qquad 0 < j < mk,$$

whence

(6.9) the Serre spectral sequence ($\mathbb{Z}$ coefficients) for the ϕ_i: $E \to B_i$ collapse.

It follows from this and (6.7) that for $i = 0, 1$ and $0 \leqslant j \leqslant mk$,

$$H^j(B_i; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = nk, 0 \leqslant n < m, \\ 0, & \text{otherwise}. \end{cases} \tag{6.10}$$

Regard $H^j(B_i; \mathbb{Z})$, $0 \leqslant j \leqslant mk$, as a subgroup of $H^j(E; \mathbb{Z})$ and let $\alpha_n \in H^{nk}(B_0; \mathbb{Z})$, $\beta_n \in H^{nk}(B_1; \mathbb{Z})$ be generators ($0 \leqslant n < m$) with $\alpha_0 = \beta_0$ the unit element. Then (6.8) and (6.9) imply that α_1 (resp. β_1) restricts to a generator of the cohomology $H^k(S^k; \mathbb{Z})$ of the fiber of ϕ_1 (resp. ϕ_0). It follows from (6.6) and (6.7) that for $1 \leqslant n < m$ each of the pairs

$$\{\alpha_n, \beta_n\}; \quad \{\alpha_n, \beta_1 \cup \alpha_{n-1}\}; \quad \{\alpha_1 \cup \beta_{n-1}, \beta_n\}$$

is a basis for $H^{nk}(E; \mathbb{Z})$. Moreover

$$\alpha_1 \cup \beta_{m-1} = \pm \beta_1 \cup \alpha_{m-1}$$

is a basis for $H^{mk}(E; \mathbb{Z})$. Replacing α_n or β_n ($n \geqslant 2$) by their negatives if necessary we obtain

$$\begin{aligned} \alpha_1 \cup \beta_1 &= \alpha_2 + \beta_2, \\ \alpha_1 \cup \beta_n &= \alpha_{n+1} + p_{n+1}\beta_{n+1}, \quad 2 \leqslant n < m - 1, \\ \beta_1 \cup \alpha_n &= \beta_{n+1} + q_{n+1}\alpha_{n+1}, \quad 2 \leqslant n < m - 1, \end{aligned}$$

with $p_{n+1}, q_{n+1} \in \mathbb{Z}$.

On the other hand it is easy to see that for $0 \neq \alpha \in H^k(A_m(k))$, $\alpha^{m-1} \neq 0$. In view of (6.6) the same holds for $\alpha \in H^k(E; \mathbb{Q})$; in particular

$$\alpha_1 \cup \alpha_n = r_{n+1}\alpha_{n+1}, \quad \beta_1 \cup \beta_n = s_{n+1}\beta_{n+1}, \quad 1 \leqslant n < m - 1,$$

with r_{n+1}, s_{n+1} nonzero integers. Replace α_1, β_1 by $-\alpha_1$, $-\beta_1$ if necessary to arrange $r_2 > 0$. Finally, since $H^{mk}(B_1; \mathbb{Z}) = 0$,

$$\alpha_n\alpha_{m-n} = \beta_n\beta_{m-n} = 0, \quad 1 \leqslant n \leqslant m - 1.$$

Now consider the cases $m = 3, 4, 6$ separately.

The case $m = 3$. Here $\alpha_1^2\beta_1 = \alpha_1(\alpha_2 + \beta_2) = \alpha_1\beta_2 = \pm\alpha_2\beta_1$ and so $\alpha_1^2 = \alpha_2$. By [2, Theorem A] applied to the $2k$-skeleton of B_0 we have $k = 2, 4$ or 8.

The case $m = 4$. Here $\alpha_1\beta_3 = \alpha_1(\beta_3 + q_3\alpha_3) = \alpha_1\beta_1\alpha_2 = \beta_2\alpha_2$. Hence $\alpha_1\beta_3 = \alpha_1\alpha_2\beta_1 = \alpha_2\beta_2 = \alpha_1\beta_1\beta_2 = \alpha_3\beta_1$. It follows that $\beta_3 = \beta_1\beta_2$ and $\alpha_3 = \alpha_1\alpha_2$; i.e. $r_3 = s_3 = 1$. On the other hand $2\alpha_2\beta_2 = (\alpha_2 + \beta_2)^2 = \alpha_1^2\beta_1^2 = r_2s_2\alpha_2\beta_2$. Thus we may suppose $r_2 = 1$, $s_2 = 2$.

Now [2, Theorem A] applied to the $2k$-skeleton of B_0 gives $k = 2, 4$ or 8 and [2, Theorem B] applied to the $3k$-skeleton gives $k \neq 8$.

The case $m = 6$. As when $m = 4$ we find

$$\alpha_1\beta_5 = \alpha_1\beta_1\alpha_4 = \beta_2\alpha_4 = \beta_2\alpha_1\beta_3 = \alpha_3\beta_3 = \alpha_2\alpha_3\beta_1 = \alpha_2\beta_4 = \beta_1\alpha_1\beta_4 = \beta_1\alpha_5.$$

Hence $\beta_5 = \beta_1\beta_4 = \beta_2\beta_3$, $\alpha_5 = \alpha_1\alpha_4 = \alpha_2\alpha_3$, whence $r_5 = s_5 = 1$, $r_2 = r_4$, and $s_2 = s_4$. From $3\alpha_2\beta_2\alpha_1\beta_1 = 3\alpha_2\beta_2(\alpha_2 + \beta_2) = (\alpha_2 + \beta_2)^3 = \alpha_1^3\beta_1^3$ we deduce $r_2 s_2 = 3$. Hence we may take $r_2 = 1$, $s_2 = 3$, and by [2, Theorem A], $k = 2$, 4 or 8.

From $r_2\alpha_2^2 = \alpha_1^2\alpha_2 = r_3 r_4 \alpha_4$ we find $\alpha_2^2 = r_3\alpha_4$, $\beta_2^2 = s_3\beta_4$. Using this in $\alpha_2\beta_2\alpha_1\beta_1 = \alpha_2^2\beta_2 + \beta_2^2\alpha_2$ yields $r_3 s_3 = r_3 + s_3$. Since $r_3 \neq 0$ this yields $r_3 = s_3 = 2$. Hence [2, Theorem B] gives $k \neq 8$.

This completes the proof of Table 1.9.

To complete the proof of 1.8 we must rule out the exceptional spaces $A_4(4) \times \Omega S^{17}$ and $A_6(4) \times \Omega S^{25}$ as possible cylinder fibers in the case that the ϕ_i are normal (linear) sphere bundles of smooth manifolds B_i and $DE \simeq S^{n+1}$, and in the case of cohomogeneity one actions. We consider these cases separately.

$DE \simeq S^{n+1}$; *ϕ_i normal sphere bundles.* Since we want to exclude only $A_m(k) \times \Omega S^{mk+1}$ as a possible cylinder fiber for (ϕ_0, ϕ_1) when $k = 4$ and $m = 4$ or 6, we may suppose that the ϕ_i are the 4-sphere bundles of normal vector bundles ν_i of rank 5. Since, moreover, $DE \simeq S^{n+1}$ the ϕ_i satisfy (6.5) and the hypotheses of 6.4. Thus all the properties developed in the proof of 6.4 apply, and we will use the notation established there without further comment.

The cohomology class $\alpha_1 \in H^4(E; \mathbb{Z})$ orients ϕ_1. Let ξ be the oriented rank 4 vector bundle over E "tangent to ϕ_1": $\xi_z = T_z(S^4_{\phi_1 z})$ where $S^4_{\phi_1 z}$ is the fiber of ϕ_1 at $\phi_1 z$. The Euler class $\chi \in H^4(E; \mathbb{Z})$ of ξ has the form $\lambda_1\alpha_1 + \lambda_2\beta_1$ ($\lambda_1, \lambda_2 \in \mathbb{Z}$) and $\lambda_1 = 2$ because α_1 (resp. β_1) restricts to the fundamental class (resp. 0) in the fibers of ϕ_1. Moreover χ^2 is the second Pontrijagin class $p_2(\xi)$ and so $\chi^2 = \phi_1^*(p_2(\nu_1))$; i.e. $\chi^2 = \lambda_3\beta_2$ for some $\lambda_3 \in \mathbb{Z}$.

On the other hand the multiplication table developed above for $H^*(E; \mathbb{Z})$ shows that $\chi^2 = 4(1 + \lambda_2)\alpha_2 + (4\lambda_2 + 2\lambda_2^2)\beta_2$ if $m = 4$ and $\chi^2 = 4(1 + \lambda_2)\alpha_2 + (4\lambda_2 + 3\lambda_2^2)\beta_2$ if $m = 6$. Hence $\lambda_2 = -1$, $\chi = 2\alpha_1 - \beta_1$, and $p_2(\nu_1) = -2\beta_2$ (resp. $-\beta_2$) if $m = 4$ (resp. 6). In particular, the fourth Stiefel-Whitney class satisfies

$$w_4(\nu_1) = w_4(\xi) = \beta_1 \pmod 2.$$

Our next objective is to show that the first Pontrjagin class, $p_1(\nu_1)$, is zero. Write $p_1(\nu_1) = s\beta_1$ ($s \in \mathbb{Z}$). Then the total tangential Pontrjagin class of B_1 is given by $1 + \Sigma p_i(B) = (1 + s\beta_1 + t\beta_2)^{-1}$ where $t = -2, -1$ if $m = 4, 6$. Again using the multiplication table for $H^*(E; \mathbb{Z})$ we apply the Hirzebruch signature theorem [12] to obtain the signature of B_1 as a polynomial in s with rational coefficients. In each case direct inspection shows $s = 0$ is the only real root. But since B_1 has no middle homology, $\operatorname{sign}(B_1) = 0$ and hence $p_1(\nu_1) = 0$.

Now consider the restriction of ν_1 to the four skeleton, S^4, of B_1. There it splits off a trivial line bundle to give a rank 4 vector bundle η over S^4 with

$\langle w_4(\eta), [S^4]\rangle = 1$ and $p_1(\eta) = 0$. The Euler class of η then satisfies $\langle \chi(\eta), [S^4]\rangle \equiv 1 \bmod (2)$. But (cf. [14, 20.10]) since $p_1(\eta) = 0$, $\langle \chi(\eta), [S^4]\rangle \equiv 0 \bmod (2)$. This contradiction rules out $A_4(4) \times \Omega S^{17}$ and $A_6(4) \times \Omega S^{25}$ in the case ϕ_i are normal bundles and $DE \simeq S^{n+1}$.

$E = G/H \subset M^{n+1}$ codimension 1 principal orbit. In the case of a group action $G \times M \to M$ of cohomogeneity one the decomposition theorem of Mostert [16] asserts that: Either all orbits are principal and $M \to M/G = S^1$ is a fibration or there are exactly two exceptional orbits $B_0 = G/K$, $B_1 = G/L$ with $H \subset K$, $L \subset G$. In the first case $F \sim \{\mathrm{pt}\}$. In the second Mostert's theorem gives linear actions of K and L on Euclidean discs D^{k+1}, D^{l+1} with $S^k = K/H$, $S^l = L/H$ and

$$M = \left(G \times_K D^{k+1}\right) \cup_{G/H} \left(G \times_L D^{L+1}\right).$$

This exhibits M as the double mapping cylinder of the maps $\phi_0, \phi_1\colon G/H \to G/K, G/L$.

Now by applying 4.5(iv) to the Sullivan models for the fibrations

$$\begin{tikzcd}
G/K \arrow[d] & G/H \arrow[l] \arrow[r] \arrow[d] & G/L \arrow[d] \\
B_K \arrow[dr] & B_H \arrow[l] \arrow[r] \arrow[d] & B_L \arrow[dl] \\
& B_G &
\end{tikzcd}$$

we may replace the projections $G/H \to G/K, G/L$ by the maps $B_H \to B_K, B_L$ of classifying spaces. In particular the group G is irrelevant.

Let H^0, K^0, L^0 be the connected components of H, K, L containing 1. The universal cover of $H^0(K^0, L^0)$ is a product of $\overline{H}(\overline{K}, \overline{L})$ and a Euclidean factor, where $\overline{H}(\overline{K}, \overline{L})$ is a compact simply connected semisimple Lie group. In particular $\overline{H} \to H$, $\overline{K} \to K$, $\overline{L} \to L$ are homotopy equivalent to the universal covers of H^0, K^0, L^0. It follows that $B\overline{H}$, $B\overline{K}$, $B\overline{L}$ are the 2-connected Postnikov fibers of BH, BK, BL.

To rule out $A_4(4) \times \Omega S^{17}$ and $A_6(4) \times \Omega S^{25}$ we need only consider the case $K/H = L/H = S^4$. But then $\pi_i(BH) \to \pi_i(BK)$, $\pi_i(BL)$ is an isomorphism for $i = 1, 2$. In particular the double mapping cylinder for $BH^0 \to BK^0$, BL^0 is the universal cover for that of $BH \to BK$, BL. This, together with Lemma 4.5(iv), applied to the Sullivan models for the fibrations

$$\begin{tikzcd}
B\overline{K} \arrow[d] & B\overline{H} \arrow[l] \arrow[r] \arrow[d] & B\overline{L} \arrow[d] \\
BK^0 \arrow[dr] & BH^0 \arrow[l] \arrow[r] \arrow[d] & BL^0 \arrow[dl] \\
& K(\pi_1, 1) &
\end{tikzcd},$$

allows us to reduce to the case H, K, $L = \overline{H}$, $\overline{K}$, $\overline{L}$ are 1-connected semisimple and compact.

In particular, each group is a product of simple factors. Moreover, because $K/H = L/H = S^4$, H is maximal in K and L and has the same rank [3]. If some factor of H is also a factor of K and of L it can be split off without affecting the cylinder fiber. It follows now from [3] that once this process has been completed the only possibilities for H, K, L are as given in the Table 6.11. Since neither of the exceptional cases appear in the right-hand column we are done.

TABLE 6.11

H	K	L	$\mathbf{Q}$ homotopy of F
$S^3 \times S^3$	Spin 5	Spin 5	$S^4 \times \Omega S^5$
$S^3 \times S^3 \times S^3$	Spin 5 $\times S^3$	$S^3 \times$ Spin 5	$A_4(4) \times \Omega S^{13}$
$S^3 \times S^3 \times S^3 \times S^3$	Spin 5 $\times S^3 \times S^3$	$S^3 \times S^3 \times$ Spin 5	$S^4 \times S^4 \times \Omega S^9$

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