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THE HEAT EQUATION SHRINKS EMBEDDED PLANE CURVES TO ROUND POINTS

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One can shorten a smooth curve immersed in a Riemannian surface by moving it in the direction of its curvature vector field. This process is known by many names, including "Curve Shortening," "Flow by Curvature," and "Heat Flow on Isometric Immersions." While this flow is defined by local information, it has many subtle and mysterious global properties. Even when the curve is immersed in the Euclidean plane, the global behavior is very difficult to analyze. Most striking are the facts that a convex curve shrinks to a point, becoming round in the limit, and that, in the absence of singularities, embedded curves remain embedded. We will add to this list the fact that embedded curves become convex without developing singularities. This fact completes the proof of the conjecture that curve shortening shrinks embedded plane curves smoothly to points, with round limiting shape.

The Main Theorem. *Let* $C(\cdot, 0)\colon S^1 \to \mathbf{R}^2$ *be a smooth embedded curve in the plane. Then* $C\colon S^1 \times [0, T) \to \mathbf{R}^2$ *exists satisfying*

$$\partial C/\partial t = \kappa \cdot \mathbf{N}, \tag{$*$}$$

where K *is the curvature of* C, *and* $\mathbf{N}$ *is its unit inward normal vector.* $C(\cdot, t)$ *is smooth for all* t, *it converges to a point as* $t \to T$, *and its limiting shape as* $t \to T$ *is a round circle, with convergence in the* C^∞ *norm.*

A more visual description of this flow is the evolution of elastic bands in honey. If the tension in the elastic is kept constant, then its behavior is determined (approximately) by equation $(*)$. For a discussion of this problem in its most general setting, the reader is referred to [5].

For the case where the initial curve is convex, this theorem was proven by M. Gage and R. Hamilton in [5].

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G. Huisken [6] generalized it to convex hypersurfaces in $\mathbf{R}^n$ flowing via mean curvature. In higher dimensions, it is generally agreed that the Main Theorem is false for nonconvex embedded hypersurfaces. A barbell with a long, thin handle develops a singularity in the middle in short time.

§1 contains useful formulas and analytic preliminaries, many of which appear in [5]. In particular, we will prove the long-term smoothness of solutions given bounded curvature.

§2 contains the proof of the *nonexistence of corners*. This theorem, due to Richard Hamilton, is a generalization of a similar theorem in [5]. It states that if the curvature blows up anywhere, then it does so along an arc which has a total curvature of at least π.

§3 contains the proof of the *δ-whisker lemma*, which is an important tool in the proof of the Main Theorem. It says that, under certain conditions, the curve cannot get too close to itself.

§4 begins the proof of the Main Theorem. There are three principal cases. In this section, we show that spirals do not collapse, and that curves which shrink to a point become convex. These are precisely the cases where the curvature is blowing up along arcs which turn through more than π.

§5 deals with the last case, that of curves which have curvature blowing up on arcs which turn through exactly π. In the end, everything has been ruled out, except the case of a curve becoming convex before it becomes singular. The Main Theorem then follows from the results in [5].

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1. Equations and existence

Throughout this section, we assume that $C\colon S^1 \times [0, T) \to \mathbf{R}^2$ is a family of smooth curves satisfying the evolution equation $(*)$. We will usually denote $C(\cdot, t)$ by $C(t)$, with the understanding that $C(t)$ is a curve in the plane. We start with the statement of short-term existence.

Theorem 1.1 [5]. *Let $C(\cdot, 0)$ be a smooth, embedded closed curve in the plane $\mathbf{R}^2$. Then $C\colon S^1 \times [0, \varepsilon) \to \mathbf{R}^2$ exists satisfying equation $(*)$. Furthermore, $C(\cdot, t)$ is smooth and embedded.*

We actually have more than smoothness after time 0. E. Calabi has shown that, for very general initial conditions, the solutions are analytic for positive time.

Theorem 1.2 [2]. *Let $C(0)$ be a piecewise C^1 plane curve, with the property that there is an $\varepsilon > 0$, such that the tangent direction to the curve changes by less than π along any arc of length ε. Then the solutions $C(t)$ of equation $(*)$ exist for short time, and are analytic for $t > 0$.*

The proof uses standard techniques in the theory of analyticity of solutions to strictly parabolic equations. Thus we can, without loss of generality, assume that the initial curve is analytic. For the proof of the Main Theorem, we need only the fact that the curve has a finite number of inflection points for all positive time. If you like, add this requirement to the hypotheses of the Main Theorem.

In the next three lemmas, we show that $C(t)$ remains smooth and embedded as long as its curvature remains bounded.

If we parametrize the curve in such a way that points are moving by their curvature vectors, then we can calculate the derivatives of curvature with respect to time and arc-length. Since arc-length is not preserved under this flow, the variables s and t are not independent.

Lemma 1.3 (*Lemma* 3.1.6 *in* [5]). *The evolution of curvature with respect to arc-length is given by*

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3.$$

Lemma 1.4 (*Lemma* 3.1.3 *in* [5]). *The time and arc-length derivatives do not commute, as motion normal to the curve affects arc-length. The relation is*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + k^2\frac{\partial}{\partial s}.$$

Lemma 1.5. *Suppose that κ is bounded for $t \in [0, t_0)$. Then for some $\varepsilon > 0$, $C(t)$ exists and is smooth for $t \in [0, t_0 + \varepsilon)$.*

Proof. Using Lemma 1.4, we get

$$\frac{\partial}{\partial t}\left[\frac{\partial \kappa}{\partial s}\right] = \frac{\partial}{\partial s^2}\left[\frac{\partial \kappa}{\partial s}\right] + 4\kappa^2\left[\frac{\partial \kappa}{\partial s}\right].$$

Even though t and s are not independent variables, this equation bounds the rate of growth of the derivative of κ to exponential. Therefore, it is bounded for all finite time. In general, repeated applications of Lemma 1.4 yield

$$\frac{\partial}{\partial t}\left[\frac{\partial^n \kappa}{\partial s^n}\right] = \frac{\partial}{\partial s^2}\left[\frac{\partial^n \kappa}{\partial s^n}\right] + (n+3)\kappa^2\left[\frac{\partial^n \kappa}{\partial s^n}\right] + \text{previously bounded terms.}$$

Thus the nth derivative of κ grows no faster than exponentially. Lemma 1.3 now shows that the time derivative of κ is bounded, so the curve converges as $t \to t_0$. Similarly, $C(t_0)$ is smooth. Therefore, by Lemma 1.1, $C(t)$ exists and is smooth for some further short time. q.e.d.

Bounded curvature not only implies that a smooth curve remains smooth, it also implies that an embedded curve remains embedded.

Lemma 1.6 (*Theorem* 3.2.1 *in* [5]). *If κ is uniformly bounded for a collection of evolving $C(t)$, $t \in [0, t_0]$, and if $C(0)$ is embedded, then $C(t)$ is also embedded for $t \in [0, t_0]$.*

We now prove some of the properties of evolving curves in the plane.

If we consider the curve in cartesian coordinates, and require that points move to fix their x-coordinates, then we get a different flow with the same point-sets as solutions, but different time-derivatives for curvature. We use $'$ to denote differentiation with respect to x.

Lemma 1.7. *Choose cartesian coordinates in $\mathbf{R}^2$ so that $C(t_0)$ is locally a graph. Then the evolution of y fixing x is given by*

$$\frac{\partial y}{\partial t} = \frac{y''}{1 + y'^2}.$$

In addition, $\theta(x, t) = \tan^{-1}(y'(x, t))$ and $\kappa(x, t)$ also evolve by equations which are strictly parabolic when $|y'|$ is bounded, namely

$$\frac{\partial \theta}{\partial t} = \frac{\theta''}{1 + y'^2}, \quad \frac{\partial \kappa}{\partial t} = \frac{\kappa''}{1 + y'^2} + \kappa^3.$$

Proof. The speed of the curve in its normal direction is κ. The correction term for vertical speed is $\sec(\theta)$,

$$\frac{\partial y}{\partial t} = \kappa \sec(\theta), \quad \kappa = \frac{y''}{\left[1 + y'^2\right]^{3/2}}, \quad \sec\theta = \left[1 + y'^2\right]^{1/2}.$$

The formula for the evolution of y follows.

For the evolution of θ we differentiate the above formula for the evolution of y to obtain

$$\frac{\partial y'}{\partial t} = \frac{[y']''}{1 + y'^2} - \frac{2y'y''^2}{\left[1 + y'^2\right]^2}.$$

An application of the chain rule yields the desired formula.

For the evolution of κ, we can either start from Lemma 1.3 and correct for the difference between the normal flow and the vertical flow, or we can find the evolution of y'' and substitute the formula for curvature in cartesian coordinates. In either calculation, most of the terms drop away and we are left with the desired formula.

Lemma 1.8 [7] (*The Maximum Principle*). *Suppose that* $F(x, t)$: $[0, \varepsilon] \times [t_0, t_0 + \varepsilon_0] \to \mathbf{R}$ *and that*

(i) *the evolution of F with respect to t is governed by a strictly parabolic differential equation.*

(ii) $F(x, t_0) \geqslant 0$, *but not* $\equiv 0$.

(iii) $F(0, t) \geqslant 0$, $F(\varepsilon, t) \geqslant 0$.

Then $F(x, t) > 0$ *for* $x \in (0, \varepsilon)$ *and* $t \in (t_0, t_0 + \varepsilon]$.

For every application of the Maximum Principle in this paper, condition (i) is either automatically satisfied, or it is equivalent to a bound on y'.

Lemmas 1.7 and 1.8 have many important implications for the evolution of the quantities y, θ, and κ.

Lemma 1.9. (i) *For a given choice of cartesian coordinates, local minima for y, θ, and κ increase with time, local maxima decrease, and terrace points for y and θ disappear instantaneously. Furthermore, the points of the curve where local maxima and minima for y, θ, and κ are realized vary continuously with time.*

(ii) *If, for some choice of cartesian coordinates, y' is uniformly bounded on the subset* $\alpha(t) = C(t) \cap (\mathbf{R}^2 \mid x \in [0, \xi])$, *then $\alpha(t)$ cannot converge to the x-axis from one side in finite time.*

(iii) *The total curvature of an arc connecting two isolated inflection points is strictly decreasing with time. In fact, the θ-intervals of tangent directions to such an arc strictly nest with time.*

(iv) *When an even number of isolated points of inflection meet, the resulting flat point on the curve disappears instantaneously.*

Proof. When applying the Maximum Principle to these equations, we should be slightly cautious, for the equations are not linear. We get around this by observing that we *have* a smooth solution, so if we know that the coefficient of F'' is bounded away from zero (in all cases, this is equivalent to saying that y' is bounded), then we can fix that coefficient to obtain a linear strictly parabolic equation which the original solution happens to satisfy. Compare with Lemma 3.4.

In all cases but (ii), where we assume that y' stays bounded, we know that the curve exists for some further time, so there is no problem choosing coordinates which guarantee that y' stays bounded for some short time. Cases (i) and (ii) now follow from Lemma 1.7 and 1.8. Cases (iii) and (iv) are immediate consequences of (i). q.e.d.

We add two more lemmas which will be useful later:

Lemma 1.10. *The time derivative of the area bounded between the curve, the x-axis, and two vertical lines is given by $\int \kappa \, ds$ over the part of the curve bounding the region.*

Proof. Differentiate under the integral sign:

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t}\int_a^b y\,dx = \int_a^b \frac{y''}{1+y'^2}\,dx = \int_a^b \frac{y''}{\left(1+y'^2\right)^{3/2}}\,ds.$$

q.e.d.

Note that $\int \kappa\, ds$ is just the difference in the angles of intersection of the curve with the two vertical lines. As a consequence, we get the formula for the time derivative of the area of the whole curve.

Lemma 1.11 (*Lemma* 3.1.7 *in* [5]). *The time derivative of the area enclosed by the curve is a constant* -2π.

2. The nonexistence of corners

The next step is to show that the curvature cannot become unbounded unless it does so on an arc which turns through an angle of less than π. The argument is a generalization of the integral and pointwise estimates in [5].

Suppose that $C(t)$ is an evolving family of curves. Following [5], we make this

Definition. *Let* $\bar{\kappa}(t) = \sup\{b\colon\ |\kappa| > b$ *on some subarc of* $C(t)$ *which has total curvature* $\pi\}$.

Theorem 2.1 (*Hamilton*). *If* $\bar{\kappa}(t)$ *is bounded for* $t \in [0, T)$, *then* κ *is uniformly bounded on the same time interval.*

Proof. Let θ denote the angle which the tangent to the curve makes with a fixed line. Away from the inflection points, θ is a good local coordinate. Throughout the proof of Theorem 2.1, $'$ will denote differentiation with respect to θ.

If $\bar{\kappa}(t)$ is bounded, then let K be an upper bound. Let $\alpha(t)$ be a subarc of $C(t)$ which connects consecutive inflection points. By Lemma 1.9, the endpoints of $\alpha(t)$ evolve continuously for all except a finite number of times. Let $R(t)$ be the subset of $\alpha(t)$ on which $\kappa > K$. Clearly, $R(t)$ avoids inflection points. Since the θ-ranges of an arc connecting inflection points form a nested sequence of intervals, we may parametrize $R(t)$ by θ. If necessary, choose K to be a regular value for curvature in the (θ, t)-plane. We will need three lemmas from [5].

Lemma 2.2 (*Lemma* 4.1.3 *in* [5]). *Let* $\alpha(t)$ *be a family of subarcs of* $C(t)$ *satisfying* $\kappa > 0$ *on the interior of* $\alpha(t)$. *Let* θ *denote the angle that the tangent to* α *makes with a fixed line. Then the evolution of* κ *fixing* θ *is given by*

$$\frac{\partial \kappa}{\partial t} = \kappa^2\kappa'' + \kappa^3.$$

Lemma 2.3 (*Lemma* 4.3.3 *in* [5]). *Let* $f(\theta)$ *be a function defined on an interval* $[a, b]$ *such that* $|a - b| \leqslant \pi$, *and* $f(a) = f(b) = 0$. *Then*

$$\int_a^b f(\theta)^2 - f'(\theta)^2 \, d\theta \leqslant 0.$$

Lemma 2.4 (*Lemma* 3.1.2 *in* [5]). *The time derivative of the length* $L(t)$ *of* $C(t)$ *is given by*

$$\frac{dL}{dt} = -\int_{C(t)} \kappa^2 \, ds.$$

We are now ready for the first integral estimate.

Proposition 2.5. *With the hypotheses of Theorem* 2.1, *the integral*

$$\int_{R(t)} \log\left|\frac{\kappa(\theta, t)}{K}\right| d\theta$$

is bounded for $t < T$.

Proof. At time t, $R(t)$ is a countable union of intervals. Using the formulas for evolution yields

$$\begin{aligned}\frac{\partial}{\partial t}\int_{R(t)} \log\left|\frac{\kappa(\theta, t)}{K}\right| d\theta &= \sum_{i=0}^{\infty} \frac{\partial}{\partial t}\int_{a_i}^{b_i} \log\left|\frac{\kappa(\theta, t)}{K}\right| d\theta \\ &= \int_{R(t)} \kappa\kappa'' + \kappa^2 \, d\theta + \sum_{i=0}^{\infty} \frac{\partial b_i}{\partial t} \log\left|\frac{\kappa(b_i)}{K}\right| - \sum_{i=0}^{\infty} \frac{\partial a_i}{\partial t} \log\left|\frac{\kappa(a_i)}{K}\right|.\end{aligned}$$

The two sums are zero for $\kappa = K$ on the boundary of R.

Integrating by parts, we get

$$\int_{R(t)} \kappa\kappa'' + \kappa^2 \, d\theta = \int_{R(t)} \kappa^2 - \kappa'^2 \, d\theta + \sum_{i=0}^{\infty} \kappa\kappa' \Big|_{a_i}^{b_i}.$$

The last term above is negative, for κ' is $\geqslant 0$ at a_i and $\leqslant 0$ at b_i. So

$$\int_{R(t)} \kappa\kappa'' + \kappa^2 \, d\theta \leqslant \int_{R(t)} \kappa^2 - \kappa'^2 \, d\theta.$$

We now use Lemma 2.3 on the function $f(\theta, t) = \kappa(\theta, t) - K$. Since $R(t)$ is composed of intervals of θ-length less than or equal to π, we conclude that

$$\int_{R(t)} f(\theta)^2 - f'(\theta)^2 \, d\theta \leqslant 0.$$

Therefore, since

$$\int_{R(t)} f^2 - f'^2 \, d\theta = \int_{R(t)} \kappa^2 - \kappa'^2 - 2K\kappa + K^2 \, d\theta,$$

we conclude that

$$\frac{\partial}{\partial t}\int_{R(t)} \log\left|\frac{\kappa(\theta,t)}{K}\right| d\theta \leqslant 2K\int_{R(t)} \kappa\, d\theta.$$

We estimate the right-hand integral as follows:

$$\int_{R(t)} \kappa\, d\theta = \int_{R(t)} \kappa^2\, ds < \int_{C(t)} \kappa^2\, ds = -\frac{dL}{dt}.$$

The last equality uses Lemma 2.4. We conclude that

$$\frac{\partial}{\partial t}\int_{R(t)} \log\left|\frac{\kappa(\theta,t)}{K}\right| d\theta < -2K\frac{dL}{dt}.$$

Integrating with respect to time yields

$$\int_{R(t)} \log\left|\frac{\kappa(\theta,t)}{K}\right| d\theta < \int_{R(0)} \log\left|\frac{\kappa(\theta,0)}{K}\right| d\theta + 2K\cdot(L(0) - L(t))$$

which gives the desired bound. q.e.d.

Next, we get L^p estimates for curvature.

Proposition 2.6. *With the hypotheses of Theorem* 2.1, *and for* $p \geqslant 0$, *the* L^p *norm of curvature is bounded for* $t < T$.

Proof. We look at the time derivative for even powers of κ:

$$\frac{\partial}{\partial t}\int_{C(t)} \kappa^{2p}\, ds = \int_{C(t)} \frac{\partial}{\partial t}\kappa^{2p} - \kappa^{2p+2}\, ds$$

$$= \int_{C(t)} 2p\kappa^{2p-1}\frac{\partial^2\kappa}{\partial s^2} + (2p-1)\kappa^{2p+2}\, ds.$$

Note the use of Lemmas 1.3 and 1.4. Integrating by parts yields

$$\int_{C(t)} -2p(2p-1)\kappa^{2p-2}\left[\frac{\partial\kappa}{\partial s}\right]^2 + (2p-1)\kappa^{2p+2}\, ds.$$

Now we restrict to the set where $\kappa^{2p+2} > K$ and substitute $d\theta = \kappa\cdot ds$. Since the first term above is negative and the length of the curve is bounded, the above integral is bounded by

$$(2p-1)\int_{R(t)} \kappa^{2p+1} - 2p\kappa^{2p-1}(\kappa')^2\, d\theta + c_1$$

for some constant c_1.

Setting $F = \kappa^{p+1/2}$, we get

$$(2p-1)\int_{R(t)} F^2 - \frac{8p}{(2p+1)^2}\left[\frac{\partial F}{\partial\theta}\right]^2 d\theta + c_1.$$

To get an inequality argument similar to Lemma 2.3, we need to pick K big enough so that the set where $F > K$ consists of intervals of length less than $\pi\sqrt{8p}/(2p+1)$. The bound from Proposition 2.5 guarantees that this can be done. As before, we get the inequalities

$$\frac{\partial}{\partial t}\int_{C(t)} \kappa^{2p}\,ds \leqslant c_2 \int_{R(t)} F\,d\theta + c_1 \leqslant c_2 \int_{C(t)} \kappa^{p+3/2}\,ds + c_1$$

for some constants c_1 and c_2. For $p \geqslant 3/2$, this bounds the growth of the L^p norm of κ to exponential. Since $C(t)$ is compact, the bound on the L^p norm implies a bound for the L^p norm for all q less than p. This proves the lemma.

Now that we have a bound on the L^p norms of κ, we try to bound the L^2 norm of the derivative of κ.

Proposition 2.7. *With the above hypotheses, the integral* $\int_{C(t)} [\partial\kappa/\partial s]^2\,ds$ *is bounded for* $t < T$.

Proof. We differentiate with respect to time, and we use the formula for interchanging the order of the derivatives:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{C(t)} \left[\frac{\partial\kappa}{\partial s}\right]^2 ds &= \int_{C(t)} 2\left[\frac{\partial\kappa}{\partial s}\right]\frac{\partial}{\partial t}\left[\frac{\partial\kappa}{\partial s}\right] - \kappa^2\left[\frac{\partial\kappa}{\partial s}\right]^2 ds\\ &= \int_{C(t)} 2\left[\frac{\partial\kappa}{\partial s}\right]\frac{\partial}{\partial s}\left[\frac{\partial\kappa}{\partial t}\right] + \kappa^2\left[\frac{\partial\kappa}{\partial s}\right]^2 ds.\end{aligned}$$

Integrating by parts twice gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_{C(t)} \left[\frac{\partial\kappa}{\partial s}\right]^2 ds &= \int_{C(t)} -2\left[\frac{\partial^2\kappa}{\partial s^2}\right]^2 - 2\kappa^3\left[\frac{\partial^2\kappa}{\partial s^2}\right] + \kappa^2\left[\frac{\partial\kappa}{\partial s}\right]^2 ds\\ &= \int_{C(t)} 7\kappa^2\left[\frac{\partial\kappa}{\partial s}\right]^2 - 2\left[\frac{\partial^2\kappa}{\partial s^2}\right]^2 ds.\end{aligned}$$

By Proposition 2.6, we can find a K such that

$$\int_{\kappa>K} \kappa^2\,ds < \frac{2}{7L},$$

where L is the length of the initial curve. On the part of the curve where $\kappa < K$, we estimate

$$\int_{\kappa<K} 7\kappa^2\left[\frac{\partial\kappa}{\partial s}\right]^2 ds < 7K^2\int_{C(t)} \left[\frac{\partial\kappa}{\partial s}\right]^2 ds;$$

where $\kappa > K$, we estimate

$$\int_{\kappa>K} 7\kappa^2\left[\frac{\partial\kappa}{\partial s}\right]^2 ds < 7\sup\left[\frac{\partial\kappa}{\partial s}\right]^2\int_{C(t)} \kappa^2\,ds < \frac{2}{L}\sup\left[\frac{\partial\kappa}{\partial s}\right]^2.$$

The supremum of a function which is somewhere zero is less than or equal to the integral of the absolute value of its derivative. Using this and the Hölder inequality, we get

$$\sup\left[\frac{\partial \kappa}{\partial s}\right]^2 \leqslant \left[\int_{C(t)} \left|\frac{\partial^2 \kappa}{\partial s^2}\right| ds\right]^2 \leqslant L \int_{C(t)} \left[\frac{\partial^2 \kappa}{\partial s^2}\right]^2 ds.$$

Combining terms, we see that the growth of the integral in question is at most exponential with exponent $7K$. q.e.d.

Proof of Theorem 2.1, *continued.* Since we are assuming that the curve $C(t)$ is not convex over the time interval in question, $\kappa = 0$ somewhere on the curve. Therefore the supremum of curvature is bounded by

$$\sup(\kappa) < \int_{C(t)} \left|\frac{\partial \kappa}{\partial s}\right| ds < L + \int_{C(t)} \left[\frac{\partial \kappa}{\partial s}\right]^2 ds$$

which is bounded by Proposition 2.7.

3. The δ-whisker lemma

The δ-whisker lemma is an important tool in the proof of the Main Theorem. It prevents the curve from getting too close to itself along subarcs which turn through at least π.

Consider the curve $C(t)$ in the plane with a choice of cartesian coordinates. We can label each critical point of the height function, $y(s)$, with a "plus" or a "minus," respectively, depending on whether the interior of the curve lies below or above the critical point. Alternatively, we can orient the curve, and assign "plus" to critical points where the tangent vector to the curve points to the left, and "minus" where it points to the right (see Figure 1).

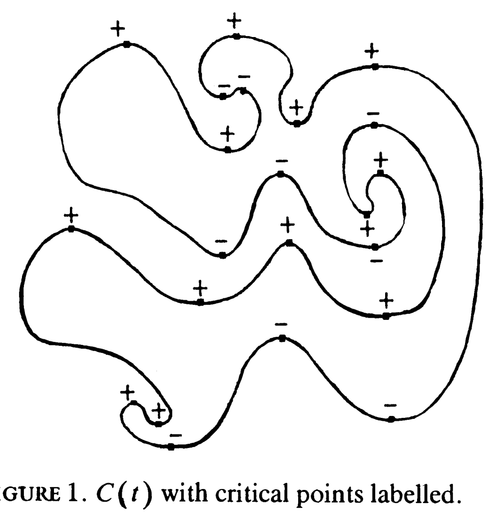

FIGURE 1. $C(t)$ with critical points labelled.

In these fixed cartesian coordinates, the curve is a union of graphs of functions. These functions can be labelled plus or minus in the same way as the critical points. From Lemma 1.9, we know that local maxima move down, local minima move up, and that terrace points disappear instantaneously. Critical points are never created, for the height difference between adjacent maxima and minima is always decreasing. Furthermore, the domains of the functions defining $C(t)$ are monotonically nesting with time. Also, from Lemma 1.9, we know that the positions of the critical points are varying continuously. These facts have the following implication.

Lemma 3.1. *A given critical point at time t_0 may be followed continuously backwards in time to a critical point on the initial curve.*

Proof. In forward time, sometimes an adjacent pair of critical points will cancel, but sometimes three or more critical points will combine to make one, as when a nonconvex curve becomes convex. In the latter case, as we follow a single critical point backwards in time, it may split into several. We choose any one of them.

Definitions. By a *subarc* of $C(t)$, we mean a family of arcs, $\alpha(t) \subset C(t)$, such that the endpoints of $\alpha(t)$ vary continuously in the plane. A subarc $\alpha(t_0)$ of $C(t_0)$ is *nice* with respect to a vector $\mathbf{v}$ if the tangent vectors which point inwards at the endpoints of $\alpha(t_0)$ both point in the same direction as $\mathbf{v}$. Note that if we make $\mathbf{v}$ horizontal, then the endpoints of $\alpha(t)$ are critical points with labels of opposite sign.

We now prove the very useful δ-whisker lemma.

Lemma 3.2. *Given $C(0)$ smooth, then there is a $\delta > 0$ such that if*

(i) *$C(t)$ exists for $t < T$,*

(ii) *$\alpha(t_0)$ is a nice subarc of $C(t_0)$ for some $t_0 < t$,*

(iii) *L is a line segment of length δ, based at a point p on $\alpha(t_0)$, such that L points in the same direction as the inward pointing tangent vectors to the endpoints of $\alpha(t_0)$,*

then L is disjoint from $\beta(t_0) = C(t_0) \setminus \alpha(t_0)$ (see Figure 2).

Furthermore, if $\alpha_1(t_0)$ and $\alpha_2(t_0)$ are disjoint nice subarcs of $C(t_0)$, then any two line segments, L_1 and L_2 satisfying condition (iii) *for $\alpha_1(t_0)$ and $\alpha_2(t_0)$ respectively, are disjoint.*

Proof. By following the endpoints of $\alpha(t_0)$ backwards in time, we can find a family of subarcs $\alpha(t)$, $0 \leqslant t < t_0$, which evolve into $\alpha(t_0)$, and where each $\alpha(t)$ connects a continuously varying $+$ critical point to a continuously varying $-$ critical point.

Break the curve $C(t)$ into the two pieces: $\alpha(t)$ and its complement $\beta(t) = C(t) \setminus \alpha(t)$, and start translating them horizontally away from each other. Let

$d(t)$ be the maximum distance, possibly infinite, which the arcs can be translated before they touch each other.

Lemma 3.3. *$d(t)$ is a nondecreasing function.*

Proof. Suppose $d(t)$ were realized at an interior point for both $\alpha(t)$ and $\beta(t)$. In this case we show that $d(t)$ would be increasing due to the following lemma and the fact that translation commutes with evolution.

This lemma is a generalization of Lemma 1.9, but it requires the added hypothesis of a bound on the second derivative of the solutions. It states that two tangent arcs which do not cross separate instantly.

Lemma 3.4. *Let $y_1(x,t)$ and $y_2(x,t)$ be functions whose graphs evolve by $(*)$ for $x \in [0,\varepsilon]$, $t \in [t_1,t_2]$. Suppose that they satisfy the following:*

(i) $|y_i'|$, $|y_i''|$ *are bounded*, $i = 1, 2$.

(ii) $y_2(x,t_1) \geqslant y_1(x,t_1)$, *but not* $\equiv$.

(iii) $y_2(0,t) \geqslant y_1(0,t)$, $y_2(\varepsilon,t) \geqslant y_1(\varepsilon,t)$.

Then $y_2(x,t_2) > y_1(x,t_2)$ *for* $x \in (0,\varepsilon)$.

Proof. Here, again, $'$ denotes differentiation with respect to x.

Consider the operator

$$D(f) = 2\frac{\partial f}{\partial t} - \left[\frac{1}{1+y_2'^2} + \frac{1}{1+y_1'^2}\right] f'' + \left[\frac{(y_2' + y_1)(y_2'' + y_1'')}{(1+y_2'^2)(1+y_1'^2)}\right] f'.$$

By hypothesis, D is strictly parabolic. A straightforward calculation using Lemma 1.7 shows that $D(y_2 - y_1) = 0$, so the Maximum Principle applies, proving the lemma.

Now consider the case where $d(t)$ is realized at an endpoint of either $\alpha(t)$ or $\beta(t)$. By Lemma 1.9, we know that the endpoints of $\alpha(t)$ and $\beta(t)$ are not

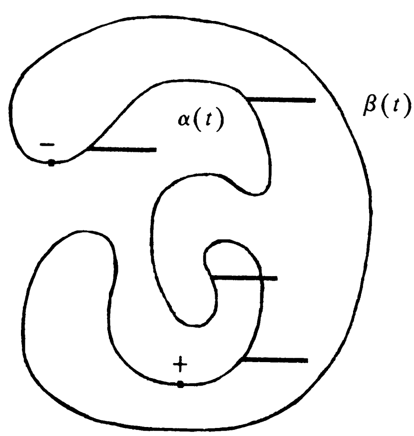

FIGURE 2. The δ-whiskers.

inflection points for any $t < t_0$. Therefore, $C(t)$ does not cross the tangents to the endpoints of $\alpha(t)$ and $\beta(t)$ in some neighborhood of the point of tangency. Thus, for example, if $\bar{\alpha}(t)$ is a translate of $\alpha(t)$ by some fixed amount d, which is tangent to an endpoint of $\beta(t)$, then the two arcs $C(t)$ and $\bar{\alpha}(t)$ are tangent, but do not cross. Therefore, the previous lemma applies, the curves are separating, and the maximum translation distance before intersection increases. The same holds true if the endpoint of the translate $\bar{\alpha}(t)$ by d is tangent to the endpoint of $\beta(t)$ (see Figure 3). In every event, $d(t)$ is increasing. q.e.d.

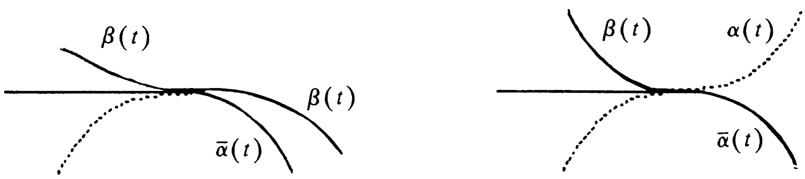

FIGURE 3

Let $\alpha_1(t_0)$ and $\alpha_2(t_0)$ be two disjoint subarcs of $C(t_0)$ which are nice with respect to the unit vectors v_1 and v_2, respectively. We can follow the endpoints of $\alpha_i(t_0)$ backwards in time to obtain families of nice subarcs $\alpha_i(t)$. The $\alpha_i(t)$ are disjoint: If v_1 and v_2 are not parallel, then the endpoints of $\alpha_i(t)$ have different tangents, and thus cannot evolve continuously past one another. If v_1 and v_2 are parallel, then we observe that disjoint critical points never collide in backwards time, for that would entail the creation of new critical points in forwards time, which is impossible by Lemma 1.9.

Consider translation of $\alpha_1(t)$ and $\alpha_2(t)$ in the directions of the inwards pointing tangent vectors to their endpoints. Let $d_{12}(t)$ be the maximum distance which the two curves can be moved before one of them bumps into either the curve $C(t) \setminus \{\alpha_1(t) \cup \alpha_2(t)\}$, or some translate of the other curve by some amount less than or equal to $d_{12}(t)$. $d_{12}(t)$ is a monotonically increasing function for the same reasons that $d(t)$ was. Translation commutes with evolution, and the first contacts are always at points of tangency where the curves do not cross. Therefore they are separatng, and at any later time, the arcs may be translated a little further.

Let δ be the minimum over all choices of disjoint nice $\alpha_1(0)$ and $\alpha_2(0)$ of $d_{12}(0)$. The lemma follows immediately.

As an application of the δ-whisker lemma, we prove that if a curve shrinks to a point, then its curvature is bounded away from $-\infty$. First we need a lemma similar to Theorem 2.1. While Theorem 2.1 prevents the curvature from blowing up unless it does so on some arc which turns through at least π, we can say something about the curvature on a subarc of the curve connecting inflection points, regardless of the behavior of the rest of the curve.

From Lemma 1.9, we see that the position of an inflection point varies continuously until it disappears. Let $p(t)$ be a continuously varying point on $C(t)$ with the property that $\kappa > 0$ at $p(t)$ for all t greater than some t_0. Let $\alpha(t)$ be the maximal subarc containing $p(t)$ such that $\kappa > 0$ restricted to $\alpha(t)$. The endpoints of $\alpha(t)$ vary continuously, except for a finite number of times when the endpoints of $\alpha(t)$ may jump outwards, including a larger arc. From Lemma 1.9, the total curvature of $\alpha(t)$ is decreasing where it is continuous.

Lemma 3.5. *Given $C(0)$, there is a constant K, such that if $\kappa > K$ at a point p_0 of $C(t_0)$, then the maximal positive curvature subarc $\alpha(t_0)$ containing p_0 has total curvature $\geqslant \pi$. In particular, $\alpha(t_0)$ is nice with respect to the inwards pointing tangent vectors at either of its endpoints.*

Proof. First consider a time interval when the endpoints of $\alpha(t)$ are evolving continuously. The total curvature of $\alpha(t)$ is then a strictly decreasing function of time. We may then consider the function κ_ω which is the maximum of κ restricted to $\alpha(t)$ at the time when the total curvature of $\alpha(t)$ is ω.

Lemma 3.6. *κ_π bounded implies that κ_ω is bounded for all $\omega < \pi$.*

Proof. By Lemma 1.3, a maximum of κ can grow no faster than its cube, so we know that for some $\varepsilon > 0$, κ_ω is bounded for $\omega > \pi - 2\varepsilon$.

Lemma 3.7. *Let $\alpha(t) \subset C(t)$, $t_1 < t < t_2$, be any continuously varying subarc connecting isolated inflection points. Suppose that the total curvature of $\alpha(t_1)$ is $\leqslant \pi - 2\varepsilon$. Let κ_t denote the maximum of curvature on $\alpha(t)$. Then $\kappa_t \leqslant \kappa_{t_1}/\sin(\varepsilon)$.*

Proof. Parametrize the curve by θ and t. By the hypotheses, the function $F(\theta) = \kappa_{t_1} \cdot \sin(\theta)/\sin(\varepsilon)$ is strictly greater than $\kappa(\theta, t_1)$ over the interval $\theta \in [\varepsilon, \pi - \varepsilon]$. $F(\theta)$ is the curvature of the parallel translating curve known as the "Grim Reaper." It is the graph of the function $c \cdot y = \operatorname{Log}(\sec(c \cdot x))$. By Lemma 2.2, $F(\theta)$ is stationary under evolution. By hypothesis, the range of θ on $\alpha(t)$ is monotonically decreasing, and so is contained within $[\varepsilon, \pi - \varepsilon]$ for all $t_1 < t < t_2$. Since κ is zero at the endpoints of $\alpha(t)$, the Maximum Principle prevents the graph of $\kappa(\theta, t)$ from crossing the graph of $F(\theta)$ for any $t > t_1$. q.e.d.

This completes the proof of Lemma 3.6.

Proof of Lemma 3.5, *continued.* Each time the total curvature of an arc connecting inflection points drops below π, we have a bound on the maximum of its curvature. At a finite number of times, however, two or more arcs may fuse to make a single arc with total curvature greater than π. Let K be the maximum of the bounds obtained in this fashion. Lemma 3.5 follows. This is where we use the finite number of inflection points hypothesis most strongly.

Lemma 3.8. *Suppose that $C(t)$ shrinks to a point as $t \to T$. Then there is an $M > -\infty$ such that $\kappa(\cdot, t) > M$ for all $t < T$.*

Note. We are using the convention that a strictly convex curve has $\kappa > 0$ and an inward pointing normal vector.

Proof. If $\kappa \to -\infty$, then, by Lemma 3.5, for all t greater than some t_0, there is an arc $\alpha(t)$ which has total curvature greater than or equal to π and $\kappa < 0$. By Lemma 3.2, there is a δ-whisker tangent to the endpoints of this arc for all $t_0 < t < T$ which meets $C(t)$ at only one point. If κ is negative on the arc, then the δ-whisker lies inside the curve, contradicting the hypothesis that the curve shrinks to a point. q.e.d.

Note. If a nice arc has total curvature exactly π, then the δ-whisker based on that arc intersects the curve only at its base point.

We can say even more about a curve which shrinks to a point.

Lemma 3.9. *Suppose that $C(t)$ shrinks to a point as $t \to T$. Then the $C(t)$ is convex in the limit, in the sense that its total curvature converges to 2π.*

Proof. If the total curvature of the curve is bounded away from 2π, then it contains a subarc whose total curvature is bounded away from zero, but on which κ is negative. Since this arc is eventually contained in an arbitrarily small ball, the minimum of its curvature must converge to $-\infty$. This contradicts Lemma 3.8.

4. The Main Theorem—Part I

Or, spirals do not collapse, curves that shrink to points become convex, and embedded curves keep bounded isoperimetric ratios.

The Main Theorem. *Let $C(\cdot, 0)$: $S^1 \to \mathbf{R}^2$ be a smooth embedded curve in the plane. Then C: $S^1 \times [0, T) \to \mathbf{R}^2$ exists satisfying*

$$\frac{\partial C}{\partial t} = \kappa \cdot \mathbf{N}.$$

$C(\cdot, t)$ is smooth for all $t < T$, and there is a $t_0 < T$ such that, for $t > t_0$, $C(\cdot, t)$ is smooth and convex, so it shrinks to a round point nicely via [5].

Proof. By Lemmas 1.5, 1.6, and 1.11, it is sufficient to show that the curvature κ remains bounded while $C(\cdot, t)$ has inflection points.

Let $\bar{\omega}$ be the supremum of all angles ω such that for any $\varepsilon > 0$, there is a $t_0 < T$ such that some subarc $\beta(t_0)$ of $C(t_0)$ has the following properties:

(i) the total curvature of $\beta(t_0)$ is ω,

(ii) the diameter of $\beta(t_0)$ is less than ε,

(iii) κ restricted to $\beta(t_0)$ is either less than K or greater than $-K$, where K is the bound from Lemma 3.5.

The third condition guarantees that an arbitrarily large percentage of the total curvature of $\beta(t_0)$ comes from arcs without inflection points, all of whose curvatures have the same sign.

By Theorem 2.1, we know that $\bar{\omega} \geqslant \pi$. Lemma 3.9 implies that, if the curve shrinks to a point, then $\bar{\omega} = 2\pi$. In this event, we wish to show that the curve first becomes convex. The remaining cases are:

I. $\bar{\omega} > \pi$, and the curve does not shrink to a point.

II. $\bar{\omega} = 2\pi$, and the curve does shrink to a point.

III. $\bar{\omega} = \pi$.

Case I. This is the first case which one suspects will yield a counterexample to the main theorem. A spiral with 10^{95} turns is going to get quite close to crushing the inflection point which is trapped in the middle. Surprisingly, it is the easiest case to rule out.

Theorem 4.1. *With the hypotheses of the Main Theorem, Case* I *does not occur.*

Proof. For any $\varepsilon > 0$, there is a t_0 and an arc $\beta(t_0)$, such that the diameter of $\beta(t_0)$ is less than ε, and the total curvature of $\beta(t_0)$ is ω, where ω is at least $(\bar{\omega} + \pi)/2$. Consider the outward pointing rays tangent to the endpoints of $\beta(t_0)$. Either they cross, or one of them crosses $\beta(t_0)$. In either event, the crossing must occur within a small neighborhood of the endpoints of $\beta(t_0)$ (where small is equal to $\max \varepsilon, \varepsilon/(\bar{\omega} - \pi)$). In other words, at least one endpoint of $\beta(t_0)$ is on a collision course with $C(t_0)$, and by an amount bounded away from zero. Since $C(t_0)$ is embedded, it must curve very fast to avoid the suggested self-intersection. The hypothesis that the curve is not shrinking to a point prevents the endpoints of $\beta(t_0)$ from being connected by a short arc. To at least one side of $\beta(t_0)$, then, and intersecting this small neighborhood, is an arc $\alpha(t_0)$ which has curvature of large magnitude and sign opposite to the prevailing curvature of $\beta(t_0)$. We may, without loss, extend $\beta(t_0)$ until it contacts $\alpha(t_0)$ at a point of inflection. If ε is chosen sufficiently small, then the magnitude of the curvature on $\alpha(t_0)$ must exceed K. By Lemma 3.5, the total curvature of $\alpha(t_0)$ exceeds π, and so $\alpha(t_0)$ must contain a nice subarc adjacent to $\beta(t_0)$, hence there is a δ-whisker tangent to the inflection point between $\alpha(t_0)$ and $\beta(t_0)$ which points away from $\beta(t_0)$. Since this whisker cannot intersect $C(t_0)$, and since "small" can be chosen smaller than δ, we conclude that $C(t_0)$ must curve very sharply at the other end of $\beta(t_0)$. By an identical argument, we conclude that there must be whiskers tangent to both endpoints of $\beta(t_0)$, pointing away from $\beta(t_0)$. As has already been mentioned, rays in these directions must intersect either each other, or the curve $C(t_0)$ in some small neighborhood. By Theorem 3.2, δ-whiskers can do neither (see Figures 4 and 5).

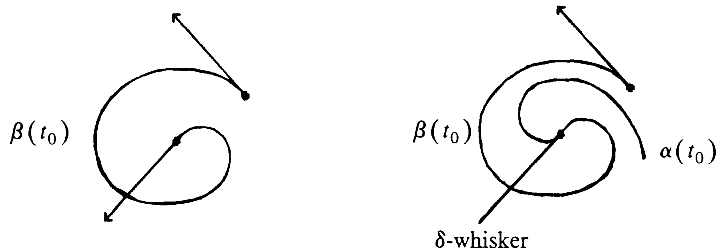

FIGURE 4. One of the rays tangent to the endpoints of $\beta(t_0)$ crosses $\beta(t_0)$.

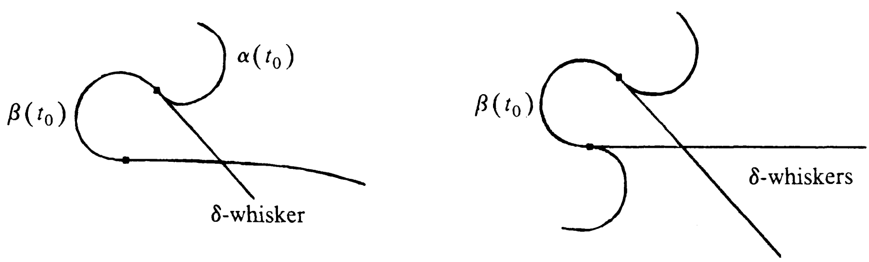

FIGURE 5. The rays tangent to the endpoints of $\beta(t_0)$ cross each other.

Case II. Since convex curves really *do* shrink to points, this case is very difficult to control. We must show that the curve becomes convex before it degenerates to a point. If the shape of the curve accumulates to some compact, unit area curve, then our task is not so difficult. If the shape is unbounded, then we must work harder. Happily, Lemma 3.8 keeps κ bounded away from $-\infty$ when the curve shrinks to a point. This is vital.

We may assume that $C(t)$ is not convex for any $t < T$, or else we are done. To study the possibilities, we need the notion of a continuous family of re-expansions of $C(t)$.

An evolving curve $C(t)$ passes through a family of shapes. We can envision another family $\tilde{C}(\tau)$ passing through the same shapes (up to homothety), but at different speeds. For example, if $\tilde{C}(0) = 5 \cdot C(0)$, then $C(t)$ evolves at 25 times the speed of $\tilde{C}(\tau)$. To see this, consider that, where $C(0)$ has curvature κ, $\tilde{C}(0)$ has curvature $\kappa/5$, and hence speed $\kappa/5$, but $5 \cdot C(t)$ moves at speed $5 \cdot \kappa$. Therefore, $\tilde{C}(25 \cdot t) = 5 \cdot C(t)$.

By the same argument, we can choose τ to be any monotonically increasing function of t, and we conclude that

$$\tilde{C}(\tau) = \left[\frac{d\tau}{dt}\right]^{1/2} C(t).$$

Example. *The area preserving expansion.* We know, from Lemma 1.11, that the area enclosed by a curve decreases at the constant rate of 2π. Suppose that the area enclosed by $C(0)$ is exactly 2π, and that $C(t)$ shrinks to a point. Let $\tau = -\ln(1 - t)$. Using the above formula for $\tilde{C}(\tau)$, and the fact that the ratio of the areas is the square of the expansion factor, we see that the area of $\tilde{C}(\tau)$ remains constant. Note that $\tilde{C}(\tau)$ exists for all $\tau > 0$.

Since the length of $C(t)$ is always decreasing, it is possible to choose a different τ so that the *length* of the expanded curve is kept constant.

Suppose that $C(t)$ converges to a point as $t \to T$, but it is never convex. From Lemma 3.8, we know that κ must be bounded away from $-\infty$. For $\tilde{C}(\tau)$, the area preserving expansion of $C(t)$, then, there are two possibilities. Either the diameter of $\tilde{C}(\tau)$ converges to infinity, or it does not. In the latter case, some sequence of expansions of $C(t)$ converges to a (not strictly) convex curve of bounded diameter, for the set of curves of fixed area, bounded diameter, and bounded total curvature is compact.

Theorem 4.2. *There is no evolving family $C(t)$ shrinking to a point which admits an expansion $\tilde{C}(\tau)$ such that:*

(i) *some subsequence $\tilde{C}(\tau_i)$ converges to a convex curve of bounded diameter and of area $= 2\pi$,*

(ii) *$\tilde{C}(\tau_i)$ is not convex for any $\tau_i < \infty$.*

Proof. Since $C(t)$ is shrinking to a point with curvature bounded away from $-\infty$, the total curvature of the arcs where $\kappa < 0$ must be going to zero. Therefore, there is, up to multiples of π, a *unique* direction tangent to every arc of negative curvature for all τ. It is easy to see that all arcs of negative curvature must eventually lie parallel to a single line, for otherwise, by Lemma 3.6, large θ-portions of $C(t)$ would have bounded curvatures. Choose cartesian coordinates so that this direction is parallel to the x-axis. Now note that there is an $M > 0$ such that for any $\varepsilon > 0$, we can choose a τ_i and cartesian coordinates such that $|x| < M/2$ and $y > -\varepsilon$ everywhere on $\tilde{C}(\tau_i)$, and $\tilde{C}(\tau_i)$ has a horizontal tangent at the origin at a point of negative curvature.

The argument, at this point, is to observe that $\tilde{C}(\tau_i)$ is contained in a large convex basket with nearly positive y-values. The evolving basket will soon have strictly positive y-values, and so, then, will $\tilde{C}(\tau)$. This contradicts the curvature assumptions on $\tilde{C}(\tau)$, proving the theorem.

Definition. An (ε, M)-basket is a convex curve B satisfying:

(i) $y > -\varepsilon$ on B.

(ii) $|y'| < 1$ on B for $x \in [-M, M]$.

(iii) $y > 1$ on B for $x = \pm M$.

Lemma 4.3. *Given $M > 0$, there is a function $f(\varepsilon)$ such that if $B = B(0)$ is any (ε, M)-basket, then for all $t > f(\varepsilon)$, $B(t)$ has strictly positive y-coordinates. Furthermore, $f(\varepsilon)$ decreases to zero as $\varepsilon \to 0$.*

Proof. For some short time $t < t_0$ on $B(t)$, $y'^2 < 2$ for x in the interval $[-M, M]$. By Lemma 2.6

$$\frac{\partial y}{\partial t} > \frac{y''}{3} \quad \text{for } x \in [-M, M],\ t < t_0.$$

The lemma follows by comparison with solutions to the heat equation with boundary conditions $y = -\varepsilon$ for $x \in [-M, M]$, $t = 0$, and $y = 1$ for $x = \pm M$. q.e.d.

With M as above, choose τ_i and its corresponding ε so that $f(\varepsilon) < \min(t_0, 1/2)$ seconds. If we suspend the expansion process, then we know that $\tilde{C}(\tau)$ will exist for another full second, i.e. for all $\tau < \tau_i + 1$. On the other hand, the evolving (ε, M)-basket containing $\tilde{C}(\tau)$ will have positive y-values in $f(\varepsilon)$ seconds. Before $\tilde{C}(\tau)$ can shrink to a point, then, it will have strictly positive y-values. By hypothesis, the curvature at the horizontal tangent to $\tilde{C}(\tau)$ stays negative. By Lemma 1.9, the y-value at that tangent must be strictly decreasing, contradicting the fact that y is soon strictly positive. q.e.d.

Now suppose that the diameters of the $\tilde{C}(\tau)$ with fixed area converge to infinity. The fact that curvature is bounded away from minus infinity implies that $\tilde{C}(\tau)$ must be straightening out, in the sense that the length preserving re-expansion must be close to a line segment for all large τ. The problem is to show that the shape of the curve is converging to a line segment with some fixed direction, and is not rotating. We know that there is a unique direction parallel for all time to the subarcs of $C(t)$ of negative curvature. Choose this direction for the x-axis. We now show that the shape of the curve is converging to a horizontal line segment.

Lemma 4.4. *If the length of $\tilde{C}(\tau)$ converges to infinity, then the length preserving expansion of $C(t)$ converges to a line segment parallel to the x-axis.*

Proof. As in Theorem 4.2, arrange each $\tilde{C}(\tau)$ so that it has a horizontal tangent at the origin at a point where $\kappa < 0$. If there exists an $M > 0$ such that for any $\varepsilon > 0$, we can find a τ such that the unit area $\tilde{C}(\tau)$ is contained in an (ε, M)-basket, then we are in the same situation as the last part of Theorem 4.2, i.e. the curve becomes convex before it can shrink to a point. Thus for *any* values of M and ε, we know that the unit area (and therefore very thin) curve $\tilde{C}(\tau)$ intersects both the origin, and some point (x_0, y_0), where $x_0 = \pm M$ and $-\varepsilon < y_0 \leqslant 1$. Remember that the total curvature of $\tilde{C}(\tau)$ is converging to 2π. So if we rescale by a factor of $1/L$, we have a nearly convex curve of area $1/L^2$, connecting the origin to a point distance ≈ 1 away within $1/L$ of the

x-axis. If the shape of $\tilde{C}(\tau)$ is close to that of a line segment, then the line segment must be approaching horizontal.

Theorem 4.5. *There is no evolving family $C(t)$ whose length preserving expansion $\tilde{C}(\tau)$ converges to a line segment.*

Proof. Suppose not. By Lemma 4.4, we can choose coordinates so that $C(t)$ converges to the origin, and $\tilde{C}(\tau)$ converges to a horizontal line segment. Since κ is bounded away from $-\infty$, and the diameter is going to zero, the total curvature of the negatively curved arcs must be going to zero. Therefore, $C(t)$ cannot double back to itself. So for t sufficiently close to T, $C(t)$ is the union of two graphs, $y_1(x,t)$ and $y_2(x,t)$, both defined on the interval $x \in [a(t), b(t)]$.

Lemma 4.6. *Let $w(t) = b(t) - a(t)$, and let $I_\varepsilon(t) = [a(t) + \varepsilon w(t), b(t) - \varepsilon w(t)]$. For all $\varepsilon > 0$, there is a $t_0 < T$, such that for all $t > t_0$ and for all $x \in I_\varepsilon(t)$,*

(i) $|y_i'(x,t)| < \varepsilon$.

(ii) *$h(x,t) = y_1 - y_2$ has a unique local maximum.*

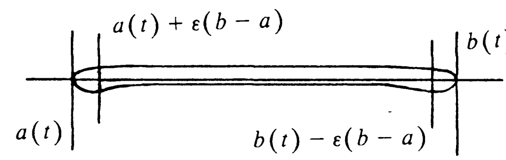

FIGURE 6

Proof. Condition (i) follows from the fact that, away from the ends, the curve is approaching a line segment. Uniform convergence follows from the fact that the expanded negative curvature is going to zero. This implies that the total curvature of the curve a bounded fraction away from the ends is converging to zero. The restriction on y' follows.

Consider the thickness function $h(x,t) = y_1 - y_2$. For t sufficiently close to T, h can have no local minima in its interior, for, by Lemma 3.4, h would be increasing at such a point, and h is converging uniformly to 0. By Lemma 1.9, no local minima can ever be created; the height difference between adjacent minima and maxima is strictly decreasing.

Corollary 4.7. *If the angle which a vertical line makes with the curve is less than $\pi - \varepsilon$, then the vertical tangent to the curve is less than $\varepsilon \cdot w(0)$ away.*

Proof. This follows from the bound on y' from Lemma 4.6, and the fact that w is decreasing with time. q.e.d.

For $\tilde{C}(\tau)$ to approach a line segment, it is necessary for the ratio of its thickness to its width to appraoch zero. We define the width of $C(t)$ by

$w(t) = b(t) - a(t)$. To define the thickness, we take the average of $h(x,t)$ over the interval $[a(t), b(t)]$, that is we define $H(t) = A(t)/w(t)$, where $A(t)$ is the area enclosed by the curve. (Of course, by Lemma 1.11, $A(t) = 2\pi \cdot (T-t)$.) So for the shape of $C(t)$ to approach a line segment, it is necessary that $H(t)/w(t)$ converge to zero, or equivalently, that $A(t)/w^2(t)$ converge to zero. This quantity actually converges to 4 divided by the standard isoperimetric ratio, so we are really saying that the isoperimetric ratio must converge to infinity.

The purpose of the average is to avoid difficulties with the possible nonuniform convergence of κ to zero in the center section. Because of the small total curvature, the *average* of κ over any fixed interval must converge uniformly to zero.

Note. In [3], M. Gage mentioned that, for a particular shape of curve, the isoperimetric ratio, L^2/A, did not improve. His example, however, had a local minimum for the thickness function in its interior. We will show, essentially, that the isoperimetric ratio improves for curves which are successfully shrinking to points, for they are eventually devoid of internal minima for thickness. We use the property of no internal minima only to obtain the estimate in the next lemma. It is both surprising and crucial.

Lemma 4.8. *Suppose that $h(x, t_0)$ has no internal minima. Let q_1 and q_3 (for first and third quartile) be the x-coordinates of the vertical lines which separate $C(t_0)$ into three pieces, the left and right ones having* $1/4$ *each of the total area, and the center piece* $1/2$ *of the total area. Then* $q_3 - q_1 \leqslant (2/3) \cdot w(t_0)$.

Proof. Let q_2 be the x-coordinate of the vertical line splitting $C(t_0)$ into two equal area pieces. The vertical lines through q_1, q_2, and q_3 divide the curve into four equal area pieces. Call these pieces P_1, P_2, P_3, and P_4, and call their widths w_1, w_2, w_3, and w_4. The maximum thickness, $h_{\max}$, of $C(t_0)$ is realized either in one of the end pieces, or in one of the central pieces. Suppose that $h_{\max}$ is realized in P_1. Then, since h decreases as we move through the other P's, we conclude that $w_2 \leqslant w_3 \leqslant w_4$. $q_3 - q_2$ is just $w_2 + w_3$, so the lemma follows in this case, for

$$w_2 + w_3 \leqslant \tfrac{2}{3}(w_2 + w_3 + w_4) \leqslant \tfrac{2}{3}(w_1 + w_2 + w_3 + w_4).$$

Suppose that $h_{\max}$ is realized in P_2. Normalize so that the four pieces have unit area. Divide P_2 into two pieces at the maximum thickness. These pieces have areas A_1 and A_3, and widths w_{21} and w_{23}. Let h_1 and h_3 be the thickness at the left and right endpoints, respectively, of P_2. Note that:

$$h \leqslant h_1 \text{ on } w_1, \quad h \geqslant h_1 \text{ on } w_{21}, \qquad h \leqslant h_3 \text{ on } w_3, \quad h \geqslant h_3 \text{ on } w_{23}.$$

Therefore, we have

$$w_{21} \leqslant A_1/h_1, \quad w_1 \geqslant 1/h_1, \qquad w_{23} \leqslant A_3/h_3, \quad w_3 \geqslant 1/h_3.$$

Hence,

$$w_{21} \leqslant A_1 \cdot w_1 \quad \text{and} \quad w_{23} \leqslant A_3 \cdot w_3.$$

Using $A_1 \leqslant 1$, $A_3 \leqslant 1$, and, as before, $w_3 \leqslant w_4$, we get

$$w_2 + w_3 = w_{21} + w_{23} + w_3 \leqslant A_1 \cdot w_1 + A_3 \cdot w_3 + w_3 < 2 \cdot w_1 + 2 \cdot w_4,$$

which implies that $3 \cdot (w_2 + w_3) < 2 \cdot (w_2 + w_3) + 2 \cdot (w_1 + w_4)$. The lemma follows.

Now blow up the curve until its area is 2π, reset time to zero, and fix the positions of q_1 and q_3 so that they divide the curve into pieces of area $\pi/2$, π, and $\pi/2$ at time zero only.

The idea of the proof is to show that the area will halve in $1/2$ second, but the width will decrease by a factor of about $1/3$. Therefore, the area A will shrink to $A/2$ and the width w will shrink to approximately $2w/3$, so $A/w^2 \to \approx (9/8)A/w^2$, which is an increase. So at each time $t_n \approx 1 - (1/2)^n$, then, A/w^2 is increasing, contradicting the hypothesis that it is converging to zero. This is not a miracle which works only for dividing the area in half. Any other fraction works as well, but with slightly different restrictions on ε.

Lemma 4.9. *At time $t_0 = (1/2 + \varepsilon/\pi)$ the vertical tangents to $C(t_0)$ are within $[a_0 - \varepsilon \cdot w(0), b_0 + \varepsilon \cdot w(0)]$. That is,*

$$w(t_0) \leqslant b_0 - a_0 + 2\varepsilon \cdot w(0) \leqslant (2/3 + 2\varepsilon) \cdot w(0).$$

Proof. From Lemma 1.10, we have that either:

(i) the area of an outside piece is decreasing at a rate in excess of $\pi - 2\varepsilon$ or,

(ii) the interior angles at the vertical line sum to less than $\pi - 2\varepsilon$.

Case (ii) for either end implies the conclusion of the lemma for that end. If we are never in case (ii) then, by Lemma 1.10, the area of an outside piece after $(1/2 + \varepsilon/\pi)$ seconds would be less than

$$\pi/2 - (\pi - \varepsilon) \cdot (1/2 + \varepsilon/\pi) = \varepsilon^2/\pi - \varepsilon/2$$

which, for small ε, is negative. This is absurd. At time t_0, then, the vertical tangents are within $\varepsilon \cdot w(0)$ of a_0 and b_0. The lemma follows.

Proof of Theorem 4.5, *continued.* With t_0 as above, we will show that

$$A(t_0)/w^2(t_0) > A(0)/w^2(0),$$

or equivalently,

$$A(t_0) \cdot w^2(0) > A(0) \cdot w^2(t_0).$$

$A(t_0) = 2\pi(1 - (1/2 + \varepsilon/\pi))$, so cross multiplying yields:

$$2\pi \cdot w^2(0)(1/2 - \varepsilon/\pi) = w^2(0) \cdot (\pi - 2\varepsilon) \quad \text{for the left-hand side,}$$

and

$$2\pi \cdot w^2(t_0) < 2\pi \cdot (2/3 + 2\varepsilon)^2 \cdot w^2(0) \quad \text{for the right-hand side.}$$

So

$$\text{L.H.S.} - \text{R.H.S.} \geqslant w^2(0) \cdot (\pi/9 - \varepsilon(2 + 16\pi/3 + 8\pi\varepsilon)).$$

For $\varepsilon < 0.01$, this is positive.

Since A/w^2 is not converging to zero, the shape of $C(t)$ is hitting a compact set infinitely often, which is a contradiction.

5. The Main Theorem—Case III

Case III. There is some sequence of arcs $\beta_i(t_i) \subset C(t_i)$, which turn through φ_i, $\varphi_i \to \pi$, such that $\operatorname{diam}(\beta_i(t_i)) \to 0$. Choose coordinates so that $[0, \pi]$ is a limit of the θ-intervals of β_i. From Theorem 2.1, we know that we can choose the $\beta_i(t_i)$ so that for any $K > 0$ and $\varepsilon > 0$ there is a $\beta_i(t_i)$ such that $\kappa > K$ on a subarc of $\beta_i(t_i)$ of total curvature $> \pi - \varepsilon$. Since θ-intervals of arcs connecting inflection points are nested with time, except for a finite number of jump discontinuities, there is a single subarc, $\beta(t) \subset C(t)$, connecting consecutive inflection points which contains infinitely many of the $\beta_i(t_i)$, and has tangent directions $\theta \in [0, \pi]$ for all sufficiently large values of t.

Lemma 5.1. *The two lines tangent to $\beta(t)$ at the $\theta = 0, \pi$ directions must be converging to the same line.*

Proof. Suppose not. Then since κ is getting large on θ-intervals $[\theta_0, \theta_0 + \pi - \varepsilon]$ close to $[0, \pi]$, and since $\beta(t)$ is contained within very-close-together lines at those angles, we see that the length of $\beta(t)$ would have to be unbounded in order to stay tangent to the horizontal lines. q.e.d.

Choose axes so that the horizontal lines tangent to $\beta(t)$ converge to the x-axis, and the vertical line tangent to $\beta(t)$ converges to the y-axis from the left.

Call the two arcs to the right of the y-axis $\beta_+(t)$ and $\beta_-(t)$. If necessary, extend $\beta_+(t)$ and $\beta_-(t)$ to the left until they contain horizontal tangents. Note that, with the current definition, β need not be disjoint from $\beta_\pm$.

Our first goal is to show that an inflection point must accumulate on the origin along one of $\beta_+(t)$ or $\beta_-(t)$.

Lemma 5.2. *Suppose that, for a fixed $\xi > 0$, $\kappa > 0$ on $\beta_+(t)$ for all $x < \xi$ and $t_0 < t < T$. Then for some $t_0 < t_1 < T$, $\beta_+(t_1)$ is below the x-axis somewhere in the interval $x \in [0, \xi]$.*

Proof. Suppose not. First, note that the height of the intersection of $\beta_+(t)$ with the y-axis must be positive for all $t < T$, for it is approaching zero and it is strictly decreasing. Therefore, the claim is that $\beta_+(t)$ is below the x-axis somewhere to the right of the horizontal tangent. Next, observe that for every $\varepsilon > 0$, there is a $\xi_0 > 0$, such that the slopes of the tangents to $\beta_+(t)$ over the interval $x \in [\xi_0/2, \xi_0]$ must eventually stay bounded between $-\varepsilon$ and ε. The lower bound follows from local convexity and the fact that the height of $\beta_+(t)$ is approaching zero. If the upper bound were false, then the total curvature of $\beta \cup \beta_+$ in an arbitrarily small ball would be greater than $\pi + \varepsilon/2$ at times arbitrarily close to T, and the proof of Case I would apply.

Let $y_+(x, t)$ be the function whose graph is $\beta_+(t)$. From the above paragraph, we know that y'_+ is bounded on the interval $[\xi_0/2, \xi_0]$ for all $t_2 < t < T$. By hypothesis, $y_+ \geqslant 0$ over the same interval and for all $t_0 < t < T$. Therefore, by Lemma 1.9, the values of y_+ cannot go uniformly to zero in finite time, which is a contradiction, since the maximum of y_+ is converging to zero. q.e.d.

If $\beta_+(t)$ has no inflection points near the origin for any $t > t_0$, then $\beta_-(t)$ must eventually have an inflection point at some point whose x-coordinate is less than ξ. Since this holds for arbitrary ξ, we see that there must be an inflection point accumulating at the origin along one of $\beta_+(t)$ and $\beta_-(t)$. If necessary, flip the picture so that the inflection point is on $\beta_-(t)$.

We are first going to rule out the case of inflection points accumulating on the origin along both $\beta_+(t)$ and $\beta_-(t)$.

Lemma 5.3. *There is no* $\beta(t) \subset C(t)$ *satisfying*:

(i) $|\beta(t)| \to \pi$ *as* $t \to T$,

(ii) $\operatorname{diam}(\beta(t)) \to 0$,

(iii) $\kappa = 0$ *at the endpoints of* $\beta(t)$.

Proof. For simplicity, restrict the $\beta_\pm$'s to be the arcs of negative curvature whose left endpoints are accumulating on the origin. Since the ranges of θ on $\beta_\pm(t)$ are eventually nesting, and since they contain horizontal directions in the limit, they must have horizontal directions for all sufficiently large time. That is, both $\beta_+(t)$ and $\beta_-(t)$ have horizontal tangents, and $\kappa < 0$ in their interiors for all $t_0 < t < T$.

Let v_0 and v_1 be the horizontal tangents to $\beta_+(t)$ and $\beta_-(t)$. We know that v_0 and v_1 converge to the same line (see Figure 7).

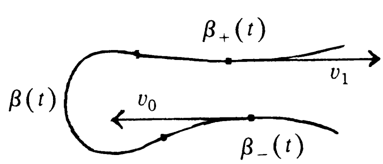

FIGURE 7

Since v_0 is moving down and v_1 is moving up, v_0 must lie above the x-axis, and v_1 must lie below it. So the picture is as depicted in Figure 8.

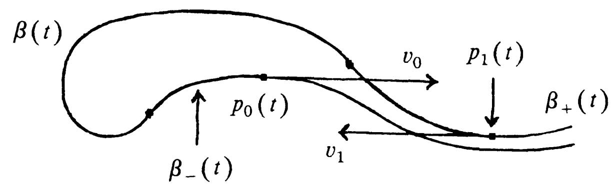

FIGURE 8

Note that the critical points to the height function, p_0 and p_1, have opposite sign. Therefore, the arc connecting p_0 with p_1 is nice, so by Lemma 3.2, the horizontal distance between the bold arcs in Figure 9 is bounded away from zero. This result is a contradiciton, as we will see in the next lemma.

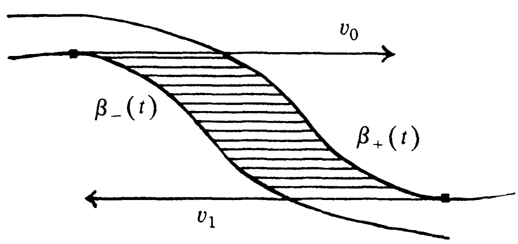

FIGURE 9

Lemma 5.4. *The horizontal distance between $\beta_-(t)$ and $\beta_+(t)$ cannot stay bounded away from zero.*

Proof. Suppose that it is, as in Lemma 5.3. Consider the function $y_+(x, t)$ whose graph is the part of $\beta(t) \cup \beta_+(t)$ which lies above the positive x-axis. Since p_0 lies above and p_1 lies below the x-axis, $y_+(x, t)$ has a positive zero for all t. But the x-coordinate of the smallest positive zero of $y_+(x, t)$ is bounded away from zero, that is, there is a $\xi > 0$ and a $t_0 < T$, such that $y_+(x, t) \geqslant 0$ for $x \in [0, \xi]$ and $t_0 < t < T$. Now y_+ must converge uniformly to zero as $t \to T$. The slope y'_+ is bounded, for $\beta(t)$ cannot turn more than ε past the horizontal before meeting the inflection point, and $\beta_+(t)$ is below the x-axis before it reaches the horizontal again. So y_+ is positive and its derivative is bounded in the interval $x \in [\xi/3, 2\xi/3]$, so it evolves by a strictly parabolic equation. Lemma 1.9 applies again; y_+ cannot converge to zero uniformly. Lemmas 5.3 and 5.4 are proved.

The only picture left is: everywhere that κ is blowing up, inflection points accumulate to the singularity on one side only.

Lemma 5.5. *The curvature of $\beta(t)$ must be positive.*

Proof. Suppose not. We could then choose cartesian coordinates so that the region between $\beta_+(t)$ and $\beta_-(t)$ is part of the outside of the curve. There is a critical point labelled $+$ at the top of $\beta_-(t)$. The lowest point on the curve is labelled with a $-$, and the arc connecting these two critical points is disjoint from $\beta_+(t)$ in a neighborhood of the x-axis. Hence Lemma 3.2 applies and the horizontal distance between $\beta_-(t)$ and $\beta_+(t)$ cannot go to zero, contradicting Lemma 5.4. q.e.d.

Since we now know that the point of intersection of $\beta_+(t)$ and the x-axis is approaching the origin, we see that the horizontal distance between $\beta_-(t)$ and $\beta_+(t)$ must be going to zero at the x-axis.

We are now in the situation that $\kappa \to +\infty$ on $\beta(t)$, and the critical point at the top of $\beta_-(t)$ is labelled $-$. By the above argument, every critical point of the height function below the x-axis must also be labelled $-$, or else Lemma 3.2 would apply, and the horizontal distance between $\beta_-(t)$ and $\beta_+(t)$ would be bounded away from zero, contradicting Lemma 5.4.

Lemma 5.6. *With the hypotheses of Case* III, *$C(t)$ approaches an arc uniformly as $t \to T$, and the horizontal width of $C(t)$ decreases monotonically as y decreasees.*

Proof. After some time t_1 there are no more $+$ critical points below the x-axis. Now consider the function $h(y,t)$ which is the length of the longest horizontal line segment at height y with endpoints on the part of $C(t)$ below, and to the right of the top of $\beta_-(t)$, and connecting it to the point on $\beta_+(t)$ at the same height. Because of the lack of $+$ critical points, the discontinuities of $h(y,t)$ for fixed t can only decrease as y decreases (see Figure 10). If h were to jump up as we decended to a particular y-value, then the new endpoint would be a local maximum for y, hence it would be labelled $+$.

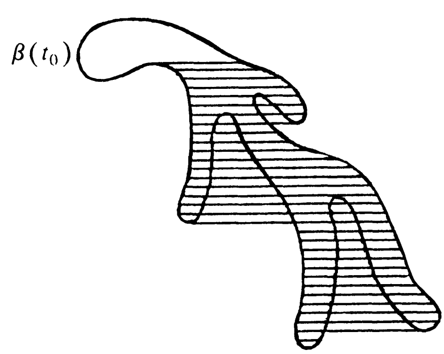

FIGURE 10. The discontinuities of $h(y, t_0)$

Fix $t = t_0$. Suppose that $h(y, t_0)$ has a discontinuity at height $y_0(t_0)$, and at no higher height. $C(t_0)$ has a $-$ critical point at height $y_0(t_0)$. Note that there must be at least one critical point at some height $y_1(t_0) > y_0(t_0)$, but that is "interior," and so does not cause a discontinuity in $h(y, t_0)$. The only way that $h(y, t_0)$ could have a local maximum at the height $y_0(t_0)$ would be for the other endpoint of the line segment to be at a $+$ critical point.

From the facts that $y_0(t)$ is increasing in t, $h(y, t)$ never has a local maximum at height $y_0(t)$, and new local maxima cannot be created (Lemma 3.4), we must eventually have no local maxima for $h(y, t)$ at any height above $y_0(t)$ and below the top of $\beta_-(t)$. Otherwise, since there is already a local maximum to $h(y, t)$ at the top of $\beta_-(t)$, we would have to have a trapped local minimum for all time. At such a minimum, Lemma 3.4 shows that h would be increasing with time, thus preventing h from approaching zero at the x-axis. Therefore, $h(y, t)$ is eventually monotonic in y above $y_0(t)$.

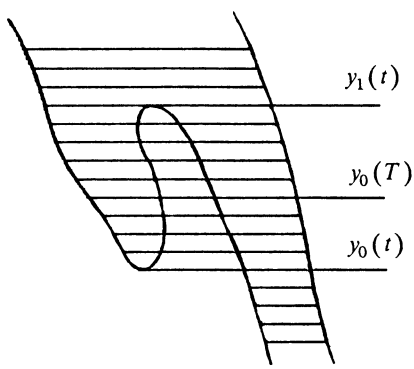

FIGURE 11

This argument can be repeated for each of the disjoint pieces of interior $(C(t)) \cap \{\mathbf{R}^2 \mid y < y_1(t)\}$. At $y = y_1(t)$, the horizontal tangent implies that the widths of the horizontal line segments to either side are decreasing as y decreases. The widths at $y = y_1(t)$ are converging to zero, for each is bounded by $h(y_1(t), t)$.

We have, then, that the length of every horizontal line segment with endpoints on $C(t)$ below the x-axis is approaching zero, and eventually, $h(y, t)$ has no local maxima below the top of $\beta_-(t)$.

Lemma 5.7. *With the hypotheses of Case* III, *the curve over the entire curve is bounded away from* $-\infty$.

Proof. Compare with Lemma 3.8. Negative unbounded curvature implies that some arc of negative curvature has total curvature at least π. By Lemma 3.2, we know that there is a δ-whisker inside the curve which intersects the curve only at its basepoint. Choose coordinates so that the whisker covers the x-axis over the interval $x \in [\delta/2, \delta]$ for all t sufficiently close to T. The area of the curve is converging to zero, hence the curve must converge to the x-axis over the interval $x \in [\delta/2, \delta]$. This is ruled out by Lemma 1.9. q.e.d.

So not only must the curve converge to an arc, but it must do so in a two-to-one fashion, with no "folding." Any arc "folded under" would have unbounded negative curvature.

Lemma 5.8. *With the hypotheses of Case* III, *the lowest point on the curve is bounded away from the x-axis.*

Proof. At some time t_0, $\beta_+(t_0)$ is below the x-axis at some point p_0 whose x-coordinate, x_0, satisfies $x_0 \in [\xi/2, \xi]$. For all $t > t_0$, $\kappa > 0$ on $\beta_+(t)$. By Lemma 1.7, then, the height of $\beta_+(t)$ at the line $x = x_0$ is monotonically decreasing with time. q.e.d.

Let Y_b be the limit as $t \to T$ of the height of the lowest point on $C(t)$. Consider the sum of the interior angles which the tangent vectors to $C(t)$ at some height y, $2Y_b/3 \leqslant y \leqslant Y_b/3$, make with the horizontal.

Lemma 5.9. *Given any $\varepsilon > 0$ there is a time $t_0 < T$ such that for any $t > t_0$ and for any $y_0 \in [2Y_b/3, Y_b/3]$, the sum of the interior angles which $C(t)$ makes with the underside of the line $y = y_0$ is between $\pi - \varepsilon$ and π.*

Proof. Consider the outermost points of intersection of $C(t)$ with the line $y = y_0$. The upper bound on the interior angles follows from the fact that, for fixed t, $h(y, t)$ decreases as y decreases. For the lower bound, consider: The distance between the endpoints of the arcs through the outermost points is going to zero, but their curvatures are bounded from below by some constant $-M$, and they do not join up for at least a distance of $|Y_b/3|$. Remember that negative curvature is the rate at which the curve is turning away from its interior, and the interior of $C(t)$ now lies between the arcs. The lemma follows from trigonometry.

To complete the proof of Case III, we use a variant of the proof of Theorem 4.5.

Define $A_1(t)$ to be the area of the intersection of the interior of $C(t)$ with the strip $2Y_b/3 \leqslant y \leqslant Y_b/3$, and $A_2(t)$ to be the area of the interior of $C(t)$ below the line $y = 2Y_b/3$.

Fix some small ε. Pick a t_0 large enough to satisfy the conclusion of Lemma 5.9, and the additional condition that $C(t)$ lies above the line $y = 4Y_b/3$. Since $h(y, t)$ is monotonic in y, and using Lemma 1.10, we get

$$A_1(t) \geqslant h(2y_b/3, t) \cdot |y_b/3| \quad \text{and} \quad dA_1/dt \geqslant -\varepsilon,$$

$$A_2(t) \leqslant 2 \cdot h(2y_b/3, t) \cdot |y_b/3|, \quad \text{but} \quad dA_2/dt \leqslant -\pi + \varepsilon.$$

$A_1(t)$ must go to zero, but $A_2(t)$ is decreasing much faster. By time $t = t_0 + h(2y_b/3, t_0) \cdot |y_b|/\pi$, $A_1(t)$ has barely decreased, but $A_2(t)$ is already negative. This is absurd. q.e.d.

We have now ruled out all behavior for an embedded curve save that of the curve becoming convex. The Main Theorem for embedded curves now follows from the Main Theorem for convex curves in [5].

Epilogue: Curves on surfaces

The fact that embedded curves in the plane evolve nicely is a strong argument for the niceness of curves evolving on a Riemannian surface. In [1], Abresch and Langer show that curvature bounded for all time implies convergence to a closed geodesic. If our techniques were generalized slightly, they should be able to show that either the curve would become convex and shrink to a point, or its curvature would remain bounded for all time. For instance, the δ-whisker lemma works with a slight modification. In the hyperbolic plane, equidistant curves flow towards each other, so δ must be allowed to decrease exponentially. The fact that the total curvature of an arc, or even the whole curve, may increase in another problem. The $1°$ North Lattitude circle on the sphere has small total curvature, yet it converges to the North Pole via higher lattitude circles whose total curvatures converge to 2π. Finally, the analysis in [5] must be repeated, for once we know that the curve is convex and shrinking to a point, we would like to show that its shape becomes round.

We are optimistic that this program will go through, as the problems are local, and the local analysis is very close to the Euclidean case. Here is one conjecture:

Conjecture. *Let $C(0)$ be a smooth curve immersed in a surface M, with the property that $C(0)$ is contained in some compact, convex set K, and the lift $\tilde{C}(0)$ of $C(0)$ to the universal cover of K is embedded. Then $C(t)$ exists satisfying the flow-by-curvature equations, and $C(t)$ either exists for infinite time and converges to a closed geodesic, or it converges to a point in finite time with round limiting shape. Convergence is in the C^∞ norm.*

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