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SPLIT RANK AND SEMISIMPLE AUTOMORPHISM GROUPS OF G-STRUCTURES

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1. Introduction

This paper is a continuation of the investigation begun in [1], [3], [4] concerning the semisimple automorphism groups of G-structures on compact manifolds. In those papers we were concerned with semisimple groups that preserve a structure which is algebraic and which defines a volume density, i.e. where the structure group G is an algebraic subgroup of $\mathrm{SL}'(n, \mathbb{R})$, the matrices with $|\det| = 1$. (For higher order structures we assumed that G is an algebraic subgroup of $\mathrm{SL}'(n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{R})^{(k)}$, the latter being the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ fixing the origin.) One of the basic conclusions in the above papers is that for any simple noncompact Lie group H preserving such a G-structure, we must have that H locally embeds in G. (In fact a stronger assertion is proven. See the above papers and Theorem 2 below.) The main goal of the present paper is to consider the situation in which H is no longer assumed to define a volume density. In this case natural examples easily show that one cannot expect a local embedding of H in G. However, our main result asserts that a basic structural invariant of H must be visible in G. More precisely, we prove:

Theorem 1. *Let H be a semisimple Lie group with finite center and suppose that H acts smoothly on a compact manifold M so as to preserve a G-structure on M, where G is a real algebraic group. Then* $\mathbb{R}\text{-rank}(H) \leqslant \mathbb{R}\text{-rank}(G)$.

We recall that the $\mathbb{R}$-rank, or split rank, of a real algebraic group is the maximal dimension of an algebraic torus that is diagonalizable over $\mathbb{R}$. For a semisimple Lie group H, $\mathrm{Ad}(H)$ will be the connected component of the identity of a real algebraic group, and the $\mathbb{R}$-rank, or split rank, of H is defined to be the split rank of this real algebraic group. We shall also clear up

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a point that was left open in [1], [3] concerning the case in which G defines a volume density. Namely, the general results in [3] were established for noncompact simple groups, not for semisimple groups. In [3], a special argument was given that clarified the semisimple situation for the case of Lorentz structures. Here we observe that a simple argument enables us to extend the results of [3] to the semisimple case in general, at least in the case of finite center.

Theorem 2 (*cf.* [1], [3], [4]). *Let H be a connected semisimple Lie group with finite center and no compact factors, and suppose that H acts on a compact n-manifold preserving a G-structure, where G is algebraic and defines a volume density. Then there is an embedding of Lie algebras $\mathfrak{h} \to \mathfrak{g}$. Furthermore, the representation $\mathfrak{h} \to \mathfrak{g} \to \mathfrak{sl}(n, \mathbb{R})$ contains $\mathrm{ad}_{\mathfrak{h}}$ as a direct summand.*

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2. Preliminaries

We establish here some preliminary information we shall need for the proofs of Theorems 1 and 2.

Proposition 3. *Let H be a connected semisimple Lie group with finite center, acting smoothly on a connected manifold M, and assume $p \in M$ is a fixed point. Let π: $H \to \mathrm{GL}(TM_p)$ be the corresponding representation at p. If π is trivial, then H acts trivially on M.*

Proof. Let $K \subset H$ be a maximal compact subgroup. It suffices to see that K acts trivially. For a compact group, any smooth action can be linearized around fixed points, so the set of invariant frames for the tangent bundle is both open and closed.

Proposition 4. *Suppose H is a connected semisimple Lie group with finite center, acting smoothly on a connected manifold M. If the set of fixed points has positive measure, then H acts trivially.*

Proof. If the set of fixed points, F, has positive measure, choose a density point p for F in the sense of Lebesgue. Then any small ball around p intersects F in a set of positive measure. The action of the maximal compact subgroup $K \subset H$ can be linearized around p, which implies that K leaves a set of vectors in TM_p invariant which has positive measure in TM_p. It follows that this linear representation of K is trivial, and the proof of Proposition 3 completes the proof.

If a Lie group H acts smoothly on a manifold M, and $m \in M$, we let H_m be the stabilizer of m in H, and $\mathfrak{h}_m \subset \mathfrak{h}$ the Lie algebra of H_m. If V is a vector space we let $\mathrm{Gr}_d V)$ be the Grassman variety of d-dimensional linear subspaces.

For a Lie group L, we let L^0 be the identity component. If a Lie group L is the identity component of an algebraically connected real algebraic group, by a rational homomorphism of L into a real algebraic group we mean the restriction of (a necessarily unique) rational homomorphism of the ambient algebraic group. The following is standard.

Lemma 5. *Suppose H is a Lie group acting smoothly on a manifold M. Let d be the minimal dimension of an H-orbit in M. Then $M_1 = \{m \in M \mid \dim(Gm) = d\}$ is closed, and the map $m \to \mathfrak{h}_m$ defines a continuous map φ: $M_1 \to \mathrm{Gr}_q(\mathfrak{h})$, where $q = \dim(H) - d$. Further, φ is an H-map, where H acts on $\mathrm{Gr}_q(\mathfrak{h})$ via $\mathrm{Ad}(H)$.*

We recall briefly the notion of the algebraic hull of a cocycle defined for an ergodic group action (see [4] or [2] for an elaboration). Suppose that H is a locally compact group acting ergodically on a standard measure space (M, μ). Suppose that G is a real algebraic group and that α: $H \times M \to G$ is a cocycle, i.e., the following identity is satisfied (for each $h_1, h_2 \in H$, and almost all $m \in M$): $\alpha(h_1 h_2, m) = \alpha(h_1, h_2 m)\alpha(h_2, m)$. We recall that two cocycles α, β are called equivalent if there is a measurable φ: $M \to G$ such that for each h and almost all m, $\beta(h, m) = \varphi(hm)^{-1}\alpha(h, m)\varphi(m)$.

Lemma 6 ([2], [4], [5]). *There is an algebraic subgroup $L \subset G$ with the following properties*:

(i) *α is equivalent to a cocycle taking all its values in L.*

(ii) *For any proper algebraic subgroup $L' \subset L$, α is not equivalent to a cocycle taking all its values in L'.*

(iii) *Up to conjugacy in G, L is the unique algebraic subgroup satisfying* (i), (ii).

(iv) *If α is equivalent to a cocycle taking all its values in some closed subgroup $L_0 \subset G$, then some conjugate of L_0 is contained in L.*

L is then called the algebraic hull of α, and it is well defined up to conjugacy in G. The following property is easily established.

Lemma 7. *Suppose p: $G_1 \to G_2$ is a rational homomorphism of real algebraic groups. If α is a G_1-valued cocycle with algebraic hull L_1, then the algebraic hull of the G_2-valued cocycle $p \circ \alpha$ is the algebraic hull of $p(L_1)$ (in which, we recall, $p(L_1)$ is a subgroup of finite index).*

3. Proof of Theorem 1

Let M_1 be as in Lemma 5. Since M_1 is a compact H-space, we can choose a minimal H-space $M_0 \subset M_1$, i.e., a closed H-invariant subset in which every orbit is dense. Then, letting φ be as in Lemma 5 as well, we have that $\varphi(M_0) \subset \mathrm{Gr}_q(\mathfrak{h})$ is minimal. However, the action of H on $\mathrm{Gr}_q(\mathfrak{h})$ is algebraic, and hence every orbit is locally closed. It follows that $\varphi(M_0)$ consists of a single compact H-orbit. Fix $x \in M_0$, and let $\mathfrak{h}_x = \mathfrak{h}_0$. Then we can consider φ as an H-map $\varphi: M_0 \to \mathrm{Ad}(H)/N(\mathfrak{h}_0) \subset \mathrm{Gr}_q(\mathfrak{h})$, where $N(\mathfrak{h}_0)$ is the normalizer of $\mathfrak{h}_0$ in $\mathrm{Ad}(H)$. In particular, the algebraic subgroup $N(\mathfrak{h}_0)$ is cocompact in $\mathrm{Ad}(H)$, and therefore we can find a maximal $\mathbb{R}$-split torus T of $\mathrm{Ad}(H)$, with $T \subset N(\mathfrak{h}_0)$.

Let $\mathfrak{n}$ be the Lie algebra of $N(\mathfrak{h}_0)$, so that $\mathfrak{h}_0 \subset \mathfrak{n}$ is an ideal. The adjoint representation yields a rational (and in particular semisimple) representation $T \to \mathrm{GL}(\mathfrak{h}/\mathfrak{h}_0)$. Let $T_0 \subset T$ be the kernel, so that T_0 is an $\mathbb{R}$-split subtorus. Since the representation of T_0 on $\mathfrak{h}$ is semisimple, we can write $\mathfrak{h} = \mathfrak{h}_0 \oplus W$, where $W \subset \mathfrak{h}$ is a subspace and T_0 acts trivially on W. In particular, $\mathfrak{h}_0$ contains all the root spaces of T_0 acting via Ad_H on $\mathfrak{h}$ corresponding to nontrivial roots. The algebra generated by the nontrivial root spaces for T_0 is an ideal, and hence $\mathfrak{h}_0$ contains an ideal of $\mathfrak{h}$, say $\mathfrak{h}_1$, containing all the nontrivial root spaces for T_0. Thus we can write $\mathfrak{h}$ as a sum of ideals, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where T_0 acts trivially on $\mathfrak{h}_2$. Since $\mathfrak{h}_2$ is semisimple, it follows that $\mathfrak{t}_0 \subset \mathfrak{h}_1 \subset \mathfrak{h}_0$. Let H_1 be the connected normal subgroup of H corresponding to $\mathfrak{h}_1$. Then $\mathfrak{h}_1$, and hence H_1, acts trivially on $\mathfrak{h}/\mathfrak{h}_0$, and by Proposition 3, H_1 acts locally faithfully on $T(M)_x/T(Hx)_x$. In particular, T_0 acts rationally and locally faithfully on $T(M)_x/T(Hx)_x$. Let T_1 be a split torus complementary to T_0 in T. We then have that $T = T_0 \times T_1$, and T_1 acts faithfully on $\mathfrak{h}/\mathfrak{h}_0$.

Now let $M_2 \subset M_0$ be a minimal $N(\mathfrak{h}_0)^0$ space. Since $(H_x)^0$ is normal in $N(\mathfrak{h}_0)$, it fixes all points of $N(\mathfrak{h}_0)x$, and hence fixes all points in the closure of this orbit, in particular all points in M_2. Since the dimension of all stabilizers in H of points in M_0 are the same, we have $\mathfrak{h}_m = \mathfrak{h}_0$ for all $m \in M_2$. Thus for $m \in M_2$, we can identify the tangent space to the H-orbit through m with $\mathfrak{h}/\mathfrak{h}_0$. The representation of H_1 on $T(M)_m/T(Hm)_m$ will vary continuously over $m \in M_2$, and since H_1 is semisimple and M_2 is connected, all these representations are equivalent. In particular, the representations of $(T_0)^0$ on these spaces are all rational, and all equivalent.

Choose a probability measure on M_2 which is invariant and ergodic under T^0 [2, Chapter 4]. Let $\alpha: T^0 \times M_2 \to \mathrm{GL}(n, \mathbb{R})$ be a cocycle corresponding to the action of T^0 on the tangent bundle of M over the space M_2 (cf. [4]). Let L be the algebraic hull of this cocycle. Since H, and in particular T^0, leaves a G-structure on M invariant, we have (up to conjugation) $L \subset G$. By our observations above, we can measurably trivialize TM over M_2 in such a way that $TM \cong M \times \mathbb{R}^n$, $\mathbb{R}^n = V_1 \oplus V_2$, $V_1 \cong \mathfrak{h}/\mathfrak{h}_0$, such that for $t \in T_0^0$, we have

$$\alpha(m, t) = \begin{pmatrix} I & 0 \\ 0 & \pi_2(t) \end{pmatrix},$$

where π_2 is a faithful rational representation, and for $t \in T_1^0$, we have

$$\alpha(m, t) = \begin{pmatrix} \pi_1(t) & * \\ 0 & * \end{pmatrix},$$

where $\pi_1(t)$ is Ad(t) acting on $\mathfrak{h}/\mathfrak{h}_0$, and, as we remarked above, is a faithful rational representation. Let β be the projection of α in $\mathrm{GL}(V_1) \times \mathrm{GL}(V/V_1)$. To prove the theorem, it suffices to see that the split rank of L is at least as large as dim(T), and by Lemma 7, to prove this it suffices to see that the split rank of the algebraic hull of β is at least dim(T). Thus, we need only see that if π is a faithful rational representation of T^0, then the algebraic hull of the cocyle $\beta(m, t) = \pi(t)$ $(m \in M_2)$ is locally isomorphic to T. Let T^* be the algebraic hull of the group $\pi(T^0)$; then T^* is a split torus, $\pi(T^0) \subset T^*$ is of finite index, and $\dim(T) = \dim(T^*)$. If β is equivalent to a cocycle into $Q \subset T^*$, then there is a measurable T^0-map φ: $M_2 \to T^*/Q$. Since there is a finite T^0-invariant measure on M_2, there is one on T^*/Q as well, and if Q is algebraic, it is clear that $\dim(Q) = \dim(T)$. This complete the proof.

4. Proof of Theorem 2

The argument of [1, Lemma 6], using the Borel density theorem, shows that the Lie algebra of the stabilizer of almost every point is an ideal. By Proposition 4, it follows that almost every point has a discrete stabilizer. The proof then follows as in the simple case, as in [3] or [4].

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