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GAUGE THEORY ON ASYMPTOTICALLY PERIODIC 4-MANIFOLDS

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1. Introduction

S. K. Donaldson's theorem on the nonexistence of certain closed, smooth 4-manifolds [8] (and see [12]) has the surprising corollary that there exists an exotic smooth structure on $\mathbf{R}^4$. This corollary was deduced by M. Freedman using his machine [13] for analyzing topological 4-manifolds. The existence proof for this exotic structure is presented in [15], [12].

Subsequently, R. Gompf proved [15] that $\mathscr{R} = \{$oriented diffeomorphism classes of smooth manifolds which are homeomorphic to $\mathbf{R}^4\}$ has at least four elements. Freedman and L. Taylor [14] have produced a fifth element, and, recently, Gompf has shown that $\mathscr{R}$ contains a countable, doubly indexed family $\{\mathbf{R}_{m,n}\}_{m,n=0}^{\infty}$ of "exotic" $\mathbf{R}^4$'s [16], where, $\mathbf{R}_{0,0}$ is $\mathbf{R}^4$ with its standard smooth structure.

The primary purpose of this paper is to prove the following theorem.

Theorem 1.1. *There exists an uncountable family of diffeomorphism classes of oriented* 4-*manifolds which are homeomorphic to* $\mathbf{R}^4$.

The proof of the preceding theorem is a two part argument; the first part is basically topological in content, and the second part is basically analytical. The topological aspects of the proof were provided to the author by R. Gompf (see [16]).

Gompf relayed to the author (after an observation of R. Kirby) that Freedman's original existence proof realized an exotic $\mathbf{R}^4$, $\mathbf{R}$, as follows. In [13], Freedman constructs a closed, oriented topological 4-manifold, $|E_8 \oplus E_8|$, which is simply connected; and whose homology intersection form is the definite, nondiagonalizable (over $\mathbf{Z}$) unimodular symmetric form $E_8 \oplus E_8$. Donaldson [8] asserts that $|E_8 \oplus E_8|$ is not smoothable, but Freedman's surgery techniques show that $V \equiv |E_8 + E_8| \setminus \text{pt.}$ is smoothable. Now, according to Freedman there exists $\mathbf{R} \subset \mathscr{R}$, compact sets $K \subset V$ and $K_1 \subset \mathbf{R}$, and a

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proper diffeomorphism

$$\phi : V \setminus K \to \mathbf{R} \setminus K_1. \tag{1.1}$$

Let $\psi : \mathbf{R} \to \mathbf{R}^4$ be a homeomorphism. For each $r > 0$, the open balls $B_r = \{x \in \mathbf{R}^4 : |x| < r\}$ are embedded topologically in $\mathbf{R}$ by ψ. Let

$$\mathbf{R}_r = \psi^{-1}(B_r). \tag{1.2}$$

Each $\mathbf{R}_r \subset \mathbf{R}$ is homeomorphic to $\mathbf{R}^4$. Each $\mathbf{R}_r$ also inherits a differentiable structure from the inclusion $\mathbf{R}_r \to \mathbf{R}$. When speaking of $\mathbf{R}_r$ as a smooth manifold, it is with reference to this inherited smooth structure.

Since $|E_8 \oplus E_8|$ is not smoothable, there exists some $r_0 < \infty$ such that for all $r > r_0$, the smooth manifold $\mathbf{R}_r$ is *not* diffeomorphic to $\mathbf{R}^4$. (This was Freedman's original observation. The number r_0 is determined by the condition $K_1 \subset \mathbf{R}_{r_0}$.) It is natural to ask whether, for a pair $r, s > r_0$, one could have $\mathbf{R}_r$ and $\mathbf{R}_s$ diffeomorphic.

It was observed by Freedman that if there is a diffeomorphic pair $\mathbf{R}_r$, $\mathbf{R}_s$ for $r, s > r_0$ and $r < s$, then there would exist a smoothing on V which has a differentially periodic end. Indeed, let $\chi : \mathbf{R}_r \to \mathbf{R}_s$ be a diffeomorphism. Let S_r^3 (S_r^3) be the topologically embedded boundary 3-spheres of $\mathbf{R}_r$ ($\mathbf{R}_s$) in $\mathbf{R}_{s+1}$. That is, $\partial \mathbf{R}_r = S_r^3$. And similarly for $\mathbf{R}_s$. Then χ maps some open collar $N_r \subset \mathbf{R}_r$ of S_r^3 (take $N_r = \psi^{-1}(B_r \setminus \overline{B}_{r-\varepsilon})$) diffeomorphically onto an open collar $N_s \subset \mathbf{R}_s$ of S_s^3.

Let $W = \mathbf{R}_s \setminus (\mathbf{R}_r \setminus N_r) = \psi^{-1}(B_s \setminus \overline{B}_{r-\varepsilon})$. W is an open submanifold of $\mathbf{R}_s$ which is homeomorphic $S^3 \times (r - \varepsilon, s)$. An exotic $\mathbf{R}^4$ with a periodic end is

$$\tilde{\mathbf{R}} = \mathbf{R}_s \cup_N W \cup_N W \cup_N \cdots, \tag{1.3}$$

where $\mathbf{R}_s \cup_N W$ is obtained from the disjoint union of $\mathbf{R}_s$ and W by identifying $N_s \subset \mathbf{R}_s$ with $N_r \subset W$ using the diffeomorphism $\chi : N_r \to N_s$. The iteration of this identification, gives the smooth manifold $\tilde{\mathbf{R}}$.

Using the map ϕ in (1.1), one obtains from $\tilde{\mathbf{R}}$ a smoothing on V which is periodic at infinity; that is, there exist compact sets $K \subset V$ and $K_1 \subset \tilde{\mathbf{R}}$ and a proper diffeomorphism

$$\hat{\phi} : V \to \mathbf{R} \setminus K_1. \tag{1.4}$$

If no such asymptotically periodic smoothing of V (as in (1.4)) exists, then necessarily each $\mathbf{R}_r$ ($r > r_0$) defines a distinct diffeomorphism class of 4-manifolds homeomorphism to $\mathbf{R}^4$. Freedman had suggested that one might prove Theorem 1.1 by using gauge theory, a la Donaldson [9], to prove that V has no end-periodic smoothings. Gompf asked that author whether such a generalization of Donaldson's arguments was possible. The answer is provided in Theorem 1.4 below, which asserts that certain smooth, end-periodic 4-manifolds do not exist.

Before stating Theorem 1.4, certain preliminary definitions are required to set the stage; in particular, a working definition of end-periodic is needed. Loosely speaking, the end of an end-periodic 4-manifold is constructed from a fundamental segment W by gluing copies of W together end to end. (Were the diffeomorphism in (1.4) to exist, then W in that case would be smooth, and homeomorphic to $(0,1) \times S^3$.)

Definition 1.2. A smooth, oriented 4-manifold M is end-periodic if the following data exists:

(1) A smooth, connected, oriented and open 4-manifold W with two ends, N_+ and N_-. W is called the fundamental segment. Thus, there exists a compact set $C \subset W$ such that $W \setminus C$ is the disjoint union of two nonempty, connected, open sets, N_+ and N_-.

(2) Suppose that there is a compact set $C_+ \subset N_+$ such that $N_+ \setminus C_+$ has two connected components, N_{++} and N_{+-}. Assume that C_+ is such that $W \setminus C_+$ is the disjoint union of $N_- \cup C \cup N_{+-}$ and N_{++}. Similarly, assume that a compact set $C_- \subset N_-$ exists such that $N_- \setminus C$ is the disjoint union of two connected components, N_{--} and N_{-+}, and that $W \setminus C_-$ is the disjoint union of N_{--} and $N_{-+} \cup C \cup N_+$. Assume that there is a diffeomorphism $i: N_+ \to N_-$ which is orientation preserving and which takes N_{++} to N_{-+} and N_{+-} to N_{--}.

(3) An open set $K \subset M$, with one end, N. Suppose that a compact set $C_0 \subset N$ exists such that $N \setminus C_0$ is the disjoint union of two open sets, N_{0-} and N_{0+}. Assume that $K \setminus C$ has two components, $(K \setminus N) \cup N_{0-}$ and N_{0+}. Require that there exists a diffeomorphism $i_-: N \to N_-$ which takes N_{0-} to N_{--} and N_{0+} to N_{-+}. Require that i_- preserve orientation.

(4) An orientation preserving diffeomorphism $\phi: M \to K \cup_N W \cup_N W \cup_N \cdots$. Here, $K \cup_N W$ is obtained from the disjoint union of K and W by identifying $N \subset K$ with $N_- \subset W$ via i_-. Also, $W \cup_N W$ is obtained from the disjoint union of two copies of $W, W_1 \cup W_2$, by identifying $N_+ \subset W_1$ with $N_- \subset W_1$ via i (see Figure 1). Identify End $M \equiv W_0 \cup_N W_1 \cup_N \cdots$.

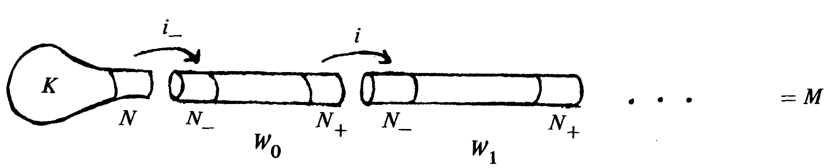


FIGURE 1

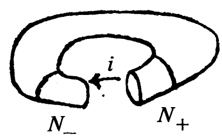
In order to define reasonable (for the present purposes) gauge theories on such end-periodic manifolds, it is necessary to restrict the allowable fundamental segments W.

Definition 1.3. An end-periodic 4-manifold M is admissible if the fundamental segment W has the following properties:

(1) $\pi_1(W)$ does not represent nontrivially in $SU(2)$.

(2) $H_1(N; \mathbf{R}) = 0$ and $H_2(N; \mathbf{R}) = 0$.

(3) Let Y be the compact, oriented 4-manifold which is obtained from W by identifying N_+ with N_- via i (see Figure 2). Require that the intersection pairing on $H_2(Y; \mathbf{R})$ be positive definite.



$= Y$

FIGURE 2

If M is an end-periodic 4-manifold such that $H_2(N; \mathbf{R}) = 0$, then the intersection pairings on $H_2(K; \mathbf{R})$ and $H_2(M; \mathbf{R})$ are nondegenerate. If also $H_1(M; \mathbf{Z}) = 0$, then both $H_2(K; \mathbf{Z})$ and $H_2(M; \mathbf{Z})$ are torsion free (see Lemma 5.7). Concerning the intersection pairings, one has

Theorem 1.4. *Let M be a smooth, end-periodic and admissible 4-manifold. Suppose that $\pi_1(M)$ has only the trivial representation into $SU(2)$. If $H_2(K; \mathbf{Z})$ has positive definite, unimodular intersection pairing, then this pairing is diagonalizable over $\mathbf{Z}$. If the intersection pairing on $H_2(K; \mathbf{Z})$ is only known to be positive definite, then the intersection pairing on $H_2(M; \mathbf{Z})$ is unimodular and diagonalizable over $\mathbf{Z}$ in the following sense: There is a sequence of free abelian groups $\Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \cdots \subseteq H_2(M; \mathbf{Z})$ with $\lim_{\to} \Lambda_n = H_2(M; \mathbf{Z})$ such that (1) $\Lambda_{-1} \otimes R = H_2(K; R)$ and (2) the intersection pairing on Λ_n is unimodular and diagonalizable.*

Theorem 1.4, as argued, implies Theorem 1.1. Additional corollaries to Theorem 1.4 have been pointed out to the author. The proofs below are due to R. Gompf.

Proposition 1.5 (*R. Gompf* [16]). *There exists a family $\{\mathbf{R}_{s,t}: s, t \in (0, \infty)\}$ of distinct elements in $\mathcal{R}$ (exotic $\mathbf{R}^4$'s). $\mathbf{R}_{s,t}$ embeds in $\mathbf{R}_{s',t'}$ if and only if $s \leqslant s'$ and $t \leqslant t'$. If $s > s'$ or if $t > t'$, there is a compact set in $\mathbf{R}_{s,t}$ which does not embed in $\mathbf{R}_{s',t'}$. No two elements of $\{\mathbf{R}_{s,t}\}$ have diffeomorphic ends.*

Proposition 1.6 (*M. Freedman*). *Let M be a closed, definite, TOP 4-manifold with nonstandard intersection form and with $\pi_1 = 1$ (e.g., E_8, $E_8 \# E_8$). Then M has no simplicial triangulation.*

This result provides examples of manifolds with no simplicial triangulation. No examples of such manifolds of dimension $\neq 4$ have been found. A. Casson [6] has independently proved that any closed TOP 4-manifold with nontrivial Kirby-Siebenmann invariant has no simplicial triangulation.

Proof. If M were triangulated, it would be PL except at a finite number of vertices. Coalesce these to a single point p (whose link is fake S^3). Now $M \setminus p$ is PL, hence smoothable; and its end is smoothly (fake $S^3) \times R$, which is periodic and contradicts Theorem 1.4.

Proposition 1.7 (*S. Akbulut*). *Let Σ be a homology 3-sphere which bounds a smooth 4-manifold with nonstandard, definite intersection form and with $\pi_1 = 1$ (e.g., the Poincaré homology 3-sphere). Then $\Sigma \# -\Sigma$ does not bound a contractible 4-manifold; or a definite 4-manifold with $\pi_1 = 1$.*

Proposition 1.8 (*R. Gompf*). *Let Σ be as in the previous proposition. Then Σ does not embed in any manifold homeomorphic to $S^3 \times S^1 \#_n \mathbf{CP}^2$ representing a generator of H_3.*

Proof. If it did, take 1/2 the universal cover and cap off to contradict Theorem 1.4.

Given as facts the technical results of §§3–10, the proof of Theorem 1.4 follows the argument of Donaldson in [9]. With extra restrictions on K, the argument of Fintushel-Stern in [11] carries over to the end-periodic category too. The Fintushel-Stern argument is given in §2. The full proof of Theorem 1.4 is given in §11. The technology that is developed in §§3–10 will, presumably, allow arguments on compact M to generally transplant to admissible, end-periodic manifolds.

On a different vein, the end-periodic technology—and especially the compactness results in §10—suggests an interesting interaction between the homotopy type of the segment W and the topology of K. For example, Freedman constructs a simply connected oriented top 4-manifold whose intersection form is the unimodular matrix E_8. By Quinn [25], $|E_8| \setminus$ pt. has a smoothing. Rohlin's theorem prevents a smooth product, $S^3 \times (0, \infty)$, from occurring as the end of $E_8 \setminus$ pt. (A. Casson [6] proves that no smooth homotopy sphere $\times$ $(0, \infty)$ can appear). Theorem 1.4 rules out periodic ends with segments homeomorphic to $S^3 \times (0, 1)$. It is known [13] that there exists a smooth, simply connected 4-manifold with intersection matrix E_8 with a product end, $\Sigma \times (0, \infty)$, where Σ = Poincaré homology sphere. Since $\pi_1(\Sigma)$ has two representations in $SU(2)/\mathrm{Ad}\, SU(2)$, Theorem 1.4 is not violated. Here, one sees the representations of $\pi_1(\Sigma)$ intimately tied to the properties of the quadratic form E_8. This suggests a nonlinear version of the index theorems of Atiyah-Singer-Patodi [5]. It should also be intimately related to Casson's recent work in [6].

Before turning to §2, the author gratefully thanks R. Gompf and M. H. Freedman for their advice and suggestions concerning the material in this article.

2. The proof of Theorem 1.1

Suppose that $V = (E_8 \# E_8) \setminus \mathrm{pt.}$ has an end-periodic smoothing, that is, End V is homeomorphic to $S^3 \times (0, \infty)$ and diffeomorphic to an exotic, periodic smoothing of $S^3 \times (0, \infty)$. As in [11], one can find a class $e \in H_2(V; \mathbf{Z})$ with $e \cdot e = 2$. As explained in §9, the class e determines an end-periodic (see §7) principal $SO(3)$-bundle, $P \to V$. This bundle has a reducible self-dual connection with square integrable curvature form (Proposition 9.1). In fact, there is a moduli space, M, of orbits under Aut P of self-dual connections on P with the following properties: M is a manifold except at the orbit of this reducible connection (Proposition 8.2); the dimension of M is 1. Also, M has one endpoint, the orbit of the reducible connection (Propositions 8.2 and 9.3). Finally, M is compact (Proposition 10.1). Thus, M is a compact 1-dimensional manifold with one endpoint.

A bit of experimentation with a length of string will convincingly demonstrate that no such M exists. Hence, $E_8 \# E_8 \setminus \mathrm{pt.}$ has no end-periodic smoothing. As argued, this implies Theorem 1.1.

Theorem 1.4 is proved in §11. The remainder of this paper contains the machinery that the proofs require. The strategy is to translate the formalism in [14], [9], [12], [11] from compact manifolds to end-periodic manifolds. Given a convenient Fredholm theory for the anti-self-dual DeRham complex (and its twisted counterparts),

$$(2.1) \qquad 0 \to C_0^\infty(M) \xrightarrow{d} C_0^\infty(T^*M) \xrightarrow{P_-d} C^\infty\big(P_- \wedge_2 T^*M\big) \to 0,$$

the formalism translates relatively easily. Here, $C_0^\infty(\cdot)$ are functions/sections with compact support. Also, $P_- = \frac{1}{2}(1 - *)$, with $*$ = the Hodge dual of an end-periodic metric on TM (see §3). §§3–4 are concerned with Fredholm theory for elliptic complexes on end-periodic manifolds. Because the expense is not too high, §§3–4 consider the theory for a quite general class of such complexes on manifolds of dimension > 2. The analysis here is modeled closely on the work of Lockhart and McOwen and their analysis of operators on manifolds with product ends [21]. However, there is a crucial difference in the two problems. The situation for end-periodic complexes is summarized in Theorem 3.1.

In §5, the specific case of the anti-self-dual DeRham complex is considered. The result in §5 is a computation of the Betti-numbers of this complex in terms of the homology of the compact piece, $K \subset M$. Here, one should compare [5] and [21].

§6 is a digression concerning the complex in (2.1) with P_- defined by a metric which is not strictly end-periodic, but only asymptotically so. The set of strictly end-periodic metrics is too specialized; for these metrics, the moduli space is not guaranteed to be manifold away from the reducible orbits.

In §§7–11, gauge theories on end-periodic 4-manifolds are discussed. §7 defines a useful Banach manifold of connections on an end-periodic principal bundle. The appropriate Banach Lie group of gauge transformations is defined in §7, and the orbit space is shown to be a smooth Banach manifold. Here the Fredholm theory of §§3–6 allows a more or less direct translation of the arguments in [24], [2], [12].

§8 proves that for a generic metric (defined in §6), the self-dual moduli spaces are smooth manifolds away from the reducible orbits. The neighborhoods of the reducible orbits are also described. Here, again, the Fredholm theory in §§3–6 allows a direct translation from [8], [12].

In §9, the moduli spaces are shown to be nonempty. The arguments from [11] (Proposition 9.1) and [28], [30] (Proposition 9.2) are modified to the end-periodic case.

In §10, the boundary of the moduli space is described. Here, the argument diverges significantly from the compact case (cf. [8], [12]). In particular, the topology of the segments which make up End M now plays a crucial role. The flat connections on these segments determine the structure of the end of the moduli space.

§11 contains the complete proof of Theorem 1.4. Here, the new argument of Donaldson in [9] is adapted to the end-periodic situation.

3. End-periodic differential operators

Suppose that M is an end-periodic n-manifold as given by Definition 1.2. To understand Fredholm theory on such M, it is convenient to introduce the furled up manifold,

$$Y = W/\sim , \tag{3.1}$$

where $\sim$ identifies N_+ with N_- via i. The manifold Y is compact, oriented with Z-fold cover

$$\tilde{Y} = \cdots \cup_N W_{-1} \cup_N W_0 \cup_N W_1 \cdots , \tag{3.2}$$

with projection $\pi: \tilde{Y} \to Y$. Here, each W_j is a copy of the fundamental domain W. End-periodicity identifies $M \setminus (K \setminus N)$ with

$$\text{End } M = W_0 \cup_N W_1 \cup_N W_2 \cup_N \cdots, \tag{3.3}$$

as a subset of Y. The deck transformations of $\tilde{Y}$ act on End M as a faithful representation of the semigroup $Z_+ = \{0, 1, 2, \cdots\}$ generated by

$$T: W_j = W_{j+1}. \tag{3.4}$$

A vector bundle, $E \to M$, will be called end-periodic if T lifts to a bundle map

$$\tilde{T}: E|_{W_j} \to E|_{W_{j+1}}. \tag{3.5}$$

Alternately, E is end-periodic if

$$E|_{\text{End } M} = \pi^* E_Y, \tag{3.6}$$

where $\pi: \text{End } M \to Y$ is the projection and $E_Y \to Y$ is a vector bundle.

In general, a geometric object on M will be called end-periodic if it transforms naturally under T of (3.4). This is equivalent to saying that it is the pull-back via π of an object on Y. Both definitions are, at times, convenient. For example, let E, F be end-periodic vector bundles over M. A differential operator $\partial: C_0^\infty(E) \to C_0^\infty(F)$ is end-periodic if

$$\partial \tilde{T} s = \tilde{T} \partial s \tag{3.7}$$

for all $s \in C^\infty(E|_{\text{End } M})$. Equivalently, ∂ is end-periodic if, under the isomorphisms $E|_{\text{End } M} = \pi^* E_Y$ and $F|_{\text{End } M} = \pi^* F_Y$, one has $\partial|_{\text{End } M} = \pi^* \partial_Y$, where $\partial_Y: C^\infty(E_Y) \to C^\infty(F_Y)$ is a differential operator over Y.

Let $\{E_j\}_{j=0}^N$ be a set of end-periodic vector bundles over M. Let $\{\partial_{j+1}: C_0^\infty(E_j) \to C_0^\infty(E_{j+1})\}_{j=0}^{(N-1)}$ be a set of degree > 0, end-periodic differential operators. This data defines an end-periodic, elliptic differential complex over M,

$$\{E, \partial\} = 0 \to C_0^\infty(E_0) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_N} C_0^\infty(E_N) \to 0, \tag{3.8}$$

if the corresponding symbol sequence is exact, and if $\partial_{j+1}\partial_j = 0$.

For such an end-periodic, elliptic, differential complex, the question arises: To what Banach space completions of $\{C_0^\infty(E_j)\}$ does $\{E, \partial\}$ extend as a Fredholm complex? Here one is looking for Banach space completions, $L(j)$, of $C_0^\infty(E_j)$ for $j \in \{0, 1, \ldots, N\}$ such that ∂_{j+1} extends to a bounded operator with closed range from $L(j) \to L(j+1)$. In addition, one requires that for $j \in \{0, , \cdots, N\}$ the induced map

$$\partial_{j+1}: L(j)/\text{Im}\,\partial_j \to L(j+1)$$

has finite dimensional kernel. Here, set $L(N+1) = L(-1) = 0$, $\partial_0 = 0$, and $\partial_{N+1} = 0$.

Lockhart and McOwen [21] answer the preceding question for a 2-step elliptic complex,

$$0 \to C_0^\infty(E) \overset{\partial}{\to} C_0^\infty(F) \to 0$$

over an n-manifold whose end has the product structure: $(n-1)$-manifold $\times$ $(0, \infty)$. Their work will be seen here to generalize.

To begin the analysis, fix a Riemannian metric on TM which is end-periodic (over End M, require that the metric be pulled back via π from Y). If $E \to M$ is an end-periodic vector bundle, give E a fiber metric which is also end-periodic. All norms, inner products, and integrals will be computed using these end-periodic metrics, unless specified otherwise. Inner products will be denoted by (,).

To measure distances on End M, one requires functions $\tau\colon \text{End } M \to (0, \infty)$ and $\rho\colon \tilde{Y} \to R$. They are defined as follows: Choose any smooth $t\colon W \to [0,1]$ such that $t|_{N_-} = 0$ and $t|_{N_+} = 1$. (Thus, dt descends to Y as a smooth 1-form.) Define ρ on $\tilde{Y}$ by setting $\rho(x) = n + t(x)$ if $x \in W_n$. Extend to M by bumping ρ to zero in N and extending by zero to the rest of K. Call the result τ.

Decay restrictions on sections of an end-periodic bundle $E \to M$ are best enforced using weights. As done in [21], for $1 < p < \infty$, and for $\delta \in \mathbf{R}$, define the weighted space $L_\delta^p(E)$ as the Banach space completion of $C_0^\infty(E)$ in the norm

$$\|s\|_{L_\delta^p} = \left[\int_M d\,\mathrm{vol}\left(e^{\delta\tau}|s|^p\right)\right]^{1/p}. \tag{3.9}$$

To define weighted Sobolev spaces which control the derivatives of sections of E, one must choose a connection on E which is periodic on End_M. If E is associated to the frame bundle of M (i.e. TM, T^*M, the bundles of p-forms, $\Lambda_p T^*M$, and the bundles of (p,q)-tensors, $\otimes_p TM \otimes_q T^*M$), by fiat, the Christoffel-Levi-Civita connection from the metric will be used. For some other vector bundle $E \to M$, a connection A is end-periodic if, under the isomorphism $E|_{\text{End } M} = \pi^* E_Y$, it is identified with the pull-back $\pi^* A_Y$ of a connection on E_Y. Thus, the covariant derivative of an end-periodic connection commutes with $\tilde{T}$ of (3.5). If E and F are end-periodic vector bundles over M with the end-periodic metrics and connections, then $E \otimes F$, $E \oplus F$ will always be given the obvious product periodic structures (unless specified to the contrary).

For $1 \leqslant p < \infty$, $0 \leqslant j < \infty$, and $\delta \in \mathbf{R}$, define the weighted Sobolev space $L^p_{j,\delta}(E)$ to be the completion of $C_0^\infty(E)$ in the norm

$$(3.10) \qquad \|s\|_{L^p_{j,\delta}} = \left[\int_M d\,\mathrm{vol}\left(e^{\tau\delta}\sum_{k=0}^{j}\left|\nabla^{(k)}s\right|^p\right)\right]^{1/p},$$

where $\nabla^{(k)} = \nabla\nabla\cdots\nabla$, k-times, and $\nabla: C_0^\infty(E \otimes_q T^*M) \to C_0^\infty(E \otimes_{q+1} T^*M)$ is the covariant derivative from the end-periodic connections on E and T^*M.

Let $\{E_j\}_{j=0}^N$ be a set of end-periodic vector bundles over M, and let $\{E, \partial\}$ be an elliptic differential complex, which is end-periodic. This extends to a sequence of bounded linear operators,

$$(3.11) \quad 0 \to L^p_{k+q,\delta}(E_0) \xrightarrow{\partial_1} L^p_{k+q-m_1,\delta}(E_1) \to \cdots \xrightarrow{\partial_N} L^p_{k,\delta}(E_N) \to 0,$$

where $m_j = \text{degree}\ \partial_j$ and $q = \sum_{j=1}^N m_j$. Here, $k \geqslant 0$.

To understand the conditions under which (3.11) is a Fredholm complex, it is necessary to consider the equivalent sequence over the compact space Y. Since $\{E_j = \pi^* E_{j,Y}\}$ and $\{\partial_j = \pi^*\partial_{j,Y}\}$ over End M, one has the following elliptic complex on Y:

$$(3.12) \qquad \{E_Y, \partial_Y\} = 0 \to C^\infty(E_{0,Y}) \xrightarrow{\partial_{1,Y}} \cdots \xrightarrow{\partial_{N,Y}} C^\infty(E_{N,Y}) \to 0.$$

For each $j \in \{1,\cdots,N\}$ let

$$(3.13) \qquad \partial^*_{j,Y}: C^\infty(E_{j+1,Y}) \to C^\infty(E_{j,Y})$$

denote the formal L^2-adjoint. Since Y is compact, the cohomology of $\{E_Y, \partial_Y\}$ is finite dimensional; that is, for each $j \in \{1,\cdots,N\}$, one has the isomorphism

$$(3.14) \quad H^j(E_Y, \partial_Y) = \left\{\psi \in C^\infty(E_{j,Y}) : \partial_{j+1,Y}\psi = 0 \text{ and } \partial^*_{j,Y}\psi = 0\right\}$$

of finite-dimensional vector spaces. The index of this complex, $\mathrm{Ind}\{E_Y, \partial_Y\}$, is the number

$$(3.15) \qquad \mathrm{Ind}\{E_Y, \partial_Y\} = \sum_{j=1}^{N} (-1)^j \dim H^j(E_Y, \partial_Y).$$

The Atiyah-Singer index theorem computes $\mathrm{Ind}\{E_Y, \partial_Y\}$ from the topology of Y [8].

A necessary condition for (3.11) to define a Fredholm complex is the vanishing of $\mathrm{Ind}\{\partial_Y\}$. There is a second condition which involves an action of $H^1_{\mathrm{DR}}(Y)$ (the first DeRham cohomology of Y) on the complex $\{E_Y, \partial_Y\}$.

Observe that the function ρ on $\tilde{Y}$ does not descend to Y. However, its exterior derivative, $d\rho$, is the pull-back from Y of a closed one-form, $\gamma \in C^\infty(T^*Y)$. The DeRham cohomology class, $[\gamma]\ (\neq 0) \in H^1_{\mathrm{DR}}(Y)$ generates the kernel of $\pi^*: H^1_{\mathrm{DR}}(Y) \to H^1_{\mathrm{DR}}(\tilde{Y})$. The one-form γ induces a sequence of differential operators $\{\sigma_{j+1}(\gamma): C^\infty(E_{j,Y}) \to C^\infty(E_{j+1,Y})\}_{j=0}^{N-1}$ via the formula

$$\sigma_{j+1}(\gamma)s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{-\varepsilon\rho}\left[\partial_{j+1,Y}, e^{\varepsilon\rho}\right]s\right), \tag{3.16}$$

with ρ of (3.8) considered as a multivalued "function" on Y. This $\sigma_{j+1}(\gamma)$ is a differential operator with C^∞ coefficients of order one less than that of $\partial_{j+1,Y}$. (If order $\partial_{j+1,Y} = 1$, then $\sigma_{j+1}(\gamma)$ is, pointwise, the symbol of $\partial_{j+1,Y}$ evaluated on the 1-form γ—thus giving a section of $\mathrm{Hom}(E_{j,Y}; E_{j+1,Y})$.

If $s \in \ker \partial_{j+1,Y}$, then

$$\sigma_{j+1}(\gamma)s = \partial_{j+1,Y}(\rho s), \tag{3.17}$$

which shows that $\sigma_{j+1}(\gamma): \ker \partial_{j,Y} \to \ker \partial_{j+1,Y}$. Also, if $s = \partial_{j,Y}f$, then

$$\sigma_{j+1}(\gamma)s = \partial_{j+1,Y}\rho\partial_{j,Y}f = -\partial_{j+1,Y}\sigma_j(\gamma)f$$

which shows that $\sigma_{j+1}(\gamma): \mathrm{Im}\,\partial_{j,Y} \to \mathrm{Im}\,\partial_{j+1,Y}$. Then, in the usual way, σ_{j+1} induces a map on the cohomology $H^*(E_Y, \partial_Y)$:

$$\bar{\sigma}_{j+1}[\gamma]: H^j(E_Y, \partial_Y) \to H^{j+1}(E_Y, \partial_Y). \tag{3.18}$$

(3.17) shows that the maps $\{\bar{\sigma}_{j+1}[\gamma]\}_{j=0}^{N-1}$ depend only on the cohomology class $[\gamma] \in H^1_{\mathrm{DR}}$.

If each operator ∂_j has degree 1, then $\bar{\sigma}_{j+1}(\gamma)\bar{\sigma}_j(\gamma) = 0$, since the symbol sequence for the complex in (3.12) is exact.

Theorem 3.1. *Let $\partial = \{\partial_{j+1}: C_0^\infty(E_j) \to C_0^\infty(E_{j+1})\}_{j=0}^N$ be an end-periodic, elliptic differential complex over an end-periodic n-manifold, M. Suppose that the associated complex $\{\partial_{j+1,Y}: C^\infty(E_{j,Y}) \to C^\infty(E_{j+1,Y})\}$ has index $= 0$ and is such that for each $j \in \{0, \cdots, N\}$, the induced map*

$$\bar{\sigma}_{j+1}: H^j(E_Y, \partial_Y)/\mathrm{im}\,\bar{\sigma}_j \to H^{j+1}(E_Y, \partial_Y)/\mathrm{Im}\,\bar{\sigma}_{j+1}\bar{\sigma}_j$$

is injective. Then there is a discrete set $D \in R$ without accumulation points such that for all $p \in [2, \infty)$, $m \geqslant q$, and $\delta \in R \setminus D$, the complex in (3.11) *is Fredholm.*

Theorem 3.1 follows from Proposition 4.1 and 4.2 of the next section.

For gauge theories, the omnipresent elliptic complex is the anti-self-dual part of the DeRham complex:

$$0 \to C_0^\infty(M) \xrightarrow{d} C_0^\infty(T^*M) \xrightarrow{P_-d} C_0^\infty(P_-\wedge_2 T^*) \to 0. \tag{3.19}$$

Here d = exterior derivative, and $P_-: \Lambda_2 T^* \to \Lambda_2 T^*$ is the anti-self-dual projection from the metric; $P_- = \frac{1}{2}(1 - *)$, where $*$ = metric Hodge "star." The associated complex on Y has cohomology

$$(3.20) \qquad \dim H^0 = 1, \quad \dim H^1 = b_1(Y), \quad \dim H^2 = b_2^-(Y).$$

Here b_1 = first Betti-number, and $b_2^- = \frac{1}{2}$ (second Betti-number minus signature (Y)).

Lemma 3.2. *For the anti-self-dual DeRham complex over Y, the conditions of Theorem* 3.1 *are equivalent to the assertion that the Euler-characteristic of* Y = *signature of* Y *and that the map* $[\gamma] \cup : H^1_{DR}(Y) \to H^2_{DR}(Y)$ *has* 1-*dimensional kernel* ($= \lambda[\gamma]$, $\lambda \in \mathbf{R}$).

Proof of Lemma 3.2. The index of the anti-self-dual DeRham complex on Y is

$$1 - b_1(Y) + b_2^-(Y) = -\tfrac{1}{2}\,(\text{Euler-characteristic}(Y)\text{-signature}(Y)).$$

Theorem 3.1 demands that this number vanish. The map $\sigma_1(\gamma)$ sends $f \in C^\infty(Y)$ to γf in $C^\infty(T^*Y)$. Since $[\gamma] \neq 0$ in $H^1_{\mathrm{DR}}(Y)$, the map $\bar{\sigma}_1(\gamma)$ is automatically injective. For $w \in C^\infty(T^*Y)$, one has $\sigma_2(\gamma)w = P_-(\gamma \wedge w)$. Suppose that $dw = 0$ and that $[P_-(\gamma \wedge w)] = 0 \in P_- H^2_{\mathrm{DR}}(Y)$. Then $P_-(\gamma \wedge w) = P_- d\alpha$ for some $\alpha \in C^\infty(T^*Y)$ which means that $\gamma \wedge w = d\alpha$. That is, $[\gamma] \cup [w] = 0 \in H^2_{\mathrm{DR}}(Y)$. Conversely, if $[\gamma] \cup [w] = 0$, then $\gamma \wedge w = d\alpha$ and $P_-(\gamma \wedge w) = P_- d\alpha$.

4. Fredholm theory

The study of end-periodic operators on M requires, eventually, the study of periodic operators on $\tilde{Y}$. This is an excision property of elliptic complexes. Periodic operators on $\tilde{Y}$ are obtained as follows: Let $\{E_{i,Y}\}_{i=0}^N$ be vector bundles over Y and let $\{\partial_{j+1,Y}: C^\infty(E_{i,Y}) \to C^\infty(E_{i+1,Y})\}$ define a periodic, elliptic complex over Y. The pull-backs $\{\tilde{E}_i = \pi^* E_{i,Y}\}$ and $\{\partial_{i+1} = \pi^*\partial_{i+1,Y}: C_0^\infty(\tilde{E}_i) \to C_0^\infty(\tilde{E}_{i+1})\}$ define a periodic elliptic complex over $\tilde{Y}$.

With the time function ρ, on $\tilde{Y}$, define the weighted Sobolev spaces $\{L^p_{k,\delta}(\tilde{E}_i)\}$ for $p \geqslant 2$, $k > 0$, and $\delta \in \mathbf{R}$. Each ∂_{i+1} extends to a bounded operator from $L^p_{k,\delta}(\tilde{E}_i)$ to $L_{k-m_{i+1},\delta}(\tilde{E}_{i+1})$ if $k \geqslant m_{i+1}$ = degree(∂_{i+1}). The relevance of this structure on $\tilde{Y}$ is tied to

Proposition 4.1. *Let M be an end-periodic manifold, let* $\{E_i\}$ *be end-periodic vector bundles over M, and let* $\{\partial_{i+1}: C_0^\infty(E_i) \to C_0^\infty(E_{i+1})\}$ *define an end-periodic elliptic complex over M. For a given* $\delta \in \mathbf{R}$, *the complex in* (3.11) *is*

Fredholm if and only if the following complex over $\tilde{Y}$ is Fredholm:

$$(4.1)\qquad 0 \to L^p_{k+q,\delta}(\tilde{E}_0) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_N} L^p_{k,\delta}(\tilde{E}_N) \to 0.$$

Proposition 4.1 is an excision assertion which extends to end-periodic manifolds an equivalent assertion for manifolds with product ends in [21, §4]. The proof here translates with little modification from [21]. The details are straightforward, tedious, and omitted.

As the operators in (4.1) are periodic, the study of (4.1) reduces to studying structure on the compact manifold Y. The result is Proposition 4.2, below. Together, Propositions 4.2 and 4.3 yield Theorem 3.1.

Proposition 4.2. *Assume that the conditions of Theorem* 3.1 *are met for the elliptic complex in* (3.12) *over* Y. *Then there exists a discrete set* $D \subset \mathbf{R}$ *with no accumulation points such that the complex in* (4.1) *is Fredholm for all* $\delta \in \mathbf{R} \setminus D$.

In studying translation invariant operators on a cylinder, $(n-1)$-manifold $\times\, \mathbf{R}$ [21], use the Fourier-Laplace transform. In the situation here, only the discrete group $\mathbf{Z}$ acts. To exploit this action, one should use the Fourier-Laplace series. Proposition 4.2 is proved using this tool. The remainder of this section contains the arguments.

Proof of Proposition 4.2. Let $\tilde{E} \to \tilde{Y}$ be a periodic vector bundle. For $\psi \in C_0^\infty(\tilde{E})$ and $z \in \mathbf{C}^*$, define the "Fourier-Laplace transform" of ψ by

$$(4.2)\qquad \hat{\psi}_z(\cdot) = \sum_{n=-\infty}^{\infty} z^n (\tilde{T}^n \psi)(\cdot),$$

where, $\tilde{T}: \tilde{E} \to \tilde{E}$ covers the deck transformation $T: \tilde{Y} \to \tilde{Y}$ (see (3.4), (3.5)). For fixed $z \in \mathbf{C}$, $\hat{\psi}$ obeys the periodicity condition

$$(4.3)\qquad (\tilde{T}\hat{\psi}_z)(\cdot) = z^{-1}\hat{\psi}_z(\cdot).$$

By restriction to W_0, $\hat{\psi}_z$ defines a smooth section over $Y = \tilde{Y}/\mathbf{Z}$ of the vector bundle

$$E_Y(z) = [\tilde{E} \otimes_{\mathbf{R}} \mathbf{C}/\mathbf{Z}],$$

where $\mathbf{Z}$ acts on $\tilde{E}_{\mathbf{C}}$ via the action sending $1 \in \mathbf{Z}$ and $(p, \lambda) \in E \otimes_{\mathbf{R}} \mathbf{C}$ to $(\tilde{T}p, z\lambda)$.

One may think of the collection

$$(4.4)\qquad E_Y = \{E_Y(z): z \in \mathbf{C}^*\}$$

as defining a smooth vector bundle over $Y \times \mathbf{C}^*$.

The Fourier-Laplace inversion formula is as follows: Let $\hat{\eta}$ be any section of E_Y over $Y \times \mathbf{C}^*$, holomorphic in $\mathbf{C}^*$. Then, if $s \in (0, \infty)$, the formula

$$(4.5)\qquad (\tilde{T}^n \eta)(x) = \frac{1}{2\pi i}\int_{|z|=s} z^{-n}\hat{\eta}_z(\pi(x))\frac{dz}{z}$$

for $x \in W_0$, and $\pi(x) \in Y$ defines a section of $\tilde{E}$ over $\tilde{Y}$. By Cauchy's integral formula, the left-hand side of (4.5) is independent of s. Cauchy's formula implies that (4.2) and (4.5) are inverses of each other.

If $\tilde{E}$, $\tilde{F}$ are periodic vector bundles over $\tilde{Y}$, and if $\partial : C_0^\infty(\tilde{E}) \to C_0^\infty(\tilde{F})$ is a periodic differential operator, then ∂ commutes with a Fourier-Laplace transform in the sense that (4.2) extends the differential operator $\partial_Y : C^\infty(E_Y) \to C^\infty(F_Y)$ to $\partial_Y : C^\infty(E_Y(z)) \to C^\infty(F_Y(z))$ via the formula

$$\partial_Y \hat{\psi}_z = \left(\widehat{\partial\psi}\right)_z. \tag{4.6}$$

For each $z \in \mathbf{C}^*$, the bundle $E_Y(z) \to Y$ is isomorphic to $E_Y(z) = E(1)$. Indeed, if $\hat{\psi}_z$ is a section over Y of $E_Y(z)$, then $z^t\hat{\psi}_z$ is a complex-valued section over Y of E_Y. Here $t: W_0 \to [0,1]$ is the time function of §3. (Fix a branch of $\ln z$ to define $z^t = e^{t \ln z}$.)

Via the preceding isomorphism, the family of operators $\{\partial_Y : C^\infty(E_Y(z)) \to C^\infty(F_Y(z))\}$ is mapped into the holomorphic family

$$\left\{\partial_Y(z) = \partial_Y + z^t\left[\partial_Y, z^{-t}\right] : C^\infty(E_Y; \mathbf{C}) \to C^\infty(F_Y; \mathbf{C})\right\}. \tag{4.7}$$

It is convenient to use both descriptions of this holomorphic family of operators.

The fundamental lemma to relate the analysis on Y to that on $\tilde{Y}$ is

Lemma 4.3. *For $p \in [2, \infty)$, $k > 0$, and $\delta \in \mathbf{R}^*$, let* (4.1) *define a periodic elliptic complex over $\tilde{Y}$. Equation* (4.1) *defines a Fredholm complex if and only if for all $z \in C^*$ with $|z| = e^{\delta/2}$, the cohomology vanishes for the complex $\{\partial_{j+1,Y}(z) : C^\infty(E_{j,Y}) \to C^\infty(E_{j+1,Y})\}_{j=0}^{N-1}$ over Y.*

Lemma 4.3 has reduced the proof of Proposition 4.2 to the study of the complex $\mathbf{F}(z) = \{\partial_{j+1,Y}(z) : C^\infty(E_{j,Y}) \to C^\infty(E_{j+1,Y})\}$ for all $z \in C = \{\omega \in \mathbf{C}^* : |\omega| = s = e^{\delta/2}\}$.

Since $\partial_{j+1,Y}(z) - \partial_{j+1,Y}$ is a compact operator for all j, the index of the complex $\mathbf{F}(z)$ and that of $\mathbf{F}(1)$ agree. Thus, for the cohomology of $\mathbf{F}(z)$ to vanish, the index of $\mathbf{F}(z) = \text{index}(\mathbf{F}(1)) = 0$. This is the first condition asserted by Theorem 3.1—it is a necessary condition.

To understand the second condition of Theorem 3.1, observe that its immediate implication is to

Lemma 4.4. *Under the conditions of Theorem* 3.1, *the cohomology of* $\mathbf{F}(z)$ *vanishes for all z in an annulus, $A_\varepsilon = \{\omega \in \mathbf{C}^* : 0 < |\omega - 1| < \varepsilon\}$ for some $\varepsilon > 0$.*

Lemma 4.4 asserts that the conditions of Theorem 3.1 imply that the cohomology of $\mathbf{F}(z)$ vanishes for all z in an open domain in $\mathbf{C}^*$. The fact that each $\partial_{j,Y}(z)$ depends holomorphically can be used to obtain

Lemma 4.5. *For z in a domain* $\Omega \subset \mathbf{C}$, *let*

$$0 \to B_0 \overset{\partial_1(z)}{\to} \cdots \overset{\partial_1(z)}{\to} B_N \to 0$$

be a Fredholm complex, where, for $j \in \{1, \ldots, N\}$, $\partial_j(z)$ *is a bounded, linear operator from the Hilbert space* B_{j-1} *to the Hilbert space* B_j. *Assume that* $\partial_{j+1}(z)\partial_j(z) = 0$ *for all j and* $z \in \Omega$. *Also, assume that each* $\partial_j(z)$ *depends holomorphically on* $z \in \Omega$, *and that for all* $z, z' \in \Omega$, $\partial_j(z) - \partial_j(z')$ *is a compact operator. If the cohomology of* $\{B, \partial(z)\}$ *vanishes at* $z = z_0 \in \Omega$, *then it vanishes for all* $z \in \Omega \setminus D$, *where D is a discrete set with no accumulation points in* $\operatorname{Int}\Omega$.

Together, Lemmas 4.3–4.5 imply Proposition 4.2 as follows: By Lemma 4.3, one need only study the cohomology of a $\mathbf{C}^*$'s worth of elliptic complexes over the compact manifold Y; precisely, for $z \in \mathbf{C}^*$,

$$0 \to L^p_{k+q}(E_{0,Y}) \overset{\partial_{1,Y}(z)}{\to} \cdots \overset{\partial_{N,Y}(z)}{\to} L^p_k(E_{N,Y}) \to 0.$$

From Lemmas 4.4 and 4.5 there exists a discrete set $D \in R^*$ with no accumulation points such that for all $\delta \in R \setminus D$, the cohomology of the above vanishes for all $z \in \mathbf{C}^*$ such that $|z| = e^{\delta/2}$.

Proof of Lemma 4.3. If the cohomology $H^*(E_Y, \partial_Y(z))$ vanishes, then for each $j \in \{0, \cdots, N-1\}$,

(4.8) $\quad \partial_{j+1,Y}(z): L^p_\cdot(E_{j,Y})/\operatorname{Im}\partial_{j,Y}(z) \to \ker \partial_{j+2,Y}(z) \subset L^p_\cdot(E_{j+1,Y})$

is an isomorphism. Let

(4.9) $$R_{j+1,Y}(z): \ker \partial_{j+2,Y}(z) \to L^p_\cdot(E_{j,Y})/\operatorname{Im}\partial_{j,Y}(z)$$

be the inverse to $\partial_{j+1,Y}(z)$ in (4.8). By assumption, $R_{j+1,Y}(z)$ is a bounded operator, defined for all $z \in C = \{\omega \in \mathbf{C}^* : |\omega| = e^{\delta/2}\}$.

For fixed $\psi \in \ker \partial_{j+2} \subset C_0^\infty(\tilde{E}_{j+1})$ and for $z \in C$,

(4.10) $$R_{j+1}(z)(z^t\hat{\psi}_z) \in L^p_\cdot(E_{j,Y})/\operatorname{Im}\partial_{j,Y}(z).$$

Let $\hat{b}_z(\psi) \in C^\infty(E_{j,Y})$ be a section which projects to $R_{j+1,Y}(z)(z^t\hat{\psi}_z)$. There is no obstruction to requiring that $\{\hat{b}_z(\psi) : z \in C\}$ defines a continuous section of $E_{j,Y}$ over $Y \times C$. The lift $\hat{b}_z(\psi)$ is unique up to

(4.11) $$\hat{b}_z(\psi) \to \hat{b}_z(\psi) + \partial_{j,Y}(z)\hat{\eta}(z)$$

with $\hat{\eta}(z)$ a continuous section of $E_{j-1,Y}$ over $Y \times C$ which is C^∞ in Y.

Let $b(\psi) \in C^\infty(\tilde{E}_j)$ be defined by the Fourier-Laplace inversion formula

(4.12) $$(\tilde{T}^n b(\psi))(x) = \frac{1}{2\pi i}\int_C \frac{dz}{z} z^{-n}\big(z^{-t}\hat{b}_z(\psi)(\pi(z))\big).$$

Notice that

$$\partial_{j+1} b(\psi) = \psi.$$

The ambiguity in the lift ((4.11)) means that $b(\psi)$ is unique up to $b(\psi) \to b(\psi) + \partial_j \eta$, where η is obtained from $z^{-t}\partial_{j,Y}(z)\eta(z)$ by Fourier-Laplace inversion as in (4.12). As in [21], [19], [22], (4.12) extends as a bounded operator,

$$\partial_{j+1}^{-1} : (\ker \partial_{j+2}) \cap L^P_{\cdot,\delta}(\tilde{E}_{j+1}) \to L^P_{\cdot,\delta}(\tilde{E}_j)/\operatorname{Im}\partial_j, \tag{4.13}$$

which inverts ∂_{j+1}. The existence of the inverses implies that the complex in (4.1) is Fredholm.

Conversely, suppose that for some $z \in C$, $H^j(E_Y(z), \partial_Y) \neq 0$. Let $\hat{\psi}_z \in C^\infty(E_j(z))$ represent a nonzero element. Via (4.3), $\hat{\psi}_z$ defines an element $\psi_z \in C^\infty(\tilde{E}_j)$. For each $q \in \{1, 2, \cdots\}$, define $\psi_q \in C_0^\infty(\tilde{E}_j)$ by

$$\psi_q(x) = \beta_q \psi_z(x)/q^{1/p}, \tag{4.14}$$

where $\beta_q \in C^\infty(\tilde{Y}; [0,1])$ obeys

$$\begin{aligned} &\beta_q(x) = 1 && \text{if } x \in W_k \text{ for } |k| < q, \\ &\beta_q(x) = 0 && \text{if } x \in W_k \text{ for } |k| > q+1, \\ &\sup|\nabla^l \beta_q| < B && \text{for all } l < 2\,(\text{degree } \partial_{j+1}) + k. \end{aligned} \tag{4.15}$$

Observe that $|\psi_q|_{L^P_{\cdot,\delta}}$ is bounded away from zero, independent of q; but it is also bounded, independent of q. Also, since

$$\partial_{j+1}\psi_q = [\partial_{j+1}, \beta_q]\,\psi_z/q^{1/p},$$

has support in $W_{\pm(q+1)}$ only,

$$\lim_{q\to\infty} \|\partial_{j+1}\psi_q\|_{L^P_{\cdot,\delta}} \to 0, \tag{4.16}$$

which implies that the complex in (4.1) cannot be Fredholm. Indeed, if so, then for each q, there exists $b_q \in \ker \partial_{j+1}$ such that

$$\lim_{q\to\infty} \|\psi_q - b_q\|_{L^P_{\cdot,\delta}} = 0 \tag{4.17}$$

(the closed range). Furthermore, $\{[b_q]\}$ converges in $L^P_{\cdot,\delta}(E_j)/\operatorname{Im}\partial_j$ (finite-dimensional cohomology). Thus, for a subsequence, there exists $b \in \ker \partial_{j+1}$ and $\{\eta_q\} \subset L^P_{\cdot,\delta}(E_{j-1})$ such that

$$\|\psi_q - b - \partial_j \eta_q\|_{L^P_{\cdot,\delta}} \to 0. \tag{4.18}$$

Now use (4.2) to observe that

$$\lim_{q\to\infty} \left\|\hat{\psi}_z - q^{(1-p)/p}\partial_{j,Y}\hat{\eta}_{qz}\right\|_{L^P(E_j(z))} = 0. \tag{4.19}$$

But Range $\partial_{j,Y} \subset L^p_\cdot(E_j(z))$ is closed, so $\hat{\psi}_z = \partial_{j,Y}\hat{\rho}_z$ for some $\hat{\rho}_z \in L^p_\cdot(E_{j-1}(z))$. This contradicts that $\hat{\psi}_z$ is nontrivial in $H^j(E(z), \partial_Y)$.

Proof of Lemma 4.4. Assume to the contrary that a sequence of points $\{1 + w_n\}_{n=1}^{\infty} \subset \mathbf{C}^*$ exists with $|w_n| \to 0$ such that for each n, $H^j(E_Y, \partial_Y(1 + w_n)) \neq 0$. Then, for each n, there exists $\psi_n \in C^\infty(E_{j,Y})$ such that

$$\|\psi_n\|_{L^2} = 1,$$

$$\partial_{j+1}\psi_n + w_n\sigma_{j+1}(\gamma)\psi_n + w_n^2 R_{j+1,n}\psi_n = 0, \tag{4.20}$$

where, $R_{j+1,n}: C^\infty(E_{j,Y}) \to C^\infty(E_{j+1,Y})$ has degree 2 less than degree ∂_{j+1}. Further,

$$\left\{ R_{j+1,n} : L^p_{k+l}\left(E_{j,Y}\right) \to L^p_k\left(E_{j+l,Y}\right)\right\}_{n+1}^{\infty}$$

is uniformly bounded when $p \in [2, \infty)$, $l =$ degree ∂_{j+1}, and $k \geqslant 0$.

Let $\partial_j^* : C^\infty(E_{j+1,Y}) \to C^\infty(E_{j,Y})$ denote the formal L^2-adjoint of ∂_j. Likewise denote $\sigma_j^*(\gamma)$ and $R_{j,n}^*$. Then it is no loss of generality to assume of $\{\psi_n\}$ that

$$\partial_{j,Y}(1 + w_n)^*\psi_n = \partial_j^*\psi_n + \bar{w}_n\sigma_j^*(\gamma)\psi_n + \bar{w}_n^2 R_{j+1,n}^*\psi_n = 0. \tag{4.21}$$

(4.20) and (4.21) imply, via standard arguments, that $\{\psi_n\}$ has a subsequence which converges in the C^∞-topology to some $0 \neq \psi \in H^j(E_Y, \partial_Y)$. Relabel this subsequence as $\{\psi_n\}$. Thus, one must have $H^j(E_Y, \partial_Y) \neq 0$. Let $\{\eta_k\}$ be an L^2-orthonormal basis for $H^{j+1}(E_Y, \partial_Y)$, the finite dimensional vector space in (3.14). Let $\{\phi_l\}$ be a similar basis for $H^{j-1}(E_Y, \partial_Y)$. From (4.20) and (4.21),

$$\begin{aligned} \langle \eta_k, \sigma_{j+1}(\gamma)\psi\rangle_{L^2} &= \langle \eta_k, \sigma_{j+1}(\gamma)(\psi - \psi_n)\rangle_{L^2} + O(|w_n|), \\ \langle \psi, \sigma_j(\gamma)\phi_l\rangle_{L^2} &= \langle \psi - \psi_n, \sigma_j(\gamma)\phi_l\rangle_{L^2} + O(|w_n|). \end{aligned} \tag{4.22}$$

As the right-hand side above tends to zero as $n \to \infty$, one sees that

$$\psi \in \ker\left(\sigma_{j+1}(\gamma): H^j(E_Y, \partial_Y) \to H^{j+1}(E_Y, \partial_Y)\right),$$
$$\psi \notin \operatorname{Im}\left(\sigma_j(\gamma): H^{j-1}(E_Y, \partial_Y) \to H^j(E_Y, \partial_Y)\right).$$

These two facts contradict the conditions of Theorem 3.1. Hence, Lemma 4.4 is true.

Proof of Lemma 4.5. For each $j \in \{1, \cdots, N\}$, $B_j = \ker\partial_{j+1}(z_0) \oplus \operatorname{Im}\partial_j(z_0)$. Thus, a partial inverse $\partial_j^{-1}(z_0): B_j \to B_{j-1}$ exists such that

$$\begin{aligned} &\partial_j^{-1}(z_0)\partial_j(z_0) = 1 \quad \text{on } \left(\ker\partial_j(z_0)\right)^{\perp} \subset B_{j-1}, \\ &\partial_j^{-1}(z_0)\,|_{(\operatorname{Im}\partial_j(z_0))^{\perp}} = 0. \end{aligned} \tag{4.23}$$

As a function of $z \in \Omega$, consider first the operator $Q_1(z): B_1 \to B_1$ given by

$$Q_1(z) = \partial_1^{-1}(z_0)\partial_1(z).$$

This operator is bounded, Fredholm, and it depends holomorphically on $z \in \Omega$ (in the sense of Chapter VII of [18]). Because $\partial(z)$ is Fredholm, the zero eigenvalue (if present) has finite multiplicity and it is isolated. Thus, by a theorem in [18, Supplementary notes to Chapter VII], the resolvent of $Q_1(z)$, $R_1(\lambda, z) = (Q_1(z) - \lambda)^{-1}$, has the property that $R_1(O, z)$ does not exist for any $z \in \Omega$; or else $R_1(O, z)$ is meromorphic in $z \in \Omega$. Since $R_1(O, z_0) = 1$, $R_1(O, z)$ is meromorphic in $z \in \Omega$. Thus, $\ker \partial_1(z) \neq \varnothing$ only for $z \in D_0$ where $D_0 \subset \Omega$ is discrete with no accumulation points in $\operatorname{Int}\Omega$. Thus, the lemma is true for $H^0(\{B, \partial(z)\})$. Now suppose for $k > 0$, a discrete set $D_{k-1} \subset \Omega$ without accumulation points in $\operatorname{Int}\Omega$ exists such that the cohomology $\{H^j(\{B, \partial(z)\})\}_{j<k} = \varnothing$. The previous argument shows that

$$Q_k = \partial_k(z_0)^{-1}\partial_k(z): (\operatorname{Im}\partial_{k-1}(z_0))^{\perp} \to (\operatorname{Im}\partial_{k-1}(z_0))^{\perp}$$

is an isomorphism for all $z \in \Omega \setminus D'$ with $D' \subset \Omega$ discrete and having no accumulation points in $\operatorname{Int}\Omega$. For $z \in \Omega$, consider $\psi \in B_k$ with $\partial_k(z))\psi = 0$. By assumption, if $z \in \Omega \setminus D'_{k+1}$ there exists $\eta \in B_{k-1}$ such that $\psi' = \psi + \partial_{k-1}(z)\eta \in (\operatorname{Im}\partial_{n-1}(z_0))^{\perp}$; for such z one can solve for $\eta \in (\operatorname{Im}\partial_{k-2}(z_0))^{\perp}$ obeying

$$\partial_{k-1}^{-1}(z_0)\partial_{k-1}(z)\eta + \partial_{k-1}^{-1}(z_0)\psi = 0.$$

Thus set $D_k = D' \cup D_{k-1}$. This set is discrete with no accumulation points in $\operatorname{Int}\Omega$ and if $z \notin D_{k'}$ then $H^k(\{B, \partial(z)\}) = \varnothing$. Lemma 4.5 follows by induction.

5. Index calculations

If M is an end-periodic 4-manifold which is admissible in the sense of Definition 1.3, then the anti-self-dual DeRham complex

$$(5.1)\qquad 0 \to C_0^\infty(M) \xrightarrow{d} C_0^\infty(T^*M) \xrightarrow{P_-d} C_0^\infty(P_- \Lambda_2 T^*M) \to 0$$

satisfies the conditions of Theorem 3.1 (cf. Lemma 3.2). Thus, for $p \geqslant 2$, $k \geqslant 0$, and all but a discrete set of $\delta \in R$, the complex $\{L^p_{\cdot,\delta}; (d, P_-d)\}$:

$$(5.2)\quad 0 \to L^p_{k+2,\delta}(M) \xrightarrow{d} L^p_{k+1,\delta}(T^*M) \xrightarrow{P_-d} L^p_{k,\delta}(P_- \Lambda_2 T^*M) \to 0$$

is Fredholm. The latter half of this paper requires information about the cohomology groups of the complex in (5.2) for δ near zero. These are computed in the following proposition.

Proposition 5.1. *Let M be an admissible, end-periodic 4-manifold. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, $p \geqslant 2$, and $k \geqslant 0$, the complex in (5.2) is Fredholm with the following cohomology:*

$$\dim H^0 = 0, \quad \dim H^1 = b_1(K), \quad \dim H^2 = b_2^-(K),$$

where $b_1(K) = \dim H_1(K, R)$ and $b_2^-(K) = \frac{1}{2}(\dim H_2(K, R) - \tau(K))$. The number $\tau(K)$ is the signature of the intersection pairing on $H_2(K, R)$.

We remark that the admissibility assumption implies that the intersection pairing on $H_2(K, R)$ is nondegenerate (see Lemma 5.7).

The remainder of this section contains the proof of Proposition 5.1.

Proof of Proposition 5.1. Theorem 3.1 and Lemma 3.2 assert that the anti-self-dual DeRham complex, $\{L^p_{\cdot,\delta}; (d, P_-d)\}$, is Fredholm for $\delta \in (0, \delta_0)$ with $\delta_0 > 0$. It remains to compute its cohomology. For simplicity, only the case $p = 2$ will be considered. The cases $p > 2$ follow from the $p = 2$ case using standard elliptic regularity plus some obvious function space inclusions.

For $\delta \geqslant 0$, the constants are not in $L^2_{0,\delta}$. Thus, for $\delta \geqslant 0$,

$$\ker\left(d : L^2_{k+2,\delta}(M) \to L^2_{k+1,\delta}(T^*M)\right) = \varnothing, \quad H^0\left(L^2_{\cdot,\delta}, (d, P_-d)\right) = 0.$$

This is true for $\delta \geqslant 0$ (and, obviously, it is false for $\delta < 0$). For $\delta > 0$, one obtains in addition the useful Sobolev inequalities below:

Lemma 5.2. *Let $\delta > 0$ and let $f \in C^\infty(M)$ obey*

$$\|f\|^2 = \int_M e^{\tau\delta}|df|^2 < \infty. \tag{5.3}$$

Then $\bar{f} \in R$ exists such that

$$\int_M e^{\tau\delta}|f - \bar{f}|^2 < Z\|f\|^2, \tag{5.4}$$

$$\left(\int_M e^{2\tau\delta}|f - \bar{f}|^4\right)^{1/2} < Z\|f\|^2, \tag{5.5}$$

with $Z = Z(\delta) < \infty$, independent of f. Further, for $p \in (2, 4)$,

$$\left[\int_M e^{4\delta\tau/(4-p)}|f - \bar{f}|^{4p/(4-p)}\right]^{(4-p)/4} < Z\int_M e^{\delta\tau}|df|^p, \tag{5.6}$$

whenever the right-hand side is finite. Here, $Z = Z(p, \delta)$ is independent of f. Finally, if $p > 4$, and if

$$\int_M e^{\delta\tau}|df|^p < \infty,$$

then $\{e^{n\delta/p}(f(T^n(\cdot)) - \bar{f})\} \in C^\infty(W_0)$ converges to 0 in the C^0-topology.

This lemma will be proved at the end of §5.

Now consider H^1 for the complex $(L^2_{\cdot,\delta}; (d, P_-d))$. Suppose

$$w \in \ker\big(P_-d : L^2_{k+1,\delta}(T^*M) \to L^2_{k,\delta}\big(P_- \Lambda_2 T^*M\big)\big).$$

As $\delta > 0$, one can use integration by parts (valid for $\delta = 0$, too) to conclude that

$$(5.7) \qquad 0 = \int_M |P_-dw|^2 = \tfrac{1}{2}\int_M |dw|^2.$$

Since $dw = 0$ and since $H^1(W, R) = 0$ (due to the admissability of M), one can write

$$(5.8) \qquad w|_{\mathrm{End}\, M} = df,$$

with $f \in C^\infty(\mathrm{End}\, M)$. By fixing a smooth, nonnegative bump function β, which is 1 on $W_0 \setminus N_-$ and 0 on $K \setminus N$, one has an almost canonical way of constructing a closed 1-form $r(w) \in C_0^\infty(T^*K)$ from $w \in \ker(P_-d)$, viz:

$$(5.9) \qquad r(w) = w - d(\beta f),$$

which becomes canonical when one uses (5.4): There is a unique $f(w) \in C^\infty(\mathrm{End}\, M)$ which obeys (5.8) plus

$$(5.10) \qquad \int_{\mathrm{End}\, M} e^{\tau\delta}|f(w)|^2 < \infty.$$

Notice that (5.8)–(5.10) imply that $\beta f(w) \in L^2_{k+2,\delta}(M)$. For this reason, the map r induces an isomorphism

$$(5.11) \qquad r : H^1\big(L^2_{\cdot,\delta}; (d, P_-d)\big) \simeq H^1_{0,\mathrm{DR}}(K),$$

where $H^*_{0,\mathrm{DR}}(K)$ denotes DeRham cohomology on K with compact supports. As K has only one end, $H^1_{0,\mathrm{DR}}(K) \to H^1_{\mathrm{DR}}(K)$ is injective, and because $H^1(N; R) = 0$, $H^1_{0,\mathrm{DR}}(K) \simeq H^1_{\mathrm{DR}}(K)$. By DeRham's theorem, this is $H^1(K; R) \simeq H_1(K; R)$ (simplicial cohomology and homology, respectively).

Now, consider the cohomology $H^2(L^2_{\cdot,\delta}; (d, P_-d))$. A class $[w]$ is represented by a C^∞ 2-form on M, w, which obeys

$$(5.12) \qquad w = -{*w}, \quad e^{-\tau\delta}d(e^{\tau\delta}w) = 0, \quad \int_M e^{\tau\delta}|w|^2 = 1.$$

Lemma 5.3. *If M is an admissible, end-periodic 4-manifold, then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, a 2-form w obeying* (5.12) *satisfies*

$$\int |e^{\tau\delta}w|^2 < \infty.$$

Lemma 5.3 will be proved shortly; for now, assume it to be true.

For $n = \{0, 1, \cdots\}$, set $K_n = K \cup_N W_0 \cup_N \cdots \cup_N W_n$. A homomorphism

$$r_n : H^2\big(L^2_{\cdot,\delta}; (d, P_-d)\big) \to H^2_{0,\mathrm{DR}}(K_n) \tag{5.13}$$

is defined as follows: Let $\beta_n \in C^\infty(M; [0,1])$ be identically 1 on $K_n \setminus W_n$, identically zero on $M \setminus K_{n-1}$, and satisfy $|d\beta_n| < B$, independent of n. Thus, $\operatorname{supp}|d\beta_n| \subset i_-(N) \subset W_n$. Because $H^2(N; \mathbf{R}) = 0$, one can write

$$e^{\tau\delta} w| = d\alpha_n \quad \text{on } W_n \cap i_-(N). \tag{5.14}$$

A careful analysis of DeRham's theorem shows that one may assume, with no loss of generality, that $\alpha_n \in C^\infty(T^*(W_n \cap i_-(N)))$ obeys a priori estimates which depend on those of $e^{\tau\delta}w$: For $m > 0$,

$$\|\alpha_n\|_{C^m} < \delta(k) \|e^{\tau\delta} w\|_{C^{m-1}(W_{n-1} \cup_N W_n)}. \tag{5.15}$$

Define

$$r_n(w) = \begin{cases} e^{\tau\delta} w & \text{on } K_n \setminus W_n, \\ d(\beta_n \alpha_n) & \text{on } i_-(N) \cap W_n, \\ 0 & \text{on } M \setminus K_{n+1}. \end{cases} \tag{5.16}$$

Concerning $r_n(w)$, one has

Lemma 5.4. *There exists $n < \infty$ such that for $m > n$, r_m of* (5.13) *is an injection.*

The proof of Lemma 5.4 requires the following technical lemma:

Lemma 5.5. *Given $\varepsilon > 0$, there exists $n(\varepsilon) < \infty$ such that each $w \in L^2_{\cdot,\delta}(P_- \Lambda_2 T^*)$ obeying* (5.12) *satisfies*

$$\left| \int_M r_m(w) \wedge r_m(w) + \int_M |e^{\tau\delta} w|^2 \right| < \varepsilon \tag{5.17}$$

for $m > n(\varepsilon)$.

Lemma 5.5 will also be proved shortly. Given Lemma 5.5, Lemma 5.4 is proved as follows: Suppose $[r_m(w)] = 0 \in H^2_{0,\mathrm{DR}}(K_n)$. Then $r_m(w) = dv_m$ with $v_m \in C_0^\infty(T^*K_n)$. But, this implies (via integration by parts)

$$\int_M r_m(w) \wedge r_m(w) = 0, \tag{5.18}$$

since

$$\int_M |e^{\tau\delta} w|^2 > \int_M e^{\tau\delta} |w|^2 = 1. \tag{5.19}$$

(5.17) and (5.18) are contradictory for $m > n(\varepsilon)$ of Lemma 5.5 so for such m, $\ker r_m = \varnothing$.

Proof of Lemma 5.5. Due to (5.12) and Lemma 5.4, the 2-form $e^{\tau\delta}w$ obeys uniform estimates: Given $\varepsilon > 0$, there exists $n'(\varepsilon) < \infty$ such that if $m > n'(\varepsilon)$,

$$\|e^{\tau\delta}w\|_{C^1(W_m)} < \varepsilon. \tag{5.20}$$

(5.15) and (5.16) and Lemma 5.3 give (5.17). (Remember, the unit sphere in $H^2(L^2_{\cdot,\delta}; (d, P_-d))$ is compact.)

Using the homomorphism r_n, the dimension of $H^2(L^2_{\cdot,\delta}; (d, P_-d))$ can be computed from $H^2_{0,\mathrm{DR}}(K)$: The assignment of 2-forms $v, u \in C_0^\infty(\Lambda_2 T^*K_m)$ to the number

$$Q(v, u) = \int_{K_m} v \wedge u \tag{5.21}$$

defines a symmetric quadratic form on $H^2_{0,\mathrm{DR}}(K_m)$: Indeed, if $du = 0$ and if $b \in C^\infty(T^*K_m)$, then

$$\int_{K_m} db \wedge u = 0.$$

Choose a basis for $H^2_{0,\mathrm{DR}}(K_m)$, $\{e_j\} \in C_0^\infty(P_- \Lambda_2 T^*K_m)$, in which Q is diagonal. Let $q = q(m)$ be the number of negative eigenvalues of Q and let $\{e_j\}_{j=1}^q$ be the eigenvectors with negative eigenvalues.

Lemma 5.6. *For n as in Lemma* 5.4 *and for $m > n$, one has*

$$\dim H^2\big(L^2_{\cdot,\delta}; (d, P_-d)\big) = q(m).$$

Proof of Lemma 5.6. Lemmas 5.4 and 5.5 assert that r_m identifies $H^2(L^2_{\cdot,\delta}; (d, P_-d))$ with a linear subspace of $H^2_{0,\mathrm{DR}}(K_m)$ on which Q is negative. Thus, $q(m) \geqslant \dim H^2(L^2_{\cdot,\delta}; (d, P_-d))$. Conversely, let $u \in \operatorname{span}\{e_j\}_{j=1}^q$; thus $du = 0$ and $Q(u, u) < 0$. As $\{L^2_{\cdot,\delta}; (d, P_-d)\}$ is Fredholm, $a \in L^2_{\cdot,\delta}(T^*M)$ exists such that

$$s(u) = P_-(u + da) \in L^2_{\cdot,\delta}\big(P_- \Lambda_2 T^*M\big)$$

obeys $de^{\tau\delta}s(u) = 0$. Also,

$$\begin{aligned} \int_M s(u) \wedge s(u) &= \int_M (u + da) \wedge (u + da) \\ &\quad - \int_M P_+(u + da) \wedge P_+(u + da) \\ &\leqslant \int_M (u + da) \wedge (u + da) \equiv Q(u) < 0. \end{aligned} \tag{5.22}$$

Therefore, s induces a homomorphism

$$s : \operatorname{span}\{e_j\}_{j=1}^q \to H^2\big(L^2_{\cdot,\delta}; (d, P_-d)\big)$$

which is injective, due to (5.22). Hence, $q(m) \leqslant \dim H^2(L^2_{\cdot,\delta}; (d, P_-d))$, too.

Lemma 5.6 implies that $q(m)$ is independent of m for $m > n$. To compute $q(m)$, one must understand how the topology of M is built up from the K and the segments of End M.

Lemma 5.7. *Let M be an end-periodic 4-manifold such that $H_1(N; \mathbf{R}) = H_2(N; \mathbf{R}) = 0$. Let $K_{-1} = K$. The inclusion homomorphisms, $H^2_{\text{comp}}(K_n; \mathbf{Z}) \to H^2(K_n; \mathbf{Z})$ for all $n \geqslant -1$, and $H^2_{\text{comp}}(K_n; \mathbf{Z}) \to H^2(M; \mathbf{Z})$ are injections. Thus, the intersection pairings on $H_2(K_n; \mathbf{R})$ for $n \geqslant -1$ and $H_2(M; \mathbf{R})$ are nondegenerate. For all $n \geqslant -1$,*

$$q(n) = \tfrac{1}{2}[\dim H_2(K_n; \mathbf{R}) - \tau(K_n)] = b^2_-(K).$$

Proof of Lemma 5.7. Consider the inclusion homomorphism over $\mathbf{R}$, $l: H^2_{\text{comp}}(K_n; \mathbf{R}) \to H^2(K_n; \mathbf{R})$. By Poincaré duality, $H^2_{\text{comp}}(K_n; \mathbf{R}) = H_2(K_n; \mathbf{R})$ and the kernel of l is the radical of the intersection pairing on $H_2(K_n; \mathbf{R})$. Also, by DeRham's theorem, $H^2_{\text{comp}}(K_n; \mathbf{R}) \simeq H^2_{0,\text{DR}}(K_n)$ and via this isomorphism, the intersection pairing on H_2 and the bilinear form, Q, on $H^2_{0,\text{DR}}$ agree. Let $[w] \in H^2_{\text{comp}}(K_n; \mathbf{R})$. Suppose that $l[w] = 0$. Represent w by a closed 2-form, w, with compact support on K_n. Thus, $w = d\alpha$ with α a 1-form on K_n. Let $\beta \in C_0^\infty(K_n)$ be identically 1 on $\operatorname{supp} w \cup (K_n \setminus i_+(N))$. Here, $i_+ : N \to W_n$ defines the end of K_n. The 2-form $w' = w - d(\beta\alpha)$ is cohomologous to w in $H^2_{\text{comp}}(K_n; \mathbf{R})$. However, $[w'] \in H^2_{\text{comp}}(i_+(N); \mathbf{R})$. By Poincaré duality $H^2_{\text{comp}}(N; \mathbf{R}) \simeq H_2(N; \mathbf{R}) = 0$. Thus, $w' = d\gamma$ with γ a 1-form with compact support in $i_+(N)$. Thus $w = d(\beta\alpha + \gamma)$ and $\beta\alpha + \gamma$ has support in K_n. Hence $[w] = 0$ in $H^2_{\text{comp}}(K_n; \mathbf{R})$. This proves that over $\mathbf{R}$, the homomorphism l is injective. The nondegeneracy of the intersection pairing and of Q is a direct corollary.

The injectivity over $\mathbf{R}$, of l means that over $\mathbf{Z}$, $\ker(l)$ is contained in the torsion subgroup of $H^2_{\text{comp}}(K_n; \mathbf{Z})$. Let $[w] \in \ker(l)$. Then a compact set $V \subset K_n$ exists such that $[w] \in H^2(K_n, K_n \setminus V)$. No generality is lost by requiring that $K_n \setminus V \subset N$ and that $N \cap V$ is connected. Consider the exact sequence

$$\to H^1(K_n) \xrightarrow{r} H^1(K_n \setminus V) \xrightarrow{\delta} H^2(K_n, K_n \setminus V) \xrightarrow{l} H^2(K_n) \to .$$

Since $K_n \setminus V \subset N$, the map r factors through $r': H^1(N; \mathbf{Z}) \to H^1(K_n \setminus V; \mathbf{Z})$. However, $H^1(N; \mathbf{Z})$ is the free group, $\operatorname{Hom}(H_1(N; \mathbf{Z}); \mathbf{Z})$, and thus $H^1(N; \mathbf{Z}) = 0$ if $H^1(N; \mathbf{R}) = 0$. This is the case, by assumption, as $H^1(N; \mathbf{R}) = \operatorname{Hom}(H_1(N; \mathbf{R}), \mathbf{R})$. Therefore, the following sequence is exact:

$$0 \to H^1(K_n \setminus V) \xrightarrow{\delta} H^2(K_n, K_n \setminus V) \xrightarrow{l} H^2(K_n) \to .$$

If $l \cdot w = 0$, then $w = \delta\alpha$ with $\alpha \in H^1(K_n \setminus V)$. The preceding sequence asserts that δ is injective, so, since $H^1(K_n \setminus V; \mathbf{Z})$ is free, $w = 0$ if w is torsion. The injectivity of $H^2_{\text{comp}}(K_n; \mathbf{Z}) \to H^2(K_n; \mathbf{Z})$ now follows, and the injectivity of $H^2_{\text{comp}}(M; \mathbf{Z}) \to H^2(M; \mathbf{Z})$ follows by taking direct limits.

To complete the proof of Lemma 5.7, it is sufficient to prove that

$$(5.23) \qquad \tfrac{1}{2}(\operatorname{rank} Q - \operatorname{signature} Q)\Big|_{H^2_{0,\mathrm{DR}}(K_m)}$$

is equal to its value on $H^2_{0,\mathrm{DR}}(K)$. To see that such is the case, observe first that the previous arguments also establish that the inclusion $K \subset K_m$ induces a monomorphism

$$H^2_{0,\mathrm{DR}}(K) \overset{i^*}{\to} H^2_{0,\mathrm{DR}}(K_m).$$

Pull-back by i induces a homomorphism

$$i^* : H^2_{0,\mathrm{DR}}(K_m) \to H^2_{\mathrm{DR}}(K).$$

From i^*, a homomorphism,

$$j : H^2_{0,\mathrm{DR}}(K_m) \to H^2_{0,\mathrm{DR}}(K)$$

is constructed as follows: Let $[w] \in H^2_{0,\mathrm{DR}}(K_m)$ be represented by a closed $w \in C_0^\infty(\Lambda_2 T^*K_m)$. Since $H^2(N; \mathbf{R}) = 0$, $w|_N = d\alpha$ for some $\alpha \in C^\infty(T^*N)$. Let $\beta \in C^\infty(M; [0,1])$ with $\beta = 1$ on $K \setminus N$ and $\beta = 0$ on $w_0 \setminus i_-(N)$ and the rest of End M. Set

$$\hat{j}(w) = \begin{cases} w & \text{on } K \setminus N, \\ d(\beta\alpha) & \text{on } N. \end{cases}$$

Thus, $\hat{j}(w) \in C_0^\infty(\Lambda_2 T^*K)$, and one can check that $j[w] = [\hat{j}(w)] \in H^2_{0,\mathrm{DR}}(K)$ is independent of the choice of $\alpha \in C^\infty(T^*N)$ and representative w for $[w]$. Using Poincaré duality plus the vanishing of $H^2(N; \mathbf{R})$, one finds

$$j \circ i_* = \mathrm{id} \quad \text{on } H^2_{0,\mathrm{DR}}(K).$$

Let $w \in C_0^\infty(\Lambda_2 T^*K_m)$ represents a class $[w] \in H^2_{0,\mathrm{DR}}(K_m)$ with $Q(w, w) < 0$. Suppose that $Q(jw, jw) > 0$. Let

$$v = w - i_*\hat{j}w = \begin{cases} w & \text{on End } M \setminus (K \cap \text{End } M), \\ d(1-\beta)\alpha & \text{on } N \cap \text{End } M. \end{cases}$$

Since $Q(v, i_*\hat{j}w) = 0$, it follows that $Q(v, v) < 0$. Now v is compactly supported on

$$(5.24) \qquad W_0 \cup_N W_1 \cup_N \cdots \cup_N W_{m+1},$$

and one can check that the assumptions concerning Y imply that the intersection pairing on (5.24) is positive definite. This gives a contradiction unless

$$Q(jw, jw) < 0 \quad \text{whenever } Q(w,w) < 0, \tag{5.25}$$

which implies that the number in (5.23) is independent of M, as claimed.

The completed proof of Proposition 5.1 requires still the proof of Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Before beginning in earnest, consider the unique $\nu \in C^\infty(Y)$ which is the harmonic 1-form on Y,

$$d\nu = 0, \qquad d^*\nu = 0, \tag{5.26}$$

cohomologous to the push forward of $d\tau|_{\mathrm{End}\,M}$ to Y. Back on End M, $\pi^*\nu = ds$ with $s \in C^\infty(\mathrm{End}\,M)$. The function s obeys

$$s|_{W_n} = s|_{W_0} + n. \tag{5.27}$$

Extend s smoothly to all of M. No generality is lost by assuming that $s > 0$. Let $\Sigma \subset W_1$ be the inverse image of a regular value, r, of s such that Σ is an embedded 3-manifold on which the restriction

$$*ds|_\Sigma \tag{5.28}$$

is a positive 3-form. Now consider $u \in C_0^\infty(M)$ and observe that integration by parts gives

$$\int_{s>r} e^{s\delta}|u|^2\, ds \wedge *ds = -\int_\Sigma e^{s\delta}|u|^2 *ds - \frac{2}{\delta}\int_{s>r} e^{s\delta} u\, du \wedge *ds,$$

where (5.26) and (5.27) have been used. Hölder's inequality plus (5.28) yield

$$\frac{1}{2}\int_{s>r} e^{s\delta}|u|^2\, ds \wedge *ds + \int_\Sigma e^{s\delta}|u|^2 *ds \leqslant \frac{2}{\delta^2}\int_{s>r} e^{s\delta}|du|^2. \tag{5.29}$$

Now, let H denote the completion of $C_0^\infty(M)$ with respect to the norm $\|\cdot\|$. Since

$$Z^{-1}(\delta)\int_M e^{\tau\delta}|f|^2 < \int_M e^{s\delta}|f|^2 < Z(\delta)\int_M e^{\tau\delta}|f|^2,$$

(5.29) implies for $u \in H$ that

$$\int_M e^{\tau\delta}|u|^2 < Z(\delta)\int_M e^{\tau\delta}|du|^2. \tag{5.30}$$

Now, consider $f \in C^\infty(M)$ with $\|f\|^2 < \infty$. A standard argument provides a unique $u \in H$ with

$$e^{-\tau\delta}d^*e^{\tau\delta}d(f-u) = 0. \tag{5.31}$$

(Minimize the functional on H which sends v to

$$s(v) = \tfrac{1}{2}\int_M e^{\tau\delta}\big(|dv|^2 + 2(dv, df)\big)$$

and use (5.30) to prove convergence.) Let $g = (f - u)$. (5.4) follows by proving that $g =$ constant. Now,

$$\int_M e^{\tau\delta}|dg|^2 < \infty, \tag{5.32}$$

so, since $\tau\,|_{W_n} \in [n, n+1]$, one can conclude that

$$\int_{W_n} |dg|^2 < e^{-n\delta}\varepsilon_n^2 \tag{5.33}$$

with $\{\varepsilon_n\}$ a Cauchy sequence with limit zero. (5.31) and (5.33) with standard a priori estimates (on compact domains) show that on W_n,

$$|dg| < Z(\delta)e^{-n\delta/2}\varepsilon_n, \quad \operatorname{Osc}(g) < Z(\delta)e^{-n\delta/2}\varepsilon_n, \tag{5.34}$$

where $\operatorname{Osc}(f) = \max(f) - \min(f)$. Let $q \in [0,1]$ be a regular value of the function t on Y. Let $\Sigma_n = \tau^{-1}(q+n) \subset W_n$. This Σ_n is a smooth, embedded 3-manifold. Set

$$\hat{K}_n = \tau^{-1}([0, q+n]) \subset K_n. \tag{5.35}$$

This manifold is compact with $\partial\hat{K}_n = \Sigma_n$. Let

$$g_n = \left(\int_{W_n} g\right)\cdot\left(\int_{W_n} 1\right)^{-1}. \tag{5.36}$$

(5.31) and an integration by parts imply that

$$\int_{\hat{K}_n} e^{\tau\delta}|dg|^2 \leqslant \int_{\Sigma_n} |g - g_n|\,|dg|e^{(n+1)\delta} \leqslant z(\delta)\varepsilon_n^2. \tag{5.37}$$

Since $\hat{K}_n \subset \hat{K}_{n+1} \subset \cdots = M$, (5.37) implies that $g =$ constant, which establishes (5.4).

The proof of (5.5) uses the dimension 4 and the local Sobolev inequality (cf. [1]) $L_1^2(\text{Ball}) \to L^4(\text{Ball})$ together with the end-periodicity of M. Equation (5.6) is proved with a similar argument using the local embedding $L_1^p(\text{Ball}) \to L^{4p/(4-p)}(\text{Ball})$ ($p \in (2,4)$). The final assertion of Lemma 5.2 uses the local Sobolev embedding, $L_1^p(\text{Ball}) \to C^0(\text{Ball})$ ($p > 4$).

Proof of Lemma 5.3. The proof will be seen to follow from the following result.

Lemma 5.8. *Let M be an admissible* 4-*manifold. There exists $\zeta > 0$ and $\delta_1 > 0$ such that on $\tilde{Y} = \cdots \cup W_{-1} \cup_N W_0 \cup_N W_1 \cup_N \cdots$, if $\delta \in [0, \delta_1)$, then*

$$\zeta(\delta) = \inf\left\{\int_{\tilde{Y}} e^{-\tau\delta}\left|(P_{-d})^* e^{\tau\delta} w\right|^2 : w \in L_{1,\delta}^2\left(P_- \Lambda_2 T^*\tilde{Y}\right), \int_{\tilde{Y}} e^{\tau\delta}|w|^2 = 1\right\}$$

obeys $\zeta(\delta) > \zeta$.

Proof of Lemma 5.3, *given Lemma* 5.8. Let $(1-\beta) \in C^\infty(M;[0,1])$ equal 0 on $K \setminus N$ and equal 1 on $W_0 \setminus N_-$ and $W_1 \cup_N W_2 \cup \cdots$. For each $n \in \{1, 2, \cdots\}$, set

$$w_n = \begin{cases} (1-\beta)e^{\tau\delta}w & \text{if } \tau < n, \\ e^{n\delta/2}e^{\tau\delta/2}w & \text{if } \tau \geqslant n. \end{cases}$$

This $w_n \in L^2_{1,0}(P_- \wedge_2 T^*Y)$ and it obeys

$$dw_n = \begin{cases} -d\beta \wedge w_n & \text{if } \tau < 1, \\ 0 & \text{if } 1 < \tau < n, \\ \dfrac{\delta}{2} d\tau \wedge w_n & \text{if } \tau > n. \end{cases}$$

Observe that

$$\int_{\tilde{Y}} |dw_n|^2 < \zeta_1 \int_{W_0} |w|^2 + \zeta_2 \frac{\delta}{2} \int_{\tau > n} |w_n|^2 \tag{5.38}$$

with ζ_1, ζ_2 constants independent of n. Lemma 5.8 implies via (5.38) that there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, then $\{w_n\}_{n=1}^\infty$ is uniformly bounded in $L^2_{0,0}(P_- \wedge_2 T^*Y)$. Since $\{w_n\}$ converges in L^2_{loc} to $e^{\tau\delta}w$, one concludes that $e^{\tau\delta}w \in L^2_{0,0}(P_- \wedge_2 T^*Y)$, as required.

Proof of Lemma 5.8. If the lemma were false at $\delta = 0$, then a sequence $\{w_j\}_{j=1}^\infty \subset C_0^\infty(P_- \wedge_2 T^*\tilde{Y})$ would exist with the property that

$$\lim_{j \to \infty} \int_{\tilde{Y}} |dw_j|^2 = 0 \quad \text{but} \int_{\tilde{Y}} |w_j|^2 = 1 \quad \text{for all } j. \tag{5.39}$$

To see that such a sequence cannot exist, return to Y and consider the cohomology of the complex

$$0 \to C^\infty(Y) \overset{d_z}{\to} C^\infty(T^*Y) \overset{P_-d_z}{\to} C^\infty(P_- \wedge_2 T^*Y) \to 0,$$

where $z \in C$ obeys $|z| = 1$ and

$$d_z f = df + z^t dz^{-t} \wedge f \tag{5.40}$$

(see (4.14)).

Lemma 5.9. *Under the assumptions of Lemma* 5.8, $\operatorname{Coker}(P_-d_z) \subset C^\infty(P_- \wedge_2 T^*Y)$ *for the complex in* (5.40) *is empty for all* $z \in C$ *with* $|z| = 1$.

Assuming Lemma 5.9 for the moment, the proof of the $\delta = 0$ case of Lemma 5.8 is completed by observing that because Y and $S^1 = \{z \in C : |z| = 1\}$ are compact, there exists $\zeta > 0$ such that

$$\int_Y \left|(P_-d_z)^*\gamma\right|^2 > \zeta \int_Y |\gamma|^2 \tag{5.41}$$

for all $\gamma \in C^\infty(P_- \Lambda_2 T^*Y)$ and for all $z \in S^1$. Here, $(P_-d_z)^*$ is the formal L^2-adjoint of P_-d_z,

$$(P_-d_z)^*\gamma = *(d\gamma + z^t dz^{-t} \wedge \gamma).$$

(5.41) plus the Fourier-Laplace inversion formulas, (4.10), (4.12), and (4.13), implies that no sequence obeying (5.39) can exist.

Proof of Lemma 5.9. The admissibility assumption asserts that $\mathrm{Coker}(P_-d_z) = \varnothing$ at $z = 1$. It further asserts that for all z, the index of the complex in (5.40) is zero. Thus $\dim \mathrm{Coker}(P_-d_z) = h^0 - h^1$, where (h^0, h^1) are the dimensions of $\mathrm{Ker}(d_z)$ in $C^\infty(Y)$ and of $\ker(P_-d_z, (d_z)^*)$ in $C^\infty(T^*Y)$. For h^0, if $d_z f = 0$, then $d(z^{-t}f) = 0$. Thus, $z^{-t}f =$ constant and $f =$ constant $\cdot\, z^t$. Thus for $z \neq 1$, $f \notin C^\infty(Y)$ unless $f = 0$. So, $h^0 = 0$. For h^1, if $P_-d_z a = 0$, then $P_-d(z^{-t}a) = 0$, and this implies that

$$d(z^{-t}a) = 0. \tag{5.42}$$

(Integrate by parts in $\int_Y |P_-d(z^{-t}a)|^2$.) Since $H^1(W, \mathbf{R}) = 0$, one concludes that

$$z^{-t}a = df \tag{5.43}$$

for $f \in C^\infty(W_0; \mathbf{C})$. Also,

$$d{*}df = 0. \tag{5.44}$$

Since a is a 1-form on Y, $i^*a = a$, where $i: N_+ \to N_-$ is the identification of Definition 1.2. Thus,

$$i^*f = zf + c \tag{5.45}$$

with $c \in \mathbf{C}$, a constant.

Choose a smoothly embedded submanifold $\Sigma \subset N_+ \subset W_0$ which separates the two ends of N_+ (so $H^1(Y \setminus \Sigma; \mathbf{R}) = 0$). Let $\hat{W}$ be the manifold with boundary that is obtained by cutting Y along Σ: $\hat{W}$ embeds in W, $\partial\hat{W} = \Sigma \cup (-\Sigma)$.

By integrating over $\hat{W}$ and using (5.45) with Stoke's theorem, one finds that

$$(z - 1)\int_\Sigma *df = 0. \tag{5.46}$$

Next, multiply (5.44) by $\bar{f}$ and integrate over $\hat{W}$. Stoke's theorem plus (5.45) yield

$$c\int_\Sigma *df + \int_{\hat{W}} |df|^2 = 0. \tag{5.47}$$

Together, (5.46) and (5.47) imply that $h^1 = 0$ unless $z = 1$. This completes the proof of Lemma 5.9.

For the cases $\delta > 0$ in Lemma 5.8, set $\sigma = e^{\tau\delta/2}w$ for $w \in L^2_{1,\delta}(P_- \Lambda_2 T^*\tilde{Y})$. Then

$$\begin{aligned}(5.48)\qquad \int_{\tilde{Y}} e^{-\tau\delta}\left|(P_-d)^* e^{\tau\delta} w\right|^2 &= \int_{\tilde{Y}} \left|(P_-d)^*\sigma + \frac{\delta}{2} *(d\tau \wedge \sigma)\right|^2 \\ &\geqslant \frac{1}{2}\int_{\tilde{Y}} \left|(P_-d)^*\sigma\right|^2 - \frac{\delta^2}{4}\int_{\tilde{Y}} |d\tau|^2|\sigma|^2.\end{aligned}$$

Since $|d\tau|$ is bounded and $\sigma \in L^2(\tilde{Y})$, the existence of $\delta_1 > 0$ and the assertion of Lemma 5.8 for $\delta \in (0, \delta_1)$ follow from (5.48) and the $\delta = 0$ case.

6. Perturbations of end-periodic structures

It is convenient to weaken the strict end-periodicity requirement for the metric on TM. The reason is that eventually certain Banach space genericity results are needed which require some latitude in the choice of metric. As in [21, Chapter 3], introduce the parameter space $C^l(\mathrm{GL}(TM))$ of C^l ($l \gg 2$), oriented, automorphisms of TM. Let g_0 be a fixed, end-periodic metric on TM. Define

$$(6.1)\qquad \mathscr{C} = \left\{ \phi \in C^l(\mathrm{GL}(TM)) : \lim_{n\to\infty}\left[\sup\left(\sum_{j=0}^{l} \left|\nabla^{(j)}(\phi^* g_0 - g_0)\right|(x) : \tau(x) > n\right)\right] = 0 \right\},$$

where ∇ is the Levi-Civita connection given by g_0. The space $\mathscr{C}$ is a Banach space.

For $\phi \in \mathscr{C}$, $\phi^* g_0$ defines a new metric on TM which approaches g_0 asymptotically on $\mathrm{End}\, M$. A metric on TM of the form $\phi^* g_0$, $\phi \in \mathscr{C}$, will be called *asymptotically periodic*. The asymptotically periodic metrics on TM provide a convenient class of metrics to use.

Let $P_-(g)$; $\Lambda_2 T^*M \to \Lambda_2 T^*M$ denote the anti-self-dual projection as defined by a metric g on TM. If $g = \phi^* g_0$ for $\phi \in \mathscr{C}$, then

$$(6.2)\qquad P_-(g) = \phi^* P_-(g_0)(\phi^{-1})^*.$$

Each $\phi \in \mathscr{C}$ defines an elliptic complex,

$$(6.3)\qquad 0 \to C_0^\infty(M) \xrightarrow{d} C_0^\infty(T^*M) \xrightarrow{\partial(\phi)} C_0^\infty(P_- \Lambda_2 T^*M) \to 0.$$

Here, for convenience of notation,

$$(6.4) \qquad \partial(\phi) = P_-(\phi^{-1})^* d$$

and $P_- = P_-(g_0)$. Note, $\partial(\phi)$ always maps into the fixed space of section of $P_- \Lambda_2 T^*M$.

As ϕ varies over $\mathscr{C}$ the complex in (6.4) changes. For $l-2>k\geqslant 0$, $p\geqslant 2$, $\delta\in\mathbf{R}$, and $\phi\in\mathscr{C}$, the complex in (6.4) extends as a bounded elliptic complex

$$(6.5) \quad 0 \to L^p_{k+2,\delta}(M) \xrightarrow{d} L^p_{k+1,\delta}(T^*M) \xrightarrow{\partial(\phi)} L^p_{k,\delta}(P_- \Lambda_2 T^*M) \to 0.$$

(These spaces are defined as in (3.10) using the fixed, end-periodic metric g_0).

Proposition 6.1. *Let M be an admissible, end-periodic 4-manifold. Let g_0 be an end-periodic metric on TM. Fix $l \gg 2$. Let $\delta_0 > 0$ be as in Proposition 5.1. For $\delta \in (0, \delta_0)$, all $p \in [2, \infty)$, $l-2>k\geqslant 0$, and for all $\phi\in\mathscr{C}$, the complex in (6.6) is Fredholm with cohomology*:

$$\dim H^0 = 0, \quad \dim H^1 = b_1(K), \quad \dim H^2 = b_2^-(K).$$

Proof of Proposition 6.1. In [21, §6], Lockhard and McOwen prove a similar result for complexes on manifolds with product ends. One can readily adapt their argument to the end-periodic case to show that (6.6) is Fredholm for $\phi\in\mathscr{C}$ whenever it is Fredholm for $\phi=1$. To compute the cohomology of (6.6), one can argue as follows: As ϕ varies through $\mathscr{C}$, the operator $\partial(\phi)$ varies continuously in the Banach space of bounded operators between $L^p_{k+1,\delta}(T^*M)$ and $L^p_{k,\delta}(P_- \Lambda_2 T^*M)$. It follows that the index of the complex in (6.5) is independent of ϕ. The arguments in §5 compute $\dim H^0 = 0$ and $\dim H^1 = b_1(K)$; then $\dim H^2$ follows by subtraction.

7. Gauge theory on M

Let G be a compact Lie group. Suppose that M is an end-periodic 4-manifold and that $P \to M$ is an end-periodic principal G-bundle. Here, it will be assumed that

$$(7.1) \qquad P|_{\mathrm{End}\, M} \overset{\psi}{\simeq} \pi^*(Y \times G).$$

If G is simple and simply connected, then all such P are globally trivial,

$$(7.2) \qquad P \simeq M \times G.$$

For G simple, but with $\pi_1(G)$ nontrivial, such P are classified by an element of $H^2(M; \pi_1(G))$ which lies in the image of the homomorphism $H^2_{\mathrm{comp}}(K \setminus \overline{N}) \to H^2(M)$. For $G = SO(n)$, this element is the second Stieffel-Whitney class of the associated R^n-bundle $E = P \times_G R^n$.

The concern of gauge theory is the space of connections on such end-periodic principal bundles. To restrict attention to end-periodic connections is too drastic and to study all connections is too ambitious. Let $\mathscr{A}(P)$ = the space of all smooth connections on P. Consider

$$\hat{\mathscr{A}} = \left\{ A \in \mathscr{A}(P) : \int_M |F_A|^2 < \infty \right\}. \tag{7.3}$$

Topologize $\hat{\mathscr{A}}$ as follows: The space $\mathscr{A}(P)$ is an affine space, since fixing $A_0 \in \mathscr{A}(P)$ identifies. $\mathscr{A}(P) \simeq C^\infty(\operatorname{Ad} P \otimes T^*M)$. ($\operatorname{Ad} P = P \otimes_G \mathfrak{G}$; $\mathfrak{G}$ = Lie alg(G). The set of sections $C^\infty(\operatorname{Ad} P \otimes T^*M)$ is topologized as a Fréchet space by the set of pseudonorms

$$\left\{ |\alpha|_{k,n} = \sup_{x \in K_n} \left| \nabla_{A_0}^{(k)} \alpha \right|(x) \right\}_{k,n=0}^{\infty}. \tag{7.4}$$

Here, $K_n = K \cup_N W_0 \cup \cdots \cup_N W_n$. Thus, $\mathscr{A}(P)$ has the structure of an affine Fréchet manifold. Topologize $\hat{\mathscr{A}}$ via the inclusion of $\mathscr{A}$ into $\mathscr{A}(P)$.

Let $\mathscr{G} = \mathscr{G}(P)$ denote the set of smooth automorphisms of P. This group has a Fréchet space structure which makes it into a Fréchet Lie group. The tangent space to $1 \in \mathscr{G}$ is isomorphic to $\Gamma(\operatorname{Ad} P)$ as a Fréchet space topologized via the pseudonorms in (7.4).

One can check readily that $\mathscr{G}$ acts smoothly on $\mathscr{A}(P)$, and $\mathscr{G}$ acts as a topological transformation group on $\hat{\mathscr{A}}$

$$\overline{\mathscr{B}} = \hat{\mathscr{A}}/\mathscr{G}, \tag{7.5}$$

and topologize with the quotient topology.

Connections in $\hat{\mathscr{A}}$ are not necessarily end-periodic, but in a weak sense, they are asymptotically so, and asymptotically flat. Here, asymptotically flat means asymptotic to a flat, periodic connection on $\tilde{Y}$. If Y is not simply connected, there may be nontrivial, flat connections to be asymptotic to. Let

$$\Gamma = \operatorname{Hom}(\pi_1(\tilde{Y}), G)/\operatorname{Ad} G.$$

This set parametrizes the set of G-equivariance classes of flat connections on $\tilde{Y} \times G$.

Since P is end-periodic, P admits end-periodic connections which are isomorphic (via ψ of (7.1)) over End M to the product connection on End M $\times$ G. Using these connections, a simple grafting argument (cf. [28] or [30]) produces connections in $\mathscr{A}$ which are asymptotic (over End M) to any chosen flat connection on $\tilde{Y} \times G$.

The present interest is with connections on P which are asymptotic to a trivial, flat connection on $\tilde{Y} \times G$. To define this notion, introduce the set

$$E\mathscr{A}(P) = \{\text{End-periodic connections on } P \text{ which are isomorphic over End } M \text{ to the product connection on End } M \times G\}.$$

Lemma 7.1. *For each $A \in E\mathscr{A}(P)$,*

$$p_1(A) = \frac{1}{8\pi^2}\int_M \operatorname{tr}(F_A \wedge F_A) \tag{7.6}$$

is an integer, where $\operatorname{tr}(\cdot)$ *is the trace on the adjoint representation of* G.

The proof of this lemma and of the assertions below will be deferred to the end of the section.

With the grafting procedure in [30, §4] one observes that $\operatorname{Im}(p_1 : E\mathscr{A}(P) \to \mathbf{Z})$ is unbounded in both directions.

For the present purposes, the set of connections $E\mathscr{A}(P)$ is too small. It is useful to enlarge $E\mathscr{A}(P)$ by introducing Banach spaces of connections with controlled decay.

For each $k \in \operatorname{Im}(p_1 : E\mathscr{A}(P) \to \mathbf{Z})$, fix $A_0 \in E\mathscr{A}(P)$ with $p_1(A_0) = k$. Fix $\delta > 0$, and let

$$\mathscr{A}_k(\delta) = \left\{ A_0 + a : a \in L^2_{2;\text{loc}}(\operatorname{Ad} P \otimes T^*M), \int_M e^{\tau\delta}\left\{ |\nabla_{A_0}\nabla_{A_0} a|^2 + |\nabla_{A_0} a|^2 + |a|^2 \right\} < \infty \right\}. \tag{7.7}$$

The affine space $\mathscr{A}_k$ has a natural Banach manifold structure from the norm

$$\|\psi\|_A^2 = \int_M e^{\tau\delta}\left\{ |\nabla_A \nabla_A a|^2 + |\nabla_A a|^2 + |a|^2 \right\} \tag{7.8}$$

with $A = A_0$. The dependence on δ is to be understood implicitly.

With the given $A_0 \in E\mathscr{A} \cap p_1^{-1}(k)$, define the "small" gauge group,

$$\mathscr{G}_k = \left\{ h \in L^2_{3;\text{loc}}(\operatorname{Aut} P) : \|\nabla_A h\|_A < \infty \right\}. \tag{7.9}$$

Topologize $\mathscr{G}_k$ as follows: Use the trivialization of (7.1) to identify $C^0(\operatorname{Aut} P|_{\text{End } M})$ with $C^0(\text{End } M; G)$. Since $L^3_{2;\text{loc}} \subset C^0$, each $h \in \mathscr{G}_k$ defines a continuous map, $h : \text{End } M \to G$. An application of Lemma 5.2 and Kato's inequality shows that for $x \in W_0$,

$$r(h) = \lim_{n \to \infty} h(T^n(x)) \in G$$

exists for all $h \in \mathscr{G}_k$; further, $r(h)$ is independent of $x \in W_0$. (Kato's inequality asserts that for any $A \in \mathscr{A}(P)$ and for any vector bundle $E \to M$, associated to P, and for any $\psi \in L^2_{1;\text{loc}}(E)$, $|\nabla_A \psi|(x) > |d|\psi|(x)|$ for a.e.

$x \in M$.) A neighborhood base for the topology of $\mathcal{G}_k$ at $h \in \mathcal{G}_k$ is given by sets of the form

$$\left\{ g \in \mathcal{G}_k;\, \|\nabla_A(h-g)\|_A^2 + \operatorname{dist}_G(r(h), r(g)) < \varepsilon \right\},$$

where $\operatorname{dist}_G(\cdot,\cdot)$ is the geodesic distance on G as measured by a bi-invariant metric.

Let $\mathcal{G}'_k$ denote the closed subgroup $\{g \in \mathcal{G}_k;\, r(g) = 1\}$.

Lemma 7.2. *Let $\delta > 0$. The spaces $\mathcal{G}_k$, $\mathcal{G}'_k$ are Banach Lie groups. The Lie algebra of $\mathcal{G}_k$ is*

$$\mathfrak{G}_k = \left\{ \sigma \in L^2_{3;\mathrm{loc}}(P \times_{\mathrm{Ad}} \mathfrak{G}) : \|\nabla_A \sigma\|_A < \infty \right\}.$$

The norm on $\mathcal{G}_k$ sends σ to $\|\nabla_A\sigma\|_A + |r(\sigma)|$. The Lie algebra of $\mathcal{G}'_k$ is $\mathfrak{G}'_k = \{\sigma \in \mathfrak{G}_k;\, r(\sigma) = 0\}$. The Lie group $\mathcal{G}_k$ acts smoothly on $\mathcal{A}_k$. The set $\{g \in L^2_{3;\mathrm{loc}}(\mathrm{Aut}\, P) : g \cdot \mathcal{A}_k = \mathcal{A}_k\}$ is $\mathcal{G}_k$. The quotient $\mathcal{G}_k/\mathcal{G}'_k = G$.

As is the case when M is compact, one has the following "slice theorem":

Lemma 7.3. *Let $\delta > 0$, and let $k \in \mathrm{Im}(p_1 : E\mathcal{A}(P) \to \mathbf{Z})$. The quotient space $\mathcal{B}'_k = \mathcal{A}_k/\mathcal{G}'_k$ is a C^∞-Banach manifold and the projection $\pi : \mathcal{A}_k \to \mathcal{B}'_k$ defines a principal $\mathcal{G}'_k$-bundle. Let $\mathcal{A}^*_k \subset \mathcal{A}_k$ denote the subset of irreducible connections. The quotient space $\mathcal{B}_k = \mathcal{A}^*_k/\mathcal{G}_k$ is a C^∞-Banach manifold such that the quotient $\pi : \mathcal{A}^*_k \to \mathcal{B}_k$ defines a principal $\mathcal{G}_k/\text{center } \mathcal{G}_k$-bundle. The tangent space to $[A] \in \mathcal{B}'_k$ is isomorphic to*

$$\left\{ a \in L^2_{2;\mathrm{loc}}(\mathrm{Ad}\, P \otimes T^*M) : \|a\|_A < \infty \text{ and } e^{-\tau\delta} d_A^* e^{\tau\delta} a = 0 \right\}.$$

The tangent space to $[A] \in \mathcal{B}_k$ is isomorphic to

$$\left\{ a \in L^2_{2;\mathrm{loc}}(\mathrm{Ad}\, P \otimes T^*M) : \|a\|_A < \infty, \right. \tag{7.10}$$
$$\left. \int_M e^{\tau\delta}(d_A\sigma, a) = 0 \text{ for all } \sigma \in \mathfrak{G}_k \right\}.$$

The manifold $\mathcal{B}_k$ is the orbit space of asymptotically periodic connections which will be used in this article.

Although a choice of $A_0 \in p_1^{-1}(k) \cap E\mathcal{A}(P)$ was required to defined $\mathcal{B}_k$, this space does not depend on the particular choice. The following lemma describes the situation.

Lemma 7.4. *Let $k \in \mathrm{Im}(p_1 : E\mathcal{A}(P) \to \mathbf{Z})$ and let $A_0, A_1 \in E\mathcal{A}(P) \cap p_1^{-1}(k)$. Define the spaces $\mathcal{A}_{k0}\mathcal{A}_{k1}$ and $\mathcal{G}_{k0}$, $\mathcal{G}_{k1}$ with A_0 and A_1, respectively. There exists $g \in \mathcal{G}$ such that*

(1) $g \cdot A_1 = A_0$ *on* End M.

(2) $g \cdot \mathcal{A}_{k1} = \mathcal{A}_{k0}$.

(3) $\mathrm{Ad}\, g^{-1} \mathcal{G}_{k1} = \mathcal{G}_{k0}$.

(4) *The quotient spaces $\mathcal{B}'_{k1}$, $\mathcal{B}_{k1}$ are isomorphic as Banach manifolds to $\mathcal{B}'_{k0}$, $\mathcal{B}_{k0}$.*

Due to Lemma 7.4, no generality is lost by henceforth referring to one space $\mathscr{B}'_k$ (or $\mathscr{B}_k$), independent of the defining connection in $E\mathscr{A}(P) \cap p_1^{-1}(k)$.

For the proofs of Lemmas 7.1–7.4, it is necessary to have the following estimates on gauge invariant norms:

Lemma 7.5. *Let $P \to M$ be an end-periodic principal G-bundle over the end-periodic 4-manifold M. Let $E \to M$ be a vector bundle over M associated to P. Let A, $A_1 \in \mathscr{A}_k$ for some $k \in \operatorname{Im}(p_1 : E\mathscr{A}(P) \to \mathbf{Z})$. There exists $\zeta < \infty$ with the following property. For all $\omega \in C^\infty(E)$ with $\|\omega\|_A < \infty$,*

$$\zeta\|\omega_{A_1}\| < \|\omega\|_A < \zeta^{-1}\|\omega\|_{A_1}.$$

For all $\omega \in C^\infty(E)$ with $\|\nabla_A\omega\|_A < \infty$,

$$\zeta\|\nabla_{A_1}\omega\|_{A_1} < \|\nabla_A\omega\|_A < \zeta^{-1}\|\nabla_{A_1}\omega\|_{A_1}.$$

The remainder of this section contains the proofs of Lemmas 7.1–7.5.

Proof of Lemma 7.1. Choose a smoothly embedded, oriented submanifold $\Sigma \subset W_0$, which separates N_- from N_+. Let W_+ denote the component of $W_0 \setminus \Sigma$ which contains N_+. Compactify $K \cup_N (W_0 \setminus W_+)$ by gluing it to $-(K \cup_N (W_0 \setminus W_+))$ across Σ. Call this compact 4-manifold Q. Let Γ denote the product connection on $W_0 \times G$. Let $A \in E\mathscr{A}(P)$. Then, by assumption, there exists $g(A) \in \operatorname{Iso}(W_0 \times G; P|_{W_0})$ such

$$g(A)^*A = \Gamma. \tag{7.11}$$

Note. $g(A)$ is unique up to $g \to h \cdot g$ with $h \in \operatorname{Aut}(W_0 \times G)$ $(= C^\infty(W_0; G))$ a constant gauge transformation. Using $g(A)$, one constructs a bundle $P(A) \to Q$ by defining

$$P(A)|_{K \cup_N (W_0 \setminus W_+)} = P,$$

$$P(A)|_{-(K \cup_N (W_0 \setminus W_+))} = -(K \cup_N (W_0 \setminus W_+)) \times G,$$

with $g(A)$ identifying the two halves over Σ. Notice that A automatically extends to a connection on $P(A)$ which is trivial over $-(K \cup_N (W_0 \setminus W_+))$. Then, by Chern-Weil [23],

$$p_1(A) = p_1(P(A) \times_{\operatorname{Ad} G} \mathfrak{G}), \tag{7.12}$$

where, on the right-hand side, $p_1(\cdot)$ is the first Pontrjagin number, an integer.

It is convenient to consider Lemma 7.5 next.

Proof of Lemma 7.5. Use Lemma 5.2, Kato's inequality, and Holder's inequality.

Proof of Lemma 7.2. Armed with Lemmas 5.2 and 7.5, the proofs here are trivial translations of the arguments in [12, Chapter 3]. The key estimate is the following: Given $\delta > 0$, there exists $\zeta > 0$, such that for all $A \in \mathscr{A}_k$ and

$\sigma \in \mathfrak{S}_k$,

$$\text{(7.13)} \qquad \int_M e^{\tau\delta}|\nabla_A\sigma|^2 \geqslant \zeta\int_M e^{\tau\delta}|\sigma - r(\sigma)|^2.$$

This uses Kato's inequality plus Lemma 5.2. The only item which does not translate from [12] is the assertion: $\mathscr{G}_k = \{g \in L^2_{3;\mathrm{loc}}(\mathrm{Aut}\, P): g^*\mathscr{A}_k = \mathscr{A}_k\}$. For this, suppose $g \in L^2_{3;\mathrm{loc}}(\mathrm{Aut}\, P)$ and that

$$g^*A_0 = A_0 + g^{-1}\nabla_{A_0}g \in \mathscr{A}_k.$$

Then Holder's inequality plus Kato's inequality plus Lemma 5.2 puts $g \in \mathscr{G}_k$.

Proof of Lemma 7.3. Again, with Lemma 5.2 and Lemma 7.5, it is a straightforward adaptation from [12, Chapter 3] and from [24] to prove the assertion about $\mathscr{B}'_k$. The key is to use (7.13). There is a residual $\overline{G} = G/\mathrm{center}(G)$ action on $\mathscr{B}'_k$ which is free on $A^*_k/\mathscr{G}'_k \subset \mathscr{B}'_k$. The quotient, $\mathscr{B}_k$, will be a manifold provided that local slices of the $\overline{G}$-action exist. To construct them, note that for all $\varepsilon > 0$, sufficiently small, a neighborhood of $[A] \in \mathscr{B}'_k$ is diffeomorphic to

$$\text{(7.14)} \quad \mathscr{D} = \left\{a \in L^2_{2;\mathrm{loc}}(\mathrm{Ad}\, P \otimes T^*M): \|a\|_A < \varepsilon \text{ and } e^{-\tau\delta}d^*_A e^{\tau\delta}a = 0\right\}.$$

For small $\varepsilon < 0$, there exists for each $a \in \mathscr{D}$ and each $\sigma \in \mathfrak{S}$, a unique $q(a; \sigma)$ obeying

$$\text{(7.15)} \qquad e^{-\tau\delta}d^*_A e^{\tau\delta}d_{A+a}q(a;\sigma) = 0 \quad \text{and} \quad r(q(a;\sigma)) = \sigma.$$

To construct q, minimize over the closed submanifold $r^{-1}(\sigma) \in \mathfrak{S}_k$ the bilinear functional $\eta \to \int_M e^{\tau\delta}(d_A\eta, d_{A+a}\eta)$. Use (7.13) plus elliptic regularity to prove that a unique minimum exists.

Using Lemmas 5.2 and 7.5, it is easy to prove that the assignment of $(a, \sigma) \in \mathscr{D} \times \mathfrak{S}$ to $q(a; \sigma) \in \mathfrak{S}_k$ defines a smooth map from $\mathscr{D} \times \mathfrak{S}$ to $\mathfrak{S}_k$ which is linear in $\mathfrak{S}$ for fixed $a \in \mathscr{D}$. By construction,

$$d_{A+a}q(a;\sigma) \in \mathscr{D} \quad \text{for } |\sigma| \ll \varepsilon.$$

Define a map $\hat{q}: \mathscr{D} \to \mathfrak{S}^*$ by sending $a \in \mathscr{D}$ to the linear functional

$$\text{(7.16)} \qquad \sigma \to \langle \hat{q}(a), \sigma\rangle = \int_M e^{\tau\delta}(d_{A+a}q(a;\sigma), a).$$

The claim is that $\hat{q}^{-1}(0) \subset \mathscr{D}$ is a smooth submanifold, which, for ε sufficiently small, is a slice of the $\overline{G}$-action.

Since A is irreducible, $A + a$ is also irreducible for $a \in \mathscr{D}$ and for ε sufficiently small. Thus, $d_{A=a}q(a;\sigma) = 0$ if and only if $\sigma = 0$. This means that the differential of $\hat{q}$ at $a \in q^{-1}(0)$ is surjective and so $\hat{q}^{-1}(0)$ is a smooth submanifold of $\mathscr{D}$.

The tangent space to the $\bar{G}$-orbit through $a \in q^{-1}(0)$ is precisely the

$$\text{Span}\{ d_{A+a}q(a;\sigma); \sigma \in \mathfrak{G} \}. \tag{7.17}$$

This subspace of $T\mathscr{D}|_a$, plus $T\hat{q}^{-1}(0)|_a$, spans $T\mathscr{D}_a$. To show that $\hat{q}^{-1}(0) \subset \mathscr{D}$ is a slice of the $\bar{G}$-action, one must show that any time $a \in \hat{q}^{-1}(0)$, $h \in \bar{G}$, and $h \cdot a \in \hat{q}^{-1}(0)$, then $h = 1$. For ε-sufficiently small, any time $h \cdot a \in \hat{q}^{-1}(0)$,

$$h = 1 + q(a;\sigma) + O(|\sigma|^2), \tag{7.18}$$

with $\sigma \in \mathfrak{G}$ and $|\sigma| \ll \varepsilon$. The $O(|\sigma|^2)$ term is $O(\varepsilon^2)$ in the norm of (7.9). Equations (7.16) and (7.18) imply that $\sigma = 0$ and $h = 1$.

Proof of Lemma 7.4. The last three assertions of the lemma follow immediately from the first assertion. To prove the first assertion, consider $A_0, A_1 \in E\mathscr{A}(P)$. On End M, one assumes that $A_0 = g \cdot A_1$ for $g \in \text{Aut}(P|_{\text{End}\,M})$ ($= C^\infty(\text{End}\,M; G)$.) Let $a = A_0 - A_1 \in C^\infty(\text{Ad}\,P \otimes T^*M)$ restricted to End M,

$$a = g\nabla_{A_1} g^{-1}. \tag{7.19}$$

Since both A_0 and A_1 are trivial over W_0, Chern-Simon's theory [8] gives

$$0 = p_1(A_1) - p_1(A_0) = \alpha(G)\langle g^*(\gamma), [\Sigma]\rangle, \tag{7.20}$$

where $\alpha(G) \neq 0$ is a group-theoretic constant, $\gamma \in H^3(G)$ ($\simeq \mathbf{Z}$) is the generator, and $[\Sigma]$ is the fundamental class of the 3-manifold Σ. Since $\dim \Sigma = 3$, (7.15) implies that $g|\Sigma$ is homotopic to the constant map $1 : \Sigma \to 1_G$. Affecting such a null homotopy in a tubular half-neighborhood of Σ, $(-\varepsilon, 0) \times \Sigma \subset W_0 \setminus W_+$ gives an extension of g to $\text{Aut}\,P|_M$. Such a g is unique up to composition with $\mathscr{G}'_{k0}$ and $\mathscr{G}'_{k1}$. Its existence proves Lemma 7.4.

8. Moduli spaces: Internal structure

Let $P \to M$ be an end-periodic, principal G-bundle over M obeying (7.1). The constructions in the previous sections provide the necessary technical machinery with which to study, as in [12], the moduli spaces of self-dual connections on P. The basic lesson in the next few sections is the following: For those $\delta \in (0, \infty)$ where the complex in (5.2) is Fredholm, the constructions on compact 4-manifolds will succeed on the admissible, end-periodic M (cf. [30], [9]).

Fix an end-periodic metric, g_0, on T^*M. Let $k \in \text{Im}(p_1 : E\mathscr{A}(P) \to \mathbf{Z})$. Define a $\mathscr{G}_k$-equivariant map from $\mathscr{A}_k \times \mathscr{C}$ into $L^2_{1;\text{loc}}(\text{Ad}\,P \otimes P_- \Lambda_2 T^*)$ by sending (A, ϕ) to

$$\mathscr{P}(A, \phi) = P_-(g_0)(\phi^{-1})^* F_A. \tag{8.1}$$

Let

$$(8.2)\quad \mathcal{k}_k^\delta = \left\{ \omega \in L^2_{1;\mathrm{loc}}(\operatorname{Ad} P \otimes T^*) : \int_M e^{\tau\delta}\left(|\nabla_{A_0}\omega|^2 + |\omega|^2\right) < \infty \right\}.$$

This linear space is a Banach space with the obvious norm. The image of $\mathscr{P}$ lies in $\mathcal{k}_k^\delta$.

As in [12, §3], one has the parametrized spaces

$$(8.3)\qquad \hat{\mathscr{M}}'_k = \mathscr{P}^{-1}(0)/\mathscr{G}'_k \quad \text{and} \quad \hat{\mathscr{M}}_k = \mathscr{P}^{-1}(0)/\mathscr{G}_k = \hat{\mathscr{M}}'_k/\overline{G}.$$

As in [12, Theorem 3.16], one has the following structure theorem.

Proposition 8.1. *Let M be an admissible, end-periodic 4-manifold and let $P \to M$ be an end-periodic principal $SU(2)$ or $SO(3)$-bundle which obeys (7.1). Let $k \in \operatorname{Im}(p_1 : E\mathscr{A}(P) \to \mathbf{Z})$. There exists $\delta_1 > 0$, such that for all $\delta \in (0, \delta_1)$, $\hat{\mathscr{M}}'_k \cap (\mathscr{A}_k^*/\mathscr{G}'_k \times \mathscr{C})$ and $\hat{\mathscr{M}}_k \cap (\mathscr{B}_k \times \mathscr{C})$ are smooth Banach manifolds.*

The projection to $\mathscr{C}$ from $\mathscr{A}_k \times \mathscr{C}$ induces projections

$$(8.4)\qquad \overline{\pi}' : \hat{\mathscr{M}}'_k \to \mathscr{C} \quad \text{and} \quad \overline{\pi} : \hat{\mathscr{M}}_k \to \mathscr{C}.$$

K. Uhlenbeck's generic metric theorem [12, Theorem 3.17] translates in the present case to

Proposition 8.2. *Make the same assumptions as in Proposition 8.1. There exists $\delta_2 > 0$ such that for all $\delta \in (0, \delta_2)$, the following holds: A Baire set of $\phi \in \mathscr{C}$ exists for which the moduli spaces $\mathscr{M}'_k(\phi) \equiv \overline{\pi}'^{-1}(\phi)$ and $\mathscr{M}_k(\phi) \equiv \overline{\pi}^{-1}(\phi)$ are such that $\mathscr{M}'_k(\phi) \cap (\mathscr{A}_k^*/\mathscr{G}'_k)$ and $\mathscr{M}_k(\phi) \cap \mathscr{B}_k$ are smooth manifolds of dimensions $2k - 3(b_2^-(K) - b_1(K))$ and $2k - 3(1 + b_2^-(K) - b_1(K))$ respectively.*

The manifold $\mathscr{M}'_k(\phi)$ admits an $SO(3)$-action which is free on the orbit of an irreducible connection. Thus, the projections $\mathscr{M}'_k(\phi) \to \mathscr{M}_k(\phi) \cap \mathscr{B}_k$ define a principal $SO(3)$ bundle.

At the orbit of a reducible connection in $\mathscr{M}'_k(\phi)$, the $SO(3)$-action will not be free. by perturbing $\mathscr{M}_k(\phi)$ as in [8], one can assume that a neighborhood of the reducible orbits in $\mathscr{M}_k(\phi)$ have a standard form. For compact M, the result is described in [12, Theorem 4.11].

Proposition 8.3. *Make the same assumptions as in Proposition 8.1, and assume that $b_1(K) = 0$. There exists $\delta_3 > 0$, such that for all $\delta \in (0, \delta_3)$, the following is true: If the intersection pairing on $H_2(K, \mathbf{Z})$ is indefinite, then for a Baire set of $\phi \in \mathscr{C}$, $\mathscr{M}_k(\phi)$ contains no orbits of reducible connections; it is a smooth manifold of dimension $2k - 3(1 + b_2^-(K) - b_1(K))$. If $H_2(K; \mathbf{Z})$ is definite, then the orbits of reducible connections in $\mathscr{M}_k(\phi)$ are isolated. There is a perturbation of $\mathscr{M}_k(\phi)$ which is compactly supported in a neighborhood of each* *such orbit so that locally about such an orbit,* $\mathcal{M}_k(\phi)$ *is homeomorphic to an open cone on* $\mathbf{CP}^l$*, with* $l = k - 2$*. The identification is a diffeomorphism off the vertex.*

The strategy for proving Propositions 8.1–8.3 will be to set up a formalism that allows the arguments in [12, Chapters 3, 4] to translate directly to the end-periodic case. The key to the formalism is in understanding the operator $P_-d_A: C^\infty(\mathrm{Ad}\, P \otimes T^*) \to C^\infty(\mathrm{Ad}\, P \otimes P_- \Lambda_2 T^*)$ when $A \in \mathcal{A}_k$. The principle result required is Lemma 8.4, below. To state the lemma, introduce the Banach space $\mathfrak{A}_k$ defined as follows: Pick $A \in \mathcal{A}_k$ and set

$$\mathfrak{A}_k = \left\{ a \in C^\infty(\mathrm{Ad}\, P \otimes T^*): \|a\|_A < \infty \right\}, \tag{8.5}$$

with norm $\|\cdot\|_A$ (see (7.8)). Due to Lemma 8.4, the choice of $A \in \mathcal{A}_k$ is immaterial to $\mathfrak{A}_k$.

Lemma 8.4. *There exists* $\delta_4 > 0$ *such that under the assumptions of Proposition* 8.1: *If* $\delta \in (0, \delta_4)$ *and if* $(A, \phi) \in \mathcal{P}^{-1}(0) \cap (\mathcal{A}_k \times \mathcal{C})$*, then*

$$0 \to \mathfrak{G}'_k \xrightarrow{d_A} \mathfrak{A}_k \xrightarrow{P_-(g_0)(\phi^{-1})^*} \mathscr{h}^\delta_k \to 0 \tag{8.6}$$

is Fredholm with index $8k - 3(b_2^-(K) - b_1(K))$*. Further,* $\ker d_A \subset \mathfrak{G}'_k$ *is empty.*

Assume for the moment that this lemma holds.

Proof of Proposition 8.1. The map $\mathcal{P}$ from $\mathcal{A}_k \times \mathcal{C}$ to $\mathscr{h}^\delta_k$ is readily seen to be smooth. One identifies $T\mathcal{A}_k|_{[A]} \simeq \mathfrak{A}_k$. The differential $d\mathcal{P}$ at $(A, \phi) \in \mathcal{P}^{-1}(0)$ splits as a direct sum $d\mathcal{P}_1 + d\mathcal{P}_2$ corresponding to $T(\mathcal{A}_k \times \mathcal{C}) \simeq T\mathcal{A}_k \oplus T\mathcal{C}$. Observe that

$$d\mathcal{P}_1|_{(A,\phi)} = P_-(g_0)(\phi^{-1})^* d_A.$$

Since (8.6) is Fredholm, $\mathrm{Coker}\, d\mathcal{P} \subset \mathrm{Coker}\, d\mathcal{P}_1$ is finite dimensional. By using the L^2-inner product of the metric $\phi^* g_0$ on TM, one can represent $\omega \in \mathrm{Coker}\, dP_1|_{(A,\phi)}$ by $\omega \in C^\infty(\mathrm{Ad}\, P \otimes P_-(\phi^* g_0)\Lambda_2 T^*M) \cap L^2$ obeying $d_A \omega = 0$. The proof is finished by copying the proof of Theorem 3.16 of [12].

Proof of Proposition 8.2. The projection $\bar{\pi}': \mathcal{M}'_k \cap (A^*_k/\mathcal{G}'_k \times \mathcal{C}) \to \mathcal{C}$ is a smooth, Fredholm map of index $2k - 3(b_2^-(K) - b_1(K))$. The index calculation is Lemma 8.4. The Smale-Sard theorem [27] establishes the assertion of Proposition 8.2 for $\mathcal{M}'_k \cap (\mathcal{A}^*_k/\mathcal{G}'_k)$. The group $\mathcal{G}_k/\mathcal{G}'_k = SU(2)$ (or $SO(3)$) acts on $\mathcal{M}'_k \cap (\mathcal{A}^*_k/\mathcal{G}'_k)$; the stabilizer of a point is $\pm 1 \subset SU(2)$ and $1 \subset SO(3)$. Thus, in both cases there is a free action of $SO(3)$. The quotient, $\mathcal{M}_k(\phi) \cap \mathcal{B}_k$ is a manifold provided that local slices of the $SO(3)$-action exist. Let $(A, \phi) \in \mathcal{M}'_k(\phi) \cap (\mathcal{A}^*_k/\mathcal{G}'_k)$. A neighborhood of (A, ϕ) is diffeomorphic to

$$\mathcal{N} = \left\{ a \in \mathfrak{A}_k : e^{-\tau\delta} *_\phi d_A *_\phi e^{\tau\delta} a = 0,\ P_-(\phi^* g_0) d_A a = 0 \text{ and } \|a\|_A < \varepsilon \right\}$$

for sufficiently small ε. Here, $*_\phi$ is the Hodge star for the metric ϕ^*g_0. In the remainder of the proof, all inner products, volume forms, and anti-self-dual projections are with respect to ϕ^*g_0. This will be implicit.

Given $a \in \mathscr{N}$ and $\sigma \in su(2)$ ($= so(3)$), construct $q(a;\sigma) \in \mathfrak{G}_k$ as in (7.15), but use the metric ϕ^*g_0. The construction is the same at the expense of a smaller $\varepsilon > 0$ in (8.6). Since $A + a$ is self-dual,

$$(8.7) \qquad d_{A+a}q(a;\sigma) \in \mathscr{N} \quad \text{for } \sigma \in su(2) \text{ and } |\sigma| \ll \varepsilon.$$

Using the metric ϕ^*g_0, construct the map $q: \mathscr{N} \to su(2)^*$ as in (7.16). The argument in §7 can now be directly appropriated to show that $q^{-1}(0) \subset \mathscr{N}$ is a slice of the $SO(3)$-action.

Proof of Proposition 8.3. The proof of Theorem 4.11 in [12] translates directly to the case here. See also the proof of Corollary 3.21 in [12]. The reader is also referred to [11].

Proof of Lemma 8.4. (7.13) implies that $d_A: \mathfrak{G}'_k \to \mathfrak{A}_k$ has no kernel and it has closed range. It follows that (8.6) is Fredholm if the two-step complex

$$(8.8) \qquad \delta(A,\phi) = \begin{matrix} P_-(g_0)(\phi^{-1})^* d_A \\ e^{-\tau\delta} d_A^* e^{\tau\delta} \end{matrix} : \mathfrak{A}_k \to \begin{pmatrix} \hbar_k^\delta \\ \hat{\mathfrak{G}}_k \end{pmatrix}$$

is Fredholm. Here,

$$(8.9) \qquad \hat{\mathfrak{G}}_k = \left\{ \sigma \in L^2_{1;\mathrm{loc}}(\mathrm{Aut}\, P) : \int_M e^{\tau\delta}\left(|\nabla_A\sigma|^2 + |\sigma|^2\right) < \infty \right\}.$$

Pick $A_0 \in E\mathscr{A} \cap \mathscr{A}_k$ and consider the operator

$$(8.10) \qquad \delta(A_0,\phi): \mathfrak{A}_k \to \begin{pmatrix} \hbar_k^\delta \\ \hat{\mathfrak{G}}_k \end{pmatrix}.$$

This operator, on End M, is isomorphic via (7.1) to

$$(8.11) \qquad \begin{aligned} \delta(0,\phi) \equiv \begin{matrix} P_-(g_0)(\phi^{-1*}d) \\ e^{-\tau\delta} d * e^{\tau\delta} \end{matrix} &: L^2_{2;\delta}(T^*M) \times su(2) \\ &\to \begin{pmatrix} L^2_{1,\delta}(P_- \Lambda_2 T^*M) \\ L^2_{1,\delta}(M) \end{pmatrix} \times su(2). \end{aligned}$$

The complex in (8.11) is Fredholm, by Proposition 6.1. Arguing as in §6 of [21], one proves that $\delta(A_0,\phi)$ is also Fredholm. Then, Lemmas 5.2 and 7.5 plus Kato's inequality show that

$$\delta(A_0,\phi) - \delta(A,\phi): \mathfrak{A}_k \to \begin{pmatrix} \hbar_k^\delta \\ \hat{\mathfrak{G}}_k \end{pmatrix}$$

is a compact operator. Thus, $\delta(A,\phi)$ is Fredholm. Further, $\operatorname{Index}(\delta(A,\phi)) = \operatorname{Index}(\delta(A_0,\phi))$. Since $\delta(A_0,\phi)$ is isomorphic to $\delta(0,\phi)$ over $\operatorname{End} M$, the excision property of the index [4] asserts that the difference, $\operatorname{Index}(\delta(A_0,\phi)) - \operatorname{Index}(\delta(0,\phi))$ depends only on K. By embedding K in the compact 4-manifold Q from the proof of Lemma 7.1, one can use the index theorem on compact manifolds to compute this difference. The answer is Lemma 8.4 (see [2]).

9. Moduli spaces: Existence

Fix an asymptotically periodic metric, ϕ^*g_0, on T^*M (as defined in §6). A connection on a principal G-bundle P over the end-periodic, admissible 4-manifold M is self-dual if $P_-F_A = 0$, with $P_- = P_-(\phi^*g_0)$. (In this section, all norms, inner products, and operator adjoints are taken using ϕ^*g_0. This will be implicit.) The purpose of this section is to prove that the moduli spaces of self-dual connections, $\mathscr{M}_k(\phi)$, can be nonempty.

Rather than discuss the existence question for admissible, end-periodic manifolds in the generality of [30], attention will be restricted to the case where $b_1(K) = b_2(K) = 0$ (see Proposition 6.1).

Consider first the existence of reducible self-dual connections on principal $SU(2)$ and $SO(3)$ bundles $P \to M$ which obey (7.1). To construct such a connection, pick a class $e \in H_2(M,\mathbf{Z})$. Through Poincaré duality, e is dual to a class $e \in H^2_{\text{comp}}(M,\mathbf{Z})$, and via the injection (Lemma 5.7) $H^2_{\text{comp}} \to H^2$, the class e defines a line bundle,

$$L(e) \to M. \tag{9.1}$$

Via DeRham's theorem, e defines a closed two-form, $\omega(e) \in H^2_{0,\text{DR}}(M)$. This two-form is $\sqrt{-1} \times$ the curvature of a connection A_e on L. By representing e by an embedded, closed two-dimensional surface $R \subset M$, one can arrange that A_e is trivial away from a tubular neighborhood of R in M.

Following [11], construct from $L(e)$ the $\mathbf{R}^3$-bundle

$$E = L(e) \oplus_{\mathbf{R}} \mathscr{E}, \tag{9.2}$$

where $\mathscr{E} = M \times \mathbf{R}$ is the trivial, real-line bundle. Put an end-periodic metric on E. Define $P = P(e)$ to be the bundle of orthonormal oriented frames on E—a principal $SO(3)$ bundle over M. Note, if the mod 2 reduction of e, e_2, is in the image $H^2_{\text{comp}}(K \setminus N; \mathbf{Z}_2) \to H^2(M;\mathbf{Z}_2)$, then $P(e)$ obeys (7.1). Indeed, its Stieffel-Whitney class will then vanish in $H^2(\operatorname{End} M;\mathbf{Z}_2)$, since $w_2(E) = e_2$. Assume that such is now the case.

The connection A_e is in $\mathscr{A}_k(P(e))$ (by Lemma 7.4) for any $\delta \geqslant 0$ and for

$$k = e \cdot e. \tag{9.3}$$

Thus, $\mathscr{A}_{(e\cdot e)}(P(e))$ has reducible connections. Note, if $e_2 = 0$, then $P(e)$ lifts to the $SU(2)$ bundle $M \times SU(2)$. Thus, $\mathscr{A}_k(M \times SU(2))$ has reducible connections whenever $k = e \cdot e$ for $e \in H_2(M, \mathbf{Z})$ with $e_2 = 0 \in H^2(M, \mathbf{Z}_2)$.

For self-dual reducible connections, one has

Proposition 9.1. *Let M be an end-periodic, admissible 4-manifold with $b_1(K) = b_2^-(K) = 0$. Let $\delta_0 > 0$ be as given in Proposition 6.1. Fix $\delta \in (0, \delta_0)$. Let $P \to M$ be a principal $G = SU(2)$ or $SO(3)$ bundle which obeys (7.1). The orbits of reducible connections in $\mathscr{M}_k(P)$ are in 1-1 correspondence with the set of pairs $\{\pm e \in H_2(M, \mathbf{Z}) : e \cdot e = k$ and $e_2 = w_2(P \otimes_{\mathrm{Ad}} \mathfrak{G})\}$.*

Proof of Proposition 9.1. Given $e \in H_2(M, \mathbf{Z})$ as above, Proposition 6.1 finds a self-dual connection A on $L(e)$ such that $a = i(A - A_e) \in L^2_{2,\delta}(T^*M)$ obeys

$$e^{-\tau\delta} d * e^{\tau\delta} a = 0.$$

It follows from Proposition 6.1 that A is unique. This connection defines a reducible self-dual connection in $\mathscr{A}_k(P)$ (since P is isomorphic to $P(e)$ and $k = e \cdot e$; see Lemmas 7.1 and 7.2). Thus, each pair

$$\{\pm e \in H_2(M, \mathbf{Z});\, e \cdot e = k \text{ and } e_2 = w_2(P \times_{\mathrm{Ad}} \mathfrak{G})\}$$

determines a reducible orbit in $\mathscr{M}_k(P)$. It follows from Proposition 5.1 (by mimicking the arguments in [11] or [12, Chapter 10]) that any such reducible orbit in $\mathscr{M}_k(P)$ comes from a pair $\{\pm e \in H_2(M, \mathbf{Z}) : e \cdot e = k,\ e_2 = w_2(P \times_{\mathrm{Ad}} \mathfrak{G})\}$.

To construct irreducible self-dual connections on $M \times SU(2)$, one adapts the construction in [28], [30]. For an end-periodic, admissible 4-manifold with positive definite intersection form on $H_2(K; \mathbf{Z})$, the construction yields nonempty $\mathscr{M}_{4l}$ for $l \in \{1, 2, \cdots\}$ (see Proposition 9.2, below). (When the intersection form on $H_2(K, \mathbf{Z})$ is indefinite, then $\mathscr{M}_{4l}$ is nonempty for all $l \geqslant l(b_2^-(K))$.) For the proof of Theorem 1.4, only the case $b_2^-(K) = 0$ and $l = 1$ is required. The general situation should be an exercise for the reader who is familiar with [30].

Definition 4.2 of [30] assigns to each $(x, \lambda) \in M \times (0,1)$ a point $T(x, \lambda) \in \mathscr{B}_4$ such that the induced map

$$T : M \times (0,1) \to \mathscr{B}_4$$

is a smooth embedding. The point $T(x, \lambda)$ has the following properties: Let $[A] = T(x, \lambda)$. For $\delta \in [0, \infty)$ and $p \in [1, \infty]$,

$$\text{(9.4)} \qquad \begin{aligned} \left(\int e^{\tau\delta} |P_- F_A|^p\right)^{1/p} &\leqslant z e^{\tau(x)\delta/p} \lambda^{2/p}, \\ \left(\int e^{\tau\delta} |F_A|^p\right)^{1/p} &\leqslant z e^{\tau(x)\delta/p} \lambda^{-2+4/p}, \end{aligned}$$

and if $y \in M$ obeys $\operatorname{dist}(y, x) \geqslant 4\sqrt{\lambda}$, then

$$(9.5) \qquad |F_A|(y) \equiv 0.$$

Also, A is irreducible.

To find self-dual connections in $\mathscr{B}_4$, one searches for such of the form $[\hat{A}(x, \lambda)]$, where

$$(9.6) \qquad \hat{A}(x, \lambda) = A(x, \lambda) + a(x, \lambda),$$

with $[A(x, \lambda)] = T(x, \lambda)$ and $a(x, \lambda) \in C^\infty(\operatorname{Ad} P \otimes T^*M)$ is small in a suitable way.

Proposition 9.2. *Let M be as in Proposition 9.1. There exists $\delta_1 > 0$ such that if $\delta \in (0, \delta_1)$, then $\mathscr{M}_4 \neq \varnothing$. In fact, $\lambda_1 > 0$ and $\zeta < \infty$ exist with the following properties: Let $\delta \in (0, \delta_1)$, $(x, \lambda) \in M \times (0, \lambda_1)$, and $[A] = T(x, \lambda)$.*

(1) *There exists $[\hat{A}(x, \lambda)] \in \mathscr{M}_4 \cap \mathscr{B}_4$ such that:*

(2) *If $y \in M$ obeys $d = \operatorname{dist}(y, x) \geqslant 8\sqrt{\lambda}$, then*

$$|F_A|(y) \leqslant \zeta\lambda^{1/2}(1 + d^{-2}).$$

(3) *The assignment of $(x, \lambda) \in M \times (0, \lambda_1)$ to $[\hat{A}(x, \lambda)] \in \mathscr{B}_4$ defines a smooth map of $M \times (0, \lambda_1)$ into $\mathscr{B}_4$ which is smoothly homotopic to the map T.*

The remainder of this section contains the proof of Proposition 9.2. The argument here is almost identical to the argument in [28] and [30] for compact M. Familiarity with [28], [30] (see also [12, Chapters 6, 7]) will be assumed.

Proof of Proposition 9.2. As in §§2 and 3 of [30], the strategy is to first find $a(x, \lambda)$ in (9.6), given appropriately chosen data $(\delta_1, \lambda_1, \zeta)$. A priori estimates from the existence proof yield assertions (2) and (3) of the lemma.

Consider first the existence question. One writes $a = *d_A\omega$ for $\omega \in C^{m+1}(\operatorname{Ad} P \otimes P_- \Lambda_2 T^*)$. If $A + *d_A\omega$ is to be self-dual, then ω must satisfy

$$(9.7) \qquad P_-d_A(P_-d_A)^*\omega + (P_-d_A)^*\omega \wedge (P_-d_A)^*\omega + P_-F_A = 0.$$

(9.7) is solved by a successive approximation scheme; one sets

$$(9.8) \qquad \omega = \sum_{j=0}^{\infty} \omega_j,$$

with

$$(9.9) \qquad P_-d_A(P_-d_A)^*\omega_0 = -P_-F_A,$$

and for $j > 0$,

$$(9.10) \qquad P_-d_A(P_-d_A)^*\omega_j = -Q_j.$$

Here,

$$(9.11)\qquad Q_j = \sum_{k=0}^{j-2} P_-\big(\nu_k \wedge \nu_{j-1} + \nu_{j-1} \wedge \nu_k\big) + P_-\big(\nu_{j-1} \wedge \nu_{j-1}\big),$$

where $\nu_j = (P_- d_A)^* \omega$.

To succeed here, the following eigenvalue estimate is crucial.

Lemma 9.3. *There exists* $\delta_1 > 0$, $\lambda_1 \geqslant 0$, *and* $\zeta > 0$ *such that if* $\delta \in [0, \delta_1)$ *and* $\lambda \in (0, \lambda_1)$, *then the following is true: Let* $A = A(x, \lambda)$ *and let* $\mathscr{h}_k^\delta$ *denote the Banach space of* $\sigma \in L^2_{1,\text{loc}}(\text{Ad}\, P \otimes P_- \wedge_2 T^*)$ *such that*

$$\int_M e^{\tau\delta}\big\{|\nabla_A \sigma|^2 + |\sigma|^2\big\} < \infty.$$

Then

$$\zeta_A \equiv \inf_{0 \neq \sigma \in \mathscr{h}_k^\delta} \left[\frac{\int_M e^{\tau\delta} |e^{-\tau\delta}(P_- d_A)^* e^{\tau\delta}\sigma|^2}{\int_M e^{\tau\delta}|\sigma|^2}\right] \geqslant \zeta.$$

Given Lemma 9.3 and the Sobolev estimates of Lemma 5.2 (and Kato's inequality), the existence proof now proceeds, virtually word for word, as a copy of the arguments in [28, §§4, 5] and [30, §§2,3]. The details are left to the reader. (Remember, the end-periodicity of M implies that M has "bounded geometry.") The end result is that one finds $\lambda_1 > 0$ such that if $(x, \lambda) \in M \times (0, \lambda_1)$, then there exists a unique $\omega \in \mathscr{h}_k^0$ such that $A + *d_A\omega$ is self-dual and

$$(9.12)\qquad \int_M \sum_{l=0}^{2} |\nabla_A^{(l)}\omega|^2 \leqslant \zeta(\lambda).$$

To put $A + *d_A\omega \in \mathscr{A}_4$, one must exploit (9.7). First of all, elliptic regularity plus (9.12) implies that $\omega \in C^{m+1}(\text{Ad}\, P \otimes P_- \wedge_2 T^*)$.

To obtain the $L^2_{\cdot,\delta}$-estimates on $*d_A\omega$ the first step is to obtain the following L^∞-estimate on ω.

Lemma 9.4. *Let M be as in Proposition* 9.1. *There exists* $\lambda_1 > 0$ *and* $\zeta < \infty$ *such that if* $\lambda \in (0, \lambda_1)$, $[A] = T(x, \lambda)$, ω *obeys* (9.7) *and* (9.12), *then*

$$\sup_{x \in M} |\omega(x)| \leqslant \zeta \lambda^{1/2}.$$

Proof of Lemma 9.4. The Weitzenbock formula for $P_- d_A (P_- d_A)^*$ (see [12, Appendix C]) implies that

$$(9.13)\qquad d*d|\omega| - r|\omega| \leqslant z\big(|*d_A\omega|^2 + |P_- F_A|\big).$$

Here, $r \geqslant 0$ is uniformly bounded due to (9.4) and the fact that M has bounded geometry. Because M has bounded geometry, the injectivity radius of M is bounded away from zero by $\rho > 0$. For $x \in M$, let $\beta_x(\cdot) \in C_0^\infty(M)$ be identically 1 if $\operatorname{dist}(x, y) < \frac{1}{4}\rho$ and zero if $\operatorname{dist}(x, y) > \frac{1}{2}\rho$. Assume that $\beta_x \geqslant 0$ and that $|d\beta_x| < 20\rho^{-1}$. Then

$$(9.14)\qquad \int_M d\operatorname{vol}(y)\cdot\beta_x(y)(\operatorname{dist}(x,y))^{-2}|{*d_A\omega}|^2(y) \leqslant \zeta\lambda.$$

Use (9.12) and Lemma A.3 of [29] to prove this. From (9.4) and Hölder's inequality:

$$(9.15)\qquad \int_M d\operatorname{vol}(y)\beta_x(y)(\operatorname{dist}(x-y))^{-2}|P_-F_A| \leqslant \zeta\lambda^{1/2}.$$

(9.12) with Lemma 5.2 and Hölder's inequality establishes that

$$(9.16)\qquad \int_M d\operatorname{vol}(y)\beta_x(y)(\operatorname{dist}(x-y))^{-2}r|\omega| \leqslant \zeta\cdot\lambda^{1/2}.$$

Now, multiply both sides of (9.13) by $\beta_x(\cdot)(\operatorname{dist}(x,\cdot))^{-2}$ and integrate over M. Use the fact that $(\operatorname{dist}(x,\cdot))^{-2}$ is (up to a constant) the Green's function for d^*d on M to order $\operatorname{dist}(x,\cdot)^{-1}$. Then integration by parts and (9.14)–(9.16) give Lemma 9.4.

Now, one can get $L^2_{\cdot,\delta}$-estimates as follows: Let

$$(9.17)\qquad q_n = \begin{cases} e^{\tau\delta} & \text{if } \tau \leqslant n, \\ e^{n\delta} & \text{if } \tau \geqslant n. \end{cases}$$

Contract both sides of (9.7) with $q_n\omega$, and integrate over M. After integration by parts, one has

$$(9.18)\qquad \int_M q_n|d_A\omega|^2 \leqslant \delta\int_M q_n|\omega||d_A\omega| + \int_M q_n|\omega||d_A\omega|^2 + \int_M q_n|\omega||P_-F_A|.$$

By Lemma 9.4, and (9.4), (9.5), there exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, then

$$(9.19)\qquad \int_M q_n|d_A\omega|^2 \leqslant \zeta\delta\int_M q_n|\omega|^2 + \zeta\cdot\lambda^2 e^{\tau(x)\delta}.$$

With Lemma 9.3, it follows that $\delta_1 > 0$ exists such that if $\delta \in [0, \delta_1)$,

$$(9.20)\qquad \int q_n|\omega|^2 \leqslant \zeta\int q_n|d_A\omega|^2.$$

(9.19) and (9.20) imply that $\lambda_1, \delta_1 > 0$ exist such that if $\delta \in [0, \delta_1]$ and $\lambda \in (0, \lambda_1)$, then

$$\int_M q_n|d_A\omega|^2 \leqslant \zeta\lambda^2 e^{\tau(x)\delta}$$

with ζ independent of A and n. Taking $n \to \infty$, above, shows that for δ and λ as above,

$$(9.21) \qquad \int_M e^{\tau\delta}|d_A\omega|^2 \leqslant \zeta\lambda^2 e^{\tau(x)\delta}.$$

The $L^2_{0,\delta}$ estimate for $\nabla_A(*d_A\omega)$ now follows by using the Weitzenboch formula with Lemmas 5.2 and 7.5 on the equation

$$0 = \int e^{\tau\delta}|P_-F_A + P_-d_A*d_A\omega + P_-(*d_A\omega \wedge *d_A\omega)|^2.$$

The details are straightforward and one finds that

$$(9.22) \qquad \int_M e^{\tau\delta}|\nabla_A(*d_A\omega)|^2 \leqslant \zeta \cdot \lambda^2 e^{\tau(x)\delta},$$

with ζ, again, independent of $(x, \lambda) \in M \times (0, \lambda_1)$. The derivation of (9.22) is left to the reader (see [28, §§4, 5]).

Assertion (2) of Proposition 9.2 is obtained by exploiting the fact that A is flat and trivial on $M_\lambda \equiv M \setminus \{y \in M : \operatorname{dist}(x, \lambda) > 2\sqrt{\lambda}\}$: Assertion (1) implies that $a = *d_A\omega$ obeys the (elliptic) system

$$P_-da + P_-a \wedge a = 0 \quad \text{and} \quad d^*a = 0$$

on M_λ. Meanwhile, (9.21) and (9.22) give uniform estimates on the $L^2_{1;\mathrm{loc}}$ norms of a on M_λ. Since M has bounded geometry, the standard bootstrap arguments give assertion (2). Here, one may have to adjust the numbers $\delta_1, \lambda_1 > 0$.

The final assertion of Proposition 9.2 is proved as in §3 of [30]. The formalism there translates word for word over here.

Thus, Proposition 9.2, at its heart, comes down to the

Proof of Lemma 9.3. The proof of Proposition 8.8 in [28] translates almost directly to the situation at hand. To use said proof, one must study the operator

$$(9.23) \quad K_{A,\delta} \equiv P_-d_A\left(e^{-\tau\delta}(P_-d_A)^* e^{\tau\delta}\right) \quad \text{on } L^2_{0,\delta}\left(\operatorname{Ad} P \otimes P_- \Lambda_2 T^*M\right).$$

Via Weitzenboch formulae and integration by parts, it is easy to show that $K_{A,\delta}$ is a closed, essentially self-adjoint, nonnegative operator on $L^2_{0,\delta}$ with dense domain $L^2_{2,\delta}$. The key fact is

Lemma 9.5. *Let M be an end-periodic, admissible 4-manifold and let $\delta \in [0, \delta_1)$ with $\delta_1 > 0$ given in Lemma 5.8. Let $P \to M$ be a principal G-bundle obeying (7.1). Let $k \in \operatorname{Im}(p_1 : E\mathscr{A}(P) \to \mathbf{Z})$ and let $A \in \mathscr{A}_k$. Then $K_{A,\delta}$ has pure point spectra with finite multiplicities in the interval $[0, \zeta(\delta))$, with $\zeta(\delta)$ as defined in Lemma 5.8.*

To prove Lemma 9.3 from Lemma 9.5, note that the only case of interest is when $\zeta_A < \zeta(\delta)$, whence Lemma 9.5 implies that ζ_A is an eigenvalue of $K_{A,\delta}$. This fact, plus Lemma 5.8, allows one to carry the proof of Proposition 8.8 in [28] to the case at hand, essentially verbatim. This is left to the reader.

Proof of Lemma 9.5. Suppose that $K_{A,\delta}$ has been shown to have pure point spectra in an interval $[0,\nu] \subset [0,\zeta(\delta))$. Let V denote the closed, linear subspace of $L^2_{0,\delta}$ which is spanned by the eigenvectors with eigenvalues in $[0,\nu]$. Let $V^\perp \subset \mathscr{k}^\delta_k$ denote the $L^2_{0,\delta}$-orthogonal complement to V. Define a bounded quadratic functional on $V^\perp$ by sending σ to

$$J(\sigma) = \int_M e^{\tau\delta}|e^{-\tau\delta}(P_-d_A)^* e^{\tau\delta}\sigma|^2. \tag{9.24}$$

Let

$$\nu_1 \equiv \inf_{0 \neq \sigma \in V^\perp} \left[\frac{J(\sigma)}{\int_M e^{\tau\delta}|\sigma|^2}\right]. \tag{9.25}$$

Let $\{\sigma_i\} \subset V^\perp$ be a sequence which obeys

$$\int_M e^{\tau\delta}|\sigma_i|^2 = 1 \quad \text{and} \quad J(\sigma_i) \to \nu_1. \tag{9.26}$$

This last condition implies via a Weitzenboch formula that

$$\left[\int_M e^{\tau\delta}\left(|\nabla_A\sigma_i|^2 + |\sigma_i|^2\right)\right]^{1/2}$$

is bounded uniformly in i. Then using (9.26) and the fact that ν_1 is an infimum, one obtains the strong convergence of a subsequence of $\{\sigma_i\}$ (denoted $\{\sigma_i\}$) in $L^2_{1;\mathrm{loc}}(\mathrm{Ad}\,P \otimes P_- \Lambda_2 T^*)$. This subsequence also converges weakly in the Banach space $V^\perp$ to some $\sigma \in V^\perp$. ($L^2_{0,\delta}$-orthogonally is preserved by weak limits.)

Given $\varepsilon > 0$, choose $n = n(\varepsilon)$ such that

$$\int_{\tau \geqslant n} e^{\tau\delta}|\sigma|^2 < \varepsilon^2. \tag{9.27}$$

Then $i(\varepsilon) < \infty$ exists such that for all $j > i(\varepsilon)$,

$$\int_{W_n} e^{\tau\delta}|\sigma_j|^2 < \varepsilon^2. \tag{9.28}$$

Let $\beta = \beta(n) \in C^\infty(\tau^{-1}(n,\infty);[0,1])$ be such that $\beta \equiv 1$ on $\tau^{-1}((n+1,\infty))$ with $\beta \equiv 0$ on $i_-(N) \subset W_n$. Make $\|d\beta\|_\infty$ independent of n. Write

$$\sigma_i = \beta\sigma_i + (1-\beta)\sigma_i,$$

and observe, using (9.27) and (9.28), that

$$(1 + z\varepsilon)J(\sigma_i) \geqslant J(\beta\sigma_i) + J((1-\beta)\sigma_i) - z\varepsilon, \tag{9.29}$$

with z independent of $\{\sigma_i\}$ and of A. (Just use Hölder's inequality.)

Write $A = A_0 + a$ with $A_0 \in E\mathscr{A}(P) \cap \mathscr{A}_k$ and with $\|a\|_{A_0} < \infty$. Then, using Kato's inequality plus Lemma 5.2 one finds $n = n(A, \varepsilon) < \infty$ such that

$$J(\beta\sigma_i) \geqslant \int_{\tilde{Y}} e^{\tau\delta}|e^{-\tau\delta}(P_-d)^* e^{\tau\delta}\beta\sigma_i|^2 - z\varepsilon \quad \text{for all } i > i(\varepsilon). \tag{9.30}$$

Then, (9.29), (9.30), and Lemma 5.8 imply that

$$J(\sigma_i) \geqslant \nu_1 \int_M e^{\tau\delta}|\beta\sigma_i|^2 + \zeta(\delta)\int_M e^{\tau\delta}|(1-\beta)\sigma_i|^2 - z\varepsilon. \tag{9.31}$$

Since $\varepsilon > 0$ was arbitrary, (9.26) and (9.31) imply the following: If $\nu_1 < \zeta(\delta)$, then given $\varepsilon > 0$, there exists $n(\varepsilon) < \infty$ such that for all $i > i(\varepsilon)$

$$\int_{\tau \geqslant n} e^{\tau\delta}|\sigma_i|^2 < \varepsilon, \tag{9.32}$$

which insures that $\{\sigma_i\}$ converges *strongly* in $L^2_{0,\delta}(\mathrm{Ad}\, P \otimes P_- \wedge_2 T^*)$. Standard arguments now show that σ is an eigenvector of $K_{A,\delta}$ with eigenvalue ν_1. Since $\sigma \in V^\perp$, one has $\nu_1 > \nu$. This proves that $K_{A,\delta}$ has discrete spectrum in $(0, \zeta(\delta))$.

The same argument proves that the eigenvalues in $[0, \zeta(\delta))$ have finite multiplicity. Indeed, the argument shows that any sequence of normalized eigenvectors with a fixed eigenvalue in $[0, \zeta(\delta)]$ has a convergent subsequence. This can happen only if the eigenspaces are finite dimensional.

10. Moduli spaces: Boundary

In the case where M is a compact, 4-manifold with definite intersection form, the moduli space used in [8] is diffeomorphic, outside a compact set, to $M \times (0,1)$. The compact analogues of (9.6) and Proposition 9.3 provide a map which induces the diffeomorphism. This is the "Collar Theorem" [12, Theorem 9.1]. Fintushel-Stern's argument [11] on compact M requires a compact moduli space. In both cases, the "boundary" of the moduli-space is the crucial issue.

For the end-periodic analogue, one has

Proposition 10.1. *Let M be an end-periodic, admissible 4-manifold with positive definite intersection form on $H_2(K; \mathbf{Z})$. Assume that $\pi_1(W)$ has no nontrivial representations in $SU(2)$. There exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1)$, the following is true: Let $\phi \in \mathscr{C}$.*

(1) *Let $P \to M$ be a principal $SO(3)$ bundle obeying* (7.1). *Let $k = 2, 3$. Then $\mathscr{M}_k(\phi)$ is compact.*

(2) *Let $P = M \times SU(2)$. Then there is an open set $\mathscr{K} \subset \mathscr{M}_4(\phi)$ with the property that for some $\lambda_1 > 0$, $\mathscr{K}$ is diffeomorphic to $M \times (0, \lambda_1)$ and isotopic in $\mathscr{B}_4$ to the image of $T: M \times (0, \lambda_1) \to \mathscr{B}_4$ of Proposition* 9.3. *If $\{[A_i]\} \subset \overline{\mathscr{M}_4(\phi) \setminus \mathscr{K}}$ has no convergent subsequence, then for all $n < \infty$,*

$$\overline{\lim_{i \to \infty}} \left[\sup\left\{ |F_{A_i}(x)| : x \in \tau^{-1}([0, n]) \right\} \right] = 0.$$

The rest of this section contains the proof. To begin, normalize the inner product on Ad P so that

$$|\sigma|^2 = -\frac{1}{8\pi^2} \operatorname{tr}_{\otimes}(\sigma \cdot \sigma). \tag{10.1}$$

This choice has the property that for $[A] \in \mathscr{M}_k(\phi)$, one has

$$\int_M |F_A|^2 = |k|, \tag{10.2}$$

if one uses the asymptotically periodic metric $\phi^* g_0$ to measure norms on TM and volumes on M. Henceforth, this will always be done, and with no explicit notation.

In general, for $k > 0$, consider $\{[A_j]\} \in \mathscr{M}_k(\phi)$, a sequence of orbits of self-dual connections. Suppose that

$$\lim_{j \to \infty} \left[\sup_{x \in M} |F_{A_j}|(x) \right] = \infty. \tag{10.3}$$

Theorem 8.31 of [12] is valid even on noncompact M as it is an essentially local theorem on M. From Theorem 8.3 of [12], (10.3) is true *only* if

$$k = \int_M |F_{A_j}|^2 \geqslant 4. \tag{10.4}$$

Suppose that $\{[A_j]\} \in \mathscr{M}_k(\phi)$ is a sequence such that (10.3) is *not* true. Either

$$\lim_{n \to \infty} \left\{ \lim_{j \to \infty} \int_{\tau \geqslant n} |F_{A_j}|^2 \right\} = 0 \tag{10.5}$$

or not.

Lemma 10.2. *Let M be an end-periodic, admissible 4-manifold with no nontrivial representations of $\pi_1(W)$ in $SU(2)$. There exists $\delta_1 > 0$ such that for $\delta \in (0, \delta_1)$, the following is true: Let $\phi \in \mathscr{C}$. Let $P \to M$ be a principal $SO(3)$ or $SU(2)$ bundle which obeys* (7.1). *Suppose that $\{[A_j]\} \in \mathscr{M}_k(\phi)$ (for $k \geqslant 0$)$\}$. There exists* (1) *a self-dual connection A on P with $\int_M |F_A|^2 \leqslant k$;* (2) *a finite set of points $\{x_\alpha\} \in M$;* (3) *a sequence $\{h_j\} \in C^\infty(\operatorname{Aut} P|_{M \setminus \{x_\alpha\}})$; and* (4) *a subsequence $\{A_j\}$ (now relabeled) such that $\{h_j A_j\}$ converges on compact* *domains in* $M \setminus \{x_\alpha\}$ *to* A *in the* C^m*-topology* (*if* ϕ *is* C^m). *Suppose that* (10.5) *holds. Now, the sequence* $\{h_j\}$ *can be chosen so that* $[A] \in \mathscr{M}_l(\phi)$ *for some* $l \in [0, k]$. *In this case,* $n < \infty$ *exists such that*

$$\lim_{j\to\infty} \int_{\tau^{-1}((n,\infty))} e^{\tau\delta}\Big\{\big|\nabla_A\nabla_A(A - H_jA_j)\big|^2 + \big|\nabla_A(A - h_jA_j)\big|^2 + \big|(A - h_jA_j)\big|^2\Big\} = 0.$$

When (10.5) *holds,* $l = k$ *if and only if* $\{x_\alpha\} = \varnothing$, *whence* $[h_jA_j]$ *converges to* $[A]$ *in* $\mathscr{M}_k(\phi)$.

Lemma 10.2 is the extension of the combination of Theorem 8.8 and Theorem 8.31 of [12] to the end-periodic case. (See also [29, Proposition 4.4].)

Lemma 10.2 describes the situation if (10.5) is satisfied. When it is not, one has

Lemma 10.3. *Make the same assumptions as in Lemma* 10.2 *concerning* M, P, *and* δ. *If* $\{[A_j]\} \in \mathscr{M}_k(\phi)$ *does not obey* (10.5), *then*

$$\lim_{n\to\infty}\left\{\lim_{j\to\infty}\int_{\tau^{-1}([n,\ \infty))} |F_{A_j}|^2\right\} \geqslant 4. \tag{10.6}$$

Lemmas 10.2 and 10.3 will be proved shortly; assume them for the moment.

Proof of Assertion (1) *of Proposition* 10.1. Since $k = 2, 3$, this is now immediate from (10.2–4) and Lemmas 10.2, 10.3.

Proof of Assertion (2) *of Proposition* 10.1. Let λ_1 be as in Proposition 9.2. For some $\lambda \in (0, \lambda_1]$, one can define, as in Chapter 8 of [12], the set of self-dual orbits with scale size $< \lambda$. This is $\mathscr{M}_{4,\lambda}$. With the techniques in §9, it is straightforward to reprove the Collar Theorem of [8] (Theorem 9.1 of [12]) in the present circumstance. The reader is encouraged to trace the argument through. The result is some $\lambda \in (0, \lambda_1)$ such that $\mathscr{M}_{4,\lambda}$ is diffeomorphic to $M \times (0, \lambda)$ and isotopic in $\mathscr{B}_4$ to the image of the map T. Set $\mathscr{K} \equiv \mathscr{M}_{4,\lambda}$. If $\{[A_i]\} \subset \mathscr{M}_4 \setminus \mathscr{K}$ has no convergent subsequence, then neither of (10.3) or (10.5) can hold. Then, Lemma 10.3 and (10.2) imply that for any $n \geqslant 0$,

$$\lim_{j\to\infty}\int_{\tau^{-1}([0,n])} |F_{A_j}|^2 = 0. \tag{10.7}$$

By Theorem 8.8 of [12], the sequence $\{A_j\}$ is gauge equivalent to one which converges in C^m of compact domains to a flat connection on M. This implies the final part of assertion (2).

Proof of Lemma 10.2. If (10.5) is satisfied, then a subsequence $\{[A_j]\}$ exists with the following property: Given $\varepsilon > 0$, there exists $n \equiv n(\varepsilon) < \infty$ such that for all j, Theorem 8.8 of [12] plus (10.6) implies that the following data exists: An integer $n < \infty$, a subsequence $\{[A_j]\}$, and a sequence $\{q_j\} \in C^\infty(\mathrm{Aut}\, P|_{\tau^{-1}((n,\infty))})$ such that $\{q_j A_j\}$ converges strongly in the C^m-topology on compact domains in $\tau^{-1}((n,\infty))$. Further, the limiting connection A' is self-dual. Theorems 8.8 and 8.31 applied to $\tau^{-1}([0, n+1]) \subset M$ imply that one has the following additional data: A smaller subsequence $\{[A_j]\}$, a finite set of points $\{x_\alpha\} \in \tau^{-1}([0, n+1))$, and a sequence $\{u_j\} \subset C^\infty(\mathrm{Aut}\, P|_{(\tau^{-1}([0,\ n+1)))\setminus\{x_\alpha\}})$ such that $\{u_j A_j\}$ converges strongly in the C^m-topology on compact domains in $\tau^{-1}([0, n+1))\setminus\{x_\alpha\}$. The limit A'' is a self-dual connection on $\tau^{-1}([0, n+1))$. (This uses the removable singularity theorem [31].) There exists $h \in C^m(\mathrm{Aut}\, P|_{\tau^{-1}((n,n+1))})$ such that

$$h \cdot A'' = A' \quad \text{on } \tau^{-1}\big((n, n+1)\big).$$

Since the Stieffel-Whitney classes of P are preserved under the limits [26], the data $(A'', A'; h)$ defines a self-dual connection A on P, and from the data $\{u_j, q_j\}$, one can construct $\{p_j \in C^\infty(\mathrm{Aut}\, P|_{M\setminus\{x_\alpha\}})$ such that $\{p_j A_j\}$ converges to A in C^m of compact domains in $M\setminus\{x_\alpha\}$. (Argue by Theorem 8.8 of [12].) The connection A is self-dual, and weak-lower semicontinuity implies that

$$\int_M |F_A|^2 \leqslant k.$$

To prove the remaining assertions of Lemma 10.2, one must use (10.5) to find a gauge for each A_j which gives uniform decay in the weighted spaces. This is a multistep bootstrapping process. The first step is the next lemma. This lemma is also crucial for the proof of Lemma 10.3.

Lemma 10.4. *Let U be an oriented open noncompact, C^m-Riemannian 4-manifold ($m \gg 2$). Let $Q \subset U$ be a smooth submanifold with compact closure, $\bar{Q} \subset U$. Let $P \to U$ be a principal G-bundle. There exists $\varepsilon < 0$ and $\zeta < \infty$ which depend on U, Q, P, and a C^m neighborhood of the Riemannian metric on U with the following significance: Let A be a self-dual connection on P with $\int_U |F_A|^2 < \varepsilon$. Then $h \in C^{m+1}(\bar{Q}; G)$ exists such that*

$$\sup_Q \left\{ \sum_{l=0}^{m} \left|\nabla_\Gamma^{(l)}(h^*A - \Gamma)\right|^2 \right\} \leqslant \zeta \int_U |F_A|^2,$$

where Γ is a flat connection on $P|_Q$.

Proof of Lemma 10.4. This is essentially Theorem 8.8 of [12], but one must keep track of the norms involved. To begin fix a locally finite, open cover of U by geodesic balls $\{B_\alpha\}$ such that the balls of $1/2$ the radius, $\{\tilde{B}_\alpha\}$, cover U. By [32], there exists $\{h_\alpha \in L_2^2(\mathrm{Iso}(B_\alpha \times G, P|_{B_\alpha}))\}$ such that

$$a_\alpha = h_\alpha^* A - \Gamma_0 \in L_1^2\big(T^*B_\alpha \times \mathfrak{G}\big)$$

obeys $d^*_{\Gamma_0} a_\alpha = 0$ and $i^*(*a_\alpha) = 0$. Here, $i: \partial B_\alpha \to B_\alpha$ is the inclusion and Γ_0 is the product connection on $B_\alpha \times G$. Further, if $B_\alpha \cap \overline{Q} \neq \varnothing$, then

$$\int_{B_\alpha} \left\{ |\nabla_{\Gamma_0} a_\alpha|^2 + |a_\alpha|^2 \right\} \leqslant \zeta \int_{B_\alpha} |F_A|^2.$$

Since $\Gamma_0 + a_\alpha$ is self-dual, a_α satisfies uniform C^m-estimates in the ball of radius $3/4$ (radius (B_α)). These C^m-estimates are bounded by

$$\zeta_n \int_{B_\alpha} |F_A|^2. \tag{10.8}$$

In $\tilde{B}_\alpha \cap \tilde{B}_\beta$,

$$a_\alpha = h_{\alpha\beta} a_\beta h_{\alpha\beta}^{-1} + h_{\alpha\beta} d h_{\alpha\beta}^{-1} \tag{10.9}$$

with $h_{\alpha\beta} \in C^{m+1}(\tilde{B}_\alpha \cap \tilde{B}_\beta; G)$. Due to (10.9), $dh_{\alpha\beta}$ obeys uniform C^m-estimates in $(\tilde{B}_\alpha \cap \tilde{B}_\beta) \cap \overline{Q}$; these with bound, (10.8). The data $\{\tilde{B}_\alpha, h_{\alpha\beta}, a_\alpha\}$ defines a pair: principal G-bundle, $P' \to Q$; connection on P'. A priori, P' is isomorphic to P (see [12, Theorem 8.8])

Arguing as in the proof of Proposition 3.2 and Corollary 3.3 of [32], one constructs $\rho_\alpha \in C^{m+1}(\tilde{B}_\alpha; G)$ for those B_α which intersect such that (1) ρ_α obeys C^{m+1}-estimates in $\tilde{B}_\alpha$ with bound by (10.8), (2) in $(\tilde{B}_\alpha \cap \tilde{B}_\beta) \cap \overline{Q}$, $\rho_\alpha h_{\alpha\beta} \rho_\beta^{-1} \equiv z_{\alpha\beta}$ is constant. If $B_\alpha \cap \overline{Q} = \varnothing$, set $\rho_\alpha \equiv 1$. The data $\{\tilde{B}_\alpha \cap \overline{Q}, z_{\alpha\beta}\}$ defines a flat connection on a bundle $P''|_Q$ isomorphic to $P'|_Q$. Call this connection $\overline{\Gamma}$. The data $\{\tilde{B} \cap \overline{Q}, z_{\alpha\beta}, \rho_\alpha a_\alpha \rho_\alpha^{-1} + \rho_\alpha d\rho_\alpha^{-1}\}$ defines a connection $\overline{A}$ on P'' which obeys

$$\sup_{\overline{Q}} \left\{ \sum_{l=0}^{m} |\nabla_{\overline{\Gamma}}^{(l)}(\overline{A} - \overline{\Gamma})|^2 \right\} \leqslant \zeta_Q \int_0 |F_A|^2;$$

this follows as all the relevant ρ_α and a_α estimates are bounded by (10.8). The pair $(P'', \overline{A})$ is isomorphic (as bundle, connection) to $(P|_Q, A)$. Pulling back via such an isomorphism gives Lemma 10.4.

To apply Lemma 10.4, take

$$U = W_{-1} \cup_N W_0 \cup_N W_1, \quad \text{and} \quad Q = W_0. \tag{10.10}$$

Use the periodic metric, g_0, for TU. Let $\varepsilon > 0$ be as given in Lemma 10.4 for $P = U \times G$, G a compact Lie group.

Now choose $k < \infty$ according to the following criteria: If $n \geqslant k$, then the asymptotically periodic metric $\phi^* g_0$ on $U_n \equiv W_{n-1} \cup_N W_n \cup_N W_{n+1}$ is C^m-close enough to g_0 for Lemma 10.4 to apply for $U = U_n$, $Q = W_n$, $P = U_n \times G$, and the given ε, above.

Lemma 10.5. *Let M be an end-periodic 4-manifold such that $\pi_1(W)$ has only the trivial representation into G. Let ε, k be as defined in the previous paragraphs. There exists $\rho < \infty$ and $\varepsilon_1 > 0$, with the following significance: Let A be a self-dual connection on* End $M \times G$. *Suppose that $n \geqslant k$ exists such that*

$$\int_{\tau \geqslant n} |F_A|^2 < \varepsilon_1.$$

Then $h \in C^{m+1}(\tau^{-1}([n+1, \infty)); G)$ exists such that for $j \geqslant n+1$,

$$\sup_{W_j} \left\{ \sum_{l=0}^{m} \left|\nabla_\Gamma^{(m)}(h^*A - \Gamma)\right|^2 \right\} \leqslant \rho \cdot \int_{U_j \cup U_{j+1}} \left|F_A\right|^2, \tag{10.11}$$

where Γ is the product connection on $\tau^{-1}([n+1, \infty)) \times G$.

Proof of Lemma 10.5. By Lemma 10.4, there exists $h_j \in C^{m+1}(W_j; G)$ for each $j \geqslant n+1$, such that (10.11) is obeyed with n_j replacing h. The assumption on $\pi_1(W)$ insures that the flat connections on $W \times G$ are trivial. On W_j, set $a_j \equiv h_j^*A - \Gamma_j$. On $W_j \cap W_{j+1}$, define $h_{j,j+1} \in C^m(W_j \cap W_{j+1}; G)$ by the cocycle condition

$$a_j = h_{j,j+1} a_{j+1} h_{j,j+1}^{-1} + h_{j,j+1} d h_{j,j+1}^{-1}. \tag{10.12}$$

(10.11) and (10.12) provide bounded C^{m+1}-estimates for each $h_{j,j+1}$ with bound

$$\zeta \int_{U_j \cup U_{j+1}} \left|F_{A_j}\right|^2. \tag{10.13}$$

For $\varepsilon_1 > 0$, and small, the argument which proved Lemma 10.4 can be repeated with the data $\{h_{j,j+1}\}$ to produce $\rho_j \in C^{m+1}(W_j; G)$ obeying C^{m+1}-estimates bounded by (10.13) such that $\rho_j h_{j,j+1} \rho_{j+1}^{-1} = z_{j,j+1}$ in $W_j \cup W_{j+1}$. Again $z_{j+1} =$ constant. Now, change ρ_j to

$$\tilde{\rho}_j = \rho_j z_{j,j+1}^{-1} \cdots z_{n+1,n}^{-1}.$$

Then $d\tilde{\rho}_j$ obeys C^m-estimates bounded by (10.13), and $\tilde{\rho}_j h_{j,j+1} \tilde{\rho}_{j+1}^{-1} = 1$. On W_j, set $h = \tilde{\rho}_j h_j$. Check that h does as required.

An important remark to make here is that the assumption on $\pi_1(W)$ is critical in the preceding argument.

To obtain the rest of Lemma 10.2, it is necessary to fine tune the gauge transformation h of Lemma 10.5 in order to obtain uniform weighted estimates. Let A, h be as in that lemma and set $a = h^*A - \Gamma$. Since $\tau: W_n \to [n, n+1]$, (10.11) proves that if $j \geqslant n+1$, then

$$\int_{\tau \geqslant j+1} e^{\tau\delta} \left\{ \sum_{l=0}^{m} \left|\nabla_\Gamma^{(l)} a\right|^2 \right\} \leqslant \zeta \int_{\tau \geqslant j} e^{\tau\delta} \left|F_{A_j}\right|^2, \tag{10.14}$$

whenever the right-hand side, above, is finite.

Lemma 10.6. *Make the same assumptions on* M *as in Lemma* 10.5. *There exists* $k < \infty$ *and* $\delta_1 > 0$, *and for* $\delta \in (0, \delta_1)$, *there exists* $\varepsilon(\delta) > 0$ *and* $z(\delta) < \infty$ *with the following significance: Let* A *be a self-dual connection on* $\operatorname{End} M \times G$ *such that for some* $n > k$,

$$\int_{\tau \geqslant n} |F_A|^2 < \varepsilon(\delta) \quad \text{and} \quad \int_{\tau \geqslant n} e^{\tau\delta} |F_A|^2 < \infty.$$

Then $s \in C^{m+1}(\tau^{-1}([n+1, \infty)); G)$ *exists such that* $a = s^*A - \Gamma \in L^2_{1,\delta}$ *and it obeys*

$$\int_{\tau \geqslant n+2} |a|^2 + \sup_{\tau \geqslant n+2} |a|^2 \leqslant z \int_{\tau \geqslant n} |F_A|^2, \tag{10.15}$$

and on $\tau([n+2, \infty))$,

$$e^{-\tau\delta} d^*_\Gamma e^{\tau\delta} a = 0. \tag{10.16}$$

Proof of Lemma 10.6. The proof is simplified under the assumption that $b_1(K) = 0$. Since the question here is on $\operatorname{End} M$, one can always arrange by surgery in $K \setminus N$ that $H_1(K; \mathbf{R}) = 0$. This will maintain the admissibility of the manifold.

First, let ε, k be as in Lemma 10.5, and make sure that $\varepsilon(\delta) < \epsilon$. Let h be as specified in Lemma 10.5. Look for s of the form qh. Let $b = h^*A - \Gamma$. Then b obeys (10.15) and a will be given as

$$a = qbq^{-1} + qdq^{-1} \quad \text{on } \tau^{-1}([n+2, \infty)).$$

(10.16) is now an equation for q. To prove that a solution to (10.15) and (10.16) exists, the continuity method will be used. Thus, consider a family $\{q_t : t \in [0,1]\} \subset C^{m+1}(M; G)$ such that

$$b_t \equiv t\beta q_t b q_t^{-1} + q_t d_\Gamma q_t^{-1} \tag{10.17}$$

is in $L^2_{2;\delta}(T^*M \otimes \mathfrak{G})$ and solves

$$\text{(a)} \quad e^{-\tau\delta} d^*_\Gamma e^{\tau\delta} b_t = 0,$$
$$\text{(b)} \quad \int_M |b_t|^2 + \sup_M |b_t|^2 \leqslant z \int_{\tau \geqslant n} |F_{A_t}|^2, \tag{10.18}$$

where $\beta \in C^\infty(M)$ is 0 on $\tau^{-1}([0, n+1])$, 1 on $\tau^{-1}((n+2, \infty))$, and $|d\beta| < 20$. The goal is to find the conditions on δ, $\varepsilon_2(\delta)$, and $z = z(\delta)$ under which (10.18) is solvable for all $t \in [0,1]$. For this purpose, set

$$\Lambda = \{t \in [0,1] : (10.17) \text{ and } (10.18) \text{ are solvable for } q_t\}.$$

At $t = 0$, $q_0 \equiv 1$ solves (10.17) and (10.18), so $\Lambda \neq \varnothing$. Elliptic regularity readily establishes that Λ is closed. If Λ is open, then $\Lambda = [0,1]$. One may assume that the right-hand side of (10.18) is nonzero. Otherwise, A is flat on

$\tau^{-1}([n,\infty))$. Then A is the trivial connection on each W_j if $j \geqslant n$ (due to the assumption on $\pi_1(W)$). Then Van-Kampen's Theorem [18] implies that A is trivial on $\tau^{-1}([n,\infty))$. Or, argue as in the proof of Lemma 10.5.

(10.14) insures that $\beta b \in L^2_{2,\delta}(T^*M \otimes \mathfrak{G})$. So, by the implicit function theorem with Lemma 5.2, one can find some $t_0 > 0$ such that $[0, t_0] \in \Lambda$. Let $0 < t \in \Lambda$. Then the implicit function theorem provides an open interval $(t-\nu, t+\nu)$ such that for all $\lambda \in (t-\nu, t+\nu)$, (10.17) and (10.18a) are solvable for q_λ, but with b_λ obeying

$$\int_M |b_\lambda|^2 + \sup_M |b_\lambda|^2 < 2z \int_{\tau^{-1}((n,\infty))} |F_A|^2. \tag{10.19}$$

For a better estimate for b_λ, one must use the self-duality equation which implies that

$$P_- d_\Gamma b_\lambda = P_-\big(-b_\lambda \wedge b_\lambda + q_\lambda(td\beta \wedge b + t\beta(t\beta - 1)b \wedge b)q_\lambda^{-1}\big). \tag{10.20}$$

Together, (10.18a) and (10.20) will provide the required estimate. To obtain the estimate, note that one can solve for $\omega \in L^2_{3,\delta}(P_- \Lambda_2 T^*M \otimes \mathfrak{G}) \cap C^{m+1}$ which is L^2_0-orthogonal to $\ker((P_- d_\Gamma)^* \cap L^2_0(P_- T^*M \otimes \mathfrak{G})$ and satisfies

$$\begin{aligned} &P_- d_\Gamma e^{-\tau\delta}(P_- d_\Gamma)^* e^{\tau\delta}\omega \\ &\qquad = P_-\big(-b_\lambda \wedge b_\lambda + q_\lambda(td\beta \wedge b + t\beta(t\beta - 1)b \wedge b)q_\lambda^{-1}\big). \end{aligned} \tag{10.21}$$

This is due to Lemma 9.5. (Here, one must choose δ small as determined in said lemma.)

The right-hand side of (10.21) is quadratic in b_λ. Thus, ω will obey λ-independent estimates. First, contract both sides of (10.21) with ω and integrate over M. Do not use a weight. Integrate by parts and use Lemma 9.5 to obtain (for δ sufficiently small)

$$\int_M \left|e^{-\tau\delta}(P_- d_\Gamma)^* e^{\tau\delta}\omega\right|^2 \leqslant \zeta\left[(4z^2+1)\left(\int_{\tau \geqslant n} |F_A|^2\right)^2 + \int_{\tau \geqslant n} |F_A|^2\right], \tag{10.22}$$

where (10.11), (10.14), and (10.19) have been used to estimate b_λ and b.

Now, bootstrapping in a straightforward fashion (as in the proof of Lemma 9.4) gives, with (10.11), (10.14), (10.19), (10.21), and (10.22), the estimate

$$\begin{aligned} &\int_M \left|e^{-\tau\delta}(P_- d_\Gamma)^* e^{\tau\delta}\omega\right|^2 + \sup_M \left|e^{-\tau\delta}(P_- d_\Gamma)^* e^{\tau\delta}\omega\right|^2 \\ &\qquad \leqslant \zeta\left\{(4z^2+1)\left(\int_{\tau \geqslant n} |F_A|^2\right)^2 + \int_{\tau \geqslant n} |F_A|^2\right\}. \end{aligned} \tag{10.23}$$

(10.19a), (10.20), and (10.21) imply that

$$(10.24)\qquad b_\lambda = e^{-\tau\delta}(P_-d_\Gamma)^* e^{\tau\delta}\omega,$$

which is a consequence of Proposition 6.1 since, $b_1(K) = 0$ by assumption.

Choose $z \equiv 2\zeta$, with ζ as in (10.23). Then, choose $\varepsilon_1 < \frac{1}{2}(4z^2+1)^{-1}$. With these choices, (10.23) and (10.24) verify that Λ is open in $[0,1]$. This proves Lemma 10.6.

Lemma 10.7. *Under the assumptions of Lemma* 10.6, *there exists* $k < \infty$ *and* $\delta_1 > 0$, *and for* $\delta \in (0, \delta_1)$, *there exist* $\varepsilon(\delta) > 0$ *and* $z(\delta) < \infty$ *with the following significance: Let A be a self-dual connection on* End $M \times G$ *such that for some* $n > k$,

$$\int_{\tau \geqslant n} |F_A|^2 < \varepsilon(\delta) \quad \textit{and} \quad \int_{\tau\geqslant n} e^{\tau\delta}|F_A|^2 < \infty.$$

Then for $Q \geqslant 4$,

$$\int_{\tau \geqslant Q+n} e^{\tau\delta}|F_A|^2 \leqslant \frac{z(\delta)}{Q^2} e^{n\delta}\int_{\tau\geqslant n}|F_A|^2.$$

Proof of Lemma 10.7. Require that k, δ_1, and $\varepsilon(\delta)$ be such that Lemma 10.6 holds. Let $a = s^*A - \Gamma$, and let $v = \beta a$, with β now obeying $\beta \equiv 1$ if $\tau \geqslant n+3$, $\beta \equiv 0$ if $\tau \leqslant n+2$, and $|d\beta| \leqslant 20$. This v obeys

$$(10.25)\quad e^{-\tau\delta}d_\Gamma^* e^{\tau\delta}v = \delta(d\beta, a), \qquad P_-d_\Gamma v = P_-(d\beta\wedge a - a \wedge v).$$

Due to Proposition 6.1 and Lemma 9.5 there exists $\zeta(\delta) > 0$ such that

$$(10.26)\qquad \int_M e^{\tau\delta}\left(|e^{-\tau\delta}d_\Gamma^* e^{\tau\delta}v| + |P_-d_\Gamma v|^2\right) \geqslant \zeta(\delta)\int_M e^{\tau\delta}|v|^2.$$

Together, (10.15), (10.25), and (10.26) imply that

$$(10.27)\qquad \left(\zeta(\delta) - \int_{\tau\geqslant n}|F_A|^2\right)\int_M e^{\tau\delta}v^2 \leqslant \zeta e^{n\delta}\int_{\tau\geqslant n}|F_A|^2.$$

Choose $e(\delta) \leqslant \frac{1}{2}\zeta(\delta)$ so that (10.27) yields the uniform estimate

$$(10.28)\qquad \int_{\tau\geqslant n+3} e^{\tau\delta}|a|^2 \leqslant \zeta_1 e^{n\delta}\int_{\tau\geqslant n}|F_A|^2.$$

Now, let $\beta_Q \in C^\infty(M)$ obey $\beta_Q \equiv 1$ if $\tau \geqslant n+Q$, $\beta_Q \equiv 0$ if $\tau \leqslant n + \frac{1}{2}Q$, and $|d\beta_Q| \leqslant 40Q^{-1}$. Here, take $Q \geqslant 4$. Then $v_Q = \beta_Q a$ obeys (10.25) with (v_Q, β_Q) replacing (v, β). In place of (10.28), one obtains

$$(10.29)\qquad \int_{\tau\geqslant n+Q} e^{\tau\delta}|a|^2 \leqslant \frac{\zeta_1}{Q^2}e^{n\delta}\int_{\tau\geqslant n}|F_A|^2.$$

From (10.25) and (10.29), one readily obtains by bootstrapping

$$(10.30)\qquad \int_{\tau \geqslant n+Q} e^{\tau\delta}|\nabla_\Gamma a|^2 \leqslant \frac{\zeta^2}{Q^2} e^{n\delta} \int_{\tau \geqslant n} |F_A|^2.$$

The lemma is a direct consequence of (10.29), (10.30), and Lemma 5.2.

Proof of Lemma 10.2, *completion.* (10.5) plus Lemmas 10.5 and 10.7 provides $p < \infty$ and for each A_i, some $q_i \in C^{m+1}(\tau^{-1}([p, \infty)); G)$ (with $G = SU(2)$ or $SO(3)$) such that $a_i = q_i^* A_i - \Gamma$ obeys for $Q \geqslant 1$

$$(10.31)\qquad \int_{\tau \geqslant p+Q} e^{\tau\delta}\left(\sum_{l=0}^{m} |\nabla_\Gamma^{(l)} a_i|^2\right) \leqslant \frac{z(\delta)e^{p\delta}}{Q^2} \int_{\tau \geqslant p} |F_{A_i}|^2,$$

where $m > 2$ is assumed. It follows from (10.31) (cf. [29]) that a subsequence of $\{A_i\}$ (now relabeled) has the property that $\{a_i\}$ converges on $\tau^{-1}([p+1, \infty))$ in the norm

$$\int_{\tau \geqslant p+1} e^{\tau\delta}\left(\sum_{l=0}^{m} |\nabla_\Gamma^{(l)} \cdot|^2\right)^{1/2}.$$

Let a denote the limit. Then $a = h^*A - \Gamma$ for $h \in C^{m+1}(\tau^{-1}([p+1, \infty)); G)$. Set $s_i = h^{-1}q_i$. The first part of Lemma 10.2 provides

$$\{u_i\} \subset C^{m+1}(\tau^{-1}([0, p+2]); G)$$

such that $\{u_i^* A_i - A\}$ converges outside of the finite set of points $\{x_\alpha\} \subset \tau^{-1}([0, p])$. By altering each s_i by a constant group element, one can arrange that $\{s_i u_i^{-1}\}$ converges in $C^{m+1}(\tau^{-1}([p+1, p+2]))$ to 1. Thus, for i sufficiently large, one can deform s_i over $\tau^{-1}([p+1, p+2])$ to equal u_i there. Call the resulting gauge transformation h_i. The set $\{h_i\}$ has the required properties.

Proof of Lemma 10.3. According to Lemma 10.2, there exists a self-dual connection A on M with

$$(10.32)\qquad \int_M |F_A|^2 < \infty,$$

and with the following additional properties: A subsequence of $\{A_j\}$ (now relabeled) plus a sequence of gauge transformations, $\{h_j\}$, exist such that $\{h_j^* A_j\}$ converges to A in $C^{m+1}(M \setminus$ finite set of points). (This is convergence on compact subsets of $M \setminus$ finite set.)

Due to (10.32), given $\varepsilon > 0$, there exists $n < \infty$ such that

$$(10.33)\qquad \int_{\tau \geqslant n} |F_A|^2 < \varepsilon.$$

Choose n so that the finite set above does not intersect $\tau^{-1}([n, n+3])$.

By assumption, $P|_{\operatorname{End} M} \simeq \operatorname{End} M \times G$ with $G = SU(2)$ or $SO(3)$. If $G = SO(3)$, lift all connections to the double cover, $\tilde{P} = \operatorname{End} M \times SU(2)$.

Lemma 10.4 provides $h \in C^{m+1}(W_{n+1}; SU(2))$ such that

$$(10.34) \qquad \sup_{W_{n+1}} \left(\sum_{l=0}^{m} \left| \nabla_\Gamma^{(l)} (h^*A - \Gamma) \right|^2 \right) < \zeta\varepsilon.$$

Here, because $\pi_1(W)$ has no nontrivial representation in $SU(2)$, Γ is the flat, product connection on $W_{n+1} \times SU(2)$.

For all j sufficiently large, (10.34) implies that

$$(10.35) \qquad \sup_{W_{n+1}} \left(\sum_{l=0}^{m} \left| \nabla_\Gamma^{(l)} \left(h^* h_j^* A_j - \Gamma \right) \right|^2 \right) < \zeta\varepsilon.$$

Now, each $A_j \in \mathscr{A}_k$ and is self-dual. Thus, given $\varepsilon > 0$ and $j < \infty$, there exists $Q(j) < \infty$ such that

$$(10.36) \qquad \int_{\tau \geqslant Q+n} e^{\tau\delta} |F_{A_j}|^2 < \varepsilon.$$

Lemma 10.4 supplies $q_j \in C^{m+1}(W_{Q+n+1}; SU(2))$ such that

$$(10.37) \qquad \sup_{W_{Q+n+1}} \left(\sum_{l=0}^{m} \left| \nabla_\Gamma^{(l)} \left(q_j^* A_j - \Gamma \right) \right|^2 \right) < \zeta\varepsilon.$$

Again, Γ is the trivial connection (by definition of $\mathscr{A}_k$, this time.) Let $\beta_n \in C^\infty(M)$ obey $\beta_{n+1} \equiv 1$ if $\tau \geqslant n + 2$, $\beta_n \equiv 0$ if $\tau \leqslant n + 1$, and $|d\beta_{n+1}| < 20$. For j large, define a connection (and principal $SU(2)$ bundle) on

$$B_\varepsilon^j \equiv W_n \cup_N W_{n+1} \cup_N \cdots \cup_N W_{n+Q} \cup_N W_{n+Q+1} \cup_N W_{n+Q+2}$$

by specifying

$$(10.38) \qquad \tilde{A}_j = \begin{cases} \beta_{n+1} (hh_j)^* A_j & \text{on } \tau^{-1}([n, n+2]), \\ A_j & \text{on } \tau^{-1}([n+2, n+Q]), \\ (1 - \beta_{n+Q+1}) q_j^* A_j & \text{on } \tau^{-1}([n+Q, n+Q+2]). \end{cases}$$

By construction, $\tilde{A}_j = \Gamma$ on W_n and on W_{n+Q+2}. Now, furl up B_ε^j to obtain the closed manifold Y_ε^j—identify N_- in W_n with N_+ in W_{q+n+2} as in Figure 3.

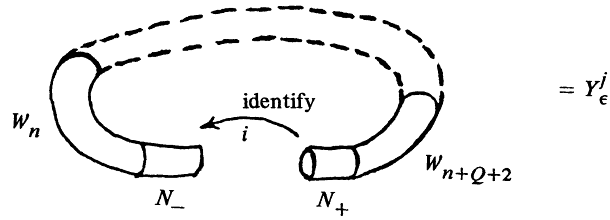


FIGURE 3

The connection, $\tilde{A}_j$ descends to Y_ε^j. Notice that for j large enough,

$$\int_{Y_\varepsilon^j} |P - F_{\tilde{A}_j}|^2 \leqslant \zeta\varepsilon, \tag{10.39}$$

$$\int_{Y_\varepsilon^j} |F_{\tilde{A}_j}|^2 \geqslant \int_{\tau \geqslant n} |F_{A_j}|^2 - \zeta\varepsilon. \tag{10.40}$$

(10.39) and (10.40) imply via the Chern-Weil formula that

$$\int_{Y_\varepsilon^j} |F_{\tilde{A}_j}|^2 \leqslant \zeta\varepsilon \mod 4. \tag{10.41}$$

(10.40) and (10.41) imply Lemma 10.3.

11. The proof of Theorem 1.4

Let $P = M \times SU(2)$ with M an admissible, end-periodic 4-manifold. Assume that $\pi_1(M)$ has only the trivial representation in $SU(2)$, and that $b_2^-(K) = 0$. For a generic, asymptotically periodic metric on TM, $\mathcal{M}_4 \subset \mathcal{B}_4$ is nonempty (Proposition 9.2). It is a smooth, 5-dimensional manifold away from the orbits of reducible connection (Proposition 8.2). By a local perturbation, one may assume that a neighborhood of each reducible orbit in $\mathcal{M}_4$ is diffeomorphic to the cone on $\mathbf{CP}^2$ (minus the vertex) (see Proposition 8.3). Proposition 9.1 catalogues which reducible orbits appear in $\mathcal{M}_4$. Finally, Proposition 10.1 describes the ends of $\mathcal{M}_4$.

Donaldson's argument in §3 of [9] will be adapted to the end-periodic case to prove Theorem 1.4. Rather than translating the constructions of §§2, 3 of [9] to this case, it is simplest to compactify M and the moduli space $\mathcal{M}_4$. Then Donaldson's argument can be used directly.

To compactify $\mathcal{M}_4$, consider the following function on $\mathcal{M}_4$: send $[A] \in \mathcal{M}_4$ to

$$f([A]) = \int_M \beta |F_A|^2, \tag{11.1}$$

where $0 \leqslant \beta \in C_0^\infty(M)$ is identically one on K and zero on End $M \setminus W_0$. It follows from Proposition 9.3 that f is nonconstant and that

$$\inf_{\mathcal{M}_4} f = 0. \tag{11.2}$$

The function f is smooth on $\mathcal{M}_4 \cap \mathcal{B}_4$ and continuous on $\mathcal{M}_4$. As the orbits in $\mathcal{M}_4$ of reducible connections are isolated (Proposition 9.1), one can find arbitrary small $\varepsilon > 0$, such that $(f^{-1}(\varepsilon) \cap \mathcal{M}_4) \subset \mathcal{M}_4 \cap \mathcal{B}_4$, where it is a smooth, 4-dimensional submanifold. Fix such an ε and set

$$\mathscr{M}^{\varepsilon} = \{[A] \in \mathscr{M}_4 : f([A]) \geqslant \varepsilon\}. \tag{11.3}$$

By construction, $\mathscr{M}^{\varepsilon}$ is a smooth 5-dimensional manifold with boundary. Note, $\mathscr{M}^{\varepsilon}$ is not compact. Lemma 11.1 makes rigorous the intuitive notion that $\mathscr{M} \setminus \mathscr{M}^{\varepsilon}$ consists of orbits of connections whose curvatures on K are pointwise small.

Lemma 11.1. *Given $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that if $[A] \in f^{-1}([0, \varepsilon_1]) \cap \mathscr{M}_4$, then*

$$\sup_{x \in K} |F_A|(x) < \varepsilon.$$

Proof of Lemma 11.1. This follows from Lemma 10.4.

Let $\Sigma \subset W_0$ be the inverse image of a regular value $q \in [0,1]$ of τ. Let $\Sigma_n = \tau^{-1}(q + n)$ be the translate of Σ. It is convenient to compactify M by first cutting along Σ_n to obtain the compact manifold with boundary,

$$U_n = \tau^{-1}([0, q + n]).$$

Set $Q_n = U_n \cup_{\Sigma} (-U_n)$. The number n will be determined from ε of Lemma 11.1.

The family of orbits of connections $\mathscr{M}^{\varepsilon}$ defines a like family on Q_n. This new family is constructed as follows: via translation, $\{L_n \equiv M \setminus U_{n-2}\}_{n=2}^{\infty}$ are all mutually diffeomorphic to L_0; the diffeomorphism is T^n.

Let Γ denote the product connection on $L_0 \times SU(2)$ and let

$$\mathscr{A}(L_0) = \{a \in L^2_{2;\delta}(T^*L_0) \times su(2)\}.$$

Let

$$\mathscr{G}(L_0) = \left\{ g \in L^2_{3;\text{loc}}(L_0; SU(2)) : \int_{L_0} e^{\tau\delta} \sum_{j=0}^{3} |\nabla g^{(l)}|^2 < \infty \right\}.$$

Fix $x \in L_0$, and let P_x denote the fiber $\{x\} \times SU(2)$. $\mathscr{G}(L_0)$ acts on P_x by $(g(\cdot), p) \to g(x)p$. Set

$$\mathscr{B}'(L_0) = (\mathscr{A}(L_0) \times P_x)/\mathscr{G}(L_0). \tag{11.4}$$

A repetition of the proofs of Lemmas 7.2 and 7.3 shows that $\mathscr{B}'(L_0)$ is a smooth Banach manifold. Note that $\mathscr{B}'(L_0)$ admits a smooth $SO(3)$ action with fixed point $[\Gamma, 1]$, $1 \in SU(2)$.

Now, back on M, let $x_n = T^n(x) \in L_n$ and let $\mathscr{M}^{\varepsilon n}$ denote the inverse image of $\mathscr{M}^{\varepsilon}$ under the projection $(\mathscr{A}_4 \times P_{x_n})/\mathscr{G}_4 \to \mathscr{A}_4/\mathscr{G}_4$ (see §7). Away from the reducible orbits, $\mathscr{M}^{\varepsilon n} \to \mathscr{M}^{\varepsilon}$ is a principal $SO(3)$ bundle.

By restriction to L_n and via pull-back by $\tilde{T}^n$, one obtains a smooth, $SO(3)$ equivariant map

$$j_n : \mathcal{M}^{\varepsilon n} \to \mathcal{B}'(L_0). \tag{11.5}$$

Lemma 11.2. *Given a neighborhood $\mathcal{D}$ of $[\Gamma, 1] \in \mathcal{B}'(L_0)$ and $\varepsilon > 0$, there exists $m < \infty$ such that for all $n \geqslant m$, $j_n(\mathcal{M}^{\varepsilon n}) \subset \mathcal{D}$.*

Proof of Lemma 11.2. This is because each $A \in \mathcal{A}_4$ is asymptotic to the trivial flat connection on End M. See also Lemma 10.4 and the proof of Lemma 10.3.

Now, choose $\mathcal{D}$ to be contractible onto $[\Gamma, 1]$. It is possible to make this retraction $SO(3)$-equivariant. Indeed, given $\nu > 0$, but small, a neighborhood $\mathcal{D}(\nu)$ of $[\Gamma, 1]$ is diffeomorphic to

$$\begin{aligned}\mathcal{D} \equiv \Big\{ a \in L^2_{2;\delta}(T^*L_0) \times su(2) : \|a\|_{L^2_{2;\delta}(T^*L_0)} < \nu, \\ e^{-\tau\delta} d^* e^{\tau\delta} a = 0 \text{ and } i^*(*a) = 0 \Big\},\end{aligned} \tag{11.6}$$

where $i : \Sigma \ (= \partial L_0) \to L_0$. The group $SO(3) = SU(2)/\{\pm 1\}$ acts on $\mathcal{D}$ by $(h, a) \to hah^{-1}$. Retract $\mathcal{D}$ onto [18] by sending $(t, a) \in [0,1] \times \mathcal{D}$ to $(1 - t)a \in \mathcal{D}$. This $SO(3)$-equivariant retraction is used to construct the compactly supported (on M) moduli space described below:

Lemma 11.3. *Given $\varepsilon > 0$, there exists a smooth homotopy $h : [0,1] \times \mathcal{M}^{\varepsilon} \to \mathcal{A}_4/\mathcal{G}_4$ and $n(\varepsilon) < \infty$ with the following properties*:

(1) *$h(0, \cdot) =$ identity.*

(2) *If $[A] \in \mathcal{M}^{\varepsilon}$, there is a lift of $h\ (\cdot, [A])$ to a path $h\ (\cdot, A) : [0,1] \to \mathcal{A}_4$ which is the constant path when restricted to $\tau^{-1}([0, n - 1])$.*

(3) *$h(1, A)$ is gauge equivalent to the trivial product connection on $\tau^{-1}([n,\infty)) \times SU(2)$.*

(4) *For each $t \in [0,1]$, $h(t, \mathcal{M}^{\varepsilon}) \cap \mathcal{B}_4$ is diffeomorphic to $\mathcal{M}^{\varepsilon} \cap B$. By implication,*

$$h\big(t, \mathcal{M}^{\varepsilon} \cap (\mathcal{A}_4 \backslash \mathcal{A}_4^*)/\mathcal{G}_4\big) \subset (\mathcal{A}_4 \backslash \mathcal{A}_4^*)/\mathcal{G}_4 \quad \textit{for all } t \in [0,1].$$

Proof of Lemma 11.3. Choose $m < \infty$, sufficiently large so that $j_n : \mathcal{M}^{\varepsilon n} \to \mathcal{D}(\nu)$ for all $n > m$ and for ν small. Let $\beta_n \in C_0^\infty(M)$ obey $\beta_n = 1$ on $\tau^{-1}([0, n - 1])$ and $\beta_n = 0$ on $\tau^{-1}((n, \infty))$. Let $t \in [0,1]$ and let $[A, l] \in \mathcal{M}^{\varepsilon n}$. Then a unique $g(A, l) \in L^3_{2;\delta}(L_n, SU(2))$ exists such that $g(A)(x_n) = l$ and

$$g(A)^* A = \Gamma + a([A]) \quad \text{on } L_n, \tag{11.7}$$

where $(T^n)^* a$ is in $\mathcal{D}$ of (11.6). Define a connection $\tilde{h}(t, (A, l))$ on $M \times SU(2)$ by setting

$$\begin{aligned}\tilde{h}(t, (A, l)) &= A \quad \text{on } \tau^{-1}([0, n - 1]), \\ \tilde{h}(t, (A, l)) &= \Gamma + (1 - t)a + t\beta_n a \quad \text{on } L_n.\end{aligned}$$

Now set $h(t,[A]) = [h(t,(A,l))]$. It is straightforward to check assertions (1)–(4) of Lemma 11.3; this is left to the reader. The only difficulty comes in checking that $h(t,[A])$ is an orbit of a reducible connection if $[A]$ is. If A is reducible, then $\phi \in L^3_{2;\mathrm{loc}}(M) \times su(2)$ exists satisfying $\nabla_A\phi = 0$. The converse also holds. Let $\psi = g(A,l)\phi g(A,l)^{-1}$ over L_n. Then $d\psi + [a,\psi] = 0$ on L_n. Since $T^{n*}a \in \mathscr{D}$, it follows that

$$i_n^*(*d\psi) = 0, \tag{11.8}$$

where $i_n : \Sigma_n \to L_n$. Also,

$$e^{-\tau\delta}d^*e^{\tau\delta}d\psi - *(a \wedge *d\psi - d\psi \wedge *a) = 0. \tag{11.9}$$

Equations (11.8) and (11.9), Lemma 5.2, and the maximum principle imply that each component of ψ (as a map of L_n into $su(2)$) is constant on L_n. Thus, $d\psi$ and $[a,\psi]$ are both zero on L_n. The conclusion is that

$$\tilde{\psi} = \phi \quad \text{on } \tau^{-1}([0, n-1]), \qquad \tilde{\psi} = \psi \quad \text{on } L_n$$

is covariantly constant for $\tilde{h}(t,(A,l))$ for all $t \in [0,1]$. Thus, $h(t,[A])$ is the orbit of a reducible connection if and only if $[A]$ is.

Given $\varepsilon > 0$, let $\tilde{\mathscr{M}}^\varepsilon = h(1, \mathscr{M}^\varepsilon)$ with h as per Lemma 11.3. This is a family of orbits of compactly supported connections in $\mathscr{B}_4$. This family defines a family of orbits of connections on a principal $SU(2)$ bundle over Q_m if $m > n(\varepsilon) + 2$ with $n(\varepsilon)$ as given in Lemma 11.3. The construction starts by constructing, as in the proof of Lemma 7.1, a principal $SU(2)$ bundle $P' \to Q_m$ with Pontrjagin number 4. ("Instanton" number 1.) Let $\mathscr{A}(P')$, $\mathscr{G}(P')$ be the space of L^2_2-connections and L^3_2-gauge transformations on P'. If $\mathscr{A}^*(P') \subset \mathscr{A}(P')$ are the irreducible connections, then

$$\mathscr{B}(P') = \mathscr{A}^*(P')/\mathscr{G}(P')$$

is a smooth Banach manifold [12, Chapter 3]. The space $\tilde{\mathscr{M}}^\varepsilon$ automatically sits inside $\mathscr{A}(P')/\mathscr{G}(P')$. Indeed, let $[A] \in \tilde{\mathscr{M}}^\varepsilon$. Then the unique (up to multiplication by a constant $h \in SU(2)$) $g(A) \in L^3_{2;\delta}(L_n, SU(2))$ exists such that

$$g(A) \cdot A = \Gamma \quad \text{on } L_{n+1}.$$

Define a bundle with connection (P'_A, A') over Q_m by writing $Q_m = K_{n+2} \cup (Q_m \setminus \overline{K}_{n+1})$ and then specifying that

$$(P'_A, A')_{K_{n+2}} \simeq (K_{n+2} \times SU(2), A),$$
$$(P'_A, A')_{Q_m \setminus \overline{K}_{n+1}} \simeq ((Q_m \setminus \overline{K}_{n+1} \times SU(2), \Gamma).$$

The clutching function is $g(A)$. The isomorphism class of (P'_A, A') of a bundle with connection defines a point in $\mathscr{A}(P')/\mathscr{G}(P')$. This defines a continuous, 1-1 map, $\overline{\Psi} : \tilde{\mathscr{M}}^\varepsilon \to \mathscr{A}(P')/\mathscr{G}(P')$, which (due to Lemma 11.3) maps orbits of reducibles to orbits of reducibles and which is an embedding away from these orbits.

For future use, it is worth digressing here to indicate which orbits of reducibles are in $\tilde{\mathscr{M}}^{\varepsilon}$. According to Proposition 9.1, the orbits of the reducibles in $\tilde{\mathscr{M}}^{\varepsilon}$ are in 1-1 correspondence with the set of pairs $\{\pm f \in H_2(M, \mathbf{Z}) : f \cdot f = 1\}$. Since an orbit in $\tilde{\mathscr{M}}^{\varepsilon}$ restricts to $\tau^{-1}([n+1, \infty))$ as the orbit of the product connection, only the pairs $\{\pm f \in H_2(K_{n+1}; \mathbf{Z}) : f \cdot f = 1\}$ can correspond to reducible orbits in $\tilde{\mathscr{M}}^{\varepsilon}$. Here, it is illuminating to remark that according to Lemma 5.7, the homomorphisms $H_2(K_l, \mathbf{Z}) \to H_2(M, \mathbf{Z})$ for $l \geqslant -1$ are injective, so there is no ambiguity involved in labeling orbits of reducibles in $\tilde{\mathscr{M}}^{\varepsilon}$ by classes in $H_2(K_l, \mathbf{Z})$. Notice that Lemma 5.7 also implies that for $l \in [-1, n+1]$, the homomorphism $H_2(K_l, \mathbf{Z}) \to H_2(Q_{n+1}; \mathbf{Z})$ is also injective; this verifies that reducible connections on M which are flat on $\tau^{-1}((n+1, \infty))$ can be extended over Q_{n+1} as reducible connections. Finally, since $\pi_1(M)$ does not have nontrivial representations in $SU(2)$, it does not have them in S^1 either. Hence, $H_1(M; \mathbf{Z}) = 0$. Thus, the group $H^2(M; \mathbf{Z})$ is free abelian. By Lemma 5.7, the inclusion homomorphisms, $H^2_{\text{comp}}(M, \mathbf{Z}) \to H^2(M; \mathbf{Z})$, $\{H^2_{\text{comp}}(K_l; \mathbf{Z}) \to H^2(M; \mathbf{Z})\}_{l \geqslant -1}$ are injections. Hence, $H_2(M; \mathbf{Z})$ and $H_2(K_l, \mathbf{Z})$ are free abelian groups too. Because the intersection pairing on $H_2(K_l, \mathbf{Z})$ is positive definite, the number of pairs $\{\pm f \in H_2(K_l, \mathbf{Z}) : f \cdot f = 1\}$ is at most rank $H_2(K_l, \mathbf{Z})$, with equality only if the intersection pairing is unimodular and diagonalizable over $\mathbf{Z}$.

From $\mathscr{M}^{\varepsilon}$, one now constructs a 5-dimensional manifold with boundary, $\mathscr{M}^{\varepsilon\lambda} \subset \mathscr{B}(P')$. Before starting, note that Proposition 10.1 provides a subset $\mathscr{K}^{\varepsilon} \subset \mathscr{M}^{\varepsilon}$ such that $\mathscr{M}^{\varepsilon} \backslash \mathscr{K}^{\varepsilon}$ is compact and $\mathscr{K}^{\varepsilon}$ is diffeomorphic to a domain with smooth boundary in $M \times (0,1)$. Proposition 9.2 provides $\lambda_2 > 0$ such that

$$K \times (0, \lambda_2) \subset \mathscr{K}^{\varepsilon}.$$

For $\lambda \in (0, \lambda_2)$, construct a manifold with boundary, $\mathscr{M}^{\varepsilon\lambda} \subset \mathscr{M}^{\varepsilon}$, in the following way: Take the set

$$(M \times \{\lambda\} \cap \mathscr{K}^{\varepsilon}) \cup (f^{-1}(\varepsilon) \cap (\mathscr{M}^{\varepsilon} \backslash \mathscr{K}^{\varepsilon})) \cup (f^{-1}(\varepsilon) \cap M \times [\lambda, 1))$$

in $\mathscr{M}^{\varepsilon}$ and smooth the corners where $f^{-1}(\varepsilon)$ intersects $M \times \{\lambda\}$ in $\mathscr{K}^{\varepsilon}$; smooth these corners away from $K \times \{\lambda\}$. The resulting space is a smooth 4-manifold which is denoted $B^{\varepsilon\lambda}$ (see Figure 4). The interior of $B^{\varepsilon\lambda}$ is a space which is a manifold away from the orbits in $\mathscr{M}^{\varepsilon}$ of reducible connections. By Lemma 11.3 and Proposition 9.1, each such orbit has a neighborhood which is diffeomorphic off of the reducible's orbit to the cone on $\mathbf{CP}^2$ minus the vertex.

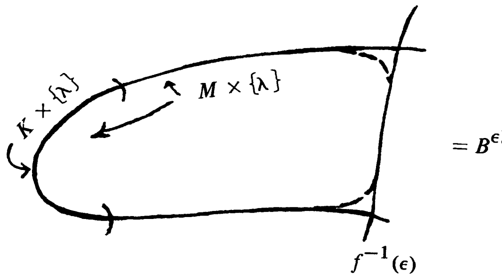


FIGURE 4

By cutting each cone, one obtains the manifold with boundary $\mathscr{M}^{\varepsilon\lambda}$. The boundary of $\mathscr{M}^{\varepsilon\lambda}$ is the disjoint union of

$$B^{\varepsilon\lambda} \sqcup_l \mathbf{CP}^2,$$

with $l \leqslant \operatorname{rank} H_2(K_{n+1}; \mathbf{Z})$.

Lemma 11.4. *For ε and $\delta > 0$, but small, and for $n < \infty$ but sufficiently large, the manifold $M^{\varepsilon\lambda}$, above, is oriented.*

The proof of Lemma 11.4, and the remaining lemmas in this section, will be deferred to the section's end.

By construction (see Proposition 9.2 and Lemma 11.3) the image of $K \times \{\lambda\}$ in $\mathscr{M}^{\varepsilon\lambda}$ is homotopic (in fact, isotopic) to the image in $\mathscr{B}(P')$ under the map T of §9. (Because the connections in the image of T are trivial on End M, T maps $K \times (0,1)$ into $\mathscr{B}(P')$.) The map $T: K \times (0,1) \to \mathscr{B}(P')$ is the obvious restriction of $T: Q_{n+1} \times (0,1) \to \mathscr{B}(P')$ (see [28], [30], [12]).

The topological significance of the map T is described in [9, §3]. In [9], Donaldson defines a map,

$$\mu: H_2(Q_{n+1}, \mathbf{Z}) \to H^2(\mathscr{B}(P'); \mathbf{Z})$$

with the property that $T^* \circ \mu: H_2(Q_{n+1}; \mathbf{Z}) \to H^2(Q_{n+1}; \mathbf{Z})$ is Poincaré duality. (Here, and henceforth, T is to be restricted to $Q_{n+1} \times \{\lambda\}$.) To exploit T and μ as did Donaldson, the relationship between the cohomology of $\partial\mathscr{M}^{\varepsilon\lambda}$ and that of K needs investigating.

Let $\alpha \in H_2(K, \mathbf{Z})$. The inclusion $i: K \to Q_{n+1}$ induces $i_*\alpha \in H_2(Q_{n+1}, \mathbf{Z})$. By Alexander duality, $T^*\mu i_*\alpha \in H^2(Q_{n+1}, \mathbf{Z})$ comes from a class $\hat{a} \in H^2_{\text{comp}}(K; \mathbf{Z})$ which is Poincaré dual to α [17].

Let $\hat{i}: K \to \partial\mathscr{M}^{\varepsilon\lambda}$ and let $J: \partial\mathscr{M}^{\varepsilon\lambda} \to \mathscr{B}(P')$ denote the obvious inclusions. Then $\hat{i}$ induces a monomorphism (Lemma 5.7),

$$\hat{i}_*: H^2_{\text{comp}}(K; \mathbf{Z}) \to H^2(B^{\varepsilon\lambda}; \mathbf{Z}).$$

Lemma 11.5. *For* ε, $\lambda > 0$, *but small, and for* $n < \infty$, *but large; one has for all* $\alpha \in H_2(K; \mathbf{Z})$,

$$\hat{i}_*\alpha = J^*\mu i_*\alpha.$$

Accept for the moment this lemma as fact. Let $\alpha, \beta \in H_2(K; \mathbf{Z})$. Then $\hat{\alpha}, \hat{\beta} \in H^2_{\mathrm{comp}}(K; \mathbf{Z})$, their Poincaré duals, can be represented by cocycles which vanish off of a compact set in K. By Poincaré and Alexander duality,

$$\alpha \cdot \beta = \langle \hat{i}_*\hat{\alpha} \cup \hat{i}_*\hat{\beta}, \zeta \rangle, \tag{11.10}$$

where ζ is the generator of $H_4(B^{\varepsilon\lambda}; \mathbf{Z})$ and $\langle \cdot, \cdot \rangle$ is the usual pairing of cohomology with homology. By Lemma 11.5, one has

$$\alpha \cdot \beta = \langle J^*\mu i_*\alpha \cup J^*\psi i_*\beta, \zeta \rangle. \tag{11.11}$$

Using the homology of $\mathscr{M}^{\varepsilon\lambda}$, (11.11) implies that

$$\alpha \cdot \beta = \sum{}' \mu(i_*\alpha) \cup \mu(i_*\beta)[\mathbf{CP}^2_e]. \tag{11.12}$$

Here, Σ' means to sum over the set of pairs $\{\pm e \in H_2(K_{n+1}; \mathbf{Z}) : e \cdot e = 1\}$ which label the orbits of the reducible connections in $\mathscr{M}^{\varepsilon}$.

By Lemma 2.27 in [9] (see also §III(ii) in [9]), (11.12) equals

$$\alpha \cdot \beta = \sum{}' (\pm)(\alpha \cdot e)(\beta \cdot e), \tag{11.13}$$

where the sign of each term is analogous (but see [10]).

Since the intersection pairing on $H_2(K; \mathbf{Z})$ is nondegenerate, the number of terms in Σ' must be at least rank $H_2(K; \mathbf{Z})$. One has additionally,

Lemma 11.6. *Let* $\Lambda_{-1} \subset H_2(K_{n+1}; \mathbf{Z})$ *denote the free abelian group that is generated by the set of pairs* $\{\pm e \in H_2(K_{n+1}; \mathbf{Z}) : e \cdot e = 1\}$ *which contribute to* Σ'. *Then* $\dim \Lambda_{-1} = \dim(H_2(K; \mathbf{Z}))$, $H_2(K; \mathbf{Z}) \subseteq \Lambda_{-1}$, *and the intersection pairing on* Λ_{-1} *is unimodular and diagonalizable over* $\mathbf{Z}$.

As a remark, if the intersection pairing on $H_2(K; \mathbf{Z})$ is unimodular, then it follows that $H_2(K; \mathbf{Z}) = \Lambda_{-1}$.

Proof of Lemma 11.6. Since $H_2(N; \mathbf{R}) = H_1(N; \mathbf{R}) = 0$, it follows by Meyer-Vietoris that

$$H_2(K_{n+1}; \mathbf{R}) \simeq H_2(K; \mathbf{R}) \oplus H_2(W_0 \cup_N \cdots \cup_N W_{n+1}; \mathbf{R})$$

and this splitting is respected by the intersection pairing. Thus

$$\Lambda_{-1} \otimes \mathbf{R} = H_2(K; \mathbf{R}). \tag{11.14}$$

Since Λ_{-1} is generated by primitive elements, it follows from (11.14) that $H_2(K; \mathbf{Z}) \subseteq \Lambda_{-1}$. Of course, from (11.14), $H_2(K; \mathbf{Z})$ and Λ_{-1} have the same dimensions. Finally, since the intersection pairing on Λ_{-1} is positive definite, and Λ_{-1} is generated by elements with square 1, the intersection form on Λ_{-1} must be unimodular and diagonalizable.

To prove that the intersection form on $H_2(M;\mathbf{Z})$ is unimodular and diagonalizable one need only replace K in the preceding arguments by $K_0 \subset K_1 \subset \cdots$ etc. This defines free abelian subgroups $\Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \cdots \subset H_2(M;\mathbf{Z})$ with the property that

$$\varinjlim \Lambda_n = H_2(M;\mathbf{Z}),$$

and, for all n, the intersection form on Λ_n is unimodular and diagonalizable. In this sense the intersection form on $H_2(M;\mathbf{Z})$ has these same properties.

The proof of Theorem 1.4 is completed now by proving Lemmas 11.4 and 11.5.

Proof of Lemma 11.4. The orientability of $\mathscr{M}^{\varepsilon\lambda}$ is measured by the first Stieffel-Whitney class of its tangent bundle. Since $\mathscr{M}^{\varepsilon\lambda}$ is diffeomorphic to an open subset of $\mathscr{M}_4$, one could show that $\mathscr{M}^{\varepsilon\lambda}$ is oriented by proving that $w_1(T\mathscr{M}_4) = 0$. By construction, $T\mathscr{M}_4$ is the restriction to $\mathscr{M}_4$ of the K-theory class (in $K(\mathscr{B}_4)$) of the index bundle for the $\mathscr{B}_4$-parametrized family of elliptic operators $\{(P_-d_A, e^{-\tau\delta}d_A^*e^{\tau\delta}) : [A] \in \mathscr{B}_4\}$ [3]. One could compute w_1 for this K-theory class directly. But, since the computation for such an index class is available on compact M [9], [12], it is simpler to argue as follows: The eigenvalue estimates from Lemmas 9.3 and 9.5 with the obstruction analysis in [30, §§3, 4] (see also [9]) when applied to the sort of grafting done here show that (when n is large) the class of $T\mathscr{M}^{\varepsilon\lambda}$ in the K-theory of $\mathscr{M}^{\varepsilon\lambda}$ is the restriction to $\mathscr{M}^{\varepsilon\lambda}$ of the class in $K(\mathscr{B}(P'))$ of

$$\operatorname{Index}\left(P_-d_{[\cdot]}, d_{[\cdot]}^*\right) + l\cdot\left(\mathscr{B}'(P') \otimes_{SO(3)} \mathbf{R}^3\right). \tag{11.15}$$

Here, index $(\cdot)$ is the formal difference of finite-dimensional vector spaces,

$$\ker\left(P_-d_A, d_A^*\right) - \operatorname{Coker}\left(P_-d_A, d_A^*\right)$$

as $[A]$ varies in $\mathscr{B}(P')$ (see [3]). The $\mathbf{R}^3$ bundle in (11.15) is associated to the canonical principal $SO(3)$ bundle

$$\begin{array}{c}\mathscr{B}'(P') = P'|_q \times_{\mathscr{G}(P')} \mathscr{A}^*(P') \\ \downarrow \\ \mathscr{B}(P')\end{array}$$

where $q \in Q_{n+1}$ is a fixed point. The integer l in (11.11) is $b_2^-(Q_{n+1}) - b_1(Q_{n+1})$, the number of anti-self-dual harmonic 2-forms on Q_{n+1} minus the number of harmonic 1-forms on Q_{n+1}. ($b_2^-(Q_{n+1})$ grows linearly with n, while $b_1(Q_{n+1})$ is independent of n.)

(11.15) implies that

$$w_1(T\mathscr{M}^{\varepsilon\lambda}) = w_1\left(\operatorname{Index}(\cdot)|_{\mathscr{M}^{\varepsilon\lambda}}\right). \tag{11.16}$$

Now notice that $\pi_1(\mathscr{B}(P')) = (1)$ [12, Chapter 5] so $w_1(\text{Index}(\cdot)) = 0$ and (11.16) implies that $\mathscr{M}^{\varepsilon\lambda}$ is orientable.

Proof of Lemma 11.5. Let $K_1 = K \cup_N W_0 \cup_N W_1$. Since $K_1 \subset Q_{n+1}$, and since $T^*\mu$ is Poincaré duality, one has

$$\hat{\alpha}|_{K_1} = J^*\mu i_*\alpha|_{K_1}.$$

In cohomology, one has the exact sequence (cf. the chapter on Alexander duality in [17])

$$(11.17) \qquad \to H^2_{\text{comp}}(K_1) \to H^2(B^{\varepsilon\lambda}) \xrightarrow{\rho} H^2(B^{\varepsilon\lambda} - K_1) \to .$$

To prove Lemma 11.5, one must show that $\rho(J^*\mu i_*\alpha) = 0$. By Alexander duality, it is sufficient to check that $\rho(J^*\mu i_*\alpha)$ pairs to zero with all classes in $H_2(B^{\varepsilon\lambda}, K_1)$. Let $V = B^{\varepsilon\lambda} - (K_1 \setminus W_1)$. Thus $V \cap K_1 = W_1$ and by excision,

$$H_2(B^{\varepsilon\lambda}, K_1) \simeq H(V, W_1).$$

Thus, a class $c \in H_2(B^{\varepsilon\lambda}, K_1)$ can be represented by an embedded 2-manifold $R \subset V$ with $\partial R \subset W_1$. However, $H_1(W, \mathbf{Z}) = 0$ since $\pi_1(W)$ has no nontrivial representations in $SU(2)$. Hence, one can assume that $\partial R = \varnothing$.

To evaluate the pairing $\langle J^*\mu i_*\alpha, R\rangle$, go to Donaldson's definition of μ in [9, 2]: let $\Sigma \subset K$ be an embedded surface which represents α in $H_2(K; \mathbf{Z})$. A connection on P' defines, by restriction, a connection on the principal $SU(2)$-bundle $P'|_\Sigma \simeq \Sigma \times SU(2)$. The manifold R parametrizes a family of orbits of irreducible connections on P'. By perturbing R, one may assume that the induced family of orbits of connections on $P'|_\Sigma$ are all irreducible. To each $[A] \in R$, one associates a Fredholm operator, the Dirac operator on Σ twisted by $P'|_\Sigma \times_{SU(2)} \mathbf{C}^2$. The association of $[A] \in R$ to this Fredholm operator defines a continuous map, $\psi: R \to BU$ [3]. Here $BU = BU(\infty)$ is $\varinjlim BU(n)$. According to Donaldson,

$$\langle J^*\mu i_*\alpha, R\rangle = \langle \psi^* c_1, R\rangle,$$

where $c_1 \in H^2(BU, \mathbf{Z})$ is the universal first Chern class. In [20], U. Koschorke proves that $\langle \psi^* c_1, R\rangle = 0$ if the kernel and cokernel of the twisted Dirac operator on Σ vanish for each $[A] \in R$. The Dirac operator in question has index zero, and the kernel and cokernel vanish for the product connection on $\Sigma \times SU(2)$. The kernel and cokernel then vanish for all connections on $\Sigma \times SU(2)$ which are C^2-close to the product connection.

Since $R \cap K = \varnothing$, the number

$$\sup_{[A]\in R} \left\{ \sup_{x \in K} |F_A|(x) \right\}$$

can be made arbitrarily small by choosing λ and ε small. This follows from Proposition 9.2 and Lemma 11.1. Since $\pi_1(M)$ does not represent nontrivially in $SU(2)$, Lemma 10.4 implies the following: Given a C^2 neighborhood $\mathcal{O}$ of the flat, product connection on $\Sigma \times SU(2)$ there exists $\varepsilon, \lambda > 0$ such that the orbits of connections parametrized by K restrict to orbits of connections on $\Sigma \times SU(2)$ which intersect $\mathcal{O}$. Thus, $\langle \psi^* c_1, R \rangle = 0$ by Koschorke, and Lemma 11.5 follows. The preceding "localization argument" to calculate the cohomology of $B^{\varepsilon\lambda}$ was modeled closely on the arguments of Donaldson in [9].

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