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POISSON GEOMETRY OF THE PRINCIPAL SERIES AND NONLINEARIZABLE STRUCTURES

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Abstract

If $\mathfrak{g}$ is a Lie algebra (over $\mathbf{R}$), then the dual space $\mathfrak{g}^*$ carries a linear Poisson structure π_0 called the *Lie-Poisson structure*. The *linearization question* is whether a Poisson structure π defined near 0 on $\mathfrak{g}^*$ and having the same 1-jet as π_0 at 0 is equivalent to π_0 on a neighborhood of 0. The answer to the linearization question is *yes* if $\mathfrak{g}$ is semisimple and π is analytic or if $\mathfrak{g}$ is semisimple of compact type and π is C^∞ (Conn), but it can be *no* if $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R})$ and π is C^∞ (Weinstein). In this paper, we show that the answer can be *no* for a C^∞ π if $\mathfrak{g}$ is any semisimple algebra of noncompact type with real rank at least two. The cases of real rank one other than $\mathfrak{sl}(2,\mathbf{R})$ are still open.

The construction of nonlinearizable examples involves an analysis of the Poisson geometry of the subset of $\mathfrak{g}^*$ corresponding to the principal series representations. This analysis, in turn, relies on showing the functorial nature of the phase space for a classical particle in a Yang-Mills field (Sternberg, Weinstein). A by-product of our results is an analog in Poisson geometry to the correspondence between certain representations of $\mathfrak{g}$ and of the associated Cartan motion algebra (Gell-Mann, Hermann, Mukunda, Mackey, Dooley, and Rice).

0. Introduction

A *Poisson structure* on a manifold M may be defined as an antisymmetric contravariant tensor field (bivector) π for which the Poisson bracket operation on $C^\infty(M)$ defined by $\{f, g\} = [df \wedge dg, \pi]$ satisfies the Jacobi identity. The local classification problem for Poisson structures was reduced by the splitting theorem in [23] to the case where $\pi = 0$ at the point m_0 of interest. Near such a point, one may truncate the Taylor series of π at first order to obtain a linear Poisson structure of the form $\pi'_{ij}(x) = \sum c_{ij}^k x_k$, where the x_i are local coordinates centered around m_0. Such a structure is called a *Lie-Poisson structure*,

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since the c_{ij}^k are the structure constants of a Lie algebra $\mathfrak{g}$, which we call the *cotangent Lie algebra* for π at m_0. The Lie-Poisson structure lives naturally on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$; it was discovered and studied by Lie [12] (see [22]).

The *linearization question* is whether a Poisson structure is locally isomorphic to the Lie-Poisson structure on the dual of its cotangent Lie algebra. The linearization question for Poisson structures having semisimple cotangent Lie algebra was raised in [23] and was answered in the affirmative for the analytic case in [1] and for the C^∞ compact case in [2]. In this paper, we will construct for each semisimple Lie algebra $\mathfrak{g}$ of noncompact type and real rank at least 2 a nonlinearizable C^∞ Poisson structure whose cotangent Lie algebra is $\mathfrak{g}$. A nonlinearizable example was already given in [23] for $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$; the remaining cases of real rank 1 remain open.

In §1, we introduce our general strategy, in which Poisson structures are modified with the aid of infinitesimal automorphisms, and we state our main result as Theorem 1.2. In §2, we construct a Poisson model for an open subset of $\mathfrak{g}^*$ by using an analogy to the construction of the principal series representations by inducing. The Poisson model is analyzed and simplified in §3 with the help of a new result in the geometry of Poisson manifolds, which states that the construction of the phase space of a classical particle in a Yang-Mills field ([17], [20]) is functorial with respect to mappings of principal bundles and Poisson manifolds. The results of §§2 and 3 are combined at the end of §3 to produce the infinitesimal automorphisms needed for the proof of Theorem 1.2.

One of the by-products of our construction may be of interest for its own sake and is discussed in §4. Given the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, we consider in addition to $\mathfrak{g}$ the "contracted" Lie algebra $\mathfrak{g}_1$ which is the semidirect product $\mathfrak{k} \times_s \mathfrak{s}$ with respect to the adjoint action of $\mathfrak{k}$ on $\mathfrak{s}$. ($\mathfrak{g}_1$ is also called the *Cartan motion algebra* associated with $\mathfrak{g}$.) We show that there are conic open subsets U and U_1 of $\mathfrak{g}^*$ and $\mathfrak{g}_1^*$ respectively and a homogeneous, $\mathfrak{k}$-equivariant Poisson automorphism ι from U to U_1. In other words, ι carries coadjoint orbits to coadjoint orbits and is a symplectomorphism on each orbit with respect to the Lie-Kirillov-Kostant-Souriau symplectic structure. From the point of view of the "orbit method," this result suggests that there should be a $\mathfrak{k}$-equivariant correspondence between certain representations of $\mathfrak{g}$ and of $\mathfrak{g}_1$. In fact, after completing part of our work, we learned from R. J. Blattner that such a correspondence was conjectured by Mackey [13] based on the analysis of some special cases, and this reference led us to the earlier work of Gell-Mann, Hermann [6], and Mukunda [16], which, along with the more recent paper of Dooley and Rice [3], has been helpful guidance for the present research. We hope that our results may be a useful step toward carrying out Mackey's program in more generality.

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1. Raising the rank of Poisson structures

To "nonlinearize" the Lie-Poisson structure π_0 on $\mathfrak{g}^*$, we will raise its rank by 2 on a set whose closure contains the origin. To do so, we will replace the bivector field π by $\pi_0 + X \wedge Y$, where X and Y are vector fields transverse to the symplectic leaves of π_0. The following lemma tells us when the new structure satisfies the Jacobi identity.

(1.1) Lemma. *If X and Y are infinitesimal automorphisms of a Poisson structure π, then $\pi + X \wedge Y$ is a Poisson structure if and only if X, Y, and $[X, Y]$ are linearly dependent (e.g. if X and Y commute).*

Proof. The Poisson bracket associated with a bivector field satisfies the Jacobi identity if and only if the Schouten bracket of the field with itself is zero [11].

If X is a vector field and π is a Poisson structure, then $[X, \pi]$ is just the Lie derivative of π by X and so is zero when X is an infinitesimal automorphism of π. By the derivation property of the Schouten bracket, it follows that $[X \wedge Y, \pi] = 0$ if X and Y are infinitesimal automorphisms of π. Finally, the Schouten bracket of a bivector $X \wedge Y$ with itself is

$$\begin{aligned}[X \wedge Y, X \wedge Y] &= [X \wedge Y, X] \wedge Y + X \wedge [X \wedge Y, Y] \\ &= X \wedge [Y, X] \wedge Y - X \wedge [X, Y] \wedge Y \\ &= -2X \wedge Y \wedge [X, Y].\end{aligned}$$

It follows that $[\pi + X \wedge Y, \pi + X \wedge Y] = -2X \wedge Y \wedge [X, Y]$, which is zero just when X, Y, and $[X, Y]$ are linearly dependent. q.e.d.

Now let $\mathfrak{g}$ be any semisimple Lie algebra of noncompact type with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. A maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{s}$ is called a *Cartan subalgebra* of $\mathfrak{s}$; its dimension d is called the *real rank* of $\mathfrak{g}$ (or sometimes the *split rank* of $\mathfrak{g}$, or the *rank of the symmetric space G/K*). In the next two sections of this paper, we will construct a d-dimensional space of commuting infinitesimal automorphisms which are linearly independent and transverse to the symplectic leaves on a set whose closure contains the origin. Once we have found such a space, we will have proven the following theorem, which is the main result of this paper.

(1.2) Theorem. *Let $\mathfrak{g}$ be any semisimple Lie algebra of noncompact type with real rank at least two. Then there is a Poisson structure whose cotangent Lie algebra is $\mathfrak{g}$ but which is not locally equivalent to the Lie-Poisson structure on $\mathfrak{g}^*$.*

2. A Poisson model for the principal series

The construction in this section is just the analog in Poisson geometry of the following construction of the principal series by the method of induced representations (see [3], [13], or [19]).

Let G be a noncompact semisimple Lie group, and KAN its Iwasawa decomposition. If M is the centralizer of A in K, then $P = MAN$ is called a (minimal) parabolic subgroup. N is normal in P, so P/N is isomorphic to the (direct) product $M \times A$ of commuting subgroups. Let η be an irreducible representation of M and let ψ be a character of A, so that $\eta \otimes \exp(2\pi i \psi)$ is an irreducible representation of $M \times A$, and hence of P. The representation of G obtained by inducing this is denoted by $\rho_{\eta\psi}$ and is called a *principal series representation*. Specifically, if H_η is the representation space for η, then the representation space for $\rho_{\eta\psi}$ consists of maps from G to H_η which are equivariant with respect to the right action of P on G. The representation of G on this space (which can also be thought of as the space of sections of an H_η bundle over G/P) is then given by the left action of G on itself.

To form the symplectic model of this construction (see also [8] or [20]) we begin with the Lie-Poisson manifold $(\mathfrak{m} \times \mathfrak{a})^* = \mathfrak{m}^* \times \mathfrak{a}^*$, which corresponds to the direct sum of all the irreducible representation spaces of $M \times A$. (Each individual representation corresponds to a single coadjoint orbit in $(\mathfrak{m} \times \mathfrak{a})^*$.) The symplectic manifold T^*G corresponds to the functions on G, and their product $Q = T^*G \times \mathfrak{m}^* \times \mathfrak{a}^*$ corresponds to the space of functions on G with values in the sum of the representation spaces. The group P acts on Q via right translations on G and the adjoint action of P on $P/N = M \times A$; what corresponds to taking the P-equivariant functions in the definition of the principal series is passing to the reduced manifold of Q at $0 \in \mathfrak{p}^*$. (We refer here to reduction of *Poisson* manifolds rather than of the symplectic manifolds to which the reduction process is usually applied; see the Appendix.) This reduced manifold, which we shall denote by $\not{p}$, then corresponds to the sum of the representation spaces for all the principal series representations.

To get a more explicit description of $\not{p}$, we begin by using the left translations in T^*G to identify Q with $G \times \mathfrak{g}^* \times \mathfrak{m}^* \times \mathfrak{a}^*$ and then, by the Iwasawa decomposition, with $G \times \mathfrak{k}^* \times \mathfrak{a}^* \times \mathfrak{n}^* \times \mathfrak{m}^* \times \mathfrak{a}^*$. The momentum map $J : Q \to p^*$ for the action of P is given by $J(g, k, a, n, m, a') = (k|_{\mathfrak{m}} - m, a - a', n)$; the reduced manifold $\not{p}$ is $J^{-1}(0)/P$.

From the description of J just given, we know that $J^{-1}(0)$ is $\{(g, k, a, 0, k|_{\mathfrak{m}}, a)\}$ as a subset of $G \times \mathfrak{g}^* \times \mathfrak{m}^* \times \mathfrak{a}^*$, so the projection of $J^{-1}(0)$ into $G \times \mathfrak{g}^*$ is an embedding. It will be convenient from this point on to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. In doing so, we must be careful to interpret the 0 in the $\mathfrak{n}^*$ slot as meaning that the element "$(k, a, 0)$" of $\mathfrak{g}^*$ corresponds to an element of the Killing-orthogonal space to $\mathfrak{n}$ in $\mathfrak{g}$. This orthogonal space is just $\mathfrak{p}$ (see [19, p. 23]), so we conclude that $J^{-1}(0)$ is diffeomorphic to $G \times \mathfrak{p}$ (and *not* $G \times \mathfrak{k} \times \mathfrak{a}$). Thus we may identify $\not{p}$ as $(G \times \mathfrak{p})/P$, the homogeneous vector bundle over G/P with fiber $\mathfrak{p}$ associated to the adjoint representation of $\mathfrak{p}$. It follows that this bundle carries a Poisson structure.

Our real interest is in understanding the Poisson structure on $\mathfrak{g}^*$ itself, using the momentum mapping from $\not{p}$ to $\mathfrak{g}^*$ obtained by passing to the reduced manifold the left action of G on T^*G (and the trivial action on $\mathfrak{m}^* \times \mathfrak{a}^*$). This momentum map is given by *right* translations, since we used left translations to trivialize T^*G above. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by using the Killing form, the final map $\mathscr{E}: \not{p} \to \mathfrak{g}$ is given by (the quotient by P of) $(g, p) \mapsto \mathrm{Ad}_g\, p$.

The problem now is to find open subsets of $\not{p}$ and $\mathfrak{g}$ between which the map $\mathscr{E}$ is a diffeomorphism. As a first step, we shall find out where the differential of $\mathscr{E}$ is an isomorphism. By G-equivariance, it suffices to look at points of the form (id, p), where the image of the differential of $\mathscr{E}$ is $\mathrm{Im}\,\mathrm{ad}_p + \mathfrak{p}$; its "Killing orthogonal" space is $\mathrm{Ker}\,\mathrm{ad}_p \cap \mathfrak{n}$, so the condition on p is that its centralizer contain no nonzero elements of $\mathfrak{n}$. Now we have the following lemma. Note that the regular elements of $\mathfrak{a}$, defined in the proof of the lemma, form an open dense subset of $\mathfrak{a}$.

(2.1) Lemma. *Let $p = m + a + n$, where $m \in \mathfrak{m}$, $a \in \mathfrak{a}$, and $n \in \mathfrak{n}$. If a is a regular element of $\mathfrak{a}$, then the centralizer of p contains no nonzero elements of $\mathfrak{n}$.*

Proof. We consider the filtration $\mathfrak{n} = \mathfrak{n}_0 \subseteq \mathfrak{n}_1 \subseteq \cdots \subseteq \mathfrak{n}_r = \{0\}$, where $\mathfrak{n}_i = [\mathfrak{n}, \mathfrak{n}_{i-1}]$. Suppose that p commutes with $n' \in \mathfrak{n}_k$. For each i, there exists a set R_i of positive roots (for the action of $\mathfrak{a}$ on $\mathfrak{g}$) such that the sum of the corresponding root spaces is a complement to $\mathfrak{n}_{i+1}$ in $\mathfrak{n}_i$; we may decompose n' as $n_{k+1} + \Sigma e_\lambda$, where each e_λ lies in a root space E_λ for some λ in R_k. Since $\mathfrak{m}$ commutes with $\mathfrak{a}$, its element m leaves invariant each $\mathfrak{n}_i$ and each of the root spaces E_λ. Now if $[p, n'] = [m, n_{k+1}] + \Sigma[m, e_\lambda] + [a, n_{k+1}] + \Sigma\lambda(a)e_\lambda + [n, n']$ is zero, then so is its component in each E_λ; i.e. $0 = [m, e_\lambda] + \lambda(a)e_\lambda$. For a to be a regular element of $\mathfrak{a}$ means that $\lambda(a) \neq 0$ for every root λ. But $\lambda(a)$ is real, and all the eigenvalues of $\mathrm{ad}(m)$ are purely imaginary because m belongs to the compact piece $\mathfrak{k}$ of the Iwasawa decomposition, so e_λ must be zero. It follows that n' lies in $\mathfrak{n}_{k+1}$ and hence, by repetition of this argument, that $n' = 0$. q.e.d.

The next result will be useful when we look at the global injectivity of $\mathscr{E}$. Let $\mathfrak{a}_r \subset \mathfrak{a}$ be the set of regular elements, and let $\mathfrak{p}_r = \mathfrak{m} + \mathfrak{a}_r + \mathfrak{n} \subset \mathfrak{p}$.

(2.2) Corollary. *Under the adjoint representation, the subgroup N acts transitively on each fiber of the projection from $\mathfrak{p}_r$ to $\mathfrak{m} + \mathfrak{a}_r$ along $\mathfrak{n}$. In particular, any element of $\mathfrak{p}_r$ is conjugate under N to an element of $\mathfrak{m} + \mathfrak{a}_r$.*

Proof. By Lemma 2.1, each orbit of the action of N on a fiber of the projection is open. Since the fiber is connected, the action is transitive. q.e.d.

We shall now find a subset of $\mathfrak{p}$ on which $\mathscr{E}$ is injective. Note that, if $\mathscr{E}(g_1, p_1) = \mathscr{E}(g_2, p_2)$, then p_1 and p_2 are conjugate by $g = g_2 g_1^{-1}$. If g belongs to P, then both (g_i, p_i) correspond to the same element of $\not{h}$, so there is no loss of uniqueness. The question then reduces to when two elements of $\mathfrak{p}$ can be conjugate by an element of G which does not belong to P. To give a satisfactory answer to this question, we define $\mathfrak{a}_+$ to be the open positive Weyl chamber, consisting of those a for which $\lambda(a) > 0$ for all positive roots, λ, and we define $\mathfrak{p}_+$ to be $\mathfrak{m} + \mathfrak{a}_+ + \mathfrak{n}$. Then we have the following result.

(2.3) Lemma. *If two elements of $\mathfrak{p}_+$ are conjugate by an element g of G, then g belongs to P.*

Proof. The proof is based on ideas from [4]. It proceeds in several steps.

Step 1. *If $a \in \mathfrak{a}_+$, $g \in K$, and* $\mathrm{Ad}_g(a) \in \mathfrak{a}_+$, *then $g \in M$.* This is a well-known fact about the isotropy representation of a symmetric space. (See [5, Chapter VII, §2. Theorem 2.12]. By duality, it does not matter whether the symmetric space is compact or noncompact.)

Step 2. *If $m + a \in \mathfrak{m} + \mathfrak{a}_+$, $g \in K$, and* $\mathrm{Ad}_g(m + a) \in \mathfrak{m} + \mathfrak{a}_+$, *then* $g \in M$. This follows from Step 1 because the adjoint action of K preserves the Cartan decomposition.

Step 3. *If $m + a \in \mathfrak{m} + \mathfrak{a}_+$, $g \in K$, and* $\mathrm{Ad}_g(m + a) \in \mathfrak{p}_+$, *then $g \in P$.* To show this, we will consider the vector fields $m_\rightarrow$ and $a_\rightarrow$ on the flag manifold $G/P = K/M$, corresponding to m and a respectively. The condition $\mathrm{Ad}_g(m + a) \in \mathfrak{p}_+$ implies that the point $g^{-1}M$ is a zero of the vector field $m_\rightarrow + a_\rightarrow$. According to [4], $a_\rightarrow$ is the gradient of an M-invariant function, so the vanishing of the sum $m_\rightarrow + a_\rightarrow$ implies the vanishing of each term. In particular, we must have $\mathrm{Ad}_g(a) \in \mathfrak{p}$; since $\mathrm{Ad}_g(a) \in \mathfrak{s}$ and $\mathfrak{s} \cap \mathfrak{p} = \mathfrak{a}$, it follows that $\mathrm{Ad}_g(a) \in \mathfrak{a}$. Now the only way that $\mathrm{Ad}_g(a + m) = \mathrm{Ad}_g(a) + \mathrm{Ad}_g(m)$ can lie in $\mathfrak{p}$ with Ad_g in $\mathfrak{a}$ is for $\mathrm{Ad}_g(m)$ to lie in $\mathfrak{m}$. Thus $\mathrm{Ad}_g(a + m) \in (\mathfrak{m} + \mathfrak{a}) \cap \mathfrak{p}_+ = \mathfrak{m} + \mathfrak{a}_+$, so by Step 2 we must have $g \in M$.

Step 4. *If $m + a \in \mathfrak{m} + \mathfrak{a}_+$, $g \in G$, and* $\mathrm{Ad}_g(m + a) \in \mathfrak{p}_+$, *then $g \in P$.* Applying the Iwasawa decomposition to g^{-1}, we may write g as $g_N g_A g_K$, where $g_K \in K$, $g_A \in A$, and $g_N \in N$. We need to show that $g_K \in M$.

Since $\mathrm{Ad}_g(m + a) = \mathrm{Ad}_{g_N}\mathrm{Ad}_{g_A}\mathrm{Ad}_{g_K}(m + a) \in \mathfrak{p}_+$, we have $\mathrm{Ad}_{g_K}(m + a) \in \mathfrak{p}_+$ as well, because the adjoint actions of N and A leave $\mathfrak{p}_+$ invariant. The result now follows from Step 3.

Step 5. *If $m + a + n \in \mathfrak{p}_+$, $g \in G$, and $\mathrm{Ad}_g(m + a + n) \in \mathfrak{p}_+$, then $g \in P$.* By Lemma 2.2, there is an element h of N such that $m + a + n = \mathrm{Ad}_h(m + a)$. Then $\mathrm{Ad}_{gh}(m + a) \in \mathfrak{p}_+$, and so $gh \in P$ by Step 4; it follows that $g \in P$ as well. q.e.d.

We may summarize the results of this section as follows.

(2.4) Theorem. *Let G be a noncompact semisimple Lie group with Iwasawa decomposition $G = KAN$ and minimal parabolic subgroup $P = MAN$. Then the reduced manifold $\not{p} = (T^*G \times \mathfrak{m}^* \times \mathfrak{a}^*)_0$ with respect to the action of P is identifiable with the homogeneous vector bundle over G/P with fiber $\mathfrak{p}$ (i.e. the adjoint bundle of the principal bundle $P \to G \to G/P$), and so this bundle has a natural Poisson structure with a hamiltonian action of G. The momentum mapping $\mathscr{E}$ of this action is given by $(g, p) \mapsto \mathrm{Ad}(g)p$. The restriction of $\mathscr{E}$ to the conic open subbundle $\not{p}^+$ with fiber $\mathfrak{p}^+ = \mathfrak{m} + \mathfrak{a}^+ + \mathfrak{n}$ is a homogeneous Poisson isomorphism of $\not{p}^+$ with an open conic subset U of $\mathfrak{g}^*$.*

3. Reduction of the structure group and introduction of infinitesimal automorphisms

In the previous section, we identified an open subset U of $\mathfrak{g}^*$ with an open subset $\not{p}_+$ in the reduced manifold $\not{p} = (T^*G \times \mathfrak{m}^* \times \mathfrak{a}^*)_0$ (reduction with respect to the action of P). The subset may be described explicitly as $\not{p}_+ = (T^*G \times \mathfrak{m}^* \times \mathfrak{a}^*_+)_0$, where $\mathfrak{a}^*_+ \subset \mathfrak{a}^*$ is the subset which corresponds to $\mathfrak{a}_+ \subset \mathfrak{a}$ under the isomorphism given by the Killing form. (Note that $\mathfrak{m}^* \times \mathfrak{a}^*_+$ is invariant under the action of P on $\mathfrak{m}^* \times \mathfrak{a}^*$.) In this section, we will analyze the Poisson manifold $\not{p}$ by considering it as a phase space for a classical particle in a Yang-Mills field ([17], [20]). After recalling the general construction of such phase spaces, we will prove and apply a functoriality property of the construction.

Let B be a principal bundle over a manifold X with structure group C, and let Π be a hamiltonian C-space; i.e. Π is a Poisson manifold with a momentum mapping $J : \Pi \to \mathfrak{c}^*$ for an action of C by Poisson automorphisms of Π. Then there is a product hamiltonian action of C on $T^*B \times \Pi$; the reduced Poisson manifold $(T^*B \times \Pi)_0$ is denoted by $Y(C, B, \Pi)$ and called the *Yang-Mills-Higgs phase space for a classical particle with configuration space X and internal phase space* Π. (See the Appendix for a brief discussion of reduction of Poisson manifolds.) $Y(C, B, \Pi)$ is a bundle over X which is a fiber product of the associated Π bundle over X with the cotangent bundle

T^*X, where the latter is considered as an affine bundle rather than as a vector bundle over X. Given a connection on the principal bundle B, $Y(C, B, \Pi)$ can be identified with the Π bundle over T^*X which is associated to the pullback of the principal bundle B under the projection $T^*X \to X$. In the construction of the symplectic model $\not{p}_+$ for the principal series, the configuration space is the generalized flag manifold $\Phi = G/P$, the principal bundle is $G \to G/P$ with structure group P, and the internal phase space is $\mathfrak{m}^* \times \mathfrak{a}_+^*$; i.e. $\not{p}_+ = Y(P, G, \mathfrak{m}^* \times \mathfrak{a}_+^*)$.

From the Iwasawa decompositions $G = KAN$ and $P = MAN$, we see that the flag manifold Φ has an alternative description as K/M. (This was already observed in the course of the proof of Lemma 2.3.) In other words, the principal bundle $P \to G \to \Phi$ admits a reduction of structure group to $M \to K \to \Phi$. Since the new structure group M inherits a hamiltonian action on $\mathfrak{m}^* \times \mathfrak{a}_+^*$, we can form the new phase space $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}_+^*)$, which has the same configuration space and internal phase space as before; only the structure group has been changed. It is reasonable to expect that this reduction of structure group does not affect the final outcome of the construction; we shall show below that this is indeed the case. The advantage of the realization of $\not{p}_+$ as $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}_+^*)$ is that the smaller structure group M acts trivially on the factor $\mathfrak{a}_+^*$ in the internal phase space $\mathfrak{m}^* \times \mathfrak{a}_+^*$; this enables us to promote arbitrary diffeomorphisms of $\mathfrak{a}_+^*$ to Poisson automorphisms of $\not{p}_+$, and vector fields on $\mathfrak{a}_+^*$ to infinitesimal Poisson automorphisms of $\not{p}_+$.

We will state and prove below a general theorem which contains all of the properties of the Yang-Mills phase space construction which we will need. It is easiest to state this theorem in terms of a category which we now describe. For convenience, we will use the letter "G" rather than "C" to denote the general structure group.

(3.1) Definition. A *classical Yang-Mills-Higgs setup* is a triple (G, B, Π), where G is a Lie group, B is a principal G-bundle (i.e. a manifold on which G acts freely from the right), and Π is a hamiltonian G-space. (Π is the internal phase space of the setup; any manifold X provided with a diffeomorphism with B/G may serve as the configuration space.) The setups are the objects in a category Γ in which a morphism from (G_1, B_1, Π_1) to (G_2, B_2, Π_2) is a triple (λ, ϕ, f) of mappings with the following properties.

(a) $\lambda : G_1 \to G_2$ is a homomorphism of Lie groups; we denote by $\lambda^* : \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ the dual of the associated homomorphism of Lie algebras.

(b) $\phi : B_1 \to B_2$ is a λ-equivariant mapping such that the induced mapping $\phi/$ from B_1/G_1 to B_2/G_2 is an embedding onto an open subset. (If λ is an identity mapping and $\phi/$ is a diffeomorphism, then ϕ is called a *gauge transformation*.)

(c) $f: \Pi_1 \to \Pi_2$ is a mapping of hamiltonian spaces; i.e. f is a λ-equivariant Poisson mapping and the diagram

$$\begin{tikzcd} \Pi_1 \arrow[r, "f"] \arrow[d, "J_1"] & \Pi_2 \arrow[d, "J_2"] \\ \mathfrak{g}_i^* & \mathfrak{g}_2^* \arrow[l, "\lambda^*"'] \end{tikzcd}$$

is commutative.[1]

(3.2) Theorem. *Y is a functor from the category Γ of classical Yang-Mills-Higgs data to the category of Poisson manifolds. Thus, for every morphism (λ, ϕ, f) from (G_1, B_1, Π_1) to (G_2, B_2, Π_2) there is an induced Poisson mapping $Y(\lambda, \phi, f)$ from $Y(G_1, B_1, \Pi_1)$ to $Y(G_2, B_2, \Pi_2)$. If $\sigma/$ and f are diffeomorphisms, then $Y(\lambda, \phi, f)$ is also a diffeomorphism.*

(3.3) Remark. The functor Y can be extended to the larger category in which the condition that $\phi/$ be an open embedding is dropped. In this case, though, the result of applying Y to a morphism may no longer be a Poisson mapping; rather, it may be a more general relation between the domain and target, as in the "symplectic category" [21]. This already happens when G and Π are trivial, in which case $B/G = B$, $\phi/ \ = \phi$, $Y(G, B, \Pi) = T^*B$, and $Y(\lambda, \phi, f)$ is the conormal bundle to the graph of ϕ.

Proof of Theorem 3.2. We shall reduce the construction and analysis of the morphisms $Y(\lambda, \phi, f)$ to some special cases. First of all, for the subcategory of Γ in which λ and f are identity mappings ("gauge transformations covering open embeddings"), the theorem follows directly from the fact that the construction of $Y(G, B, \Pi)$ involves no arbitrary choices.

We look next at trivialized bundles. Since reduction by the trivial group G_0 has no effect at all, $Y(G_0, X, \Pi)$ is simply the product Poisson manifold $T^*X \times \Pi$. (Here, and elsewhere, we identify a set with the set of its singleton subsets.) Given an object (G, B, Π) of Γ, a *local cross section* of B is a map $\sigma: X \to B$, where X is a manifold and $\sigma/ : X \to B/G$ is required to be an open embedding. Each cross section σ may be considered as a morphism $(1_G, \sigma, \mathrm{id}) : (G_0, X, \Pi) \mapsto (G, B, \Pi)$, where 1_G is the (inclusion of the point in

[1] Note that if f is any λ-equivariant Poisson mapping, then $\lambda^* \circ J_2 \circ f = J_1$ is necessarily a momentum mapping for the action of G_1 on Π_1 (Lemma 3.2 of [14]), but it could differ from J_1 by a mapping $\delta : \Pi_1 \to g_1^*$ which is constant on symplectic leaves and whose image lies in the fixed point set of the coadjoint representation of G. This fixed point set is zero for many Lie algebras, including all the semisimple ones, in which case any equivariant Poisson mapping is a mapping of hamiltonian spaces.

G_0 to the) identity element of G. In accordance with what is expected from the statement of the theorem, we will construct in terms of σ a Poisson mapping $Y(1_G, \sigma, \mathrm{id})$ from $T^*X \times \Pi$ to $Y(G, B, \Pi)$ which is an open embedding (and a diffeomorphism if the open embedding $\sigma/ : X \to B/G$ is a diffeomorphism).

To construct the Poisson mapping associated with a cross section, note first that $(1_G, \sigma, \mathrm{id})$ extends naturally to a morphism $(\mathrm{id}, \tau, \mathrm{id}) : (G, X \times G, \Pi) \mapsto (G, B, \Pi)$; τ is just the local trivialization induced by the local section σ. According to the first paragraph of this proof, we have a Poisson mapping $Y(\mathrm{id}, \tau, \mathrm{id}) : Y(G, X \times G, \Pi) \mapsto Y(G, B, \Pi)$ which is an open embedding, so that it suffices to find a Poisson isomorphism $Y(1_G, \sigma_X, \Pi)$ from $T^*X \times \Pi = Y(G_0, X, \Pi)$ to $Y(G, X \times G, \Pi)$; here, σ_X is the distinguished cross section of the trivial bundle $X \times G$.

The space $Y(G, X \times G, \Pi)$ is, by definition, the reduction at 0 of $T^*(X \times G) \times \Pi$ by the action of G. Since $T^*(X \times G)$ is naturally symplectomorphic to $T^*X \times T^*G$, and G acts trivially on X, $Y(G, X \times G, \Pi) \approx T^*X \times (T^*G \times \Pi)_0$, and so it remains to find a Poisson isomorphism from Π to $(T^*G \times \Pi)_0$. Such an isomorphism (actually its inverse) is described in Proposition A.4 in the Appendix. Using this isomorphism, we can immediately construct $Y(1_G, \sigma_X, \Pi) : T^*X \times \Pi \to Y(G, X \times G, \Pi)$.

Next, we shall study $Y(\lambda, \phi, f)$ in the case when the principal bundles B_1 and B_2 have suitably related trivializations. Namely, let $\sigma_1 : X \to B_1$ be a local section defined on some manifold X, inducing the local trivialization $\tau_1 : X \times G_1 \to B_1$. If $\phi : B_1 \to B_2$ is a bundle mapping, then $\sigma_2 = \phi \circ \sigma_1$ is a local section of B_2. Then σ_2 induces a local trivialization $\tau_2 : X \times G_2 \to B_2$ such that $\tau_2^{-1} \circ \phi \circ \tau_1 : X \times G_1 \to X \times G_2$ is given simply by $(u, g) \mapsto (u, \lambda(g))$.

Now, as we have just seen, $Y(G_i, X \times G_i, \Pi_i)$ is naturally isomorphic to the Poisson manifold product $T^*X \times \Pi_i$. Since G_i has "disappeared" here, we simply define $Y(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f) : T^*X \times \Pi_1 \mapsto T^*X \times \Pi_2$ to be the mapping $(x, \xi, \theta) \mapsto (x, \xi, f(\theta))$ for $x \in X$, $\xi \in T_x^*X$, and $\theta \in \Pi_1$. This is obviously a Poisson mapping. We wish to use it to define $Y(\lambda, \phi, f)$ between the open subsets of $Y(G_1, B_1, \Pi_1)$ and $Y(G_2, B_2, \Pi_2)$ lying over $\sigma_1/(X)$ and $\sigma_2/(X)$ respectively by the formula

$$Y(\lambda, \phi, f) = Y(\mathrm{id}, \tau_2, \mathrm{id}) \circ Y\big(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f\big) \circ Y\big(\mathrm{id}, \tau_1^{-1}, \mathrm{id}\big).$$

What remains to be shown is that this definition is independent of the choice of the local cross section σ.

To verify this independence, we take another cross section $\underline{\sigma}_1$ of B_1, defined on the same manifold X as σ_1 and with $\underline{\sigma}_1/ = \sigma_1/$. (These assumptions on $\underline{\sigma}_1$ are not essential but are made for simplicity of notation.) Then, as above, we define the cross section $\underline{\sigma}_2 = \phi \circ \underline{\sigma}_1$ of B_2 and the corresponding local trivializations $\underline{\tau}: X \times G_i \to B_i$. We need to show that

$$Y(\mathrm{id}, \tau_2, \mathrm{id}) \circ Y\big(\lambda, \underline{\tau}_2^{-1} \circ \phi \circ \underline{\tau}_1, f\big) \circ Y\big(\mathrm{id}, \underline{\tau}_1^{-1}, \mathrm{id}\big)$$
$$= Y(\mathrm{id}, \tau_2, \mathrm{id}) \circ Y\big(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f\big) \circ Y\big(\mathrm{id}, \tau_1^{-1}, \mathrm{id}\big).$$

For each i, the local trivializations τ_i and $\underline{\tau}_i$ are related by the gauge transformation $\gamma_i = \tau_i^{-1}\underline{\tau}_i : X \times G_i \to X \times G_i$, which may also be considered as a map $\tilde{\gamma}_i$ from X to G_i. The relations $\sigma_2 = \phi \circ \sigma_1$ and $\underline{\sigma}_2 = \phi \circ \underline{\sigma}_1$ and the λ-equivariance of ϕ imply that $\tilde{\gamma}_2 = \lambda \circ \tilde{\gamma}_1$. Each γ_i induces a Poisson automorphism $Y(\mathrm{id}, \gamma_i, \mathrm{id})$ of $Y(G_i, X \times G_i, \Pi_i) = T^*X \times \Pi_i$. Since by construction $\underline{\tau}_2^{-1} \circ \phi \circ \underline{\tau}_1 = \tau_2^{-1} \circ \phi \circ \tau_1$, the functoriality of Y when applied to gauge transformations implies that the equation to be proven (at the end of the preceding paragraph) is equivalent to

$$Y(\mathrm{id}, \gamma_2, \mathrm{id}) \circ Y\big(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f\big) = Y\big(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f\big) \circ Y(\mathrm{id}, \gamma_1, \mathrm{id}).$$

Recalling that $Y(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f): T^*X \times \Pi_1 \to T^*X \times \Pi_2$ is defined as the mapping $(x, \xi, \theta) \mapsto (x, \xi, f(\theta))$, it remains to obtain an explicit formula for the lifted gauge transformations $Y(\mathrm{id}, \gamma_i, \mathrm{id}): T^*X \times \Pi_i \to T^*X \times \Pi_i$.

$Y(G, X \times G, \Pi)$ is defined as $(T^*(X \times G) \times \Pi)_0 = T^*X \times (T^*G \times \Pi)_0$. According to the appendix, it is isomorphic to $T^*X \times \Pi$ by the map $(x, \xi, g, -l^*_{(g-1)}J(\theta), \theta) \mapsto (x, \xi, g\theta)$. Given a gauge transformation $\gamma: (x, g) \mapsto (x, \tilde{\gamma}(x)g)$ of $X \times G$, we will lift it to $T^*X \times T^*G \times \Pi$ and then push it down to the reduced manifold. The lift of γ to the *tangent* bundle $TX \times TG$ is given by

$$(x, \delta x, g, \delta) \mapsto \big(x, \delta x, \tilde{\gamma}(x)g, Tl_{\tilde{\gamma}(x)}(\delta g) + Tr_g \circ T\tilde{\gamma}(\delta x)\big).$$

Similarly, the lift of γ^{-1} is given by

$$(x, \delta x, g, \delta g) \mapsto \big(x, \delta x, \tilde{\gamma}^{-1}(x)g, Tl_{\tilde{\gamma}^{-1}(x)}(\delta g) + Tr_g \circ T\tilde{\gamma}^{-1}(\delta x)\big).$$

In particular, for the lift of γ^{-1},

$$(x, \delta x, \tilde{\gamma}(x), \delta g) \mapsto \big(x, \delta x, 1_G, Tl_{\tilde{\gamma}^{-1}(x)}(\delta g) + Tr_{\tilde{\gamma}^{-1}(x)} \circ T_x\tilde{\gamma}(\delta x)\big).$$

It is convenient to denote the composite derivative $Tr_{\tilde{\gamma}^{-1}(x)} \circ T_x\tilde{\gamma} : T_xX \mapsto \mathfrak{g}$ by $\tilde{\gamma}'(x)$. Now pulling back cotangent vectors by the dual of the lift of γ^{-1} and forming the product with the identity map on Π gives, for the lift of the gauge transformation γ to $T^*(X \times G) \times \Pi$,

$$(x, \xi, 1_G, \mu, \theta) \mapsto \big(x, \xi + [\tilde{\gamma}'(x)]^*\mu, \tilde{\gamma}(x), l^*_{\tilde{\gamma}^{-1}(x)}(\mu), \theta\big).$$

To apply the natural isomorphism from $Y(G, X \times G, \Pi)$ to $T^*G \times \Pi$, we must set $\mu = -J(\theta)$. Then $(x, \xi, 1_G, -J(\theta), \theta)$ goes to (x, ξ, θ), while

$$\big(x, \xi + [\tilde{\gamma}'(x)]^*\big(-J(\theta), \tilde{\gamma}(x), l^*_{\tilde{\gamma}^{-1}(x)}(-J(\theta)), \theta\big)$$

goes to $(x, \xi + [\tilde{\gamma}'(x)]^*(-J(\theta)), \tilde{\gamma}(x)\theta)$. In other words, $Y(\mathrm{id}, \gamma, \mathrm{id}) : T^*X \times \Pi \mapsto T^*X \times \Pi$ is given by $(x, \xi, \theta) \mapsto (x, \xi + [\tilde{\gamma}'(x)]^*(-J(\theta)), \tilde{\gamma}(x)\theta)$.

We are ready to prove that

$$Y(\mathrm{id}, \gamma_2, \mathrm{id}) \circ Y\left(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f\right) = Y\left(\lambda, \tau_2^{-1} \circ \phi \circ \tau_1, f\right) \circ Y(\mathrm{id}, \gamma_1, \mathrm{id}).$$

The left-hand side maps (x, ξ, θ) to $(x, \xi + [\tilde{\gamma}_2'(x)]^*(-J_2(f(\theta))), \tilde{\gamma}_2(x)f(\theta))$, while the right-hand side maps (x, ξ, θ) to $(x, \xi + [\tilde{\gamma}_1'(x)]^*(-J_1(\theta)), \tilde{\gamma}_1(x)f(\theta))$.

The first components of the two sides of the equation are both equal to x. The λ-equivariance of f and the fact that $\tilde{\gamma}_2 = \lambda \circ \tilde{\gamma}_1$ yield immediately the equality of the third components. For the second components, we use the relation $\tilde{\gamma}_2 = \lambda \circ \tilde{\gamma}_1$ once again. Since λ is a homomorphism,

$$r_{\tilde{\gamma}_2^{-1}(x)} \circ \tilde{\gamma}_2 = r_{\tilde{\gamma}_1^{-1}(x)} \circ \lambda \circ \tilde{\gamma}_1 = \lambda \circ r_{\tilde{\gamma}_1^{-1}(x)} \circ \tilde{\gamma}_1.$$

Differentiating at x and transposing yields $[\tilde{\gamma}_2'(x)]^* = [\tilde{\gamma}_1'(x)]^* \circ \lambda^*$. Combining this with the equation $\lambda^* \circ J_2 \circ f = J_1$ (f is a mapping of hamiltonian spaces) gives $[\tilde{\gamma}_2'(x)]^*(-J_2(f(\theta)) = [\tilde{\gamma}_1'(x)]^*(-J_1(\theta))$, which implies the equality of the second components.

This completes the verification that $Y(\lambda, \phi, f)$, as constructed in terms of local trivializations, is well defined. The properties of $Y(\lambda, \phi, f)$ follow immediately from the construction. q.e.d.

All the ingredients are now in place to complete the proof of Theorem 12. Let $\chi_1, \cdots, \chi_d$ be commuting vector fields with compact support on $\mathbf{R}^d$ which are linearly independent at some point. (They can be constructed by pulling back the coordinate vector fields via a diffeomorphism with rapidly growing derivatives from a cube to all of $\mathbf{R}^d$.) Next, let $U_1, U_2, \cdots$ be a sequence of open cubes with disjoint closures in $\mathfrak{a}_+^*$ such that the origin is in the closure of the union of the U_i's. For each i, let $\alpha_{i,1}, \cdots, \alpha_{i,d}$ be commuting vector fields with compact support in U_i which are linearly independent at some point. Then if one chooses a sequence of numbers $c_1, c_2, \cdots$ which vanishes sufficiently rapidly, the commuting vector fields $\alpha_j = c_1\alpha_{1,j} + c_2\alpha_{2,j} + \cdots$ extend smoothly to $\mathfrak{a}^*$ and are linearly independent on a set whose closure contains the origin.

The vector fields $\alpha_1, \cdots, \alpha_d$ may be lifted in the obvious way to $\mathfrak{m}^* \times \mathfrak{a}^*$, where they are commuting infinitesimal Poisson automorphisms equivariant with respect to the action of M. By Theorem 3.2, these vector fields may be lifted to commuting infinitesimal Poisson automorphisms $\beta_1 \cdots \beta_d$ of $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}^*)$ whose support is contained in the closure of $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}_+^*)$ and which are linearly independent on a set whose closure contains $Y(M, K, \mathfrak{m}^* \times \{0\})$. Since, by Theorem 3.2 once again, there is a Poisson isomorphism from $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}^*)$ to $Y(P, G, \mathfrak{m}^* \times \mathfrak{a}^*)$ associated with the inclusions $M \subset P$ and $K \subset G$, $\beta_1, \cdots, \beta_d$ may be transferred to commuting infinitesimal Poisson automorphisms $\gamma_1, \cdots, \gamma_d$ of $\not{p} = Y(P, G, \mathfrak{m}^* \times \mathfrak{a}^*)$ whose support is the closure of $\not{p}_+ = Y(P, G, \mathfrak{m}^* \times \mathfrak{a}_+^*)$ and which are linearly independent on a set whose closure contains $Y(P, G, \mathfrak{m}^* \times \{0\})$.

Now, by Theorem 2.4, there is a diffeomorphism $\mathscr{E}$ from $\not{p}_+$ to a conic open subset of $\mathfrak{g}^*$. Using $\mathscr{E}$, we may transfer $\gamma_1, \cdots, \gamma_d$ to commuting infinitesimal Poisson automorphisms of $\mathscr{E}(\not{p}_+) \subset \mathfrak{g}^*$; as long as the sequence $c_1, c_2, \cdots$ decreases rapidly enough, these extend smoothly to infinitesimal automorphisms $\delta_1, \cdots, \delta_d$ of $\mathfrak{g}^*$.

In fact, the vector fields $\delta_1, \cdots, \delta_d$ are linearly independent modulo the tangent spaces of the symplectic leaves on a subset of $\mathfrak{g}^*$ whose closure contains the origin. To see this, it suffices to verify the same thing for $\gamma_1, \cdots, \gamma_d$ in $\not{p}_+$, or for $\beta_1, \cdots, \beta_d$ in $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}_+^*)$. Now in the local representation of $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}_+^*)$ as a product $T^*X \times \mathfrak{m}^* \times \mathfrak{a}_+^*$, the symplectic leaves are products of the form $T^*X \times$ (symplectic leaf in $\mathfrak{m}^*$) $\times$ (point in $\mathfrak{a}_+^*$), while the vector fields $\beta_1, \cdots, \beta_d$ point in the direction of the $\mathfrak{a}_+^*$ factor and are given by $\alpha_1, \cdots, \alpha_d$. It is now obvious that, on an open set whose closure contains the origin, these vector fields are linearly independent modulo the tangent spaces to the symplectic leaves.

Let π be the bivector field on $\mathfrak{g}^*$ representing the Lie-Poisson structure. For $d \geqslant 2$, we may form the bivector field $\pi + [\delta_1 \wedge \delta_2]$, which has the same 1-jet at 0 as π. According to the discussion in §2, $\pi + [\delta_1 \wedge \delta_2]$ is a Poisson structure whose rank is 2 greater than the rank of π on an open set whose closure contains the origin, so it cannot be locally equivalent to π. The proof of Theorem 1.2 is thus complete.

4. Equivalence with the Cartan motion algebra

In this section, we will prove and discuss the following theorem, which is a consequence of Theorem 3.2.

(4.1) Theorem. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition of a semisimple Lie algebra of noncompact type, and let $\mathfrak{g}_1 = \mathfrak{k} \times_s \mathfrak{s}$ be the corresponding Cartan motion algebra. Then there is a homogeneous K-equivariant Poisson isomorphism ι between conic open subsets $U \subset \mathfrak{g}^*$ and $U_1 \subset \mathfrak{g}_1^*$.*

Proof. The subset U is just the image $\mathscr{E}(\not{p}_+)$ discussed in the previous two sections. Recalling that $\not{p}_+ = Y(G, P, \mathfrak{m}^* \times \mathfrak{a}_+^*)$, we will now carry out a similar construction starting with $\mathfrak{g}_1$ instead of $\mathfrak{g}$, which will arrive once again at $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}_+^*)$.

Corresponding to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, we have the decomposition $\mathfrak{g}_1 = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_1$, where $\mathfrak{n}_1$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{s}$ with respect to the (K-invariant) Killing form of $\mathfrak{g}$. Notice that, by projection along $\mathfrak{k}$, $\mathfrak{n}_1$ is K-equivariantly isomorphic to $\mathfrak{n}$, and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}_1$ is K-equivariantly isomorphic to $\mathfrak{a} \oplus \mathfrak{n}$. Also note that $\mathfrak{m} \subset \mathfrak{k}$ is the centralizer of (the generic element of) $\mathfrak{a}$ whether we are looking in $\mathfrak{g}_1$ or in $\mathfrak{g}$.

Next, we form the "parabolic" subalgebra $\mathfrak{p}_1 = \mathfrak{m} \oplus \mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_1 \subset \mathfrak{g}_1$, which has the structure of the semidirect product $\mathfrak{m} \times_s \mathfrak{s}$. By analogy with what we did in $\mathfrak{g}$, we form the phase space $\mathscr{P}_{1+} = Y(P_1, G_1, \mathfrak{m}^* \times \mathfrak{a}^*_+)$, which admits a momentum mapping $\mathscr{E}_1$ into $\mathfrak{g}^*_1$. Since G_1/P_1 is naturally diffeomorphic to K/M, just as G/P was, Theorem 3.2 tells us that the Poisson manifold $Y(P_1, G_1, \mathfrak{m}^* \times \mathfrak{a}^*_+)$ is isomorphic to $Y(M, K, \mathfrak{m}^* \times \mathfrak{a}^*_+)$ and hence to $\mathscr{P}_+$ and to $\mathscr{E}(\mathscr{P}_+)$; all these isomorphisms are easily checked to be K-equivariant. Thus, if we let U_1 be $\mathscr{E}_1(\mathscr{P}_{1+})$, the appropriate composition of the isomorphisms just mentioned, we will have proven the theorem once we verify that $\mathscr{E}_1$ is an embedding.

This time it is simple enough to work directly with the coadjoint representation on $\mathfrak{g}^*_1$ itself. (In any case, we do not have a nondegenerate Killing form available to take us equivariantly over to $\mathfrak{g}_1$.) The construction of $Y(P_1, G_1, \mathfrak{m}^* \times \mathfrak{a}^*_+)$ begins, similarly to that of $Y(P, G, \mathfrak{m}^* \times \mathfrak{a}^*_+)$, with the Poisson manifold product $Q_1 = T^*G_1 \times \mathfrak{m}^* \times \mathfrak{a}^*_+$. Using the left translations in T^*G_1, we may identify Q_1 with $G_1 \times \mathfrak{g}^*_1 \times \mathfrak{m}^* \times \mathfrak{a}^*_+$ and then with $G_1 \times \mathfrak{k}^* \times \mathfrak{a}^* \times \mathfrak{n}^*_1 \times \mathfrak{m}^* \times \mathfrak{a}^*_+$. The momentum map for the action of P_1 on this product is $J_1 : (g, k, a, n, m, a') \mapsto (k|_{\mathfrak{m}} - m, a - a', n)$, from which we see that $J_1^{-1}(0) = \{(g, k, a, 0, k|_{\mathfrak{m}}, a)\}$ as a subset of $G_1 \times \mathfrak{g}^*_1 \times \mathfrak{m}^* \times \mathfrak{a}^*_+$. Thus the projection of $J_1^{-1}(0)$ into $G_1 \times \mathfrak{g}^*_1$ is an embedding onto $G_1 \times \mathfrak{k}^* \times \mathfrak{a}^*_+ \times \{0\}$. The momentum mapping $\mathscr{E}_1$ for the left action of G_1, given by *right* translations in T^*G_1, is (the quotient by P_1 of) $(g, \kappa, \alpha) \mapsto \mathrm{Ad}^*_g(\kappa, \alpha)$.

The injectivity of $\mathscr{E}_1$ on $Y(P_1, G_1, \mathfrak{m}^* \times \mathfrak{a}^*_+)$ is equivalent to Lemma 4.2 below. An infinitesimal version of the proof of the lemma also shows that the differential of $\mathscr{E}_1$ is an isomorphism at each point, so $\mathscr{E}_1$ is an embedding.

(4.2) Lemma. *If* $\mathrm{Ad}^*_g(\kappa + \alpha) = (\kappa' + \alpha')$ *for elements* (κ, α) *and* (κ', α') *of* $\mathfrak{m}^* \times \mathfrak{a}^*_+$ *and* $g \in G_1$, *then* g *belongs to* P_1.

Proof. The projection of $\mathfrak{g}^*_1$ onto $\mathfrak{s}^* = \mathfrak{a}^* \times \mathfrak{n}^*$, the dual of an ideal, is equivariant with respect to the coadjoint action, so we must have $\mathrm{Ad}^*_g(\alpha) = \alpha'$. We decompose g as ks, with $k \in K$ and $s \in S$ (the subgroup corresponding to $\mathfrak{s}$); since S is abelian, we must have $\mathrm{Ad}^*_k(\alpha) = \alpha'$. Since both α and α' belong to $\mathfrak{a}^*_+$, it follows from well-known properties of the action of K on $\mathfrak{s}^* \approx \mathfrak{s}$ (see [5, Chapter VII, §2]) that k belongs to M, and so $g = ks \in MS = P_1$.

Remarks. 1. The subset U_1 consists of all elements of $\mathfrak{g}_1^*$ whose projection into $\mathfrak{s}^*$ is regular, so it is dense. On the other hand, U (or, more precisely, its counterpart in $\mathfrak{g}$) consists just of those regular elements lying on maximally split Cartan subalgebras, and so it is generally not dense. The simplest example is given by $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, in which U is the exterior of the cone of nilpotent elements. The corresponding Cartan motion algebra is the euclidean algebra $\mathfrak{e}(2)$, in which U_1 is the complement of a line. The Poisson isomorphism ι maps hyperboloids of one sheet in $\mathfrak{sl}(2, \mathbf{R})$ symplectically onto cylinders in $\mathfrak{e}(2)^*$.

2. Using Theorem 4.1, we can give another description of the infinitesimal Poisson automorphisms which are used in the proof of Theorem 1.2. As is noted in the Appendix, the Lie-Poisson structure on the dual $\mathfrak{g}_1^* = (\mathfrak{k} \times_s \mathfrak{s})^*$ may be thought of as the semidirect product of $\mathfrak{k}^*$ with the trivial Poisson manifold $\mathfrak{s}^*$, so that the entire group of K-equivariant diffeomorphisms of $\mathfrak{s}^*$ acts as automorphisms of $(\mathfrak{k} \times_s \mathfrak{s})^*$, and so K-equivariant vector fields on $\mathfrak{s}^*$ lift to infinitesimal automorphisms of $(\mathfrak{k} \times_s \mathfrak{s})^*$. Since the generic K orbit in $\mathfrak{s}^*$ has codimension equal to the real rank d of $\mathfrak{g}$, one can find d commuting and linearly independent K-equivariant vector fields supported in the set of regular elements.

3. The map ι seems to be closely related to the "Gell-Mann formula" ([6], [13], [16]) which produces representations of $\mathfrak{g}$ from representations of $\mathfrak{g}_1$. Let $X_1, \cdots, X_a, Z_1, \cdots, Z_b$ be a basis of $\mathfrak{g}$, where the "compact generators" $X_1, \cdots, X_a$ lie in $\mathfrak{k}$ and the "noncompact generators" $Z_1, \cdots, Z_b$ lie in $\mathfrak{s}$. Let $Z_1', \cdots, Z_b'$ be a corresponding set of "abelian generators," so that $X_1, \cdots, X_a$, $Z_1', \cdots, Z_b'$ form a basis of $\mathfrak{g}_1$. A *Gell-Mann formula* for a particular representation ρ_1 of $\mathfrak{g}_1$ consists of a collection of polynomials $z_i = \zeta_i(x, z')$ with the property that, if $x_i = \rho_1(X_i)$ and $z_i' = \rho_1(Z_i')$ are the operators generating ρ_1, then the operators $x_1, \cdots, x_a, \zeta_1(x, z'), \cdots, \zeta_b(x, z')$ generate a representation ρ of $\mathfrak{g}$.

If we think of (x, z') and (x, z) as coordinates on $\mathfrak{g}_1^*$ and $\mathfrak{g}^*$ respectively, then $(x, z') \mapsto (x, \zeta(x, z'))$ becomes a map from $\mathfrak{g}_1^*$ to $\mathfrak{g}^*$. In fact, the map should be thought of as defined only on the coadjoint orbit in $\mathfrak{g}_1^*$ corresponding to the representation ρ_1. The polynomials used for different representations are different, but the various Gell-Mann formulas will combine to give a smooth mapping from a subset of $\mathfrak{g}_1^*$ to $\mathfrak{g}^*$. This should be a Poisson mapping, corresponding to the fact that the Gell-Mann formulas take representations to representations.

Theorem 4.1 produces, by geometric methods, a Poisson mapping of the kind just described. It would be very interesting to have an explicit formula for this mapping and to see if it could be converted into Gell-Mann formulas. It is likely that there will occur some problems of "quantization" type, since some information is lost when one passes from the universal enveloping algebra of $\mathfrak{g}$ to the Poisson algebra of functions on $\mathfrak{g}^*$. In fact, the work of Hermann [6] suggests that Gell-Mann formulas should exist only for certain $\mathfrak{g}$, while our Poisson map exists for all semisimple Lie algebras of noncompact type.

In the same vein, our results seem to be related to the work of Joseph [7] concerning the Gelfand-Kirillov conjecture on the structure of the quotient skew-field of a universal enveloping algebra. An isomorphism between two such fields is the "quantized" version of a rational isomorphism between open sets in Lie-Poisson manifolds.

4. Tudor Ratiu has suggested that the $\mathfrak{g}^* \leftrightarrow \mathfrak{g}_l^*$ Poisson correspondence may also be useful for understanding some of the basic integrability theorems for hamiltonian systems of Toda type, such as that of Kostant [9] and Symes [18].

Appendix. Poisson reduction

The reader is referred to [10] and [14] for further details regarding the material in this Appendix.

We begin with a definition of reduction for Poisson manifolds. For our purposes, it will be sufficient to consider reduction only at $0 \in \mathfrak{g}^*$. (The general case may be reduced with some effort to this one with the aid of the "shifting lemma"; see [15].)

(A.1) Lemma. *Let Ξ be a hamiltonian G-space with momentum mapping $K: \Xi \to \mathfrak{g}^*$. Then the restriction to $K^{-1}(0)$ of the Poisson bracket of any two G-invariant functions on Ξ depends only on the restriction of the functions to $K^{-1}(0)$.*

Proof. It suffices to show that the hamiltonian vector field for every G-invariant hamiltonian function on Ξ is tangent to $K^{-1}(0)$, but that is an immediate consequence of Noether's theorem (the conservation of K for any G-invariant hamiltonian system.) q.e.d.

If 0 is a weakly regular value of K so that $K^{-1}(0)$ is a manifold, and the action of G on $K^{-1}(0)$ is sufficiently regular so that $K^{-1}(0)/G$ is also a manifold, then $K^{-1}(0)/G$ with its Poisson structure inherited from Ξ/G (thanks to Lemma A.1) is denoted by Ξ_0. It is called the *reduced Poisson manifold of* Ξ *at* 0 (with respect to the action of G).

Poisson semidirect products. In §3, we need to analyze reduced manifolds of the form $(T^*G \times \Pi)_0$, where Π is a hamiltonian G space. It is interesting to consider for a moment the more general situation where Π is any Poisson manifold on which G acts by automorphisms. The quotient Poisson manifold $(T^*G \times \Pi)/G$, where G acts on the factor T^*G by the lifts of right translations, is diffeomorphic to $\mathfrak{g}^* \times \Pi$ under the correspondence $\mathscr{F}: [(g, \mu, \theta)] \mapsto (r_g^* \mu, g^{-1}\theta)$. The induced Poisson structure on $\mathfrak{g}^* \times \Pi$ is not the product structure, unless the action of G is trivial. In fact, the induced bracket is given by

$$\{A, B\}(\mu, \theta) = \langle \mu, [\delta A/\delta\mu, \delta B/\delta\mu] \rangle + \pi(\delta A/\delta\theta, \delta B/\delta\theta) \\ + (\delta A/\delta\mu)_\Pi \cdot B - (\delta B/\delta\mu)_\Pi \cdot A.$$

The π in the second term on the right is the tensor giving the Poisson structure on Π. In the last two terms, which represent the deviation of the bracket from the product bracket, $(\delta A/\delta\mu)_\Pi$ and $(\delta B/\delta\mu)_\Pi$ are the vector fields on Π corresponding to $\delta A/\delta\mu$ and $\delta B/\delta\mu$ in $\mathfrak{g}$. We denote the Poisson manifold $\mathfrak{g}^* \times \Pi$ with this induced Poisson bracket by $\mathfrak{g}^* \times_s \Pi$ and call it the *semidirect product of* $\mathfrak{g}^*$ *and* Π with respect to the action of G on Π. (Actually, it depends only on the infinitesimal action of $\mathfrak{g}$.) If $\mathfrak{h}$ is a Lie algebra on which $\mathfrak{g}$ acts by derivations, so that the dual action on $\mathfrak{h}^*$ is a Poisson action, then the Poisson semidirect product $\mathfrak{g}^* \times_s \mathfrak{h}^*$ is naturally isomorphic to the dual of the semidirect product Lie algebra $(\mathfrak{g} \times_s \mathfrak{h})^*$. A direct consequence of this observation is that the entire group of G-equivariant Poisson automorphisms of $\mathfrak{h}^*$ lifts to a group of Poisson automorphisms of $(\mathfrak{g} \times_s \mathfrak{h})^*$, so that the linear structure of $\mathfrak{h}^*$ is irrelevant for the semidirect product construction. In particular, if $\mathfrak{h}$ is abelian, then all the G-equivariant diffeomorphisms on $\mathfrak{h}^*$ lift to Poisson automorphisms of the semidirect product.

The next proposition (due to Krishnaprasad and Marsden [10]) shows that a momentum mapping serves to "decouple" the factors in a Poisson semidirect product.

(A.2) Proposition. *Let* Π *be a hamiltonian G-space with momentum mapping* $J: \Pi \to \mathfrak{g}^*$. *Then the diffeomorphism* $J': \mathfrak{g}^* \times_s \Pi \to \mathfrak{g}^* \times \Pi$ *defined by* $J'(\mu, \theta) = (\mu + J(\theta), \theta)$ *is an isomorphism between the semidirect product and direct product Poisson structures. Thus, the composite* $J^* \circ \mathscr{F}$ *is a Poisson isomorphism between the quotient manifold* $(T^*G \times \Pi)/G$ *and the product* $\mathfrak{g}^* \times \Pi$.

(A.3) Corollary. *If the Lie algebra* $\mathfrak{g}$ *acts on the Lie algebra* $\mathfrak{h}$ *by inner derivations via a homomorphism from* $\mathfrak{g}$ *to* $\mathfrak{h}$, *then the semidirect product algebra* $\mathfrak{g} \times_s \mathfrak{h}$ *is isomorphic to the direct product* $\mathfrak{g} \times \mathfrak{h}$.

It is now quite simple to analyze the reduced manifold $(T^*G \times \Pi)_0$. The momentum mapping $K: T^*G \times \Pi \to \mathfrak{g}^*$ is given by $K(g, \mu, \theta) = l_g^*\mu + J(\theta)$, so $K^{-1}(0)$ is defined by the equation $l_g^*\mu = -J(\theta)$. Under the correspondence $\mathscr{F}$, $(T^*G \times \Pi)_0 = K^{-1}(0)/G$ goes to the submanifold of $\mathfrak{g}^* \times_s \Pi$ defined by $\mu = -J(\theta)$. Finally, applying the Poisson isomorphism J' takes $(T^*G \times \Pi)_0$ to the submanifold of $\mathfrak{g}^* \times \Pi$ defined by $\mu = 0$. This is obviously a Poisson submanifold (corroborating Lemma A.1) isomorphic to Π. Thus we have proven:

(A.4) Proposition. *Let Π be a hamiltonian G-space with momentum mapping J. Then the reduced manifold $(T^*G \times \Pi)_0$ is naturally isomorphic to Π under the mapping $(g, -l^*_{(g-1)}J(\theta), \theta) \mapsto g\theta$.*

References

[1] J. F. Conn, *Normal forms for analytic Poisson structures*, Ann. of Math. (2) **119** (1984) 576–601.

[2] ______, *Normal forms for smooth Poisson structures*, Ann. of Math. (2) **121** (1985) 565–593.

[3] A. H. Dooley & J. W. Rice, *On contractions of semisimple Lie groups*, Trans. Amer. Math. Soc. **289** (1985) 185–202.

[4] J. J. Duistermaat, J. A. C. Kolk & V. S. Varadarajan, *Functions, flows, and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups*, Compositio Math. **49** (1981) 309–398.

[5] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[6] R. Hermann, *The Gell-Mann formula for representations of semisimple groups*, Comm. Math. Phys. **2** (1966) 155–164.

[7] A. Joseph, *Symplectic structure in the enveloping algebra of a Lie algebra*, Bull. Soc. Math. France **102** (1974) 75–83.

[8] D. Kazhdan, B Kostant & S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978) 481–508.

[9] B. Kostant, *On the solution to a generalized Toda lattice and representation theory*, Advances in Math. **34** (1979) 195–338.

[10] P. S. Krishnaprasad & J. E. Marsden, *Hamiltonian structures and stability for rigid bodies with flexible attachments*. I, Arch. Rat. Mech. Anal., to appear.

[11] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Differential Geometry **12** (1977) 253–300.

[12] S. Lie, *Theorie der Transformationsgruppen*, Leipzig, Teubner (Zweiter Abschnitt, unter Mitwirkung von Prof. Dr. Friedrich Engel), 1890.

[13] G. W. Mackey, *On the analogy between semisimple Lie groups and certain related semi-direct product groups*, *Lie groups and their representations* (I. M.Gelfand, ed.), Hilger, London, 1975, 339–363.

[14] J. Marsden & T. Ratiu, *Reduction of Poisson manifolds*, Lett. Math. Phys. **11** (1986) 161–169.

[15] J. Marsden, T. Ratiu & A. Weinstein, *Semidirect products and reduction in mechanics*, Trans. Amer. Math. Soc. **281** (1984) 147–177.

[16] N. Mukunda, *Expansion of Lie groups and representations of* SL(3, C), J. Math. Phys. **10** (1969) 897–911.

[17] S. Sternberg, *Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977) 5253–5254.

[18] W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math. **59** (1980) 13–51.

[19] G. Warner, *Harmonic analysis on semisimple Lie groups*. I, Springer, New York, 1972.

[20] A. Weinstein, *A universal phase space for particles in Yang-Mills fields*, Lett. Math. Phys. **2** (1978) 417–420.

[21] ______, *Symplectic geometry*, Bull. Amer. Math. Soc. (N.S.) **5** (1981) 1–13.

[22] ______, *Sophus Lie and symplectic geometry*, Expositiones Math. **1** (1983) 95–96.

[23] ______, *The local structure of Poisson manifolds*, J. Differential Geometry **18** (1983) 523–557.

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