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FOUR-MANIFOLDS WITH POSITIVE CURVATURE OPERATOR

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1. A compact surface with positive mean scalar curvature must be diffeomorphic to the sphere S^2 or the real projective space RP^2. A compact three-manifold with positive Ricci curvature must be diffeomorphic to the sphere S^3 or a quotient of it by a finite group of fixed point free isometries in the standard metric, such as the real projective space RP^3 or a lens space $L^3_{p,q}$. This was proven in [1]. Our main result is the following generalization to four dimensions.

1.1. Theorem. *A compact four-manifold with a positive curvature operator is diffeomorphic to the sphere S^4 or the real projective space RP^4.*

Here we regard the Riemannian curvature tensor $\mathrm{Rm} = \{R_{ijkl}\}$ as a symmetric bilinear form on the two-forms Λ^2 by letting

$$\mathrm{Rm}(\phi, \psi) = R_{ijkl}\phi_{ij}\psi_{kl}.$$

We say the manifold has a positive curvature operator if $\mathrm{Rm}(\phi, \phi) > 0$ for all two-forms $\phi \neq 0$, and a nonnegative curvature operator if $\mathrm{Rm}(\phi, \phi) \geqslant 0$ for all ϕ.

These results extend to the case of nonnegative curvature. A compact surface with nonnegative mean scalar curvature must be diffeomorphic to a quotient of the sphere S^2 or the plane R^2 by a group of fixed-point free isometries in the standard metrics. The examples are the sphere S^2, the real projective space RP^2, the torus $T^2 = S^1 \times S^1$, and the Klein bottle $K^2 = RP^2 \# RP^2$ (where # denotes the connected sum).

1.2. Theorem. *A compact three-manifold with nonnegative Ricci curvature is diffeomorphic to a quotient of one of the spaces S^3 or $S^2 \times R^1$ or R^3 by a group of fixed point free isometries in the standard metrics.*

The quotients of $S^2 \times R^1$ include $S^2 \times S^1$, $RP^2 \times S^1$, the unoriented S^2 bundle over S^1, and the connected sum $K^3 = RP^3 \# RP^3$. The quotients of R^3 are the torus T^3 and five other flat three-manifolds.

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1.3. Theorem. *A compact four-manifold with nonnegative curvature operator is diffeomorphic to a quotient of one of the spaces S^4 or $\mathbf{C}P^2$ or $S^3 \times R^1$ or $S^2 \times S^2$ or $S^2 \times R^2$ or R^4 by a finite group of fixed point free isometries in the standard metric.*

The only quotient of S^4 is RP^4, and there are no quotients of the complex projective space $\mathbf{C}P^2$. The quotients of $S^3 \times R^1$ include $L^3_{p,q} \times S^1$, where $L^3_{p,q}$ is a lens space, and this gives an infinite number of examples, and also include the unoriented S^3 bundle over S^1 and the connected sum $K^4 = RP^4 \# RP^4$. The quotients of $S^2 \times S^2$ include $S^2 \times RP^2$ and $RP^2 \times RP^2$ and another space where Z_2 acts on $S^2 \times S^2$ as the antipodal map on each factor simultaneously. The quotients of $S^2 \times R^2$ include $RP^2 \times T^2$, $S^2 \times K^2$, $RP^2 \times K^2$, and unoriented S^2 bundles over T^2 and K^2. The quotients of R^4 give the torus T^4 and all the other flat four-manifolds. This gives a rich variety of examples. It should be possible to find a complete classification.

These theorems are all proved by considering the parabolic Einstein equation

$$\frac{\partial}{\partial t} g = \frac{2}{n} r g - 2\mathrm{Rc}$$

on a compact manifold N, where $r = \int R / \int 1$ is the mean scalar curvature, Rc is the Ricci tensor, and n is the dimension. For a compact three-manifold with positive Ricci curvature or a compact four-manifold with positive curvature operator, the solution exists for all time t and converges as $t \to \infty$ to a metric of a constant Riemannian curvature. For nonnegative curvature we find that unless the curvature becomes strictly positive, the metric has a restricted holonomy group whose Lie algebra is the image of the curvature operator, and this allows us to identify the manifold.

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2. Most of the proof proceeds as in [1] and its generalization by Huisken [2] to higher dimensions. The critical new part is to show that under the unnormalized evolution

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc} \tag{2.1}$$

the Riemannian curvature tensor must pinch toward a multiple of the identity as the scalar curvature R blows up, in the sense that $|\widetilde{\mathrm{Rm}}| \leqslant C R^{1-\delta}$ for some $\delta > 0$, where $\widetilde{\mathrm{Rm}}$ is the traceless part of the Riemannian curvature tensor. When this estimate holds, the rest of the proof goes through unchanged.

To see that this happens, we must study the evolution of the curvature tensor. From [1] we have

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right)$$
$$-\left(R_{pi}R_{pjkl} + R_{pj}R_{ipkl} + R_{pk}R_{ijpl} + R_{pl}R_{ijkp}\right),$$

where $B_{ijkl} = R_{ipjq}R_{kplq}$.

The first step in simplifying these equations is a trick we learned from Karen Uhlenbeck. We pick an abstract vector bundle V isomorphic to the tangent bundle TN, but with a fixed metric h_{ab} on the fibers. Then we choose (in an arbitrary manner) an isometry $u = \{u_a^i\}$ between V and TM at time $t = 0$. We let the isometry evolve by the equation

$$\frac{\partial}{\partial t}u_a^i = g^{ij}R_{jk}u_a^k.$$

Then the pull-back metric $h_{ab} = g_{ij}u_a^i u_b^j$ remains constant in time, since its time derivative is zero, and u remains an isometry between the varying metric g on TN and the fixed metric h on V. Then we use u to pull back the curvature tensor to a tensor on V

$$R_{abcd} = R_{ijkl}u_a^i u_b^j u_c^k u_d^l.$$

We can also pull back the Levi-Civita connection $\Gamma = \{\Gamma_{jk}^i\}$ on TN to a connection $\Delta = \{\Delta_{jc}^a\}$ on V, where the covariant derivative of a section $v = \{v^a\}$ of V is given locally by

$$D_i v^a = \frac{\partial}{\partial x^i}v^a + \Delta_{ib}^a v^b.$$

Then we may take the covariant derivative of any tensors of V and TM. In particular we have $D_i u_a^j = 0$ and $D_i h_{ab} = 0$. We form the Laplacian of any tensor as the trace of the second covariant derivative. Thus

$$\Delta R_{abcd} = g^{ij}D_i D_j R_{abcd}.$$

Under this transformation something magical happens—the last terms in the curvature evolution equation disappear! We get

$$\frac{\partial}{\partial t}R_{abcd} = \Delta R_{abcd} + 2\left(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}\right),$$

where $B_{abcd} = R_{aebf}R_{cedf}$. This transformation enlarges the invariance group of the equation. We now have not only the diffeomorphism group of the manifold M but also the gauge group of isometries of the bundle V. Any change in the initial isometry u will just be tracked by all the subsequent isometries. Needless to say this small increase in conceptual difficulty is well rewarded by the great decrease in the difficulties of computation.

Next we need to understand the quadratic terms. Using the first Bianchi identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

we get

$$R_{abef}R_{cdef} = 2(B_{abcd} - B_{abdc}).$$

Hence we recognize the first part of the quadratic terms as being just the square of the curvature operator.

In order to understand the other part, we need to regard the two-forms Λ^2 on V as the Lie algebra $so(n)$ of the Lie group of rotations of V. Choose a local chart on V where h_{ab} is the identity. The metric on Λ^2 is given by $|\phi|^2 = \langle\phi, \phi\rangle$, where

$$\langle\phi, \psi\rangle = \phi_{ab}\psi_{ab}$$

and the Lie bracket is given by

$$[\phi, \psi]_{ab} = \phi_{ac}\psi_{bc} - \psi_{ac}\phi_{bc}.$$

It is easy to check that the tri-linear form $\langle[\phi, \psi], \omega\rangle$ is fully antisymmetric. Choose an orthonormal basis $\phi^\alpha = \{\phi^\alpha_{ab}\}$ for the 2-forms on V. Then the inner product on $\Lambda^2(V)$

$$h_{\alpha\beta} = \langle\phi^\alpha, \phi^\beta\rangle$$

is the identity matrix in the local chart. The Lie bracket is given by

$$[\phi^\alpha, \phi^\beta] = c^{\alpha\beta}_\gamma\phi^\gamma,$$

where the $c^{\alpha\beta}_\gamma$ are the Lie structure constants relative to this basis. Note that $c^{\alpha\beta\gamma} = c^{\alpha\beta}_\delta h^{\gamma\delta}$ is fully antisymmetric since

$$c^{\alpha\beta\gamma} = \langle[\phi^\alpha, \phi^\beta], \phi^\gamma\rangle.$$

The tensor R_{abcd} on V may be regarded as a symmetric bilinear form $M_{\alpha\beta}$ on $\Lambda^2(V)$, where

$$R_{abcd} = M_{\alpha\beta}\phi^\alpha_{ab}\phi^\beta_{cd}.$$

There is a bilinear operation on the $M_{\alpha\beta}$ given by

$$(M\#N)_{\alpha\beta} = c_{\alpha\gamma\eta}c_{\beta\delta\theta}M_{\gamma\delta}N_{\eta\theta}.$$

Clearly $M\#N = N\#M$. In terms of the Lie algebra it is uniquely determined by the condition

$$(\phi \otimes \phi) \# (\psi \otimes \psi) = [\phi, \psi] \otimes [\phi, \psi].$$

Let us write $M^\# = M\#M$ for simplicity. Then $M^\#$ corresponds to the tensor

$$R^\#_{abcd} = M^\#_{\alpha\beta}\phi^\alpha_{ab}\phi^\beta_{cd},$$

where $R^{\#}_{abcd} = 2(B_{acbd} - B_{adbc})$. Consequently we can write the equation for the evolution of the curvature tensor as

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + R^2_{abcd} + R^{\#}_{abcd}$$

or equivalently as

$$\frac{\partial}{\partial t} M_{\alpha\beta} = \Delta M_{\alpha\beta} + M^2_{\alpha\beta} + M^{\#}_{\alpha\beta}.$$

We abbreviate the last equation as

$$\frac{\partial}{\partial t} M = \Delta M + M^2 + M^{\#}.$$

Notice that while neither M^2 nor $M^{\#}$ satisfies the Bianchi identity, their sum does.

We can get a better feel for this equation by considering the operation $M^{\#}$ in dimensions 3 and 4 in more detail. In dimension 3 the Lie structure constants $c_{\alpha\beta\gamma}$ are given by $1/\sqrt{2}$ times the volume form of the metric. Hence the matrix $M^{\#}$ is just the adjoint matrix

$$M^{\#} = \det M \cdot {}^{t}M^{-1}$$

or to be explicit

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}^{\#} = \begin{pmatrix} ek - fh & fg - dk & dh - eg \\ ch - bk & ak - cg & bg - ah \\ bf - ce & cd - af & ae - bd \end{pmatrix}.$$

In dimension 4 the Lie algebra $so(4)$ splits as a direct sum of two copies of $so(3)$. The volume form μ_{abcd} induces the star operation

$$\omega^*_{ab} = \tfrac{1}{2}\mu_{abcd}\omega_{cd}.$$

Since $\omega^{**} = \omega$, we get an orthogonal decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ into the eigenspaces of star with eigenvalues ± 1. This gives a block decomposition of M as

$$M = \begin{pmatrix} A & B \\ {}^{t}B & C \end{pmatrix}$$

and we can compute

$$M^{\#} = 2\begin{pmatrix} A^{\#} & B^{\#} \\ {}^{t}B^{\#} & C^{\#} \end{pmatrix},$$

where $A^\#$, $B^\#$, $C^\#$ are the adjoints of the 3×3 submatrices as before. This makes the evolution of the curvature break up into the three equations

$$(*) \qquad \begin{aligned} \frac{\partial}{\partial t} A &= \Delta A + A^2 + 2A^\# + B{}^tB, \\ \frac{\partial}{\partial t} B &= \Delta B + AB + BC + 2B^\#, \\ \frac{\partial}{\partial t} C &= \Delta C + C^2 + 2C^\# + {}^tBB. \end{aligned}$$

The reason we can handle dimension 4 is because we have this explicit decomposition. Note that the Bianchi identity says that $\operatorname{tr} A = \operatorname{tr} C$. Since $\operatorname{tr} A^2 + 2 \operatorname{tr} A^\# = (\operatorname{tr} A)^2$ and $\operatorname{tr} B{}^tB = |B|^2 = \operatorname{tr} {}^tBB$, it is easy to see that the Bianchi identity is preserved. Indeed,

$$\frac{\partial}{\partial t} \operatorname{tr} A = \Delta \operatorname{tr} A + (\operatorname{tr} A)^2 + |B|^2, \qquad \frac{\partial}{\partial t} \operatorname{tr} C = \Delta \operatorname{tr} C + (\operatorname{tr} C)^2 + |B|^2$$

so $\operatorname{tr} A$ and $\operatorname{tr} C$ satisfy the same equation.

3. Before proceeding, we make some remarks we shall need later on functions which are not quite differentiable. If $f(t)$ is a Lipshitz function of t, we say

$$\frac{df}{dt} \leqslant c \quad \text{if } \limsup_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \leqslant c,$$

where we take the lim sup of all forward difference quotients. Likewise we say

$$\frac{df}{dt} \geqslant c \quad \text{if } \liminf_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \geqslant c$$

taking the lim inf of all forward difference quotients. This is a useful notion because of the following result.

3.1. Lemma. *If $f(a) \leqslant 0$ and $df/dt \leqslant 0$ when $f \geqslant 0$ on $a \leqslant t \leqslant b$, then $f(b) \leqslant 0$.*

Proof. Pick $\varepsilon > 0$. We shall show $f(t) \leqslant \varepsilon t$. Since

$$\limsup_{h \searrow 0} \frac{f(h) - f(c)}{h} \leqslant 0,$$

there must be some interval $0 \leqslant t < \delta$ on which $f(t) \leqslant \varepsilon t$. Let $0 \leqslant t < c$ be the largest such interval with $c \leqslant b$. Then by continuity $f(t) \leqslant \varepsilon t$ on the closed interval $0 \leqslant t \leqslant c$. If $c < b$, we can find $\delta > 0$ with $f(t) \leqslant \varepsilon t$ on $0 \leqslant t < c + \delta$, since

$$\limsup_{h \searrow 0} \frac{f(c+h) - f(c)}{h} \leqslant 0.$$

Therefore $c = b$. Since $f(t) \leqslant \varepsilon t$ on $0 \leqslant t \leqslant b$ for all $\varepsilon > 0$, we have $f(b) \leqslant 0$.

3.2. Corollary. *If $f(a) \geqslant 0$ and $df/dt \geqslant 0$ on $a \leqslant t \leqslant b$, then $f(b) \geqslant 0$.*

3.3. Corollary. *If $f(a) \leqslant 0$ and $df/dt \leqslant cf$ on $a \leqslant t \leqslant b$, then $f(b) \leqslant 0$.*

Proof. Let $g = e^{-ct}f$. Then $dg/dt \leqslant 0$.

We say $df/dt \leqslant dg/dt$ if

$$\limsup_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \leqslant \liminf_{h \searrow 0} \frac{g(t+h) - g(t)}{h}.$$

3.4. Corollary. *If $f(a) \leqslant g(a)$ and $df/dt \leqslant dg/dt$ on $a \leqslant t \leqslant b$, then $f(b) \leqslant g(b)$.*

Proof. Let $h = f - g$. Let g be a smooth function of a real variable t and another variable $y \in R^k$ and let $f(t) = \sup\{g(t, y)\colon y \in Y\}$, where Y is a compact set. Then $f(t)$ is Lipshitz, and we have the following very useful estimate on its derivative.

3.5. Lemma.

$$\frac{d}{dt} f(t) \leqslant \sup\left\{\frac{\partial}{\partial t} g(t, y)\colon y \in Y(t)\right\},$$

where $Y(t) = \{y\colon g(t, y) = f(t)\}$.

Proof. Choose a sequence of times t_j decreasing to t for which

$$\lim_{t_j \searrow t} \frac{f(t_j) - f(t)}{t_j - t}$$

equals the lim sup. Choose $y_j \in Y$ with $f(t_j) = g(t_j, y_j)$. This is possible, for since Y is compact the maximum is attained. By passing to a subsequence we can assume $y_j \to y$. By continuity $f(t) = g(t, y)$, so $y \in Y(t)$. Since $g(t, y_j) \leqslant g(t, y)$ we have

$$f(t_j) - f(t) \leqslant g(t_j, y_j) - g(t, y_j).$$

By the mean value theorem we can find T_j between t_j and t with

$$\frac{g(t_j, y_j) - g(t, y_j)}{t_j - t} = \frac{\partial}{\partial t} g(T_j, y_j).$$

Since $T_j \to t$ also we have

$$\lim_{t_j \searrow t} \frac{f(t_j) - f(t)}{t_j - t} \leqslant \frac{\partial}{\partial t} g(t, y).$$

This proves the result.

4. To show the pinching result for the curvature tensor we need to use a form of the maximum principle for systems of equations. The basic idea here was suggested to us by Moe Hirsch. The effect of the heat equation is to average out the system of functions. Hence if the system lies in a convex set to start, it will remain there. We start with a simple version.

Let M be a compact manifold with a Riemannian metric g, and let $f = \{f^\alpha\}$ be a system of k functions on M. We regard f as a map of M into R^k. Let U be an open subset of R^k and let ϕ: $U \subset R^k \to R^k$ be a smooth vector field on U. We let g and ϕ depend on time also. Then we consider the nonlinear heat equation

$$\text{(PDE)} \qquad \frac{\partial f}{\partial t} = \Delta f + \phi(f)$$

with $f = f_0$ at $t = 0$, and we suppose it has a solution for some time interval $0 \leqslant t \leqslant T$. We let X be a closed convex subset of $U \subset R^k$ containing the initial data f_0, and ask when the solution remains in X. To answer this we study the ordinary differential equation

$$\text{(ODE)} \qquad \frac{df}{dt} = \phi(f)$$

on $U \subseteq R^k$, and ask when its solutions remain in X. We define the tangent cone $T_f X$ to the closed convex set X at a point $f \in \partial X$ as the smallest closed convex cone with vertex at f which contains X. It is the intersection of all the closed half-spaces containing X with f on the boundary of the half-space.

4.1. Lemma. *The solutions of the ODE $df/dt = \phi(f)$ which start in the closed convex set X will remain in X if and only if $\phi(f) \in T_f X$ for all $f \in \partial X$.*

Proof. We say that a linear function l on R^k is a support function for X at $f \in \partial X$ and write $l \in S_f X$ if $|l| = 1$ and $l(f) \geqslant l(k)$ for all other $k \in X$. Then $\phi(f) \in T_f X$ if and only if $l(\phi(f)) \leqslant 0$ for all $l \in S_f X$. Suppose $l(\phi(f)) > 0$ for some $l \in S_f X$. Then

$$\frac{d}{dt} l(f) = l\left(\frac{df}{dt}\right) = l(\phi(f)) > 0$$

so $l(f)$ is increasing and f cannot remain in X.

To see the converse, first note that we may assume X is compact. This is because we can modify the vector field $\phi(f)$ by multiplying by a cutoff function which is everywhere nonnegative, equals one on a large ball, and equals zero on a larger ball. The paths of solutions are unchanged inside the first ball. Then we can intersect X with the second ball to make X convex and compact. If there were a counterexample before the modification there would still be one afterward.

Let $s(f)$ be the distance from f to X, with $s(f) = 0$ if $f \in X$. Then

$$s(f) = \sup\{l(f - k)\},$$

where the sup is over all $k \in \partial X$ and all $l \in S_k X$. This defines a compact subset Y of $R^k \times R^k$. Hence by Lemma 3.5

$$\frac{d}{dt} s(f) \leqslant \sup\{l(\phi(f))\},$$

where the sup is over all pairs (k, l) with $k \in \partial X$, $l \in S_k X$, and

$$s(f) = l(f - k).$$

This can happen only when k is the unique closest point in X to f and l is the linear function of length 1 with gradient in the direction $f - k$.

We now use the fact that ϕ is smooth to estimate

$$|\phi(f) - \phi(k)| \leqslant C|f - k|$$

for some constant C and all f and k in X. Then since $l(\phi(k)) \leqslant 0$ by hypothesis and $|f - k| = s(f)$ we have $(d/dt)s(f) \leqslant Cs(f)$. Since $s(f) = 0$ to start, it must remain 0. This proves the lemma.

Now we prove the following result.

4.2. Theorem. *If the solution of the ODE stays in X, then the solution of the PDE stays in X.*

Proof. As before we may assume X is compact. Again we let $s(f)$ be the distance of $f \in R^k$ from X and let

$$s(t) = \sup_x s(f(x,t)) = \sup l(f(x,t) - k),$$

where the latter sup is over all $x \in N$, all $k \in \partial X$, and all $l \in S_k X$. Since this set is compact, we can use Lemma 3.5 to see that

$$\frac{d}{dt} s(t) \leqslant \sup \frac{d}{dt} l(f(x,t) - k),$$

where the sup is over all x, k, l as above with $l(f(x,t) - k) = s(t)$. Then x is some point in N where $f(x,t)$ is furthest from X, k is the unique closest point in X to $f(x,t)$, and l is the linear function of length 1 with gradient in the direction from k to $f(x,t)$. Now

$$\frac{d}{dt} l(f(x,t) - k) = l(\Delta f) + l(\phi(f)).$$

Since $l(f(x,t))$ has its maximum at x, the term $l(\Delta f) = \Delta l(f) \leqslant 0$. As before $l(\phi(k)) \leqslant 0$ and

$$l(\phi(f)) \leqslant |\phi(f) - \phi(k)| \leqslant C|f - k| = Cs(t)$$

for some constant C. Thus $(d/dt)s(t) \leqslant Cs(t)$ and since $s(t) = 0$ to start it remains so. But this shows that $f(x,t)$ remains in X.

Next we generalize this result to vector bundles. Let V be a vector bundle over a compact manifold M, and suppose V has a fixed metric h. Let g be a metric on M, and A a connection on V compatible with h. Both g and A may depend on time t. We can form the Laplacian of a section f of V as the trace of the second covariant derivative with respect to g, using the connection A on V and the Levi-Civita connection Γ on TM. Let V be an open subset of V and let $\phi(f)$ be a vector field on V tangent to the fibers. Then we can form the nonlinear heat equation

$$\text{(PDE)} \qquad \frac{\partial f}{\partial t} = \Delta f + \phi(f).$$

Let X be a closed subset of $U \subseteq V$. We ask when solutions of the PDE which start in X will remain in X. We need to impose the conditions that X is invariant under parallel translation by the connection A at each time, and that each fiber of X is convex. Then we can judge the behavior of the PDE by comparing to the ODE's

$$\text{(ODE)} \qquad \frac{df}{dt} = \phi(f)$$

in each fiber.

4.3. Theorem. *If the solutions of the ODE's in each fiber remain in X, then the solutions of the PDE remain in X.*

Proof. Again modifying the equation we can assume X is compact. Using the metric h in the fiber and writing $|f - k|$ for the Euclidean distance from f to k in the metric h, we let $s(t)$ be the maximum distance of any $f(x,t)$ from the set X. Then

$$s(t) = \sup l(f(x,t) - k),$$

where the sup is taken over all $x \in N$, all $k \in \partial X$ in the fiber over x, and all support functions $l \in S_k X$ at k in the fiber at x. The set of all such pairs (k, l) is a compact subset of $V \oplus V^*$. Then as before

$$\frac{d}{dt} s(t) \leqslant \sup \frac{d}{dt} l(f(x,t) - k),$$

where the sup is over all x where the distance in the fiber from $f(x,t)$ to x is maximal, k is the unique closest point in X to $f(x,t)$, and l is the linear function of length 1 on the fiber of V at x with gradient in the direction from k to $f(x,t)$. Again

$$\frac{d}{dt} l(f(x,t) - k) = l(\Delta f) + l(\phi(f))$$

and since $l(\phi(h)) \leqslant 0$ by hypothesis

$$l(\phi(f)) \leqslant |\phi(f) - \phi(h)| \leqslant C(f-h) = Cs(t),$$

where C is some constant bounding the first derivative of ϕ on a neighborhood of X. We claim that $l(\Delta f) \leqslant 0$ also. This will show

$$\frac{d}{dt}s(t) \leqslant Cs(t)$$

and completes the proof.

If we extend a vector in a bundle from a point x by parallel translation along geodesics emanating radially out of x, we get a smooth section of the bundle such that all the symmetrized covariant derivatives at x are zero. We extend $k \in V$ and $l \in V^*$ in this manner. Since the metric in V is invariant we continue to have $|l| = 1$, and since X is invariant under parallel translation we continue to have $k \in \partial X$ and l a support function for X at k. Therefore

$$l(f(x,t) - k) \leqslant s(t)$$

in the neighborhood. It follows that $l(f(x,t) - k)$ has its maximum at x, so

$$\Delta l(f(x,t) - k) \leqslant 0$$

at x. But k and l have all their symmetrized covariant derivatives equal to zero at x, so $l(\Delta f) \leqslant 0$ at x. This completes the proof.

In our applications we have a principal G-bundle P over M where G is a compact Lie group, E is a vector space with a metric and G acts on E preserving the metric, ϕ is a G-invariant vector field on E, and Z is a closed convex subset of E invarient under ϕ. Then solutions of the equation

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f)$$

for vectors f in $P \times_G E$ remain in the set $X = P \times_G Z$.

5. By the results in §2 we can reduce the study of the evolution equation for the metric to the study of the equation

$$\frac{\partial M}{\partial t} = \Delta M + M^2 + M^\#$$

for the symmetric bilinear form M on the Lie algebra $so(n)$. All we need is to show that if $\tilde{M}$ is the traceless part of M, then $|\tilde{M}| \leqslant C|M|^{1-\delta}$ for some $\delta > 0$ and some constant c.

Let E be the vector space of symmetric bilinear forms M on the Lie algebra $so(n)$.

5.1. Definition. *We say that a subset $Z \subseteq E$ is a pinching set if*

(1) *Z is closed and convex.*

(2) *Z is invariant under the action of the Lie group $O(n)$.*

(3) *Z is invariant under the flow of the ODE*

$$\frac{dM}{dt} = M^2 + M^\#.$$

(4) $|\tilde{M}| \leqslant C|M|^{1-\delta}$ *for some C and all $M \in Z$.*

We say that an open subset U in $so(n)$ satisfies the pinching condition if every compact subset of U is contained in a pinching set Z as above.

5.2. Convergence criterion. If U satisfies the pinching condition, and $\operatorname{tr} M > 0$ for all $M \in Z$, then every metric whose curvature lies in U will evolve as $t \to \infty$ to a metric of constant positive curvature.

Proof. Let P be the principal tangent bundle and form the associated bundle $V = P \times_G E$, where $G = O(n)$ and E is the vector space of symmetric bilinear forms on $so(n)$. Since the manifold is compact, we can find a pinching set Z as above such that at time $t = 0$ the curvature operator M lies in $X = P \times_G Z$. Then it will remain in X by the argument in §4. This gives us the required pinching estimate $|\tilde{M}| \leqslant C|M|^{1-\delta}$.

To demonstrate how this works in practice, we shall first reprove the pinching result for three-manifolds.

Let us study the ordinary differential equation

$$\frac{d}{dt}M = M^2 + M^{\#}$$

on 3×3 symmetric matrices M. We can diagonalize M with eigenvalues $m_1 \leqslant m_2 \leqslant m_3$. Then M^2 and $M^{\#}$ are also diagonal, so M remains diagonal. We get the three equations

$$\frac{d}{dt}m_1 = m_1^2 + m_2 m_3, \quad \frac{d}{dt}m_2 = m_2^2 + m_1 m_3, \quad \frac{d}{dt}m_3 = m_3^2 + m_1 m_2.$$

Note that

$$\frac{d}{dt}(m_2 - m_1) = (m_2 - m_1)(m_2 + m_1 - m_3)$$

so that if $m_1 \leqslant m_2$ to start it remains so. Hence the inequalities $m_1 \leqslant m_2 \leqslant m_3$ persist. Nonnegative sectional curvature corresponds to $m_1 \geqslant 0$, and this inequality is clearly preserved since if $0 \leqslant m_1 \leqslant m_2 \leqslant m_3$, then

$$\frac{d}{dt}m_1 = m_1^2 + m_2 m_3 \geqslant 0.$$

Nonnegative Ricci curvature corresponds to the inequality $m_1 + m_2 \geqslant 0$, and this inequality is also preserved. For note that $2m_2 \geqslant m_1 + m_2 \geqslant 0$ so $m_3 \geqslant m_2 \geqslant 0$. Then

$$\frac{d}{dt}(m_1 + m_2) = m_1^2 + m_2^2 + m_3(m_1 + m_2) \geqslant 0.$$

Theorem. *For any C we can choose $\delta > 0$ small enough so that for any K the closed convex set defined by the inequalities*

(a) $m_1 + m_2 \geqslant 0$,

(b) $m_2 + m_3 \leqslant C(m_1 + m_2)$,

(c) $m_3 - m_1 \leqslant K(m_1 + m_2 + m_3)^{1-\delta}$.

is preserved by the flow of the differential equation.

Proof. We have already seen that (a) is preserved by itself. For (b) we compute

$$\frac{d}{dt}\log(m_1 + m_2) = m_1 + m_3 + \frac{m_2(m_2 - m_1)}{m_1 + m_2} \geqslant m_1 + m_3,$$

$$\frac{d}{dt}\log(m_2 + m_3) = m_1 + m_3 - \frac{m_2(m_3 - m_2)}{m_2 + m_3} \leqslant m_1 + m_3.$$

This shows that the ratio of $m_1 + m_2$ to $m_2 + m_3$ improves.

To see that (c) is preserved, we compute

$$\frac{d}{dt}(m_3 - m_1) = (m_3 - m_1)(m_3 + m_1 - m_2),$$

$$\frac{d}{dt}(m_1 + m_2 + m_3) = (m_1 + m_2 + m_3)(m_3 + m_1 - m_2) + m_2^2 + m_2(m_1 + m_2) + m_3(m_2 - m_1).$$

Now $m_2 \geqslant 0$, $m_1 + m_2 \geqslant 0$, $m_3 \geqslant 0$, and $m_2 - m_1 \geqslant 0$. Therefore

$$\frac{d}{dt}\log(m_3 - m_1) = m_3 + m_1 - m_2,$$

$$\frac{d}{dt}\log(m_1 + m_2 + m_3) \geqslant m_3 + m_1 - m_2 + \frac{m_2^2}{m_1 + m_2 + m_3}.$$

When (b) holds

$$m_3 \leqslant m_2 + m_3 \leqslant C(m_1 + m_2) \leqslant 2Cm_2,$$

$$m_3 + m_1 - m_2 \leqslant m_1 + m_2 + m_3 \leqslant 6Cm_2$$

and hence with $\varepsilon = 1/36C^2$

$$\frac{d}{dt}\log(m_1 + m_2 + m_3) \geqslant (1 + \varepsilon)(m_3 + m_1 - m_2).$$

Therefore with $1 - \delta = 1/(1 + \varepsilon)$ the ratio of $m_3 - m_1$ to $(m_1 + m_2 + m_3)^{1-\delta}$ improves.

For any compact set of M with $m_1 + m_2 > 0$ we can find a set of the form (a)–(c) which contains it. Then the inequality $|\tilde{M}| \leqslant C|M|^{1-\delta}$ holds on this set.

6. Now we consider the case of a four-manifold. The Lie algebra decomposes as $so(4) = so(3) \oplus so(3)$ corresponding to $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, and the matrix M decomposes as

$$M = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}.$$

The evolution equation for M,

$$\frac{dM}{dt} = M^2 + M^{\#},$$

gives rise to the system of equations

$$\frac{dA}{dt} = A^2 + B\,{}^t\!B + 2A^{\#},$$
$$\frac{dB}{dt} = AB + BC + 2B^{\#},$$
$$\frac{dC}{dt} = C^2 + {}^t\!BB + 2C^{\#},$$

where A, B, and C are all 3×3 matrices and $A^{\#} = (\det A)\ {}^t\!A^{-1}$ is the adjoint. Recall that $\operatorname{tr} A = \operatorname{tr} C$ by the Bianchi identity.

We introduce the eigenvalues $a_1 \leqslant a_2 \leqslant a_3$ of A and $b_1 \leqslant b_2 \leqslant b_3$ of B and $c_1 \leqslant c_2 \leqslant c_3$ of C. Since B is not symmetric and does not map a vector space to itself, we need to explain the eigenvalues of B. For an appropriate choice of an orthonormal basis x_1, x_2, x_3 of Λ^2_+ the matrix

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

Then x_1, x_2, x_3 are eigenvectors with eigenvalues a_1, a_2, a_3. For an appropriate choice of orthonormal bases y_1^+, y_2^+, y_3^+ of Λ^2_+ and y_1^-, y_2^-, y_3^- of Λ^2_- the matrix

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$$

with $0 \leqslant b_1 \leqslant b_2 \leqslant b_3$, and this property uniquely determines the eigenvalues of B. In fact they are the eigenvalues of the symmetric matrices $\sqrt{B\,{}^t\!B}$ or $\sqrt{{}^t\!BB}$. We call y_1^+, y_2^+, y_3^+ and y_1^-, y_2^-, y_3^- eigenvectors of B. And for an appropriate choice of an orthonormal basis z_1, z_2, z_3 of Λ^2_- the matrix

$$C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

Then z_1, z_2, z_3 are eigenvectors of C with eigenvalues c_1, c_2, c_3.

We shall make our estimates using the functions $a_1, a_3, b_2 + b_3, c_1, c_3$. We shall also use the function $a - 2b + c$, where $a = a_1 + a_2 + a_3 = c = c_1 + c_2 + c_3$ and $b = b_1 + b_2 + b_3$. Note that

$$\begin{aligned}
a_1 &= \inf\{A(x_1, x_1)\colon |x_1| = 1\},\\
a_3 &= \sup\{A(x_3, x_3)\colon |x_3| = 1\},\\
b_2 + b_3 &= \sup\{B(y_2^+, y_2^-) + B(y_3^+, y_3^-);\ |y_2^+| = |y_3^+| = 1,\\
&\qquad\qquad y_2^+ \perp y_3^+,\ |y_2^-| = |y_3^-| = 1,\ y_2^- \perp y_3^-\},\\
c_1 &= \inf\{C(z_1, z_1)\colon |z_1| = 1\},\\
c_3 &= \sup\{C(z_3, z_3)\colon |z_3| = 1\}.
\end{aligned}$$

Moreover the inf or sup is attained when the various vectors are eigenvectors in some orthonormal bases with the corresponding eigenvalues. As a result we see that a_1 and c_1 are concave while a_3 and c_3 and $b_2 + b_3$ are convex functions of the matrices A, B, and C. Moreover we can compute their derivatives by Lemma 3.5. We assume $M \geqslant 0$.

6.1. Lemma.

$$\begin{aligned}
&\frac{d}{dt} a_1 \geqslant a_1^2 + b_1^2 + 2a_2a_3,\\
&\frac{d}{dt} a_3 \leqslant a_3^2 + b_3^2 + 2a_1a_2,\\
&\frac{d}{dt}(b_2 + b_3) \leqslant a_2b_2 + a_3b_3 + b_2c_2 + b_3c_3 + 2b_1b_2 + 2b_1b_3,\\
&\frac{d}{dt} c_1 \geqslant c_1^2 + b_1^2 + 2c_2c_3,\\
&\frac{d}{dt} c_3 \leqslant c_3^2 + b_3^2 + 2c_1c_2.
\end{aligned}$$

Proof. To estimate the derivative of a_1 we have to evaluate

$$A^2 + B^tB + 2A^\#$$

at a unit eigenvector x_1 where $A(x_1, x_1) = a_1$. Then $A^2(x_1, x_1) = a_1^2$, and $B^tB(x_1, x_1) \geqslant b_1^2$ since b_1^2 is the smallest eigenvalue of B^tb, and $A^\#(x_1, x_1) = 2a_2a_3$. To estimate the derivative of a_3 we have to evaluate $A^2 + B^tB + 2A^\#$ at a unit eigenvector x_3 where $A(x_3, x_3) = a_3$. Then $A^2(x_3, x_3) = a_3^2$, and $B^tB(x_3, x_3) \leqslant b_3^2$ since b_3^2 is the largest eigenvector of B^tB, and $A^\#(x_3, x_3) = 2a_1a_2$.

To estimate the derivative of $b_2 + b_3$ we must evaluate

$$\frac{dB}{dt}(y_2^+, y_2^-) + \frac{dB}{dt}(y_3^+, y_3^-),$$

where y_2^+, y_2^-, y_3^+, y_3^- are unit eigenvectors of B with eigenvalues b_2 and b_3, so $B(y_2^+, y_2^-) = b_2$ and $B(y_3^+, y_3^-) = b_3$. The supremum is also obtained when the bases (y_2^+, y_3^+) and (y_2^-, y_3^-) are rotated by the same angle, but this leaves the computation unchanged. Now

$$\frac{dB}{dt} = AB + BC + 2B^{\#}.$$

Since we identify matrices and bilinear forms by writing $P(x, y) = \langle x, Py \rangle$ in general, and since $By_2^- = b_2 y_2^+$ and $By_3^- = b_3 y_3^+$, we have

$$AB(y_2^+, y_2^-) = \langle Ay_2^+, By_2^- \rangle = b_2 A(y_2^+, y_2^+),$$
$$AB(y_3^+, y_3^-) = \langle Ay_3^+, By_3^- \rangle = b_3 A(y_3^+, y_3^-).$$

Write $\tilde{a}_2 = A(y_2^+, y_2^+)$ and $\tilde{a}_3 = A(y_3^+, y_3^+)$. Then $\tilde{a}_2 + \tilde{a}_3 \leqslant a_2 + a_3$, and consequently since all the eigenvalues are nonnegative

$$\tilde{a}_2 b_2 + \tilde{a}_3 b_3 \leqslant a_2 b_2 + a_3 b_3.$$

This inequality follows directly from

$$(a_3 - \tilde{a}_3)(b_3 - b_2) + (a_2 + a_3 - \tilde{a}_2 - \tilde{a}_3) b_2 \geqslant 0.$$

Therefore

$$AB(y_2^+, y_2^-) + AB(y_3^+, y_3^-) \leqslant a_2 b_2 + a_3 b_3.$$

Likewise

$$BC(y_2^+, y_2^-) + BC(y_3^+, y_3^-) \leqslant b_2 c_2 + b_3 c_3,$$

and finally $B^{\#}(y_2^+, y_2^-) = 2b_1 b_3$ and $B^{\#}(y_3^+, y_3^-) = 2b_1 b_2$. The estimates for c_1 and c_3 are the same as for a_1 and a_3. This proves the lemma.

Finally we must estimate the derivative of the function $a - 2b + c$. The function $a = \operatorname{tr} A = c = \operatorname{tr} C$ is linear. The function $b = b_1 + b_2 + b_3$ is not linear but it is convex, since $b = \sup \operatorname{tr} BT$, where the sup is taken over all orthogonal transformations T: $\Lambda_+^2 \to \Lambda_-^2$. To see this is true, choose coordinates in which B is diagonal with entries $b_1 \leqslant b_2 \leqslant b_3$. Then

$$\operatorname{tr} BT = b_1 t_{11} + b_2 t_{22} + b_3 t_{33}$$

and $t_{11}, t_{22}, t_{33} \leqslant 1$ with equality only when T is the identity. Therefore the function $a - 2b + c$ is concave.

6.2. Lemma.

$$\frac{d}{dt}(a - 2b + c) \geqslant (a_1 + 2b_1 + c_1)(a - 2b + c).$$

Proof. By Lemma 3.5

$$\frac{d}{dt}(a - 2b + c) \geqslant \operatorname{tr}\left(\frac{dA}{dt} - 2\frac{dB}{dt} + \frac{dC}{dt} \right)$$

evaluated in those coordinates where B is diagonal as above. Now

$$\frac{dA}{dt} - 2\frac{dB}{dt} + \frac{dC}{dt} = (A-B)^2 + (C-B)^2 + 2(A^\# - 2B^\# + C^\#).$$

Recalling that $P^\# = P\#P$, where in general $P\#Q$ is defined by

$$(P\#Q)_{\alpha\zeta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\zeta\eta\theta}P_{\beta\eta}Q_{\gamma\theta}$$

with $\varepsilon_{\alpha\beta\gamma}$ the permutation tensor, we see that $P\#Q$ gives a symmetric bilinear operation on matrices, and

$$2(A^\# - 2B^\# + C^\#) = (A-C)^\# + (A+2B+C)\#(A-2B+C).$$

Next we use the fact that for 3×3 matrices P

$$(\operatorname{tr} P)^2 = \operatorname{tr} P^2 + 2\operatorname{tr} P^\#.$$

If $P = A - C$, then $\operatorname{tr} P = 0$, so

$$\operatorname{tr}(A-C)^\# = -\tfrac{1}{2}\operatorname{tr}(A-C)^2.$$

Now the trace of the square of a symmetric matrix is the sum of the squares of its entries. Using the parallelogram law

$$|p+q|^2 + |p-q|^2 = 2\left(|p|^2 + |q|^2\right)$$

we see that

$$\operatorname{tr}(A-B)^2 + \operatorname{tr}(C-B)^2 - \tfrac{1}{2}\operatorname{tr}(A-C)^2 = \tfrac{1}{2}\operatorname{tr}(A-2B+C)^2 \geqslant 0.$$

Therefore

$$\frac{d}{dt}(a - 2b + c) \geqslant \operatorname{tr}(A+2B+C)\#(A-2B+C).$$

When $M \geqslant 0$ and

$$M = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$$

we see that $A + 2B + C \geqslant 0$ and $A - 2B + C \geqslant 0$, by applying M to the vectors (x, x) and $(x, -x)$. If P and Q are two symmetric 3×3 matrices with $P, Q \geqslant 0$, and p_1 is the smallest eigenvalue of P while q is the trace of Q, then

$$\operatorname{tr} P\#Q \geqslant p_1 q.$$

To see this, choose coordinates where Q is diagonal with eigenvalues $q_1 \leqslant q_2 \leqslant q_3$. Then

$$\operatorname{tr} P\#Q = \tfrac{1}{2}(p_{22} + p_{33})q_1 + \tfrac{1}{2}(p_{11} + p_{33})q_2 + \tfrac{1}{2}(p_{11} + p_{22})q_3$$

and $p_{11}, p_{22}, p_{33} \geqslant p_1$. Applying this with $P = A + 2B + C$ and

$$Q = A - 2B + C,$$

we see that the smallest eigenvalue $p_1 \geqslant a_1 + 2b_1 + c_1$ while the trace $q = a - 2b + c$. This finishes the proof of the lemma.

In the subsequent discussion we will need to know when certain sets are convex. In general if f is a convex function and g is a concave function, the set where $f \leqslant g$ is convex. The following result will be useful.

6.3. Lemma. *If f and g are concave and positive and if $\alpha + \beta = 1$, then $f^\alpha g^\beta$ is concave.*

Proof. Let $z = x^\alpha y^\beta$ on $x, y \geqslant 0$. The matrix of second partial derivatives is

$$-\alpha\beta x^{\alpha-2} y^{\beta-2} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

which has one zero eigenvalue and one negative. Therefore $z = x^\alpha y^\beta$ is a concave function of x and y. Let $h = f^\alpha g^\beta$ and put

$$\begin{aligned} x_1 &= f(v_1), & x_2 &= f(v_2), & x &= f((v_1 + v_2)/2), \\ y_1 &= g(v_1), & y_2 &= g(v_2), & y &= g((v_1 + v_2)/2), \\ z_1 &= h(v_1), & z_2 &= h(v_2), & z &= h((v_1 + v_2)/2). \end{aligned}$$

Then $x \geqslant (x_1 + x_2)/2$ and $y \geqslant (y_1 + y_2)/2$ so

$$z = x^\alpha y^\beta \geqslant \left(\frac{x_1 + x_2}{2}\right)^\alpha \left(\frac{y_1 + y_2}{2}\right)^\beta \geqslant \frac{x_1^\alpha y_1^\beta + x_2^\alpha y_2^\beta}{2} = \frac{z_1 + z_2}{2}$$

so h is concave also.

7. We are now in a position to verify that the open set $U = \{M > 0\}$ of positive curvature operators satisfies the pinching condition of 5.2.

7.1. Theorem. *If we choose successively constants G large enough, H large enough, δ small enough, J large enough, ε small enough, K large enough, θ small enough, and L large enough, with each depending on those chosen before, then the set Z of $M \geqslant 0$ defined by the inequalities*

(1) $(b_2 + b_3)^2 \leqslant Ga_1c_1$,

(2) $a_3 \leqslant Ha_1$ *and* $c_3 \leqslant Hc_1$,

(3) $(b_2 + b_3)^{2+\delta} \leqslant Ja_1c_1(a - 2b + c)^\delta$,

(4) $(b_2 + b_3)^{2+\varepsilon} \leqslant Ka_1c_1$,

(5) $a_3 \leqslant a_1 + La_1^{1-\theta}$ *and* $c_3 \leqslant c_1 + Lc_1^{1-\theta}$

is a pinching set for the flow of the ODE in the sense of Definition 5.1. *Moreover every compact subset of $U = \{M > 0\}$ lies in some such pinching set Z.*

Proof. Clearly Z is closed. Given any compact subset of U, we can make the large constants large enough to contain it, since there will be a lower bound on a_1, c_1, and $a - 2b + c$. This follows since if $M > 0$, then $A > 0$, $C > 0$, and $A - 2B + C > 0$ (adding the matrices after identifying Λ^2_+ and Λ^2_- by an isometry). That Z is convex follows from Lemma 6.3. It is clear that Z is invariant under the action of $O(4)$ on the space of matrices M, because this action first induces rotations on Λ^2_+ and Λ^2_- and these leave the eigenvalues unchanged. All that remains to be shown is that Z is invariant under the flow of the ODE. We shall show in fact that the sets defined by each successive group of inequalities is preserved. Note first that $M \geqslant 0$ is preserved, so we can assume $0 \leqslant a_1 \leqslant a_2 \leqslant a_3$, $0 \leqslant b_1 \leqslant b_2 \leqslant b_3$, $0 \leqslant c_1 \leqslant c_2 \leqslant c_3$.

We start by estimating the time derivative of the logarithm of the different functions.

7.2. Lemma.

$$\frac{d}{dt}\log a_1 \geqslant 2b_1 + 2a_3 + \frac{(a_1 - b_1)^2}{a_1} + 2\frac{a_3}{a_1}(a_2 - a_1),$$

$$\frac{d}{dt}\log a_3 \leqslant a_3 + 2a_1 + \frac{b_3^2}{a_3} - \frac{2a_1}{a_3}(a_3 - a_2),$$

$$\frac{d}{dt}\log(b_2 + b_3) \leqslant 2b_1 + a_3 + c_3 - \frac{b_2}{b_2 + b_3}\left[(a_3 - a_2) + (c_3 - c_2)\right],$$

$$\frac{d}{dt}\log c_1 \geqslant 2b_1 + 2c_3 + \frac{(c_1 - b_1)^2}{c_1} + 2\frac{c_3}{c_1}(c_2 - c_1),$$

$$\frac{d}{dt}\log c_3 \leqslant c_3 + 2c_1 + \frac{b_3^2}{c_3} - \frac{2c_1}{c_3}(c_3 - c_2),$$

$$\frac{d}{dt}\log(a - 2b + c) \geqslant a_1 + 2b_1 + c_1.$$

Proof. This follows just by rewriting Lemmas 6.1 and 6.2. Of course there may be a problem taking the logarithm of zero. But this will not concern us anyway, because if $b_3 = 0$ then $B = 0$ and B will remain zero; while if $a_3 = 0$ then $A = 0$ and (since $M \geqslant 0$) $B = 0$ also and then A remains 0, and $C = 0$ also since $\operatorname{tr} A = \operatorname{tr} C$.

We see immediately that

$$\frac{d}{dt}\log\frac{a_1 c_1}{(b_2 + b_3)^2} \geqslant 0$$

and hence the inequaltiy $(b_2 + b_3)^2 \leqslant G a_1 c_1$ is preserved for any constant G. Then $b_3^2 \leqslant G a_1 c_1$. Since $\operatorname{tr} A = \operatorname{tr} C$

$$c_1 \leqslant c_1 + c_2 + c_3 = a_1 + a_2 + a_3 \leqslant 3a_3$$

which shows $b_3^2/a_3 \leqslant 3Ga_1$. Then

$$\frac{d}{dt}\log a_1 \geqslant 2a_3, \qquad \frac{d}{dt}\log a_3 \leqslant a_3 + (3G + 2)a_1,$$

so if $H \geqslant 3G + 2$

$$\frac{d}{dt} \log \frac{a_3}{a_1} \leqslant Ha_1 - a_3.$$

Then if we let $f = a_3/a_1 - H$ we have $df/dt + a_3 f \leqslant 0$ which shows if $f \leqslant 0$ to start it remains so. Thus the inequalities $a_3 \leqslant Ha_1$ and likewise $c_3 \leqslant Hc_1$ are preserved.

For the third inequality we need to use the terms we threw away in the first. We start with this estimate.

7.3. Lemma. *If* $(b_2 + b_3)^2 \leqslant Ga_1c_1$ *and* $a_3 \leqslant Ha_1$, *and if* $\delta \leqslant \min(1/4H, 1/\sqrt{3GH})$, *then*

$$\frac{(a_1 - b_1)^2}{a_1} + \frac{2b_2}{b_2 + b_3}(a_3 - a_2) \geqslant \delta(a_3 - a_2).$$

Proof. We consider two cases:

Case 1: $b_1 \leqslant a_1/2$. In this case we have

$$\frac{(a_1 - b_1)^2}{a_1} \geqslant \frac{a_1}{4} \geqslant \frac{1}{4H} a_3 \geqslant \delta(a_3 - a_2).$$

Case 2: $b_1 \geqslant a_1/2$. In this case, since $c_1 \leqslant 3a_3 \leqslant 3Ha_1$, and $b_2 + b_3 \leqslant \sqrt{Ga_1c_1} \leqslant \sqrt{3GH}\, a_1$, we have

$$\frac{2b_2}{b_2 + b_3} \geqslant \frac{2b_2}{a_1}\delta \geqslant \delta$$

and this completes the proof.

As a consequence we see that (if $\delta \leqslant 2$)

$$\frac{d}{dt} \log \frac{a_1c_1}{(b_2 + b_3)^2} \geqslant \delta(a_3 - a_1) + \delta(c_3 - c_1). \tag{7.4}$$

Since we also have

$$\frac{d}{dt} \log \frac{b_2 + b_3}{a - 2b + c} \leqslant (a_3 - a_1) + (c_3 - c_1)$$

we conclude that the inequality

$$(b_2 + b_3)^{2+\delta} \leqslant Ja_1c_1(a - 2b + c)^{\delta}$$

will be preserved for any constant J.

7.5. Corollary. *There exists* $\eta > 0$ *such that on the set defined by the inequality* (3) *we have* $b \leqslant (1 - \eta)a$.

Proof. If $b \leqslant a/2 = c/2$, this is trivial. If $b \geqslant a/2$, then $b_2 + b_3 \geqslant \frac{2}{3}b \geqslant \frac{1}{3}a$ and for some constant k we have

$$a^{2+\delta} \leqslant ka^2(a - b)^{\delta}$$

which makes $a \leqslant k(a - b)$ for some k, or $b \leqslant (1 - \eta)a$ for some η.

7.6. Corollary. *There exists $\lambda > 0$ such that on the set defined by inequality* (3) *we have*

$$\max\left\{\frac{(a_1 - b_1)^2}{a_1}, \delta(a_3 - a_1)\right\} \geqslant \lambda a.$$

Proof. Since $\eta a \leqslant a - b \leqslant 3(a_3 - b_1)$, we must have either $a_3 - a_1 \geqslant \frac{1}{6}\eta a$ or $a_1 - b_1 \geqslant \frac{1}{6}\eta a$. The result follows.

Now we already saw from 7.4 that

$$\frac{d}{dt}\log\frac{a_1 c_1}{(b_2 + b_3)^2} \geqslant \delta(a_3 - a_1)$$

and we also have

$$\frac{d}{dt}\log\frac{a_1 c_1}{(b_2 + b_3)^2} \geqslant \frac{(a_1 - b_1)^2}{a_1},$$

so it follows that

$$\frac{d}{dt}\log\frac{a_1 c_1}{(b_2 + b_3)^2} \geqslant \lambda a.$$

But it is easy to bound

$$\frac{d}{dt}\log(b_2 + b_3) \leqslant 2b_1 + a_3 + c_3 \leqslant 4a$$

since in fact $2b_1 \leqslant a_3 + c_3 \leqslant 2a$. Then if ε is small enough

$$\frac{d}{dt}\log\frac{a_1 c_1}{(b_2 + b_3)^{2+\varepsilon}} \geqslant 0$$

and it follows that the inequality $(b_2 + b_3)^{2+\varepsilon} \leqslant K a_1 c_1$ is preserved for any K.

7.7. Corollary. *For some constant k and some $\theta > 0$ we have $b_3^2 \leqslant k a_1^{1-\theta} a_3$ on the previous set.*

Proof. Use $a_1 \leqslant a_3 \leqslant H a_1$ and $c_1 \leqslant c = a \leqslant 3a_3$.

We can now show that the last inequalities (5) are preserved.

7.8. Lemma. *Let $f = (a_1 + La_1^{1-\theta})/a_3$. If $\theta > 0$ is made small enough, and if L is then made large enough, we will have $df/dt \geqslant 0$ for $f \leqslant 1$. Consequently the set $f \geqslant 1$ is preserved.*

Proof. We have from 7.2 $(d/dt)\log a_1 \geqslant a_1 + 2a_3$ and therefore

$$\frac{d}{dt}\log\left(a_1 + La_1^{1-\theta}\right) \geqslant \frac{a_1 + (1-\theta)La_1^{1-\theta}}{a_1 + La_1^{1-\theta}}(a_1 + 2a_3).$$

Also from 7.2 and using $b_3^2 \leqslant k a_1^{1-\theta} a_3$ we have

$$\frac{d}{dt}\log a_3 \leqslant a_3 + 2a_1 + k a_1^{1-\theta}.$$

We can use

$$\frac{a_1 + (1-\theta)La_1^{1-\theta}}{a_1 + La_1^{1-\theta}} \geqslant 1 - \theta La_1^{1-\theta}$$

and $a_1 + 2a_3 \leqslant 3a_3 \leqslant 3Ha_1$ to write

$$\frac{d}{dt}\log f \geqslant (a_3 - a_1) - (3\theta HL + k)a_1^{1-\theta}.$$

Now if $f \leqslant 1$, then $a_3 - a_1 \geqslant La_1^{1-\theta}$, so $df/dt \geqslant 0$ provided $L \geqslant 3\theta HL + k$. This will hold if we first make θ so small that $3\theta H \leqslant \frac{1}{2}$, and then make L so large that $L \geqslant 2k$. A similar argument works for the inequality in c. This completes the proof of Theorem 7.1.

8. We now study nonnegative curvature. Let N be a compact manifold with a metric $g = \{g_{ij}\}$, V a vector bundle over N with a metric $h = \{h_{\alpha\beta}\}$ and a connection $A = \{A_{i\beta}^{\alpha}\}$, and suppose h is fixed but g and A may vary with time t. We form the Laplacian of a section f of V as $\Delta f = g^{ij}D_iD_jf$ using the metric g on TM and the Levi-Civita connection and the connection A.

8.1. Lemma. *Suppose $\partial f/\partial t = \Delta f + \phi(f)$. Let $s(f)$ be a convex function on the bundle invariant under parallel translation whose level curves $s(f) \leqslant c$ are preserved by the ODE $df/dt = \phi(f)$. Then the inequality $s(f) \leqslant c$ is preserved by the PDE for any constant c. Furthermore if $s(f) < c$ at one point at time $t = 0$, then $s(f) < c$ everywhere on M for all $t > 0$.*

Proof. Let h be a function on M with $s(f) \leqslant h \leqslant c$, and if $s(f) < c$ at some point p we can make $h < c$ at that point. Then we solve the system for the pair (f, h)

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f), \qquad \frac{\partial h}{\partial t} = \Delta h.$$

The set $X = \{(f, h)\colon s(f) \leqslant h\}$ is closed and convex since if $s(f_1) \leqslant h_1$ and $s(f_2) \leqslant h_2$,

$$s\left(\frac{f_1 + f_2}{2}\right) \leqslant \frac{s(f_1) + s(f_2)}{2} \leqslant \frac{h_1 + h_2}{2},$$

and X is invariant under parallel translation. Therefore X is preserved, and $s(f) \leqslant c$. If $h < c$ at one point at $t = 0$, then $h < c$ everywhere for $t > 0$ by the strong maximum principle, so $s(f) < c$ for $t > 0$.

8.2. Lemma. *Let M be a symmetric bilinear form on V. Suppose M satisfies a heat equation $\partial M/\partial t = \Delta M + \phi(M)$, where the matrix $\phi(M) \geqslant 0$ for all $M \geqslant 0$. Then if $M \geqslant 0$ at time $t = 0$, it remains so for $t \geqslant 0$. Moreover there exists an interval $0 < t < \delta$ on which the rank of M is constant and the null space of M is invariant under parallel translation and invariant in time and also lies in the null space of $\phi(M)$.*

Proof. The convex cone $M \geqslant 0$ is invariant under parallel translation, and if $\phi(M) \geqslant 0$ then the ODE $dM/dt = \phi(M)$ preserves the cone $M \geqslant 0$. Hence so does the PDE. Let $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$ be the eigenvalues of M. Then $m_1 + \cdots + m_k$ is a concave function of M and is invariant under parallel translation, since

$$m_1 + \cdots + m_k = \inf\{ \operatorname{tr} M|P\colon P \subset V \text{ is a subspace of } \dim k \}.$$

Note that $\dim \operatorname{null} M \geqslant k \Leftrightarrow m_1 + \cdots + m_k = 0$. If $m_1 + \cdots + m_k > 0$ at one point at $t = 0$, it will be greater than 0 everywhere at all subsequent times. It follows that rank M remains constant on some interval $0 < t < \delta$.

Let v be any smooth section of V in null M on $0 < t < \delta$. Then

$$0 = \frac{\partial}{\partial t}\left(M_{\alpha\beta}v^\alpha v^\beta\right) = \left(\frac{\partial}{\partial t}M_{\alpha\beta}\right)v^\alpha v^\beta + 2M_{\alpha\beta}v^\alpha\frac{\partial v^\beta}{\partial t}.$$

Since $M_{\alpha\beta}v^\alpha = 0$, the last term disappears. Also

$$\begin{aligned} 0 = \Delta\left(M_{\alpha\beta}v^\alpha v^\beta\right) &= \left(\Delta M_{\alpha\beta}\right)v^\alpha v^\beta + 4g^{kl}D_kM_{\alpha\beta}v^\alpha D_lv^\beta \\ &\quad + 2M_{\alpha\beta}g^{kl}D_kv^\alpha D_lv^\beta + 2M_{\alpha\beta}v^\alpha\Delta v^\beta \end{aligned}$$

and again the last term vanishes. Since

$$0 = D_k\left(M_{\alpha\beta}v^\alpha\right) = \left(D_kM_{\alpha\beta}\right)v^\alpha + M_{\alpha\beta}D_kv^\alpha$$

we get from the evolution equation

$$2M_{\alpha\beta}g^{kl}D_kv^\alpha D_lv^\beta + \phi(M)_{\alpha\beta}v^\alpha v^\beta = 0.$$

Since $M \geqslant 0$ and $\phi(M) \geqslant 0$, we must have $v \in \operatorname{null}\phi(M)$ and $D_kv^\alpha \in \operatorname{null} M$ for all k. This shows $\operatorname{null} M \subseteq \operatorname{null}\phi(M)$ and null M is invariant under parallel translation.

To see null M is also invariant in time, note first that Δv^α lies in null M, since it is invariant under parallel translation. Then

$$0 = g^{kl}D_k\left(M_{\alpha\beta}D_lv^\alpha\right) = g^{kl}D_kM_{\alpha\beta}D_lv^\alpha + M_{\alpha\beta}\Delta v^\alpha$$

and so $g^{kl}D_kM_{\alpha\beta}D_lv^\alpha = 0$. Then

$$0 = \Delta\left(M_{\alpha\beta}v^\alpha\right) = \Delta M_{\alpha\beta}v^\alpha + 2g^{kl}D_kM_{\alpha\beta}D_lv^\alpha + M_{\alpha\beta}\Delta v^\alpha$$

and hence $(\Delta M_{\alpha\beta})v^\alpha = 0$. Then

$$0 = \frac{\partial}{\partial t}\left(M_{\alpha\beta}v^\alpha\right) = M_{\alpha\beta}\frac{\partial v^\alpha}{\partial t} + \left(\Delta M_{\alpha\beta} + \phi(M)_{\alpha\beta}\right)v^\alpha.$$

Now $\operatorname{null} M \subseteq \operatorname{null}\phi(M)$, so $\phi(M)_{\alpha\beta}v^\alpha = 0$ also. Thus $M_{\alpha\beta}\partial v^\alpha/\partial t = 0$, and $\partial v/\partial t$ lies in null M also whenever v does. This shows null M is invariant in time.

We apply this to the heat equation for the curvature tensor

$$\frac{\partial M}{\partial t} = \Delta M + M^2 + M^\#.$$

Note that if $M \geqslant 0$, then $M^2 \geqslant 0$ and $M^\# \geqslant 0$ since

$$M^\#_{\alpha\zeta} = c_{\alpha\beta\gamma} c_{\zeta\eta\theta} M_{\beta\eta} M_{\gamma\theta}.$$

Hence for $0 < t < \delta$ the null space of M has constant rank and is invariant in time and under parallel translation. Moreover the null space of M must also lie in the null space of $M^\#$. The image of M is everything perpendicular to the null space. Diagonalize M so that $M_{\alpha\beta} = 0$ if $\alpha \leqslant k$ and $M_{\alpha\alpha} > 0$ if $\alpha > k$. Then we must have $M^\#_{\alpha\alpha} = 0$ also for $\alpha \leqslant k$, so $c_{\alpha\beta\gamma} = 0$ if $\alpha \leqslant k$ and $\beta, \gamma > k$. This first says that the image of M is a Lie subalgebra. (In fact it will be the subalgebra of the restricted holonomy group.) This proves the following result.

8.3. Theorem. *If $M \geqslant 0$ at $t = 0$, then for some interval $0 < t < \delta$ the image of M is a Lie subalgebra of constant rank invariant under parallel translation and invariant in time.*

For example, if W is the Kähler manifold $\mathbf{C}P^n$, then the image of M is $u(n) \subseteq so(2n)$.

9. Now we apply this result in dimensions 3 and 4. In dimension 3 we have

$$\frac{\partial M}{\partial t} = \Delta M + M^2 + M^\#,$$

where $M^\#$ is the adjoint matrix. The only Lie subalgebras of $so(3)$ are $\{0\}$, $so(3)$ itself, and any one-dimensional subspace. The first case is where $M = 0$ and the second where M is invertible. In the third case, the null space of M is spanned by a translation-invariant 2-form ϕ_{ij}. We can write $\phi_{ij} = \mu_{ijk} v^k$ for a translation invariant vector v^k. This gives locally a product decomposition by the following result.

Lemma. *If the tangent bundle TM has an orthogonal decomposition $TM = V_1 \oplus V_2$, where V_1 and V_2 are invariant under parallel translation, then locally there is a product decomposition $M = M_1 \times M_2$ such that the metric on M is the product of metrics on M_1 and M_2 and $V_1 = TM_1$, $V_2 = TM_2$.*

Proof. Since the subbundles are invariant under parallel translation, they must satisfy the Fröbenius integrability condition, because if $X, Y \in V_1$, then

$$[X, Y] = D_X Y - D_Y X \in V_1.$$

Choose coordinates $\{x^i, x^\alpha\}$ corresponding to the induced product decomposition. Then since the tangent space decomposition is invariant under parallel translation, all the Christoffel symbols $\Gamma^i_{j\alpha}$, $\Gamma^i_{\alpha\beta}$, Γ^α_{ij}, $\Gamma^\alpha_{i\beta}$ must vanish. This makes $(\partial/\partial x^\alpha)g_{ij} = 0$ and $(\partial/\partial x^i)g_{\alpha\beta} = 0$. Hence g_{ij} is a function only of the x^k and $g_{\alpha\beta}$ only of the x^γ. Also $g_{i\alpha} = 0$ since $TM_1 \perp TM_2$. Hence the metric is a product metric.

Even in case the three-manifold has only nonnegative Ricci curvature R_{ij} we still obtain the same decomposition. The sectional curvatures are the eigenvalues of the Einstein tensor E_{ij} where

$$E_{ij} = \tfrac{1}{2}Rg_{ij} - R_{ij} \quad \text{or} \quad R_{ij} = Eg_{ij} - E_{ij},$$
$$R = g^{ij}R_{ij} \quad \text{and} \quad E = g^{ij}E_{ij}.$$

Since E_{ij} corresponds to $M_{\alpha\beta}$, we write $P_{\alpha\beta} = (h^{\gamma\delta}M_{\gamma\delta})h_{\alpha\beta} - M_{\alpha\beta}$ corresponding to the Ricci tensor. Then $P_{\alpha\beta}$ satisfies the evolution equation

$$\frac{\partial}{\partial t}P_{\alpha\beta} = \Delta P_{\alpha\beta} + Q_{\alpha\beta},$$

where $Q_{\alpha\beta}$ is quadratic in $P_{\alpha\beta}$. In fact when $P_{\alpha\beta}$ is diagonal, then $Q_{\alpha\beta}$ is also, and if

$$P_{\alpha\beta} = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}, \qquad Q_{\alpha\beta} = \begin{pmatrix} \rho & & \\ & \sigma & \\ & & \tau \end{pmatrix}$$

we can compute

$$\rho = \tfrac{1}{2}\left[(\mu - \nu)^2 + \lambda(\mu + \nu)\right],$$
$$\sigma = \tfrac{1}{2}\left[(\lambda - \nu)^2 + \mu(\lambda + \nu)\right],$$
$$\tau = \tfrac{1}{2}\left[(\lambda - \mu)^2 + \nu(\lambda + \mu)\right].$$

Clearly $Q \geqslant 0$ if $P \geqslant 0$. Moreover if P has a nontrivial null space, say $\lambda = 0$, the corresponding term $\rho = 0$ only if $\mu = \nu$ also. But in this case the sectional curvature is already nonnegative, and the previous argument applies.

Then the manifold splits locally as a product $M^3 = M^2 \times R^1$, where M^2 is a surface of positive curvature and R^1 is flat. Since the curvature on the two-dimensional leaves is bounded below (by the compactness of M) we see each leaf is compact (by Myer's theorem) and either a sphere or a projective space.

We claim the universal cover $\tilde{M}$ is isometric to a product $S^2 \times R^1$ with some (possibly nonstandard) positive curvature metric on S^2. To see this, pick one leaf, and if it is a projective space take its double cover. Call this S^2. The local product decomposition gives an isometry of $S^2 \times R^1$ to N^3 by analytic continuation. Since $S^2 \times R^1$ is complete it must be the universal cover, and N^3 is a quotient of $S^2 \times R^1$ by isometries. If the metric on S^2 is nonstandard, we can replace it by a conformally equivalent metric with constant curvature. This will not change the group of isometries.

Here are the possibilities in dimension 4, classified by the holonomy group.

1. $\mathfrak{g} = \{o\}$.

Here M is flat so $\tilde{M} = R^4$.

2. $\mathfrak{g} = so(2)$.

Note $\mathfrak{g}$ cannot embed in just one factor of $so(3) \times so(3)$. This is because $\operatorname{tr} A = \operatorname{tr} C$ by the Bianchi identity, so $A = 0$ if and only if $C = 0$. Here the image of M is spanned by a single two-form ϕ, and the Bianchi identity guarantees ϕ comes from a two-plane. This gives locally a product decomposition $M = P^2 \times R^2$, where the leaves P^2 have positive curvature and R^2 is flat. Since M is compact, the curvature of each leaf is bounded below, so if each leaf is compact then the universal cover of each leaf is S^2. This gives an isometry $S^2 \times R^2 \to M$, so $S^2 \times R^2$ is the universal cover. If we replace the metric on S^2 by the conformal one with constant positive curvature, the group of isometries is unchanged. Hence M is a quotient of $S^2 \times R^2$ by standard isometries.

3. $\mathfrak{g} = so(2) \times so(2)$.

In this case there are two invariant 2-forms $\phi \in \Lambda^2_+$ and $\psi \in \Lambda^2_-$. If we take them to have unit length, then $\phi + \psi$ and $\phi - \psi$ are two perpendicular 2-planes. Here M splits locally as a product $M = P^2 \times Q^2$, where P^2 and Q^2 are two surfaces of positive curvature. Again each leaf is compact and has positive curvature, and there is an isometry $S^2 \times S^2 \to M$ which identifies $S^2 \times S^2$ as the universal cover. If we replace each metric on S^2 by the conformal one of constant curvature the group of isometries is unchanged. Hence M is a quotient of $S^2 \times S^2$ by standard isometries.

4. $\mathfrak{g} = so(3)$.

Note again $\mathfrak{g}$ cannot be just one factor of $so(3) \times so(3)$ by the Bianchi identity. In fact, since any Lie algebra preserving map of $so(3)$ to $so(3)$ must be a multiple of the identity, and since $\operatorname{tr} A = \operatorname{tr} C$, we see that $\mathfrak{g}$ embeds as $\{\phi + P\phi\}$, where P is an isometry of Λ^2_+ to Λ^2_-. But if $\phi \in \Lambda^2_+$ and $\psi \in \Lambda^2_-$, then $\phi + \psi$ is a two-plane if and only if $|\phi| = |\psi|$. Thus $\mathfrak{g} = \operatorname{Im} M$ is a three-dimensional space of two-planes. The Lie bracket of two planes is zero if they are perpendicular. Therefore $\mathfrak{g}$ consists of all two-planes contained in a three dimensional subspace. This shows locally $M = P^3 \times R^1$, where P^3 has positive sectional curvature. As before we see that the leaves are all compact, and the universal cover of M is $S^3 \times R^1$, where S^3 is a three-sphere with some metric of positive sectional curvature. Moreover if we deform the metric on S^3 to the constant curvature metric by the heat equation, the isometry group is unchanged. Hence M is a quotient of $S^3 \times R^1$ by standard isometries.

5. $\mathfrak{g} = so(3) \times so(2)$.

Each Lie algebra preserving map $so(3) \to so(3)$ and $so(2) \to so(3)$ is either zero or an isometry. It follows that $so(3)$ maps to one factor in $so(3) \times so(3)$ and $so(2)$ maps into the other. Suppose the $so(2)$ factor is included in Λ^2_+. (The other case is similar.) Then there is a uniquely determined 2-form $\omega \in \Lambda^2_+$ of length 1 which is invariant under parallel translation. This gives M the structure of a Kähler manifold. Since the holomorphic bisectional curvature is positive, M is biholomorphic to the complex projective space $\mathbf{C}P^2$ by Yau's proof of the Frankel conjecture.

6. $\mathfrak{g} = so(3) \times so(3)$.

Here $M > 0$ so the manifold is S^4 or RP^4 by our previous result.

References

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