A UNIVERSAL SMOOTHING OF FOUR-SPACE

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Except in dimension four, smooth structures can be classified up to ε-isotopy by bundle reductions. Since R^n is contractible, this implies that any smooth structure Γ on R^n, $n \neq 4$, is ε-isotopic to the standard one. In contrast, R^4 has many distinct smoothings (even up to diffeomorphism.) We construct a certain smoothing of the half-space, $\frac{1}{2}R^4 = \{(x_1, x_2, x_3, x_4) | x_4 \geqslant 0\}$ and write H for this half-space together with its smooth structure. H contains all other smoothings of $\frac{1}{2}R^4$ (see Theorem 1) and is unique with respect to this property (Corollary A). H is the universal half-space. The interior $\dot{H} = U$ is naturally identified (replace x_4 with $\ln x_4$) with a smoothing of R^4. Corollary B states that U contains every smoothing of R^4 imbedded within it. Thus, we say U is a universal R^4. The construction of U is unambiguous but we do not claim that any R^4_Γ into which all smooth R^4's imbed is diffeomorphic to U. This is not known.

Copies of H may be used to "corrupt" the differential structure of any open manifold near its ends. This process irons out any unstable differences (i.e., those not persisting into dimension five under product with the real line) in differentiable structure and may be thought of as giving the unique worst structure (in a given stable class) on the manifold (Theorems 2 and 3).

Theorem 4 says that U cannot arise as the interior of a topologically flat cell in any smooth 4-manifold. This nonimbedding result leads, by an observation of R. Gompf, to countably many structures on R^4 [7]. A somewhat formal derivation of this is given in the proof of Corollary D.

In smooth 4-manifold topology, Casson handles [1], [5] are often constructable from homotopy (or ε-homotopy) information where smoothly imbedded disks are not present. The proof of Quinn's 5-dimensional controlled h-cobordism theorem [10] may be followed just to the point where smoothness is about to be lost. This is where isotopies are defined along Casson handles

according to their topological coordinates as 2-handles. Stopping the argument before the topological isotopy we have the following statement.

Lemma (*Quinn*). *Let* $(W, \partial^+ W, \partial^- W)$ *be a smooth 5-dimensional h-cobordism which is topologically a product,* $W \cong_{\text{Top}} \partial^{\pm} W \times I$. *Fix a metric space structure on* $\partial^- W$. *Then, for every* $\varepsilon > 0$, *W has a smooth handle structure* (rel $\partial^- W$) *consisting of 2- and 3-handles. The boundary intersection matrix is* δ_{ij}. *In the middle level there may be extra pairs of intersection points between the ascending and descending spheres but Whitney circles through these are spanned by disjoint 0-framed Casson handles. And finally, the diameters of the 2-handles, 3-handles, and Casson handles are all less than* ε *when measured* (*after topological projection*) *in* $\partial^- W$.

Addendum. The Casson handles above may always be chosen from a countable collection $\mathscr{C}$ of Casson handles.

Proof. By the reimbedding theorem [4], any Casson handles may always be imbedded in a six-stage[1] tower. The latter are determined by twelve positive countable integers (the number of $\pm$ kinks at each stage) and are therefore countable. Thus, any Casson handle may be trimmed down to a fixed representative (in $\mathscr{C}$) contained in its first six stages.

Let $CH \in \mathscr{C}$. It is known [4, Theorem] that CH contains a flat core disk Δ which is smooth except at one interior point p. Consider a small sphere S_p in CH centered at p and (smoothly) transverse to Δ. We see a link $\Delta \cap S_p = L \subset S_p$ which bounds a planar flat surface Σ in the ball B_p ($\partial B_p = S_p$) which may be impossible to smooth compatibly with B_p. In fact, the proof of the theorem quoted above may be used to arrange that L be some iterated-ramified-Whitehead link and that Σ be a disjoint union of disks, but this level of detail is not important to our discussion. In any case, we now fix in each Casson handle some disk Δ and a distinguished (B_p, Σ) as above.

For each $CH_i \in \mathscr{C}$ one obtains an associated "link slice problem" L_i for which there is a topological solution Σ_i but, in general, no smooth solution in B^4. Our plan is to construct H by smoothing theory so that the Σ_i will appear smooth.

Define annuli $A_k = \{v \in R^3 | k < \|v\| < k + 1\}$ and $\overline{A}_k = \{v \in \frac{1}{2} R^4 | k < \|v\| < k + 1\}$ for $k = 0, 1, 2, \cdots$. Enumerate the Casson handles $\mathscr{C} = \{CH_i\}$; to each $k = 0, 1, 2, \cdots$, assign $i(k)$ so that each index i occurs for infinitely many k. For each k pick a diffeomorphism $d_k : ((B_p^4)_{i(k)} - \text{pt}) \to \frac{1}{2} R^4$ from the distinguished 4-ball (minus a boundary point) in $CH_{i(k)}$ to $\frac{1}{2} R^4$ so that $d_k(\Sigma_{i(k)}, \partial \Sigma_{i(k)}) \subset (\overline{A}_k, A_k)$. This creates a family of topologically flat surfaces

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[1]Improved to 5-stage [7] and later to 4-stage [5].

imbedded in $\frac{1}{2}R^4$ heading out toward infinity. The topological product neighborhoods of the surfaces in $\frac{1}{2}R^4$ have their standard (product) smooth structures which match the smooth structure on $R^3 \times [0, \varepsilon)$ and (in general) disagree with the smooth structure induced from $\frac{1}{2}R^4$ by inclusion. By taking the union of the matching smooth structures we define a smooth structure on a neighborhood of $X = R^3 \cup (\cup_k d_k(\Sigma_{i(k)}))$. The obstructions to extending this smoothing to $\frac{1}{2}R^4$ lie in $H^*(\frac{1}{2}R^4, X; \pi_{*-1}\text{Top}(4)/O(4))$, $* = 1, 2, 3$, and are zero because the coefficient groups vanish ([9] and [10]). The extension [10] of the immersion theory of smoothings (which applies to local handles of index less than the manifold dimension) to open 4-manifolds may now be applied to extend the smoothing near X to a smoothing of $\frac{1}{2}R^4$. Call the result H.

Theorem 1. *Let W be any smooth manifold homeomorphic to H. There exists a smooth (not generally proper) imbedding $e: (W, \partial) \hookrightarrow (H, \partial)$ which, on ∂W, is arbitrarily close to the given homeomorphism.*

Proof. Let $h: W \to H$ be the homeomorphism. By the 3-dimensional Haupvermutung we may approximate h by $h': W \to H$ which is a diffeomorphism over ∂H. We may (temporarily) forget our carefully constructed structure and consider the mapping cylinder Y of $h: W \to \frac{1}{2}R^4_{\text{std}}$. The space Y is a topological manifold, with a product structure and a smoothing near top $(W \times [0, \varepsilon])$ and bottom $(\frac{1}{2}R^4_{\text{std}} \times [1 - \varepsilon, 1])$. The smoothing obstruction in $H^4(Y, \text{top} \cup \text{bottom}; \pi_3(\text{Top}/PL)) \cong 0$ vanishes. Write $\overline{Y}$ for Y with the smooth structure near top and bottom extended over all of Y. Apply the lemma (with addendum) to $\overline{Y}$.

In the middle level M of $\overline{Y}$ we see a locally finite collection of Casson handles $CH_j \in \mathscr{C}$. We let j index the Casson handles in M. Inside each CH_j is a distinguished pair $(B, \Sigma)_j$, (4-ball surface) as noted above. Let γ be a half-infinite ray properly imbedded in M disjoint from the ascending and descending spheres and the Casson handles. A closed tubular neighborhood $\mathscr{N}$ of γ admits a diffeomorphism $d: \frac{1}{2}R^4_{\text{std}} \to \mathscr{N}$. The gradient-like flow on $\overline{Y}$ imbeds $(\mathscr{N} \times [0, 1]; (\mathscr{N} \times 0) \cup (\mathscr{N} \times 1))$ in $(\overline{Y}; \text{bottom} \cup \text{top})$ where it inherits the structure $(\frac{1}{2}R^4)_{\text{std}} \times [0, 1]_{\text{std}}$. Resmooth $\overline{Y}$ (to obtain $\overline{\overline{Y}}$) by replacing this standard structure on $\mathscr{N} \times [0, 1]_{\text{std}}$ by the exotic half-space structure cross identity, $d^*(H) \times [0, 1]_{\text{std}}$.

Now in the middle level M, $\mathscr{N}$ has been resmoothed to look like a copy of H. Construct a locally finite collection of disjoint imbedded closed arcs $\{\alpha_j\}$ in M-interior$(\mathscr{N}) \cup (\cup_j B_j)$ each joining the frontier of a distinguished 4-ball B_j to an annulus $d(A_K)$ on $\partial \mathscr{N}$. The annulus A_K must be chosen to be one containing exactly the link slice problem L_i which occurs on ∂B_j but this presents no conflict with the local finiteness condition since each L_i occurs in

infinitely many rings. Also take the arcs to be disjoint from all the ascending spheres, and disks $\Delta_j \subset \bigcup_j CH_j$. Let $\mathcal{N}$ denote a closed regular neighborhood of $\mathcal{N} \cup (\bigcup_j B_j) \cup (\bigcup_j \alpha_j)$, $\mathcal{N}$ with balls and arcs attached.

It is not difficult to see a topological isotopy $\mathcal{I}_t$ of $\mathcal{N}$ (equal to the identity on the $\partial \mathcal{N}$ and smooth near $\partial \mathcal{N}$) which carries the core disks Δ_j to smooth disks. It does so by carrying $\Sigma_j \subset \Delta_j$ to the copy of Σ_j in $d(\bar{A}_k)$ which was given the standard product smoothing in the construction of H.
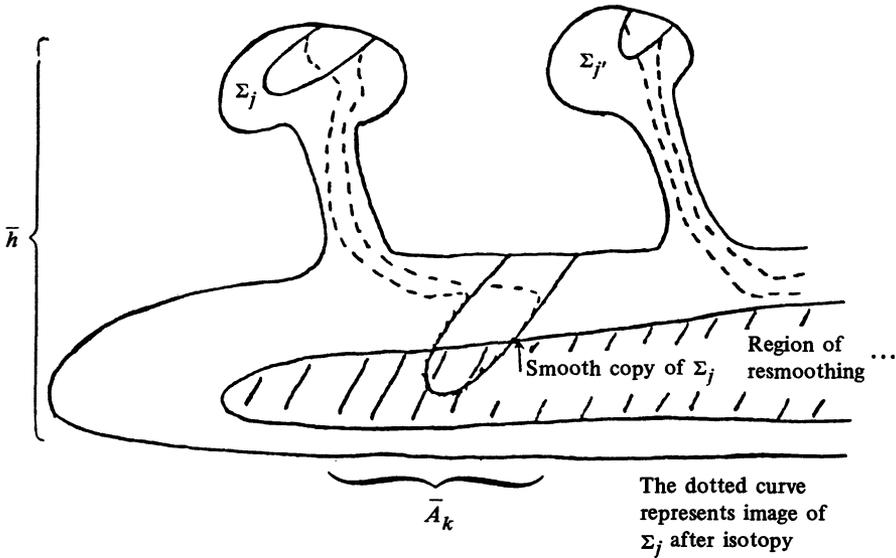
Diagram 1

Note. In the original (before our resmoothing) smooth structure on $\mathcal{N}$ the isotopy above is entirely smooth.

After our resmoothing, the conclusion of the lemma may be improved by discarding the Casson handles in favor of the smooth Whitney disks $\mathcal{I}_1(\bigsqcup_j \Delta_j)$. At the same time, ε-control is lost (the isotopy is long). The resulting data, ascending and descending spheres paired algebraically δ_{ij} with smooth and locally finite Whitney disks cancelling excess geometric intersections, is precisely what is needed to complete the usual proof of the proper-h-cobordism theorem.

Thus, $\bar{\bar{Y}}$ has a smooth product structure. The top $\partial^+ \bar{\bar{Y}}$ certainly contains a copy of

$$W \underset{\text{diff}}{\cong} \partial^+ \bar{Y} \underset{\text{diff}}{\cong} \partial^+ \bar{Y} - \mathcal{N} \times 1 \underset{\text{diff}}{\cong} \partial^+ \bar{\bar{Y}} - \mathcal{N} \times 1,$$

and the bottom may be described as

$$\partial^{-}\overline{\overline{Y}} \underset{\text{diff}}{\cong} \left(\tfrac{1}{2}R^{4} - \mathcal{N} \underset{\partial}{\bigcup} \mathcal{N}H \underset{\text{diff}}{\cong} R^{3} \times I \underset{R^{3} \times 1}{\bigcup} H \underset{\text{diff}}{\cong} H \right).$$

So the product structure on $\overline{\overline{Y}}$ smoothly imbeds W into H.

Scholium. Let W be a smooth manifold homeomorphic to $\tfrac{1}{2}R^{4}$. Suppose W is resmoothed near a proper smooth ray γ by replacing the structure in a tubular neighborhood $\mathcal{N}(\gamma)$ with a copy of H. Then the result is diffeomorphic to H. In other words, H is absorptive. This diffeomorphism may be written without confusion as $W + H \cong H$. We postpone, for the moment, the consideration of spaces with more than one end.

Theorem 2. *Let W^{5} be a smooth five-dimensional h-cobordism with one end. Suppose there are smooth, proper imbeddings $e_{\pm}\colon H \to \partial^{\pm}W$. Then W^{5} is a smooth product iff W^{5} is a topological product.*

Proof. Assume W^{5} is a topological product. Choose a properly imbedded smooth half-open ray $\gamma \subset H$. The topological product structure may be altered, if necessary, so that $e_{+}(\gamma)$ projects near $e_{-}(\gamma)$, and in particular into $e_{-}(H)$. After a controlled cancelling of handles of indices zero and one, the gradient-like flow determines a smooth imbedding of $\mathcal{N} \times I$, a neighborhood of γ in $\partial^{+}W$ crossed with $I = [-1, 1]$, with $\mathcal{N} \times 1 \subset e_{+}(H) \subset \partial^{+}W^{5}$ and $\mathcal{N} \times -1 \subset e_{-}(H) \subset \partial^{-}W^{5}$.

Resmooth W^{5} (to obtain $\overline{W}$) exactly as in the proof of Theorem 1. (Briefly, $\mathcal{N}_{t}$ is replaced with H_{t} for $t \in [-1, 1]$.) We claim that $\overline{W}$ is diffeomorphic to W. By the scholium $H + H = H$, that is there is a diffeomorphism extending the identity on ∂H between H and the resmoothing $H + H$. Consequently, $\partial^{\pm}W^{5} \underset{\text{diff}}{\cong} \partial^{\pm}\overline{W}$: in fact, the diffeomorphisms are isotopic to the identity. Tapering this isotopy in a collar of ∂W^{5} we arrive at a homeomorphism $1d'$ which is a diffeomorphism except over a region which is homeomorphic to a half-open collar $R^{4} \times [0, \infty]$ properly imbedded in interior W^{5}. Clearly, the obstruction group to the Haupvermutung H^{3} (relative; Z_{2}) vanishes and $1d'$ is isotopic to a diffeomorphism $1d''\colon W^{5} \to \overline{W}$.

By replacing W^{5} with $\overline{W}$ we see the vertical strip $H \times [-1, 1] \subset \overline{W}$. The penultimate step in the high-dimensional h-cobordism theorem is the construction of Whitney disks. As in the proof of Theorem 1, this is carried out by isotoping into the universal example, $H \times 0 \subset$ middle level, where smooth disks are plentiful. Thus, a smooth product structure is constructed on $\overline{W} \underset{\text{diff}}{\cong} W^{5}$.

We call a homeomorphism $h\colon M_{0} \to M_{1}$ between two smooth 4-manifolds *stably isotopic* to a diffeomorphism if $h \times 1d\colon M_{0} \times R \to M_{1} \times R$ is isotopic

(equivalently ε-isotopic) to a diffeomorphism. The vanishing of the Kirby-Siebenmann obstruction $[h] \in H^3(M_1; Z_2)$ is necessary and sufficient for h to be stably isotopic to a diffeomorphism.

Theorem 3. *Let M_0 and M_1 be smooth 4-manifolds with one end. Suppose there are smooth proper imbeddings $e_i: H \to M_i$, $i = 0, 1$. Let $h: M_0 \to M_1$ be a homeomorphism. Then h is isotopic to a diffeomorphism iff h is stably isotopic to a diffeomorphism.*

Remarks. Let M_1 be a 4-manifold and let $\alpha \in H^3(M_1, \partial M_1; Z_2)$ be given. By [8, Theorem B], there exists a smooth M_0 and a homeomorphism $h: M_0 \to M_1$ such that $[h] = \alpha$ (and h is a diffeomorphism along the boundary. If M_1 has one end we can assume that there is a proper imbedding $e_0: H \to M_0$.

R. Edwards has shown that we cannot replace isotopic with ε-isotopic in the conclusion of Theorem 3 (or Corollary C below) [2].

Proof. Form the mapping cylinder Y of h, Y is a topological manifold with a natural smoothing near its top $\partial^+ Y \cong_{\text{diff}} M_0$ and bottom $\partial^- Y \cong_{\text{diff}} M_1$. $[h]$ is the only obstruction to extending this structure over Y. If $[h] = 0$, then extend and apply Theorem 2.

Remark. H is characterized by any one of the above theorems. In fact, the following trivial identity (on sums of $\frac{1}{2}$ spaces) shows any two absorptive structures H and J on $\frac{1}{2}R^4$ are diffeomorphic:

$$J \cong H + J \cong H \bigcup_{\frac{1}{2}R^3} J \cong J + H \cong H.$$
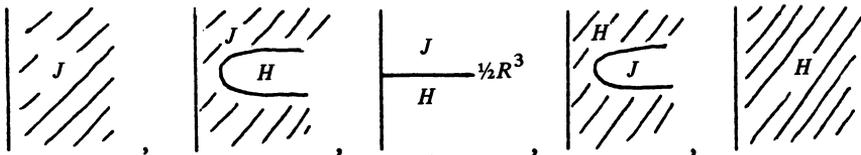
Diagram 2

Extensions. Theorem 2 holds, relative to sub-h-cobordisms which already have smooth product structures, e.g., for bounded cases. Theorem 3 therefore also holds for bounded 4-manifolds. Theorems 2 and 3 also have generalizations without assuming there is but one end of W^5 (and M^4 respectively). If there are no more than countably many ends, the correct hypothesis is that one copy of H is imbedded running toward each end of $\partial^{\pm} W^5$ (and M_i, $i = 0, 1$

respectively). In general, the space of ends $\mathscr{E}$ of a connected metrizable manifold is a subspace of the cantor set since it is defined as the inverse limit of the finite sets: π_0 (manifold compactum). Thus, $\mathscr{E}$ is separable and we may find a countable dense subset $\mathscr{E}' \subset \mathscr{E}$. The general hypothesis for Theorems 2 and 3 is that any two countable dense sets $\mathscr{E}^{\pm} \subset \mathscr{E} = \text{ends}(\partial^{\pm} W^5)$ ($= $ ends M_i, $i = 0, 1$, respectively) be selected and one copy of H is imbedded running toward each end of $\mathscr{E}^{\pm}$. Then the conclusions of Theorems 2 and 3 hold as stated. The generalization is straightforward if $\mathscr{E}^+ = \mathscr{E}^-$. Suppose that $\mathscr{E}'$ and $\mathscr{E}''$ are countable dense subsets of ends(M^4) and that M^4 contains disjoint copies of H running to each end of $\mathscr{E}'$. We show that M will also contain disjoint copies of H running to the ends of $\mathscr{E}''$.

Let $Z = (\frac{1}{2}R^4 \times [-\varepsilon, 0])_\Gamma$ be a concordance between $\frac{1}{2}R_{\text{std}}$ and H. Let

$$W^5 = (M^4 \times [0,1]) \bigcup_{M^4 \times 0} \left(\bigcup_{i=1}^{\infty} Z_i \right)$$

be formed by attaching copies of Z to neighborhoods of disjoint rays running toward $\mathscr{E}''$. W^5 is a product except over countably many sub-proper h-cobordisms with base H. The key obstacle to smoothly cancelling the handles of these is the absence of smooth Whitney disks in the middle level N. Given a proper radius function r: $N \to R^+ \cup 0$, the hypothesis on copies of H in M approaching $\mathscr{E}'$ implies that any topological Whitney disk $\Delta_j \subset CH_j$ can be smoothed by dragging it *outward* with respect to r. Thus the Whitney tricks may be performed and handles cancelled in a locally finite way giving W^5 a smooth product structure. This establishes the copies of H in M running towards $\mathscr{E}''$.

Let U be the smooth manifold which is the interior of H. We have some immediate corollaries.

Corollary A. *Let W^4 be any smoothing of R^4. There exists a smooth proper imbedding e: $H \to W^4$ iff W^4 is diffeomorphic to U.*

Proof. W^4 is $p - h$-cobordant to U and by [4] this $p - h$-cobordism is a product W^5 satisfying the hypothesis of Theorem 2.

Corollary B. *If W^4 is any smoothing of R^4, then there exists a smooth (but not usually proper) imbedding e: $W^4 \to U$.*

Proof. Remove the interior of a tubular neighborhood of some proper ray in W^4 and apply Theorem 1 to imbed in H. Now glue a standard $\frac{1}{2}R^4$ to domain and range to obtain the desired imbedding.

We call U a universal R^4 because of Corollary B. Since simply connected ends of smooth 4-manifolds are $p - h$-cobordant to either $S^3_{\text{std}} \times [0, \infty)$ or the end of the "Fake $S^3 \times R$" [3], we may deduce the following corollary from Theorem 3.

Corollary C. *Let M be a compact (connected) topological four-manifold without boundary whose (Kirby-Siebenmann) triangulation obstruction $[M] \in H^4(M; Z_2)$ vanishes. Then $M_0 = M - \text{pt}$ can be smoothed so that the smoothing of the end is diffeomorphic to the end of U. Any two such smoothings in the same stable isotopy class are isotopic. Similarly, if $[M] \neq 0$, then M_0 can be smoothed so as to be diffeomorphic near infinity to the sum (an end of "Fake $S^3 \times R$") + U. Any two such smoothings in the same stable isotopy class are isotopic.*

Thus the universal space U, by being maximally complicated, achieves a certain simplicity. Corollary C is an example of this maxim. Another is provided by considering knotted S^2's in R^4.

The first author showed [5] that any topological locally-flat imbedding e: $S^2 \to R^4$ with $\pi_1(R^4 - e(S^2)) \simeq Z$ was unknotted. Since Quinn [10] proved the annulus conjecture for R^4, we can take as our definition of unknotted that there exists an ambient isotopy of R^4 throwing e onto the standard unknotted S^2 in R^4.

The question of topologically unknotted smooth knots in R^4 (or S^4) is intimately related to smoothings of $S^3 \times S^1$. Given a smooth imbedding e: $S^2 \to S^4$, 0-framed surgery on e yields a smooth 4-manifold, $W(e)$. If e is topologically unknotted, $W(e)$ is homeomorphic to $S^3 \times S^1$. Given two topologically unknotted smooth knots e_1, e_2: $S^2 \to S^4$, $W(e_1)$ is diffeomorphic to $W(e_2)$ iff the pair $(S^4, e_1(S^2)$ is diffeomorphic to the pair $(S^4, e_2(S^2))$.

If we carry out the same program for knotted spheres in U^4, surgery yields manifolds homeomorphic to $S^3 \times S^1 - \text{pt}$ whose ends are diffeomorphic to the end of U. It is an easy exercise to see that, in U^4, there are, up to smooth isotopy, precisely two topologically unknotted, smooth 2-spheres. Moreover, the two knots are not even diffeomorphic as pairs.

Historically, U was the first smoothing of R^4 which was known not to be "compactifiable" in the following sense.

Theorem 4. *Let e: $U \to W$ be a smooth (not necessarily proper) imbedding of U into a smooth 4-manifold. There does **not** exist any topological imbedding h: $D^4 \to W$ of the 4-cell into W with closure $(e(U)) \subset h(\text{interior } D^4)$.*

Remarks. Gompf, in the next article, constructs countably many smoothings with this property. Theorem 4 was discovered while trying to prove:

Conjecture. There does not exist a smooth imbedding of U in any compact smooth 4-manifold.

Proof of Theorem 4. Suppose h: $D^4 \hookrightarrow W$ exists. Trim $h(D^4)$ along a smooth submanifold near $h\partial D^4$ to obtain a smooth imbedding of U in a compact smooth orientable manifold with boundary W^- with the image of U

contained in a smaller topologically collared 4-cell C. Doubling W^- we obtain a closed manifold W' containing U smoothly and inside C. Perform 1-surgery away from C to make W' simply connected and add copies of $S^2 \times S^2$ and CP^2. Eventually, Wall's stable classification of smooth 4-manifolds [10] tells us that $W' \cong_{\text{diff}} k(CP^2)\sharp l(S^2 \times S^2)$.

We would like to arrange that $k = 0$ in the above formula without altering the condition U inside the cell C. As long as $k > 1$ any CP^1-generator in $H_2(W'; Z)$ is "ordinary" (i.e., not dual to the Stiefel-Whitney class w_2) and may be represented in $W' - C$ by an imbedded surface. The Arf invariant is indeterminate and may be chosen to be zero (see [12]). Paired surgery of (W', surface) along a subkernel increases l (above) and modifies the surface into a smoothly imbedded 2-sphere. Blow this sphere down to reduce k until $k = 1$. When $k = 1$, the CP^2 generator α is characteristic (i.e., dual to w_2). Now

$$\frac{\text{Sign}\, W' - \alpha \cdot \alpha}{8} = \frac{1 - 1}{8} = 0$$

is the well-defined Arf invariant of any surface in $W' - C$ representing α. Applying the above procedure once more reduces k to zero.

By Casson's original construction [1], the natural generating set $\alpha_1, \beta_1, \cdots, \alpha_l, \beta_l$ for $H_2(W' - C; Z)$ can be represented by disjoint (smooth) imbeddings of $P_1, \cdots, P_l$, where each P_i is a proper homotopy $S^2 \times S^2 - \text{pt}$ made by attaching to $\mathring{B}^4$ two Casson handles (and these may be chosen from the class $\mathscr{C}$) with zero framing to the Hopf link in ∂B^4. Using the core disk $\Delta_j \subset CH_j$ we represent the $H_2(W' - C; Z)$ by disjoint topological imbeddings of $S^2 \vee S^2$. The imbeddings are smooth (and transverse) except near a point on each sphere where the singularity (B^4, Σ_j) is among the types smoothed during the construction of H.

As in the proof of Theorem 1, we can construct a topological isotopy of W' which carries the l disjoint topological imbeddings of $S^2 \vee S^2$ to smooth (and transverse) imbeddings by sliding the distinguished ball neighborhood of each singularity to superimpose the topological surface Σ on one of the smooth surfaces of U built from the same singularity. By construction, each singularity Σ_j occurs (infinitely often) smoothed outside any compact subset K of U. Thus the smooth surfaces $l(S^2 \vee S^2)$ may all be taken disjoint from any $K \subset U$. Now $W' - \text{neib}(l(S^2 \vee S^2)) \cup D^4$ is a smooth homotopy sphere containing any preassigned compactum $K \subset U$ of U. This is certainly impossible. The first fake R^4 resulting from the work of S. Donaldson and the first author contains compact submanifolds which do not smoothly imbed in any homotopy sphere (see [6] for an exposition). By the universal property of U

this compact submanifold would also lie in U, contradicting our construction. This proves Theorem 4. We can now formally derive:

Corollary D (*Gompf* [7]).   *There are at least countably many nondiffeomorphic smooth structures on R^4.*

Proof.   Let $B_1, B_2, B_3, \cdots$ be an exhaustion of U by topologically collared balls. If Corollary D fails, some infinite subsequence $B_{l(1)}, B_{l(2)}, B_{l(3)}, \cdots$ have diffeomorphic interiors: $V \cong \dot{B}_{l(1)} \cong \dot{B}_{l(2)} \cong \dot{B}_{l(3)}, \cdots$. Notice that V is very much like U; V contains (disjointly) any compactum K in U and any desired (but finite) number of smooth copies of any desired (but finite) collection of singularities (B, Σ_j). Thus the proof of Theorem 4 applies equally well to V as U. So V cannot be the interior of a topologically collared ball in a smooth 4-manifold. But $V \cong \dot{B}_{l(1)}$. This contradiction shows that no infinite collection of the interiors $\dot{B}_{l(1)}, \dot{B}_{l(2)}, \dot{B}_{l(3)}, \cdots$ can all be diffeomorphic to each other.

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