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AN INTEGRAL FORMULA FOR THE MEASURE OF RAYS ON COMPLETE OPEN SURFACES

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Dedicated to Professor Herbert Busemann on his 80th birthday

1. Introduction

It is interesting to investigate the influence of total curvature of a complete, noncompact, oriented and finitely connected Riemannian 2-manifold on the Riemannian metric. The total curvature of such an M is defined to be an improper integral of the Gaussian curvature G with respect to the area element dM induced from the Riemannian metric, and expressed as

$$c(M) = \int_M G\,dM.$$

The pioneering work on total curvature was done by Cohn-Vossen in [2], which states that if M admits total curvature, then $c(M) \leqslant 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. He also proved in [3] that if a Riemannian plane M (e.g., M is a complete Riemannian manifold homeomorphic to R^2) admits total curvature and if there exists a straight line (defined in the next paragraph) on M, then $c(M) \leqslant 0$.

It is the nature of completeness and noncompactness of a Riemannian plane M that through every point on M there passes at least a ray γ: $[0, \infty) \to M$, where it is a unit speed geodesic satisfying $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 > 0$, and d is the distance function induced from the Riemannian metric. A unit speed geodesic γ: $R \to M$ is called a straight line if $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in R$. Recall that M is said to be *finitely connected* if it is homeomorphic to a compact 2-manifold (without boundary) with finitely many points removed.

As was shown by Maeda [7], [8] and by Shiga [10], the total curvature of a Riemannian plane M imposes strong restrictions to the mass of rays emanating from an arbitrary fixed point on M. For a point p on M let S_p be the unit

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circle centered at the origin of the tangent space M_p to M at p. Let $A(p) \subset S_p$ be the set of all unit vectors tangent to rays emanating from p. S_p is equipped with the natural measure μ which is induced from the Riemannian metric of M at p. $A(p)$ is closed in S_p. If $\{p_j\}$ is a sequence of points on M converging to a point p, then $\limsup_{j \to \infty} A(p_j) \subset A(p)$ holds in the unit circle bundle over M. Therefore the function $p \mapsto \mu \circ A(p)$ is upper semicontinuous and takes value in $[0, 2\pi]$. It follows that $\mu \circ A$ is integrable in the sense of Lebesgue. It was proved in [7] that if a Riemannian plane has nonnegative Gaussian curvature, then for all $p \in M$

$$\mu \circ A(p) \geqslant 2\pi - c(M),$$

and moreover we have

$$\inf_M \mu \circ A = 2\pi - c(M).$$

Shiga proved in [10] that if a Riemannian plane M admits total curvature, then

$$2\pi - \int_M G_+ \, dM \leqslant \inf_M \mu \circ A \leqslant 2\pi - c(M).$$

One of our results will be stated as follows.

Theorem 1. *Assume that a Riemannian plane M admits positive total curvature. If $\{K_j\}$ is a monotone increasing sequence of compact sets such that $\lim_{j \to \infty} K_j = M$, then*

$$\lim_{j \to \infty} \frac{\int_{K_j} \mu \circ A \, dM}{\int_{K_j} dM} = 2\pi - c(M).$$

We begin the proof of Theorem 1 by showing a general estimate for the function $\mu \circ A$ around an end point. In fact it is proved that for any $\varepsilon > 0$ there exists a compact set K such that

$$2\pi - c(M) - \varepsilon \leqslant \mu \circ A(p) \leqslant 2\pi - c(M) + \varepsilon$$

holds for all $p \in M \setminus K$. The above estimate is obtained by an essential use of the fact that M contains no straight line, and this fact is guaranteed by the assumption that the total curvature of M is positive.

Thus the proof of Theorem 1 is divided into two cases. In the first case, where the total volume of M is unbounded, the proof is immediate from the above estimate.

In the second case, where the total volume of M is bounded, Theorem 1 is a direct consequence of the following Theorem 2.

Theorem 2. *Assume that a Riemannian plane M has a bounded total volume and that it admits total curvature. Then there exists a measure zero set E_0 of M such that through every point on $M \setminus E_0$ there passes a unique ray.*

The set E_0 stated in Theorem 2 is defined as follows. Let $\gamma\colon [0, \infty) \to M$ be an arbitrary fixed ray. The Buseman function $F_\gamma\colon M \to R$ with respect to γ (for the definition see §1) has the property that it is Lipschitz continuous. Let E_0 be the set of all nondifferentiable points for F_γ. The Lipschitz continuity for F_γ implies that E_0 is of measure zero. Therefore $\mu \circ A\colon M \to R$ is 0 on $M \setminus E_0$, and the proof of Theorem 1 in this case is straightforward. It turns out that E_0 is independent of the choice of γ, as stated in the final paragraph of §3.

Fundamental properties of Buseman functions will be needed for the proof of Theorem 2, and they are summarized in §1 (for details see [1]). Roughly speaking, E_0 is contained in the cut locus E to the end point. More precisely, a portion $E \cap F_\gamma^{-1}((-\infty, t])$ of E for a fixed $t \in F_\gamma(M)$ may be viewed as the cut locus to the fixed level set $F_\gamma^{-1}(\{t\})$ of F_γ. The difficulty in dealing with the cut locus of $F_\gamma^{-1}(\{t\})$ occurs because F_γ is not differentiable of class C^2 but differentiable almost everywhere. Note that E is not closed in general (see Nasu [9, p161]) and that it is not certain if E is of measure zero in M.

One problem is left open. We do not know if Theorem 1 remains valid without the positivity assumption for the total curvature. Clearly it does not hold when $c(M) < 0$. Thus the problem is if it is valid under the assumption that $c(M) = 0$.

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1. Review of known results

From now on let M be a Riemannian plane. The basic tools used for the proofs of our results are summarized as Lemmas 1.1–1.3, and the proofs are omitted here.

Lemma 1.1 (*compare* [8, *Lemma* C]). *Assume that γ contains no straight line. Then, for every compact set $K \subset M$, there exists a number $R(K) > 0$ such that if $p \in M$ satisfies $d(p, K) > R(K)$, then all rays emanating from p do not pass through any point on K.*

Lemma 1.2 (*compare* [11, *Theorem* A, (4)]). *Assume that M admits total curvature. Let $p \in M$ have the property that $M \setminus \{\exp_p tu\colon u \in A(p),\ t > 0\} \neq \emptyset$ and let D be a component of this set. If $u, v \in A(p)$ are tangent to the rays consisting of the boundary of D and if $\sphericalangle(u, v)$ is the angle measured with respect to D, then*

$$c(D) := \int_D G\, dM = \sphericalangle(u, v).$$

It should be noted that if there is a unique ray emanating from p, then $u = v$ and the angle in Lemma 1.2 is understood as $\sphericalangle(u, v) = 2\pi$.

Let $p \in M$ be an arbitrary fixed point. For each $t \geqslant 0$ set $S(t) := \{x \in M: d(x, p) = t\}$ and $B(t) := \{x \in M: d(x, p) \leqslant t\}$. It was proved by Hartman (see [4, Proposition 6.1]) that $S(t)$ for almost all $t \geqslant 0$ becomes a finite union of piecewise smooth closed curves. Moreover, it was proved by the author (see [11, Theorem B]) that if M admits total curvature, then there exists a $T_0 > 0$ such that $S(t)$ is homeomorphic to a circle for all $t > G_0$. In other words, the distance function to p has no critical point on $M \setminus B(T_0)$.

Lemma 1.3 (*see* [11, *Theorem* D]). *Assume that M admits total curvature. If $L(t)$ and $A(t)$ are the length of $S(t)$ and the area of $B(t)$ respectively, then we have*

$$\lim_{t \to \infty} \frac{L(t)}{t} = \lim_{t \to \infty} \frac{2A(t)}{t^2} = 2\pi - c(M).$$

Let γ: $[0, \infty) \to M$ be an arbitrary fixed ray. The *Busemann function* F_γ: $M \to R$ *with respect to* γ is defined by

$$F_\gamma(x) := \lim_{t \to \infty} [t - d(x, \gamma(t))].$$

The right side of the above equation converges uniformly on every compact set. The Lipschitz continuity of F_γ follows from $|F_\gamma(x) - F_\gamma(y)| \leqslant d(x, y)$ for all $x, y \in M$. Thus F_γ is differentiable except at a set E_0 of measure zero on M. A ray σ: $[0, \infty) \to M$ is by definition *asymptotic* to γ if there exists a sequence $\{\sigma_j: [0, l_j] \to M\}$ of minimizing geodesics such that the sequence $\{\dot\sigma_j(0)\}$ of initial unit vectors converges to $\dot\sigma(0)$ and that the sequence $\{\sigma_j(l_j)\}$ of terminal points is a monotone divergent sequence on $\gamma([0, \infty))$. It follows from the definition that through every point on M there passes at least a ray which is asymptotic to γ. It follows that if σ is an asymptotic ray to γ, then $F_\gamma \circ \sigma(s) = F_\gamma \circ \sigma(0) + s$ for all $s \geqslant 0$, and that F_γ is differentiable at points on $\sigma((0, \infty))$. An asymptotic ray to γ is said to be *maximal* if it is not contained as a proper subarc of any other ray which is asymptotic to γ. Let $E \subset M$ be the set of all initial points of maximal asymptotic rays to γ. If M contains no straight line, then every asymptotic ray meets E. It follows that if $c(M) > 0$, then $E \neq \varnothing$. Every point on $M \setminus E$ is an interior of some asymptotic ray to γ, and hence F_γ is differentiable on $M \setminus E$. The gradient vector ∇F_γ to F_γ at a point on $M \setminus E$ is the unit vector tangent to the asymptotic ray to γ. Thus the set E_0 of all nondifferentiable points of F_γ is contained entirely in E (for details, see [5, Theorem 2]).

A point $x \in M$ is said to be *noncritical* for F_γ if there exists an open half space of M_p which contains $A(x)$. A point $x \in M$ is said to be a *critical point* for F_γ if for any unit vector $v \in S_x$ there exists an asymptotic ray σ to γ such that $\sigma(0) = x$ and $\langle \dot\sigma(0), v\rangle \geqslant 0$.

2. The proof of Theorem 1 in the case where M has unbounded volume

Let M satisfy the assumptions in Theorem 1. As is stated in the introduction, the assumption $c(M) > 0$ implies that M contains no straight line. It follows from the Cohn-Vossen theorem that $c(M) \leqslant 2\pi$, and hence $\int_M |G|\,dM < \infty$. For any $\varepsilon > 0$ there exists a compact set $K \subset M$ such that

$$\int_K G_+\,dM > \int_M G_+\,dM - \varepsilon/3 \quad \text{and} \quad \left|\int_K G\,dM - c(M)\right| < \varepsilon/3.$$

It follows from Lemma 1.1 that there exists for this K an $R(K) > 0$ such that if $p \in M$ satisfies $d(p, K) > R(K)$, then every ray emanating from p does not pass through any point on K. Let D be a component of $M \setminus \{\exp_p tu\colon t > 0,\ u \in A(p)\}$ such that $K \subset D$ and let $u, v \in A(p)$ be as in Lemma 1.2 for D. We have

$$c(M) - \varepsilon \leqslant c(D) \leqslant c(M) + 2\varepsilon/3,$$

and hence

$$\mu \circ A(p) \leqslant 2\pi - \sphericalangle(u, v) \leqslant 2\pi - c(M) + \varepsilon.$$

A lower bound for $\mu \circ A(p)$ is obtained as follows. The set

$$M \setminus \{\exp_p tu\colon t \geqslant 0,\ u \in A(p)\}$$

is expressed as a disjoint union $\bigcup_{\lambda \in \Lambda} D_\lambda$ of countable open sets $\{D_\lambda\colon \lambda \in \Lambda\}$, and each D_λ is bounded by two rays emanating from p, where Λ is a countable index set. It follows that

$$2\pi - \mu \circ A(p) = \sum_{\lambda \in \Lambda} c(D_\lambda) \leqslant c(D) + \int_{M \setminus D} G_+\,dM \leqslant c(M) + \varepsilon.$$

Therefore we have

$$\mu \circ A(p) \geqslant 2\pi - c(M) - \varepsilon.$$

If the total volume of M is unbounded, then the proof of Theorem 1 is straightforward from the above estimates for $\mu \circ A(p)$ with $d(p, K) > R(K)$. Thus the proof of Theorem 1 in this case is complete.

3. The proof of Theorem 2

Let M be a Riemannian plane and $\gamma\colon [0, \infty) \to M$ an arbitrary fixed ray, and set $p = \gamma(0)$. It follows from Lemma 1.3 that $c(M) = 2\pi$ holds under the assumptions of Theorem 2.

Lemma 3.1. *Assume that M admits total curvature and that the total volume of M is bounded. Then, every point $q \in M \setminus E$ has the property that there is a unique ray emanating from q.*

Proof. Since M admits total curvature, we have $A(t) = \int_0^t L(u)\,du < \mathrm{vol}(M) < \infty$ for all $t > 0$. Therefore there exists a monotone divergent sequence $\{t_j\}$ of positive numbers such that $\{L(t_j)\}$ is strictly monotone decreasing and $\lim_{j\to\infty} L(t_j) = 0$.

Since q is an interior of some maximal asymptotic ray $\sigma\colon [0,\infty) \to M$ to γ such that $\sigma(0) \in E$, there is a positive number κ with $\sigma(\kappa) = q$. Suppose that there exists a ray $\tau\colon [0,\infty) \to M$ with $\tau(0) = q$, $\dot\tau(0) \neq \dot\sigma(\kappa)$. For a small number h with $0 \leqslant h < \min\{\kappa$, the convexity radius at $q\}$ there is an $\varepsilon > 0$ such that

$$d(\sigma(\kappa - h), \tau(h)) + \varepsilon < d(\sigma(\kappa - h), q) + d(\tau(h), q).$$

For a large number j with $L(t_j) < \varepsilon/2$ let y and x be points on the intersections of $\sigma([\kappa,\infty)) \cap S(t_j)$ and $\tau([0,\infty)) \cap S(t_j)$ respectively. It follows from the triangle inequality that

$$\begin{aligned} d(\sigma(\kappa - h), y) &< d(\sigma(\kappa - h), \tau(h)) + d(\tau(h), x) + d(x, y) \\ &< (d(\sigma(\kappa - h), q) + d(q, \tau(h)) - \varepsilon) + d(\tau(h), x) + d(x, y) \\ &< d(\sigma(\kappa - h), q) + (d(q, y) + d(y, x)) - \varepsilon + d(x, y) \\ &= d(\sigma(\kappa - h), y) + 2d(x, y) - \varepsilon < d(\sigma(\kappa - h), y) - \varepsilon/2. \end{aligned}$$

This is a contradiction, and the proof is complete.

The following Lemma 3.2 is not used for the proof of Theorem 2, but it is stated here because it contains an independent interest.

Lemma 3.2. *Assume that M admits total curvature and that the volume of M is bounded. Then there exists $T_1 > 0$ such that $F_\gamma^{-1}(\{t\})$ is homeomorphic to a circle and freely homotopic to $S(T_0)$ for all $t > T_1$. Moreover, F_γ has no critical point on $M \setminus F_\gamma^{-1}((-\infty, T_1])$.*

Proof. First of all we shall prove that there exists a $T_1' > 0$ such that $F_\gamma^{-1}(\{t\})$ is arcwise connected for all $t > T_1'$. If otherwise supposed, then there is a monotone divergent sequence $\{s'_j\}$ such that $F_\gamma^{-1}(\{s_j'\})$ is not arcwise connected for each j. Note that $F_\gamma^{-1}((-\infty, c])$ for every $c \in R$ is compact and hence $F_\gamma^{-1}(\{t\})$ is compact for all $t \in R$. For each j let W_j' be a component of $F_\gamma^{-1}((-\infty, s_j']]$ such that $\gamma([0, s_j')) \cap W_j' = \varnothing$. If W_j' has an empty interior, then we replace s_j' by an $s_j < s_j'$ sufficiently close to s_j' such that $F_\gamma^{-1}(\{s_j\})$ is not arcwise connected and such that there is a component W_j of $F_\gamma^{-1}((-\infty, s_j]]$ which contains W_j' in its interior and is homeomorphic to a closed 2-disk.

We may consider that $F_\gamma^{-1}(\{s_j\})$ is contained in $M \setminus B(T_0)$ for all j and that $M \setminus B(T_0)$ is homeomorphic to $S^1 \times [0, \infty)$. It is elementary that if $\partial B_r(x)$ denotes the boundary of the metric r-ball centered at x, then $F_\gamma^{-1}(\{t\}) = \lim_{s \to \infty} \partial B_{s-t}(\gamma(s))$ holds for all $t \in F_\gamma(M)$. In view of $\lim_{j \to \infty} L(t_j) = 0$, we find for each $k = 1, 2, \cdots$ a number j_k such that if x is any point on $S(t_{j_k})$, then the distance function to x has the following properties: (1) $d(x, \cdot)$ takes a local maximum at an interior of W_k, (2) $M \setminus B_{t_{j_k} - s_k}(x)$ has at least two compact components, and if $W_k(x)$ is a compact component of $M \setminus B_{t_{j_k} - s_k}(x)$ with $W_k(x) \cap \gamma([0, \infty)) = \emptyset$ and $W_k(x) \supset W_k$, then $W_k(x)$ is homeomorphic to a closed 2-disk, (3) $\partial W_k(x)$ lies in a small neighborhood of ∂W_k and $\partial W_k(x)$ is freely homotopic to ∂W_k in $M \setminus B(T_0)$ Then there exists a point x on $S(t_{j_k})$ and two distinct minimizing geodesics α_1, α_2 joining x to a point on $\partial W_k(x)$ such that the geodesic biangle $\alpha_1 \cup \alpha_2$ is freely homotopic to $\partial W_k(x)$ in $M \setminus B(T_0)$. The standard length-decreasing deformation proceeds to $\alpha_1 \cup \alpha_2$ in $M \setminus (B(T_0) \cup W_k(x))$ to obtain a geodesic loop α_k at x which has the minimum length among all closed curves with base point x in $M \setminus (B(T_0) \cup W_k(x))$ and they are freely homotopic to $\partial W_k(x)$. If D_k is the disk bounded by α_k, then $D_k \supset W_k(x)$ and $D_k \subset B(t_{j_k}) \setminus B(s_k)$. Thus there exists a disjoint infinite sequence $\{D_k\}$ of disks in $M \setminus B(T_0)$, and $\sum_k c(D_k) = +\infty$ leads to a contradiction that M admits total curvature. This proves the existence of T_1'.

It follows from $c(M) = 2\pi$ that $\int_M |G|\, dM < \infty$, and hence for every $\varepsilon > 0$ there exists a $t(\varepsilon) > 0$ such that

$$\int_{\overline{B}(t)} |G|\, dM > \int_M |G|\, dM - \varepsilon$$

holds for all $t > t(\varepsilon)$. We shall prove that if $T_1 = \max\{T_1', t(\pi)\}$, then every $F_\gamma^{-1}(\{t\})$ with $t > T_1$ is homeomorphic to a circle. Suppose that there is a $t' > T_1$ such that $F_\gamma^{-1}(\{t'\})$ is not homeomorphic to S^1. There is a simply closed curve $\Gamma(t')$ in $F_\gamma^{-1}(\{t'\})$ such that $\Gamma(t')$ bounds the open unbounded set $M \setminus F_\gamma^{-1}((-\infty, t'])$. Since $F_\gamma^{-1}(\{t'\})$ is arcwise connected, there exists a nontrivial curve b: $[0,1] \to F_\gamma^{-1}(\{t'\}) \setminus \Gamma(t')$ such that at every point on the curve F_γ takes a local minimum. It turns out that for each point q on $b((0,1))$, there exists two distinct rays τ_1, τ_2: $[0, \infty) \to M$ each of which is asymptotic to γ and $\tau_1(0) = \tau_2(0) = q$ and $\dot\tau_1(0) + \dot\tau_2(0) = 0$. Let $V_q \subset M \setminus B(T_0)$ be the domain bounded by $\tau_1([0, \infty)) \cup \tau_2([0, \infty))$ such that V_q does not contain $\gamma([0, \infty))$. It follows from $\lim_{j \to \infty} L(t_j) = 0$ and the Gauss-Bonnet theorem that $c(V_q) = \sphericalangle(\dot\tau_1(0)), \dot\tau_2(0)) = \pi$. However since $t' > T_1$, $M \setminus B(t') \supset V_q$ and $t' > t(\pi)$ imply that

$$c(V_q) < \int_{M \setminus B(t')} |G|\, dM < \pi.$$

This is a contradiction.

Finally, if $q \in M \setminus B(T_0)$ with $F_\gamma(q) > T_1$ is a critical point of F_γ, then the same arguments as developed above lead to a contradiction. This completes the proof of Lemma 3.2.

Proof of Theorem 2. Let $q \in M \setminus E_0$. Then the gradient vector $\nabla F_\gamma(q)$ to F_γ at q is a unit vector tangent to a ray σ: $[0, \infty) \to M$ which is asymptotic to γ and $\sigma(0) = q$. Suppose that there exists another ray τ: $[0, \infty) \to M$ with $\tau(0) = q$. Let $\{\varepsilon_j\}$ be a monotone decreasing sequence of positive numbers such that $\lim \varepsilon_j = 0$ and set $q_j = \tau(\varepsilon_j)$. It follows from the argument developed in the proof of Lemma 3.1 that emanating from each q_j there exists a unique ray which is asymptotic to γ. This fact shows that $\tau|[\varepsilon_j, \infty)$ is asymptotic to γ, and hence letting $j \to \infty$ we observe that τ is asymptotic to γ. Since F_γ is differentiable at q, we have $\dot{\tau}(0) = \nabla F_\gamma(q)$. This proves the desired uniqueness.

Notice that under the assumptions in Theorem 2 all rays are asymptotic to each other. A maximal asymptotic ray to γ is in general contained as a proper subarc of some ray (see [6]). But in our case every maximal asymptotic ray to γ is not contained as a proper subarc of any ray, and hence E_0, and E as well, are independent of the choice of γ.

4. Extensions of Theorems 1 and 2

As is seen in the proofs of our Theorems 1 and 2, the nonexistence of straight lines on a Riemannian plane plays an essential role. The nonexistence property for a general complete open surface requires that it has exactly one end. Indeed there exists at least one straight line on every complete noncompact Riemannian manifold having more than one end.

In this section let M be a connected, complete, noncompact, oriented and finitely connected Riemannian 2-manifold having one end. It is not difficult to verify that if such an M admits total curvature and if M contains a straight line, then

$$c(M) \leqslant 2\pi(\chi(M) - 1).$$

Lemma 1.1 remains valid for such an M. Lemma 1.2 can be extended to M as follows: If M admits total curvature and if $D \subset M$ is a domain bounded by two rays emanating from a point $p \in \partial D$ such that any ray starting from p does not interest D and if $M \setminus D$ is homeomorphic to a closed half-plane, then

$$c(D) = 2\pi(\chi(M) - 1) + \sphericalangle(u, v),$$

where $u, v \in A(p)$ are tangent to the rays lying in the boundary of D.

The following results are slight extensions of Theorems 1 and 2. The proofs are omitted here since they are obtained by the same method as stated in the previous §§2 and 3.

Theorem 3. *If the total curvature of M satisfies $c(M) > 2\pi(\chi(M) - 1)$ and if $\{K_j\}$ is a monotone increasing sequence of compact sets with $\lim K_j = M$, then we have*

$$\lim_{j\to\infty} \frac{\int_{K_j} \mu \circ A \, dM}{\int_{K_j} dM} = 2\pi\chi(M) - c(M).$$

Theorem 4. *Let M have a bounded total volume and admit total curvature. Then there exists a measure zero set E_0 of M such that through every point on $M \setminus E_0$ there passes a unique ray.*

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