

THE NORMALIZED CURVE SHORTENING FLOW AND HOMOTHETIC SOLUTIONS

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The curve shortening problem, by now widely known, is to understand the evolution of regular closed curves $\gamma: \mathbf{R}/\mathbf{Z} \rightarrow M$ moving according to the curvature normal vector: $\partial\gamma/\partial t = kN = -$ “the L^2 -gradient of arc length”. One motivation for this problem has been the view expressed in this connection by C. Croke, H. Gluck, W. Ziller, and others: it would be desirable to improve on some complicated and ad hoc constructions that have been used in the theory of closed geodesics to iteratively shorten curves.

As a test case it has been a goal to prove the conjecture that kN generates a flow on the space of simple closed curves in the plane, preserving embeddedness and making such curves circular asymptotically as length approaches zero. However, the evolution equation for the curvature of γ_t turns out to be quite subtle, and the conjecture is not yet settled. Indeed, in the nonsimple case one generally expects singular behavior, and part of the intrinsic interest of the problem lies in the fact that the global condition of embeddedness is apparently recognized by the “near-sighted” equation.

What is known thus far is that the conjecture is true for convex curves, that simple curves do in fact remain simple (provided curvature stays bounded), and that short time solutions to the equations exist in full generality; these results are due to M. Gage and R. Hamilton (see [1], [2], [3]).

1. Main results

The starting point for the present investigation is a modification of the usual curve shortening flow; the flow is geometrically unchanged, but a tangential field bT is added to kN to maintain constant speed $\alpha = |\partial\gamma_t/\partial\sigma|$ along the curve.

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In terms of a new time parameter τ satisfying $dt/d\tau = \alpha^2$, a normalized curvature function $\kappa(\sigma, \tau) = \alpha(\tau)k(\sigma, \tau)$, and an auxiliary function $\beta(\sigma, \tau) = \alpha(\tau) \cdot b(\sigma, \tau)$ whose average over $\sigma \in \mathbf{R}/\mathbf{Z}$ is zero, the evolution of a curve γ in a 2-manifold M is then described by

$$(1) \quad \frac{\partial \kappa}{\partial \tau} = \kappa'' + (\beta \kappa)' + \kappa R|_{\gamma}, \quad \beta' = \kappa^2 - \int_0^1 \kappa^2 ds,$$

where primes denote derivatives with respect to σ and $\alpha^{-2}R|_{\gamma}$ is the curvature of M along γ .

The restriction to 2-manifolds is clearly inessential for normalization, but simplifies the appearance of (1) significantly. One may choose to interpret κ and R as the actual curvatures of γ and M , respectively, by changing the metric on (M, g_0) to the time-dependent metric $g = \alpha^{-2}g_0$ (so γ always has unit length).

An apparent benefit of the normalization is that the leading term in the evolution equation is no longer the ‘‘Laplace-Beltrami’’ operator (which is troublesome because it ‘‘changes’’ with the curve) but rather the ordinary Laplacian. The new time parameter τ makes the flow equivariant with respect to dilations and also has the circle in \mathbf{E}^2 collapsing to a point in *infinite* rather than finite time.

Equation (1) is used to obtain the results of (A), (B) and (C) below.

(A) As the nonlinear evolution equations of curve shortening evidently do not tend themselves very well to a general partial differential equations approach, the possibility of writing down some special solutions analytically is of particular interest. Indeed, regarding the overall behavior of the flow, much of the picture which emerges in (B), (C) is related to the *homothetic solutions*—trajectories of the planar curve shortening flow for which κ does not depend on time—whose classification is the content of

Theorem A. *Let $\gamma: \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{E}^2$ be a unit speed closed curve representing a homothetic solution of the curve shortening flow. Then γ is an m -covered circle γ_m , or γ is a member of the family of transcendental curves $\{\gamma_{m,n}\}$ having the following description: if m and n are positive integers satisfying $1/2 < m/n < \sqrt{2}/2$ there is (up to congruence) a unique unit speed curve $\gamma_{m,n}: \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{E}^2$ having rotation index m and closing up in n periods of its curvature function $\kappa > 0$, a solution to the equations*

$$(2) \quad B'' + 2\lambda^2(e^B - 1) = 0, \quad B = 2 \ln \kappa/\lambda,$$

for some constant λ .

If (r, θ) are polar coordinates with origin at the center of mass of $\gamma_{m,n}$, then κ and r are related by $\kappa = C \exp(\frac{1}{2}\lambda^2 r^2)$ for some constant C .

Note that a particular case of Theorem A is the assertion that the circle is the only *simple* homothetic solution, in agreement with the above-mentioned

conjecture. The simplest noncircular solution, $\gamma_{2,3}$, is pictured in Figure 1(c), below, and five others appear in Figure 2 in §3.

To obtain such complete information about the homothetic solutions one makes use of the fact that equation (2) has a first integral:

$$(3) \quad \frac{1}{2}(B')^2 + 2\lambda^2V(B) = 2\lambda^2\eta, \quad V(B) = e^B - B - 1,$$

with η a nonnegative constant which depends only on m and n and should be thought of as determining the *shape* of $\gamma_{m,n}$. On the other hand, as explained in §3, λ is just a scaling constant.

(B) As mentioned earlier, markedly different behavior is expected for various trajectories of the curve shortening flow, depending on global properties of the initial curves. However, in terms of function space topologies the following theorem shows that the range of possible behavior is rather narrowly limited.

Theorem B. *Let γ_τ be a trajectory of the curve shortening flow in the plane and let κ_τ be the normalized curvature of γ_τ (so κ_τ satisfies equation (1)).*

- (i) κ_τ stays bounded in $L^1(\mathbf{R}/\mathbf{Z})$ (in fact $\|\kappa_\tau\|_{L^1}$ is nonincreasing in τ).
- (ii) If κ_τ stays bounded in $L^2(\mathbf{R}/\mathbf{Z})$, then so do all derivatives $\partial^i\kappa_\tau/\partial\sigma^i$.
- (iii) If κ_τ converges in L^1 as $\tau \rightarrow \infty$, then γ_τ converges to a point and is asymptotic in the $C^1(\mathbf{R}/\mathbf{Z})$ topology to a homothetic solution. The same statement holds for an arbitrary surface M^2 in place of \mathbf{E}^2 except that, instead of converging to a point, γ_τ may also converge to a geodesic.

Regarding part (ii) of Theorem B, more precise asymptotic bounds are given in §4. Such bounds reflect the partially smoothing nature of the flow, a phenomenon due to the Laplacian on the right-hand side of equation (1). To this extent comparison with the behavior of the heat equation is appropriate; however, the nonlinear lower order term generally disrupts the familiar total smoothing phenomenon. In fact, the existence of the (noncircular) homothetic solutions shows that one cannot expect the derivatives $\partial^i\kappa_\tau/\partial\sigma^i$ to decay to zero even if κ_τ remains L^2 -bounded. In this sense, the asymptotic bounds obtained here provide the best possible general set of estimates.

The utility of part (iii) is enhanced by the observation that a trajectory of the curve shortening flow is a *regular homotopy*. Thus, e.g., if γ_0 is simple and κ_τ converges in L^1 , then κ_τ is asymptotic to a circle, as conjectured. On the other hand, if γ_0 has rotation index zero (e.g., if γ_0 looks like a figure eight), then κ_τ cannot possibly converge in L^1 .

The *nonconvergence result* just mentioned, together with parts (i) and (ii) of Theorem B, suggest that the limiting curvature for a symmetrical figure eight might be a sum of two Dirac measures $\delta = \pi(\delta_0 - \delta_{1/2})$.

(C) Perhaps the main challenge of the curve shortening problem is to distinguish curves with a nonsingular future from those with a singular future.

The homothetic solutions not only provide nonconvex examples belonging to the former category, but they also appear to locate part of the boundary between the two categories; hence, there is reason to regard them as comparison solutions for the flow.

In fact, the very existence of the homothetic solutions—including the rather arbitrary looking numerical condition $1/2 < m/n < \sqrt{2}/2$ in the classification—can be understood by regarding the homothetic curves as saddle points lying between circles, on one side, and singular curves, on the other (this viewpoint helps simultaneously to explain the following rather curious coincidence: the classification of closed free elasticae in the hyperbolic plane follows the very same arithmetic condition and qualitative description [4]).

To explain the above more concretely, observe first that $\gamma_{m,n}$ is “fixed” by the group $G = G(m, n) = \langle g \rangle \cong \mathbf{Z}_n$, where g corresponds to rotation by $\theta = 2\pi m/n$. Figure 1 describes, for the case $m = 2, n = 3$, a g -equivariant regular homotopy beginning at γ_m , passing through $\gamma_{m,n}$, and tending to a singular curve $\Gamma_{m,n}$.

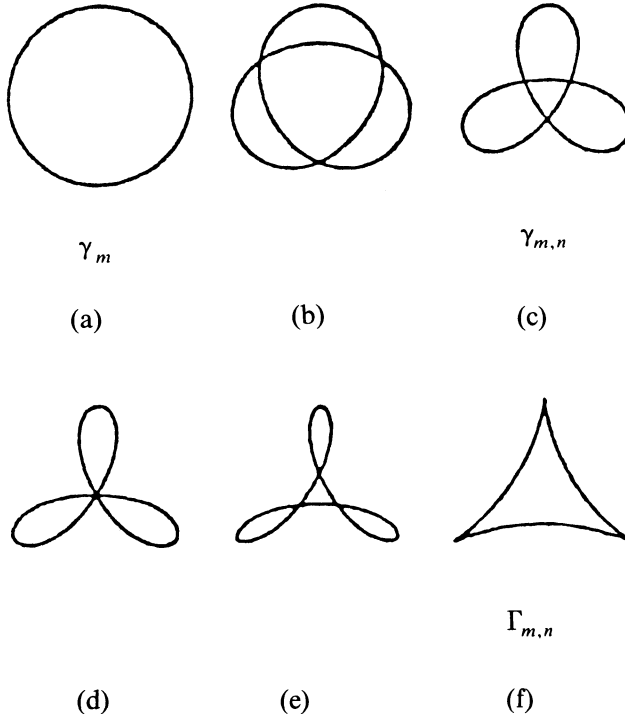


FIG. 1.

A reasonable conjecture associated with this picture is that if $\epsilon > 0$ is a small number and N is the outward pointing normal along $\gamma_{m,n}$, then the trajectories through $\gamma_+ = \gamma_{m,n} + \epsilon N$ and $\gamma_- = \gamma_{m,n} - \epsilon N$ are asymptotic to γ_m and a singular curve resembling $\Gamma_{m,n}$, respectively.

The evidence for such a conjecture comes in two parts. On the one hand, for somewhat larger ϵ , it can be proved rigorously that γ_- does indeed become singular; this follows from a general *area criterion for divergence* proved in §5. On the other hand stability computations suggest that γ_m ought to attract curves resembling (b) of Figure 1 (even though γ_m is linearly *unstable* for $m > 1$).

More specifically, let $\Omega = \{\kappa \in C^\infty(\mathbf{R}/\mathbf{Z}): \text{if } \gamma(s) \text{ has unit speed and curvature } \kappa(s), \text{ then } \gamma \text{ is a regular closed curve}\}$, let G act on Ω by $g\kappa(s) = \kappa(s - 2\pi m/n)$, and set $\Omega^G = \{\kappa \in \Omega: g\kappa = \kappa\}$, the fixed point set of G . Note that the evolution equation (1) induces a flow on Ω^G . Set $\kappa_m \equiv 2\pi m \in \Omega^G$, the curvature of γ_m , and let L be the linearization of the flow at κ_m . Then it is shown in §5 that *the operator L on $T_{\kappa_m} \Omega^G(m,n)$ has strictly negative spectrum precisely when $|m/n| < \sqrt{2}/2$.*

Note that the other inequality $1/2 < m/n$ is implicit in Figure 1; for each of the n petals must contribute at least π to the total rotation $2\pi m$. Thus, the arithmetic condition of the classification is heuristically explained.

2. The normalized flow

The following propositions refer to the notation of §1.

Proposition 2.1. *Let $\gamma: [0, t_1) \times \mathbf{R}/\mathbf{Z} \rightarrow M^n$ evolve according to $\partial\gamma/\partial t = hT + kN$, where $h: [0, t_1) \times \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}$ is some smooth function. Then each γ_t has constant speed if and only if γ_0 has constant speed and h satisfies $h'/\alpha = k^2 - \int_0^1 k^2 d\sigma$.*

Proof. The Frenet equations yield

$$\partial\alpha^2/\partial t = 2\langle \nabla_W \gamma', \gamma' \rangle = 2\langle \nabla_{\gamma'} W, \gamma' \rangle = 2\alpha(h' - \alpha k^2),$$

hence, $\partial^2 \ln \alpha / \partial t \partial \sigma = \partial(h'/\alpha - k^2) / \partial \sigma$. The proposition follows easily from the latter equation, together with the periodicity requirement on h . *q.e.d.*

We can now define the *normalized flow* on the space of constant speed immersions of \mathbf{R}/\mathbf{Z} into M by $\partial\gamma/\partial\tau = (\partial t/\partial\tau)W = \alpha^2(bT + kN)$, where b is defined to have average value zero and satisfy $b'/\alpha = k^2 - \int_0^1 k^2 d\sigma$.

Proposition 2.2. *Let R_0 be the curvature of the 2-manifold M^2 and let γ evolve according to the normalized flow. Then the speed α and curvature k of γ evolve according to $d\alpha/d\tau = -\alpha^3 \int_0^1 k^2 d\sigma$ and $\partial k/\partial\tau = k'' + \alpha b k' + \alpha^2 k^3 + \alpha^2 k R_0|_\gamma$, respectively.*

Proof. Both equations are straightforward to derive; the latter is somewhat longer, but follows quickly from the useful formula giving the evolution of k under the assumption $\partial\gamma/\partial\tau = V$, where V is an arbitrary vector field along $\gamma \subset M^n$ (see [4]):

$$\partial k^2/\partial\tau = 2\langle \nabla_T \nabla_T V, \nabla_T T \rangle - 4k^2 \langle \nabla_T V, T \rangle + 2\langle R(V, T)T, \nabla_T T \rangle. \quad \text{q.e.d.}$$

Using $\kappa = \alpha k$ and $R = \alpha^2 R_0$, Propositions 2.1 and 2.2 directly yield equations (1).

3. Homothetic solutions

The proof of Theorem A, our major goal in this section, will be subdivided into three steps:

Proposition 3.1. (i) *A constant speed parametrized closed curve $\gamma: \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{E}^2$ represents a homothetic solution of the curve shortening flow if and only if its normalized curvature function κ obeys*

$$(3.1) \quad \kappa' = -\beta\kappa, \quad \beta' = \kappa^2 - \lambda^2,$$

where β is the same auxiliary function as in (1) and λ is some positive constant.

(ii) *Considering (3.1) as a system of differential equations on the real line, all of its solutions exist globally. Obviously $\kappa = \text{const} \cdot e^{-\int\beta}$ does not change sign.*

Note that an arbitrary solution on \mathbf{R} to (3.1) need not factor over \mathbf{R}/\mathbf{Z} , as required in (i), and even then the corresponding curve γ with prescribed length $\alpha(\tau)$, a solution to the Frenet equations, need not close up.

Proof. (i) Since by normalization $\kappa(\tau, \sigma)$ does not change under dilations, it follows directly from (1) that homothetic solutions are represented precisely by those closed curves with periodic κ and β obeying:

$$(*) \quad \kappa' = -\beta\kappa - \mu_1, \quad \beta' = \kappa^2 - \mu_2.$$

Just because of periodicity it is clear that μ_2 is the square of some other constant $\lambda > 0$. (In case $\mu_2 = 0$ we could conclude $\kappa \equiv 0$, a contradiction.) To prove $\mu_1 = 0$ we first note that taking an antiderivative $\int\beta$ preserves periodicity since the average of β over a period vanishes. The derivative of the periodic function $f := \kappa e^{\int\beta}$ is computed to be $f' = (\kappa' + \beta\kappa)e^{\int\beta} = -\mu_1 e^{\int\beta}$, i.e., some strictly negative factor times μ_1 . Hence $\mu_1 = 0$.

(ii) Global existence and uniqueness already hold for system (*), since the differential inequality $|(\kappa^2 + \beta^2)'| = 2|\mu_1\kappa + \mu_2\beta| \leq \sqrt{\mu_1^2 + \mu_2^2} \sqrt{\kappa^2 + \beta^2}$ gives rise to a global a priori growth estimate on $\sqrt{\kappa^2 + \beta^2}$. q.e.d.

Substituting $\kappa = \lambda e^{B/2}$ and $\beta = -\frac{1}{2}B'$, we see that equations (3.1) and equations (2) in Theorem A are equivalent. Hence equations (2) admit only global solutions. Their periodicity and closedness properties are discussed most easily using the first integral given in (3). For $\eta \geq 0$ the strictly convex potential function $V(B)$ determines precisely two numbers $B_- = B_-(\eta) \leq 0 \leq B_+ = B_+(\eta)$ such that $V(B_{\pm}) = \eta$.

Propositions 3.2. *Consider an arbitrary solution B of equation (2) lifted to the real line:*

(i) *Up to translations, B is uniquely determined by its integral η and the parameter λ in the differential equation.*

(ii) *B is even with respect to all its extremal points and therefore oscillates between its minimum $B_-(\eta)$ and its maximum $B_+(\eta)$ with*

$$\text{period}(\eta, \lambda) = \frac{1}{\lambda} \text{period}(\eta, 1) = \frac{1}{\lambda} \int_{B_-}^{B_+} \frac{dB}{\sqrt{\eta - V(B)}}.$$

(iii) *The period of B can be estimated in terms of the positive, monotone, convex function $T(B) = (e^B - 1)/B$:*

$$\sqrt{2} \pi T(B_+)^{-1/2} \leq \lambda \text{period}(\eta, \lambda) \leq \sqrt{2} \pi T(B_-)^{-1/2}.$$

(iv) *The tangent vector of the associated solution $\gamma: \mathbf{R} \rightarrow \mathbf{E}^2$ to the Frenet equations rotates within a period of κ by*

$$\Theta(\eta, \lambda) = \Theta(\eta) := \int_{B_-}^{B_+} \frac{dB}{\sqrt{e^{-B}(\eta - V(B))}}.$$

(v) *The function $\Theta: (0, \infty) \rightarrow \mathbf{R}$, $\eta \rightarrow \Theta(\eta)$ defined in (iv) is strictly monotone decreasing and has range $(\pi, \pi\sqrt{2})$.*

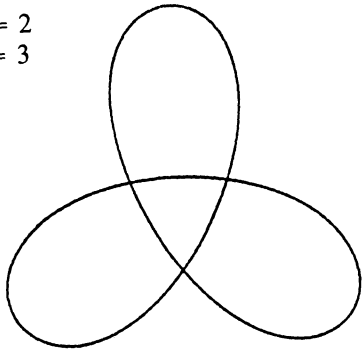
As indicated above, the claims (i), (ii), and (iv) easily follow from the first integral given in (3). In order to see (iii), we rewrite equation (2) as $B'' + 2\lambda^2 T(B)B = 0$ and use the Sturm comparison theorem. (v) is proven in the appendix.

It seems worthwhile to point out that the λ -dependence of $\text{period}(\eta, \lambda)$ and $\Theta(\eta, \lambda)$ just reflects the possibility of *scaling homothetic solutions*. Indeed we can pass to any domain $\mathbf{R}/l\mathbf{Z}$, $l > 0$, and consider the curve $\gamma_l(s) = l\gamma(s/l)$; equations (2), (3), and (3.1) continue to hold when κ , β , B and λ are replaced by

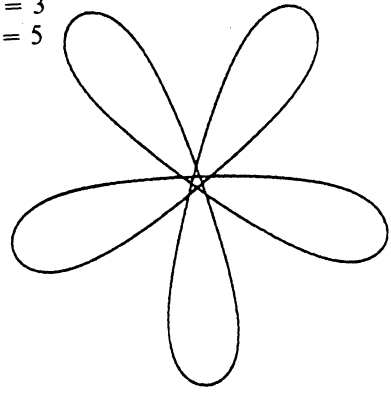
$$\kappa_l(s) = \frac{1}{l} \kappa\left(\frac{s}{l}\right), \quad \beta_l(s) = \frac{1}{l} \beta\left(\frac{s}{l}\right), \quad B_l(s) = B\left(\frac{s}{l}\right) \quad \text{and} \quad \lambda_l = \frac{\lambda}{l},$$

respectively. Thus, one may think of λ as corresponding to the scale of the curve, after the shape of the homothetic solution has been fixed by η (cf. Proposition 3.2(iv)).

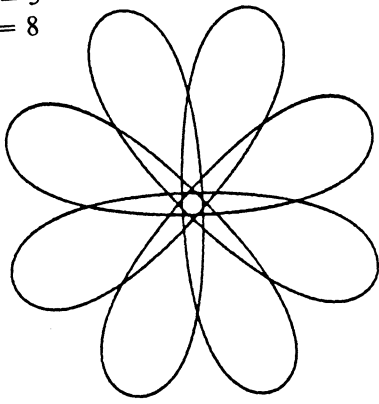
$m = 2$
 $n = 3$



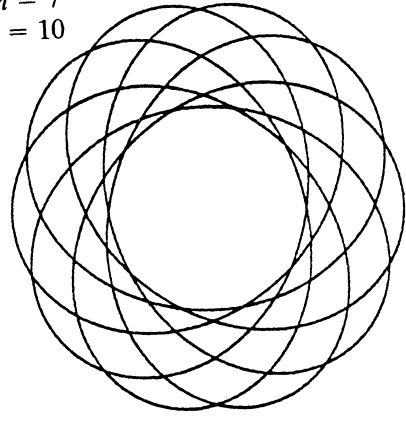
$m = 3$
 $n = 5$



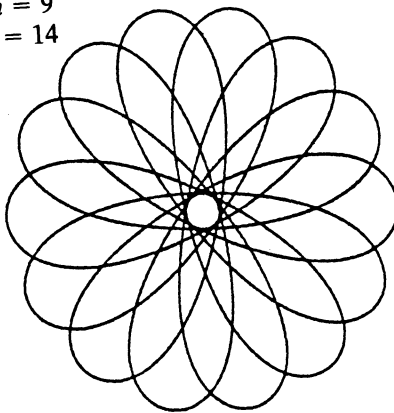
$m = 5$
 $n = 8$



$m = 7$
 $n = 10$



$m = 9$
 $n = 14$



$m = 12$
 $n = 17$

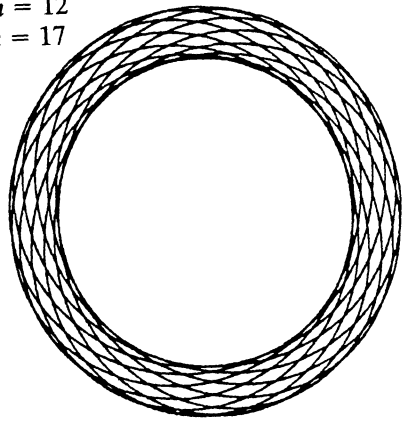


FIG. 2.

Proof of the classification statement in Theorem A. Because of 3.2(ii) it is standard Frenet theory that the curve γ is symmetric about its normal lines at points where the value of κ is extremal. The closing up of γ is easily discussed in terms of the Coxeter group generated by the reflections about the normal lines through two adjacent extrema of the curvature function. Since $\Theta(\eta)/\pi$ is always nonintegral, this group has precisely one fixed point, the center of mass of γ . The curve therefore closes up iff, for some nonzero integer n , $n\Theta(\eta)$ is an integral multiple of 2π , the rotation index m . The numerical conditions stated in terms of m and n follow directly from 3.2(v). (Our normalization of the period of γ is met putting $\lambda = n \cdot \text{period}(\eta, 1)$.)

Six of the solutions $\gamma_{m,n}$ are pictured in Figure 2. The curves were generated by computer by solving $\Theta(\eta) = m/n \cdot 2\pi$ and then integrating system (3.1) together with the Frenet equations, using the initial conditions $\beta(0) = 0$ and $\kappa(0) = \kappa_{\min}(\eta)$ (cf. equation (3)).

It remains to calculate the *transcendental relation* between κ and r for unit speed homothetic solutions (i.e.: $\alpha = 1, \kappa = k$). This readily implies the claim on the transcendence of the noncircular ones; otherwise, since r^2 is an algebraic parameter on any noncircular algebraic curve, κ would be an algebraic function of r^2 , a contradiction.

It is useful to introduce polar coordinates (r, θ) about the center of mass of γ .

Proposition 3.3. *The Killing field $\partial/\partial\theta$ is known explicitly along γ :*

$$\frac{\partial}{\partial\theta} = \lambda^{-2}J, \quad \text{where } J = \kappa T - \beta N.$$

Hence the extremal points of κ and r coincide, and $\kappa_{\min} = \lambda^2 r_{\min}$ and $\kappa_{\max} = \lambda^2 r_{\max}$.

Proof. Clearly the center of mass is preserved under the curve shortening flow. Hence $\partial/\partial r$ is parallel to $\partial\gamma/\partial t = W$ (cf. Proposition 2.1), and $\partial/\partial\theta$ is parallel to J . In fact, since $\langle \nabla_T J, T \rangle = 0$, they must be proportional. In order to determine the factor, we observe that $T = J/|J| = |\partial/\partial\theta|^{-1} \partial/\partial\theta$ at the extremals of κ and compute:

$$\nabla_T J = \lambda^2 N, \quad \text{whereas } \nabla_T \frac{\partial}{\partial\theta} = \left| \frac{\partial}{\partial\theta} \right|^{-1} \nabla_{\partial/\partial\theta} \frac{\partial}{\partial\theta} = N. \quad \text{q.e.d.}$$

Next we observe that equations (2), (3), and (3.1) yield $\beta^2 + \kappa^2 = \lambda^2(\eta + 1 + 2 \ln(\kappa/\lambda))$. Hence choosing our initial point so that $r(0) = r_{\min}$

and noting that $\partial/\partial r$ is a unit vector perpendicular to J , we have

$$\begin{aligned} r - r_{\min} &= \int_0^s \left\langle T, \frac{\partial}{\partial r} \right\rangle d\sigma = \int_0^s \frac{\kappa' d\sigma}{\kappa\sqrt{\kappa^2 + \beta^2}} \\ &= \int_{\kappa_{\min}}^{\kappa} \frac{d\kappa}{\lambda\sqrt{\eta + 1 + 2\ln(\kappa/\lambda)}} = \frac{1}{\lambda} \sqrt{\eta + 1 + 2\ln(\kappa/\lambda)} - \frac{\kappa_{\min}}{\lambda^2}. \end{aligned}$$

Again using Proposition 3.3, we can solve for κ as desired:

$$\kappa = \lambda \exp\left(\frac{1}{2}\lambda^2 r^2 - \frac{\eta+1}{2}\right).$$

4. Bounds and convergence for general trajectories

This section is devoted mostly to the proof of Theorem B, with more precise estimates. We will make repeated use of the evolution equations (1) for the planar case.

For instance, part (i) of Theorem B follows almost at once from $\partial\kappa/\partial\tau = (\kappa' + \beta\kappa)'$. For each time τ we subdivide γ at all jumping points of $\text{sgn} \circ \kappa: \mathbf{R}/\mathbf{Z} \rightarrow \{-1, 0, 1\}$, and obtain at most countably many pieces $\gamma_i: [a_i, b_i] \rightarrow \mathbf{E}^2$. Where $\kappa \equiv 0$ we clearly have $(\partial/\partial\tau)\text{sgn}(\kappa) = 0$ and, on the interior of an interval for which κ vanishes identically, the evolution equation implies $0 = \partial\kappa/\partial\tau = \partial|\kappa|/\partial\tau$. Thus, letting $\text{sgn}(i) = \text{sgn}(\kappa(\sigma))$, $\sigma \in (a_i, b_i)$, we can write

$$\begin{aligned} \frac{d}{d\tau} \|\kappa\|_{L^1} &= \frac{d}{d\tau} \int_0^1 \text{sgn}(\kappa) \frac{\partial\kappa}{\partial\tau} d\sigma = \sum_i \text{sgn}(i) \int_{a_i}^{b_i} \frac{\partial}{\partial\sigma} (\kappa' + \beta\kappa) d\sigma \\ &= \sum_i \text{sgn}(i) (\kappa'(b_i) - \kappa'(a_i)). \end{aligned}$$

Clearly, no term in the last sum is positive. In fact the sum is negative unless $\kappa'(a_i) = \kappa'(b_i) = 0$ for all i (note that if κ does not change sign there is only one piece and the above sum vanishes).

Part (ii) of the theorem is included in the following propositions on the behavior of the Sobolev seminorms of κ . We write $x_j = \int_0^1 (\partial^j\kappa/\partial\sigma^j)^2 d\sigma$ and $m = \sqrt{x_0x_1}$.

Proposition 4.1. *The time derivatives $\dot{x}_j = dx_j/d\tau$ and $\dot{m} = dm/d\tau$ satisfy the estimates*

- (i) $\dot{x}_0 \leq -2x_1 + mx_0,$
- (ii) $\dot{x}_j \leq -2x_{j+1} + x_j((3^{j+1} - 2)m + 2^{j+1}x_0),$
- (iii) $\dot{m} \leq 2m\left(-\frac{m^2}{x_0^2} + 2m + x_0\right).$

Corollary. *The solution to the evolution equations exists at least as long as $\tau < \tau_E = 4x_0(0)^{-2}$.*

Proposition 4.2. *Define $C(\tau) = \max\{x_0(\tau'): 0 \leq \tau' \leq \tau\}$ (note that $C(\tau) \geq 4\pi^2 > 32$). The functions $m(\tau)$ and $x_j(\tau)$, $j \geq 1$, are bounded on the whole existence interval in terms of $C(\tau)$ and the initial data $m(0)$ and $x_1(0), \dots, x_j(0)$, respectively. Asymptotically, i.e., for $\tau \cdot C(\tau)$, $C(\tau)$ sufficiently large, one has the following bounds depending only on $C(\tau)$:*

- (i) $m(\tau) \leq 3C(\tau)^2$,
- (ii) $x_j(\tau) \leq 3^{j(j+5)/2}C(\tau)^{2j+1}$.

When x_0 is not bounded—so the flow is approaching a singularity—the above estimates are still adequately describing the behaviour of the x_j . This is the essential point of the complementary

Proposition 4.3. *Letting $\|\cdot\|_1$ denote the L^1 -norm, one has*

- (i) $\int |\kappa|^p \geq \|\kappa\|_1^{2-p} x_0^{p-1}$ for $p \geq 2$,
- (ii) $x_j \geq x_0(x_0/\|\kappa\|_1^2 - 1)^{2j}$ for all $j \geq 1$.

Before proving the above propositions we proceed to the proof of part (iii) of Theorem B.

Let $\gamma_\tau \subset M^2$ have L^1 -convergent normalized curvature κ_τ and suppose γ_τ does not converge to a geodesic. Then there exist constants τ_0 and $C > 0$ such that $\tau > \tau_0$ implies $\int_0^1 \kappa_\tau^2 d\sigma > C$. Hence, Proposition 2.2 implies $d\alpha/d\tau > C\alpha^3$. It follows that the length of γ_τ tends to zero.

We claim that in fact γ_τ converges to a point $p \in M^2$. To see this, we consider $\tau_i \rightarrow \infty$ and, for each i , a “strongly convex set” $\Gamma_i \subset M^2$ (i.e. $\partial\Gamma_i$ has strictly positive inward curvature) which contains γ_{τ_i} and has diameter a bounded multiple of $\text{diam}(\gamma_{\tau_i})$. For $\tau > \tau_i$, the curve γ_τ must remain inside the fixed set Γ_i ; if γ_τ ever touches $\partial\Gamma_i$, the curvature normal kN is actually pointing inward in a small neighborhood of any point of first order contact. So γ_τ can never cross $\partial\Gamma_i$. We thus have a nested family of compact sets whose diameters tend to zero.

Next we integrate equation (1) twice:

$$\frac{\partial}{\partial \tau} \int_0^\sigma \int_0^u \kappa \, dv \, du = \kappa + \int_0^\sigma \beta \kappa \, du + \int_0^\sigma \int_0^u \kappa R \, dv \, du + C\sigma + D.$$

Since κ converges in L^1 , the right-hand side of the above equation clearly converges in L^1 to some function $H(\sigma)$. We claim that in fact $H(\sigma) \equiv 0$. Suppose $H(\sigma_0) = H_0 \neq 0$ for some σ_0 ; then for all sufficiently large τ ,

$$\frac{2}{H_0} \frac{\partial}{\partial \tau} \int_0^{\sigma_0} \int_0^u \kappa \, dv \, du > 1,$$

hence $\int_0^{\sigma_0} \int_0^u \kappa \, dv \, du$ must tend to infinity, contradicting convergence of κ .

Setting the right-hand side of the above equation equal to zero, it follows by induction that κ is C^∞ . Differentiating twice and observing that κR tends to zero in L^1 as $\tau \rightarrow \infty$ we recover the equation for homothetic solutions. Since $(M, g = \alpha^{-2}g_0)$ is locally asymptotic to the plane the claim follows. \square

Proof of Proposition 4.1. We will use the notational shorthand $f = \int_0^1 d\sigma$ and follow the convention that the derivative operator $\partial^j = \partial^j/\partial s^j$ is applied only to the term immediately to the right.

To begin with we have by interpolation

$$(4.1) \quad x_\mu x_j \leq x_{\mu-\nu} x_{j+\nu} \quad \text{for } 0 \leq \nu \leq \mu \leq j.$$

Secondly, since

$$\begin{aligned} \|\partial^\mu(\kappa^2 - x_0)\|_\infty &\leq \frac{1}{2} \int |\partial^{\mu+1}(\kappa^2 - x_0)| \\ &\leq \frac{1}{2} \sum_{\nu=0}^{\mu+1} \binom{\mu+1}{\nu} \int |\partial^\nu \kappa| |\partial^{\mu+1-\nu} \kappa| \leq \frac{1}{2} \sum_{\nu=0}^{\mu+1} \binom{\mu+1}{\nu} \sqrt{x_\nu} \sqrt{x_{\mu+1-\nu}}, \end{aligned}$$

inequality (4.1) gives, for all $\mu \geq 0$,

$$(4.2) \quad \|\partial^\mu(\kappa^2 - x_0)\|_\infty \leq 2^\mu \sqrt{x_0 x_{\mu+1}}.$$

Similarly, since

$$\|\kappa \partial^\nu \kappa\|_\infty \leq \left| \int \kappa \partial^\nu \kappa \right| + \frac{1}{2} \int |\partial(\kappa \partial^\nu \kappa)| \leq \sqrt{x_0 x_\nu} + \frac{1}{2} \sqrt{x_1 x_\nu} + \frac{1}{2} \sqrt{x_0 x_{\nu+1}},$$

we also obtain

$$(4.3) \quad \|\kappa \partial^\nu \kappa\|_\infty \leq \sqrt{x_0 x_\nu} + \sqrt{x_0 x_{\nu+1}}.$$

At this point we invoke the flow equations (1) to compute

$$\begin{aligned} \dot{x}_j &= 2 \int \partial^j \kappa \partial^j \kappa = -2x_{j+1} + 2 \sum_{\nu=0}^{j+1} \binom{j+1}{\nu} \int \partial^\nu \beta \partial^{j+1-\nu} \kappa \partial^j \kappa \\ &= -2x_{j+1} + \int \beta \partial(\partial^j \kappa)^2 ds + 2(j+1) \int \partial \beta (\partial^j \kappa)^2 ds + 2\varphi_j \\ &= -2x_{j+1} + (2j+1) \int (\kappa^2 - x_0)(\partial^j \kappa)^2 ds + 2\varphi_j, \end{aligned}$$

where we have set

$$(4.4) \quad \varphi_j = \sum_{\nu=2}^{j+1} \binom{j+1}{\nu} \int \partial^{\nu-1}(\kappa^2 - x_0) \partial^{j+1-\nu} \kappa \partial^j \kappa.$$

Because of (4.2) the above yields

$$(4.5) \quad \dot{x}_j \leq -2x_{j+1} + (2j+1)\sqrt{x_0 x_1} x_j + 2\varphi_j.$$

In particular, this gives inequality (i) of Proposition 4.1. To obtain the second inequality it remains to bound φ_j for $j \geq 1$:

$$\begin{aligned} \varphi_j &= \sum_{\mu=1}^{j-1} \binom{j+1}{\mu+1} \int \partial^\mu(\kappa^2 - x_0) \partial^{j-\mu} \kappa \partial^j \kappa \\ &\quad + \sum_{\nu=0}^{j-1} \binom{j}{\nu} \int \kappa \partial^\nu \kappa \partial^{j-\nu} \kappa \partial^j \kappa + \int \kappa^2 (\partial^j \kappa)^2 \\ &\leq \sum_{\mu=1}^{j-1} \binom{j+1}{\mu+1} \|\partial^\mu(\kappa^2 - x_0)\|_\infty \sqrt{x_{j-\mu}} \sqrt{x_j} \\ &\quad + \sum_{\nu=0}^{j-1} \binom{j}{\nu} \|\kappa \partial^\nu \kappa\|_\infty \sqrt{x_{j-\nu}} \sqrt{x_j} + x_0 x_j + \|\kappa^2 - x_0\|_\infty x_j. \end{aligned}$$

Using the estimates (4.1) and (4.2), and (4.1) and (4.3), respectively, we bound each term separately:

$$\begin{aligned} 2\varphi_j &\leq 2 \sum_{\mu=1}^{j-1} \binom{j+1}{\mu+1} 2^\mu \sqrt{x_0 x_1} x_j \\ &\quad + 2 \sum_{\nu=0}^{j-1} \binom{j}{\nu} (x_0 + \sqrt{x_0 x_1}) x_j + 2(x_0 + \sqrt{x_0 x_1}) x_j \\ &= (3^{j+1} - (2j + 3)) \sqrt{x_0 x_1} x_j + 2^{j+1} x_0 x_j. \end{aligned}$$

Part (ii) of the proposition now follows from (4.5).

To prove the remaining inequality (iii) we compute in a straightforward manner from (i) and (ii):

$$2m\dot{m} = x_0 \dot{x}_1 + \dot{x}_0 x_1 \leq 4m^2 \left(-\frac{m^2}{x_0^2} + 2m + x_0 \right).$$

Proof of Corollary. From Proposition 4.1(i) and the interpolation inequality $mx_0 \leq 2x_1 + \frac{1}{8}x_0^3$ we conclude $\dot{x}_0 \leq \frac{1}{8}x_0^3$, hence, the result.

Proof of Proposition 4.2. Observe first that the right-hand sides of both the inequalities are nondecreasing. Thus it will suffice to prove that exponential decay holds for $m(\tau)$ as long as (i) is violated, and to argue similarly for x_j .

(i) A direct calculation shows that $-m^2 + 2x_0^2 m + x_0^3 \leq -\frac{1}{4}x_0^2$ as long as $m \leq 2x_0^2 + \frac{1}{2}x_0$ fails. Therefore, it follows from Proposition 4.1(iii) that one has $\dot{m} \leq -\frac{1}{2}m$ as long as $m(\tau) \geq 3C(\tau)^2 (> 2x_0(\tau)^2 + \frac{1}{2}x_0(\tau))$.

(ii) By Proposition 4.1(ii) and the interpolation inequality $x_{j+1} \geq x_j^2/x_{j-1}$ one has $\dot{x}_j \leq -2^{j+10}x_j$ as long as $x_j \leq x_{j-1}(\frac{1}{2} \cdot 3^{j+1}m + 2^{j+1}C(\tau)^2)$ fails. By induction, the boundedness of x_j , $j \geq 1$, now follows from the bounds on m and x_{j-1} .

In order to obtain the desired estimate we may assume $\tau C(\tau)$ large enough so that $m(\tau) \leq 3C(\tau)^2$. One then argues inductively using $\frac{1}{2} \cdot 3^{j+1}m + 2^{j+1}C(\tau)^2 \leq 3^{j+2}C(\tau)^2$.

Proof of Proposition 4.3. Part (i) follows directly from Hölder’s inequality. Using this estimate with $p = 4$ and (4.2) one obtains

$$x_0^2 \left(\frac{x_0}{\|\kappa\|_1^2} - 1 \right) \leq \int (\kappa^2 - x_0) \kappa^2 \leq \|\kappa^2 - x_0\|_\infty x_0 \leq x_0 \sqrt{x_0 x_1}.$$

This proves part (ii) in case $j = 1$. Inequality (4.1) now yields the general result by induction.

5. Divergence and stability results

For a precise statement of the divergence result mentioned in (C) of §1 we first recall that the algebraic area of a closed curve $\gamma: \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{E}^2$ is defined in terms of an integral of the winding number $N(x, \gamma)$ over almost all points $x \in \mathbf{E}^2$: $A(\gamma) = \int_{\mathbf{E}^2} N(x, \gamma) dx = \frac{1}{2} \int_\gamma \det(\gamma, \gamma') d\sigma$. The criterion will also involve the rotation index $m = \text{Ind}(\gamma) = 1/2\pi \int_\gamma k ds$.

Proposition 5.1. *Suppose γ is an initial curve satisfying one of:*

- (i) $A(\gamma) \neq 0$ and $A(\gamma) \cdot \text{Ind}(\gamma) \leq 0$,
- (ii) $k \geq 0$ along γ and $N(p, \gamma) < 0$ for some point $p \in \mathbf{E}^2$.

Then along the trajectory γ_τ through $\gamma = \gamma_0$ the L^2 -norm of the normalized curvature κ_τ diverges before the length of γ_τ approaches zero, hence, within finite time τ .

Proof. (i) The derivative of algebraic area with respect to τ is straightforward to compute using $\partial\gamma/\partial\tau = \alpha^2(bT + kN)$:

$$\frac{d}{d\tau} A(\gamma_\tau) = - \int_0^1 \det\left(\gamma'_\tau, \frac{\partial}{\partial\tau} \gamma_\tau\right) d\sigma = -\alpha^2 \int_{\gamma_\tau} k ds = -2\pi m \alpha^2.$$

By assumption it follows that $|A(\gamma_\tau)| \geq |A(\gamma)| > 0$.

Since part (i) of Theorem B gives a bound on the winding numbers,

$$2\pi |N(x, \gamma_\tau)| \leq \int_0^1 |\kappa_\tau| ds \leq \int_0^1 |k| ds \quad \forall x \in \mathbf{E}^2,$$

we deduce a uniform positive lower bound on the enclosed area, i.e., on $\text{area}\{x: N(x, \gamma_\tau) \neq 0\}$. Thus, the isoperimetric inequality yields a uniform positive lower bound on the length of γ_τ .

However, according to the proof of the part (iii) of Theorem B, the length would have to approach zero if the flow existed for all $\tau > 0$. In view of part (ii) of Theorem B, the L^2 -norm of κ must therefore diverge.

(ii) In this case we consider the set $F_\tau = \{x \in \mathbb{E}^2: N(x, \gamma_\tau) \leq -1\}$ and its area $a_\tau = \text{area}(F_\tau)$. By hypothesis $a_0 > 0$. Our goal is to show that a_τ is nondecreasing, then again use the isoperimetric inequality and finish the proof as above.

Notice that for an arbitrary smooth family of curves $c_\tau: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{E}^2$ the function $\tau \rightarrow a_\tau$ is only Lipschitz rather than C^1 (differentiability may be lost at nontransversal intersections). The lower Dini derivative is given by

$$D_\tau^- a_\tau = \int_{\partial \bar{F}_\tau} \min\{\det(c'_\tau/|c'_\tau|, \partial c_\tau/\partial \tau)(\sigma) : \sigma \in c_\tau^{-1}\{p\}\} ds(p).$$

In our case, since $k_\tau \geq 0$ (because of part (i) of Theorem B), we obtain the desired estimate:

$$D_\tau^- a_\tau = \int_{\partial \bar{F}_\tau} \min\{k_\tau(\sigma) : \sigma \in \gamma_\tau^{-1}\{p\}\} ds(p) \geq 0.$$

Actually, since k_τ is constant in almost all fibres $\gamma_\tau^{-1}\{p\}$, a_τ is differentiable for trajectories of the curve shortening flow. q.e.d.

Figure 3 shows some initial curves which must develop singularities by the preceding criterion: Similarly, a curve resembling Figure 1(d) or (e) must become singular (though the proposition does not directly apply to (d), the direction of the curvature normal vector kN implies that after a short time $\tau > 0$ the hypothesis will be satisfied).

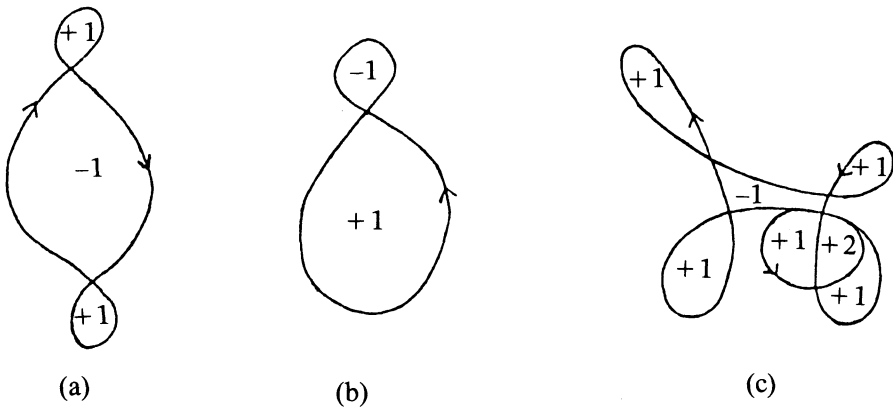


FIG. 3.

We turn now to the *linear stability analysis*, referring to the notation introduced in (C) of §1. Setting $\alpha(u) = \int_0^u \kappa(s) ds$, we can write $\Omega = \{\kappa \in C^\infty(\mathbf{R}/\mathbf{Z}): \int_0^1 \kappa ds = 2\pi m, m \text{ an integer, and } \int_0^1 e^{i\alpha(s)} ds = 0\}$. It follows that the tangent space to Ω at $\kappa \in \Omega$ can be identified with $T_\kappa\Omega = \{h \in C^\infty(\mathbf{R}/\mathbf{Z}): 0 = \int_0^1 h(s) ds = \int_0^1 \int_0^s h(u) \text{ due}^{i\alpha(s)} ds\} = \{h \in C^\infty(\mathbf{R}/\mathbf{Z}): 0 = \int_0^1 h(s) ds = \int_0^1 h(s) \int_0^s e^{i\alpha(u)} du ds\}$. In particular, the tangent space at the m -fold circle is given by

$$(5.1) \quad T_{\kappa_m}\Omega = \left\{ h \in C^\infty(\mathbf{R}/\mathbf{Z}): 0 = \int_0^1 h ds = \int_0^1 h(s) e^{i\kappa_m s} ds \right\}.$$

We wish to consider now the linearization of the flow $\dot{\kappa} = \kappa'' + (\beta\kappa)' \equiv P(\kappa)$ at some fixed κ in Ω : $\dot{h} = DP(\kappa)h = h'' + (\beta h)' + (D\beta(\kappa)h \cdot \kappa)' \equiv Lh$, where $\frac{1}{2}D\beta(\kappa)h = \int_0^s h\kappa d\sigma - s \int_0^1 h\kappa d\sigma + \int_0^1 (\sigma - \frac{1}{2})\kappa h d\sigma$.

For the special case $\kappa = \kappa_m \equiv 2\pi m$, the facts $\beta(\kappa_m) = 0$ and $\int_0^1 h ds = 0$ imply that the linear map $L: T_{\kappa_m}\Omega \rightarrow T_{\kappa_m}\Omega$ is given by:

$$(5.2) \quad Lh = h'' + 2\kappa_m^2 h.$$

Proposition 5.2. *The multiple circles κ_m , $|m| > 1$, are linearly unstable critical points of the flow on Ω .*

Proof. Set $h(s) = \cos(2\pi s)$. Then for $m \neq \pm 1$, $h \in T_{\kappa_m}\Omega$. Thus we have found a positive eigenvalue: $Lh = 4\pi^2 h$. q.e.d.

We note that the above proposition has a simple *geometric interpretation*. Consider, e.g., the case $m = 2$. Then varying $\kappa_2 \equiv 4\pi$ in the direction of $h = \cos 2\pi s$ corresponds to shrinking one circle of γ_2 while enlarging the other (one should picture a pair of tangent circles of slightly different radii, one inside the other). The flow does not tend to restore such perturbations to circularity, but rather, it amplifies the inequality in size. This shows once again the striking difference between the simple and nonsimple cases of the curve shortening problem.

On the other hand, consider the restriction of L to the “*symmetric variations*”, i.e., to the tangent space $T_\kappa\Omega^G = \{h \in T_\kappa\Omega: gh = h\}$ of Ω^G . It follows from equations (5.1) and (5.2) that any $h \in T_{\kappa_m}\Omega^G$ has Fourier series representation of the form

$$h(s) = \sum_{j=1}^{\infty} a_j \cos(2\pi jns) + b_j \sin(2\pi jns)$$

(in fact by closedness $a_j = b_j = 0$ in case $jn = m$). Substitution of this series into equation (5.2) yields

$$Lh = (2\pi)^2 \sum_{j=1}^{\infty} (2m^2 - j^2n^2)(a_j \cos(2\pi jns) + b_j \sin(2\pi jns)),$$

which gives at once

Proposition 5.3. *The flow on $\Omega^{G(m,n)}$ is linearly stable at κ_m exactly when $|m/n| < \sqrt{2}/2$.*

6. Appendix

Here we discuss the function $\Theta: (0, \infty) \rightarrow \mathbf{R}$ which arose in Proposition 3.2:

$$\Theta(\eta) = \int_{B_-}^{B_+} \frac{dB}{\sqrt{e^{-B}(\eta - V(B))}}.$$

We recall that the convex potential function $V(B) = e^B - B - 1$ assumes any value $\eta \geq 0$ at precisely two points $B_-(\eta) \leq 0 \leq B_+(\eta)$. It thus defines a bijection $\frac{1}{2}(B_+ - B_-): [0, \infty) \rightarrow [0, \infty)$, the inverse of which we shall denote by ρ . Since ρ is a monotonic bijection, claim 3.2(v) can equivalently be established for the function $\Theta \circ \rho$. This reparametrization will be useful, since by some straightforward calculation it provides explicit formulae:

$$(6.1) \quad \begin{aligned} \eta &= \rho(w) = w \coth w - 1 + \ln \sigma(w^2), \\ B_{\pm} &= B_{\pm} \circ \rho(w) = \pm w - \ln \sigma(w^2), \end{aligned}$$

where we have made use of the analytic function

$$(6.1') \quad \sigma(x) = \frac{\sinh \sqrt{x}}{\sqrt{x}} = \sum_{k \geq 0} \frac{x^k}{(2k + 1)!}$$

Introducing in addition the function

$$(6.2) \quad \begin{aligned} h(\eta) &= \sqrt{\eta} \left(\frac{e^{B_+/2}}{V'(B_+)} - \frac{e^{B_-/2}}{V'(B_-)} \right) \\ &= \frac{\sqrt{\eta}}{2} \cdot \left(\left| \sinh \frac{B_+}{2} \right|^{-1} + \left| \sinh \frac{B_-}{2} \right|^{-1} \right), \end{aligned}$$

we can calculate:

$$(6.3) \quad \begin{aligned} \Theta \circ \rho(w) &= \left(\int_{B_-}^0 + \int_0^{B_+} \right) \frac{e^{B/2} dB}{\sqrt{\rho(w) - V(B)}} \\ &= \int_0^{\rho(w)} \frac{h(\eta) d\eta}{\sqrt{\eta(\rho(w) - \eta)}} = \int_0^1 \frac{h(z\rho(w)) dz}{\sqrt{z(1-z)}}. \end{aligned}$$

Computer plots of the functions $1/\pi \cdot \Theta \circ \rho$ and $h \circ \rho$ in Figure 4 will provide some intuition for further analysis.

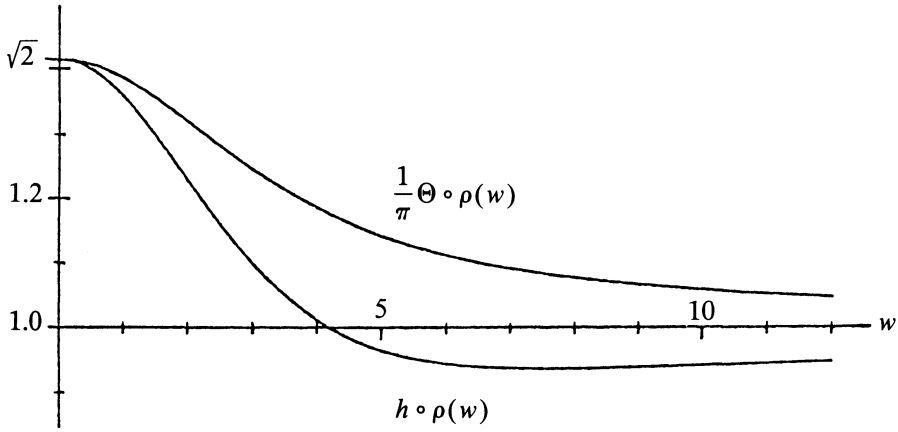


FIG. 4.

The main results of this appendix, which directly imply 3.2(v), are

Proposition 6.1. (i) $\lim_{\eta \rightarrow 0} \Theta(\eta) = \lim_{w \rightarrow 0} \Theta \circ \rho(w) = \pi\sqrt{2}$, $\lim_{\eta \rightarrow \infty} \Theta(\eta) = \lim_{w \rightarrow \infty} \Theta \circ \rho(w) = \pi$.

(ii) The functions $h \circ \rho$ and $\Theta \circ \rho$ are decaying on $[0, 5.22]$.

(iii) The function $\Theta \circ \rho$ is monotone decreasing and hence $> \pi$ on the interval $[5.22, \infty)$.

Of course in (ii) it is sufficient to prove decay for $h \circ \rho$; because of formula (6.3) the result extends to $\Theta \circ \rho$. Notice however, that $h \circ \rho$ has a minimum, approximately at 7.53, whereas $\Theta \circ \rho$ continues to decay (cf. (iii)).

Proof of (i). Clearly $\frac{1}{2}\sqrt{V(B)} |\sinh(B/2)|^{-1} \rightarrow 0, \frac{1}{2}\sqrt{2}, 1$ as $B \rightarrow -\infty, 0, \infty$, respectively. It follows that $h(\eta) \rightarrow \sqrt{2}, 1$ as $\eta \rightarrow 0, \infty$, respectively. This yields the claim since $\int_0^1 dz / \sqrt{z(1-z)} = \pi$.

Proof of (ii). Here the basic idea is to show that $H'(x)/H(x) < 0$ and $H(x) > 0$ for $\sqrt{x} \in [0, 5.22]$, $H(x)$ standing for the function $h \circ \rho(\sqrt{x})$. To begin with we list some properties of the function σ introduced in (6.1):

$$\begin{aligned}
 \sigma'(x) &= \sum_{k=0}^{\infty} \frac{k+1}{(2k+3)!} x^k, \\
 2x\sigma'(x) &= \cosh\sqrt{x} - \sigma(x), \\
 \sigma(4x) - 1 &= 4x \sum_{k=0}^{\infty} \frac{1}{(2k+3)!} (4x)^k, \\
 \sigma(4x) - \sigma(x)^2 &= 4x \sum_{k=0}^{\infty} \frac{2k+2}{(2k+4)!} (4x)^k
 \end{aligned}
 \tag{6.4}$$

and then calculate from (6.1) and (6.2):

$$\begin{aligned} h \circ \rho(w) &= \sqrt{\rho(w)} \sqrt{\sigma(w^2)} \left(\frac{e^{-w/2}}{1 - \sigma(w^2)e^{-w}} + \frac{e^{w/2}}{\sigma(w^2)e^w - 1} \right) \\ &= [\cosh w + \sigma(w^2)(-1 + \ln \sigma(w^2))]^{1/2} \\ &\quad \cdot 2 \sinh \frac{w}{2} \cdot \frac{1 + \sigma(w^2)}{\sinh(2w)/w - 1 - \sigma(w^2)^2}, \\ H(x) &= x \cdot \sqrt{2\sigma' + \sigma \frac{\ln \sigma}{x}} \cdot \sigma\left(\frac{x}{4}\right) \cdot \frac{1 + \sigma}{2\sigma(4x) - 1 - \sigma(x)^2}, \end{aligned}$$

where we have adopted the convention that σ means $\sigma(x)$, σ' means $\sigma'(x)$, etc. Clearly $H(x) > 0$, as required, and its logarithmic derivative can be written as follows:

$$(6.5) \quad \frac{H'}{H} = f_1 + f_2 + (f_{3a} - f_{3b}) - f_4,$$

where

$$\begin{aligned} f_1 &= \frac{1}{2} \left[\ln \left(2\sigma' + \sigma \frac{\ln \sigma}{x} \right) \right]', & f_2 &= \frac{1}{4} \frac{\sigma'(x/4)}{\sigma(x/4)}, \\ f_{3a} &= \frac{1}{5} \frac{1 + 5\sigma'}{1 + \sigma}, & f_{3b} &= \frac{1}{5} \frac{1}{1 + \sigma}, \\ f_4 &= \left(\ln \frac{2\sigma(4x) - 1 - \sigma(x)^2}{x} \right)' = \left(\ln \sum_{k=0}^{\infty} \frac{2k + 3}{(2k + 4)!} (4x)^k \right)'. \end{aligned}$$

Step 1. The functions $f_2, f_{3a}, f_{3b}, f_4, \sigma'/\sigma$, and σ''/σ' are positive and nonincreasing.

This follows directly from the previous formulae, the power series representations for σ, σ' , and σ'' , and the

Lemma on Power Series. Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ be real power series with coefficients $a_k, b_k > 0$, converging on $D \subset \mathbb{R}$. If a_k/b_k is a nonincreasing sequence, then the function $A(x)/B(x)$ is nonincreasing on $D \cap [0, \infty)$.

In order to control H'/H , it will be useful also to decompose the remaining summand f_1 in terms of monotonic functions. We introduce the auxiliary functions

$$(6.6) \quad \begin{aligned} r(x) &= \left(\frac{\sigma'}{\sigma} \right)^{-1} \frac{\ln \sigma}{x} - 1 = \frac{1}{30}x + O(x^2), \\ \varphi(x) &= 1 + x \left(\frac{\sigma''}{\sigma'} - \frac{\sigma'}{\sigma} \right) = 1 + x \left(\ln \frac{\sigma'}{\sigma} \right)' = 1 - \frac{1}{15}x + O(x^2). \end{aligned}$$

A straightforward calculation yields:

$$\left(2\sigma' + \sigma \frac{\ln \sigma}{x}\right)' = 2\sigma'' + (1+r) \frac{\sigma'^2}{\sigma} - r \frac{\sigma'}{x};$$

hence:

$$(6.7) \quad 2f_1 = \frac{2}{3+r} \frac{\sigma''}{\sigma'} + \frac{1+r}{3+r} \frac{\sigma'}{\sigma} - \frac{r}{3+r} \frac{1}{x}.$$

Step 2. φ is monotonic and $0 < \varphi(x) \leq \varphi(0) = 1$ for $x \geq 0$. Using formulae (6.4) and the definition of σ , we calculate ($x = w^2$):

$$\begin{aligned} \varphi(w^2) &= 1 + \frac{w}{2} \frac{d}{dw} \left(\ln \frac{\sigma'}{\sigma}(w^2) \right) = 1 + \frac{w}{2} \frac{d}{dw} \ln \left(\frac{1}{2w} \left(\coth w - \frac{1}{w} \right) \right) \\ &= \frac{1}{2} \cdot \frac{\sigma(4w^2) - 1}{\sigma(4w^2) - \sigma(w^2)^2} > 0. \end{aligned}$$

Because of the expansions given in (6.4) for numerator and denominator the above lemma on power series yields the monotonicity of φ as required.

Step 3. $1 \leq 1+r \leq \varphi^{-1}$. Moreover both the functions r and $r/(3+r)$ are nondecreasing. In order to show the positivity of r , we observe that the function $\Psi(x) = \ln \sigma - x \cdot \sigma'/\sigma$ vanishes at $x=0$ and has derivative $\Psi' = -x(\sigma'/\sigma)' \geq 0$. We compute that $\tau' = x^{-1}(1 - (1+r) \cdot \varphi)$. Note that $r'(0) = 1/30$, and by continuity of r' we have $(1+r) \cdot \varphi < 1$ for small positive x . Since by Step 2 φ is monotone decreasing, this inequality continues to hold for all x , and all the remaining claims follow.

Step 4. Given $0 \leq u \leq x \leq v$, one has

$$(6.8) \quad \frac{H'}{H}(x) \leq F(u, v) = F_+(u, v) - F_-(u, v),$$

where

$$F_+(u, v) = \frac{1}{3+r} \left(\frac{\sigma''}{\sigma'} + \frac{1}{2} \frac{\sigma'}{\sigma} \right)(u) + (f_2 + f_{3a})(u) + \frac{1}{2} \frac{r}{3+r}(v) \frac{\sigma'}{\sigma}(u),$$

$$F_-(u, v) = (f_{3b} + f_4)(v) + \frac{1}{2} \frac{r}{3+r}(u) \frac{1}{v}.$$

Using Step 1 and Step 3, this estimate follows directly from the formulae (6.5) and (6.7).

Step 5. Inequality (6.8) provides a numerical criterion for proving negativity of $H'(w^2)/H(w^2)$, $w \in [0, 5.22]$, and thus establishing part (ii) of Proposition 3.3. In fact we have the table:

w_i	$F_-(w_i^2, w_{i+1}^2)$	$F(w_i^2, w_{i+1}^2) \cdot 10^4$
0		
1.6502	.2883	-.349
2.3296	.2618	-.373
2.82	.2397	-.439
3.2076	.2222	-.328
3.5278	.2083	-.314
3.7996	.1972	-.309
4.0352	.1881	-.305
4.2426	.1806	-.364
4.4268	.1743	-.334
4.5924	.169	-.304
4.7418	.1644	-.358
4.8778	.1604	-.344
5.0024	.1569	-.348
5.117	.1538	-.331
5.2226	.1511	-.331

Proof of (iii). Rather than making use of (6.3), we split the integral for $\Theta \circ \rho$ at $B_- + 1$; for large w this number gets arbitrarily close to $-\eta = -\rho(w)$, the place where the unique maximum of the concave function $B \rightarrow e^{-B}(\eta - V(B))$ lies.

(6.9)

$$\Theta_1(\eta) = \int_{B_-}^{B_-+1} \frac{dB}{\sqrt{e^{-B}(\eta - V(B))}}, \quad \Theta_2(\eta) = \int_{B_-+1}^{B_+} \frac{dB}{\sqrt{e^{-B}(\eta - V(B))}}.$$

Step 1. Θ_1 is monotone decreasing and converges to 0 for $\eta \rightarrow \infty$. Substituting $z = B - B_-$, a direct calculation yields

$$\Theta_1 \circ \rho(w) = \int_0^1 \frac{dz}{\sqrt{(1 + ze^{-B_-})e^{-z} - 1}} = \int_0^1 \frac{dz}{\sqrt{(1 + zT(2w))e^{-z} - 1}}.$$

The claim follows, since the function $T(2w) = (e^{2w} - 1)/2w$ is monotonic and tends to infinity with w .

Step 2. On $[1.5, \infty)$ one has the differential inequality:

$$(6.10) \quad (\Theta_2 \circ \rho)'(w) \leq -\frac{1}{4w^2} \Theta_2 \circ \rho(w) + \left(\frac{1}{2w} e^{2w-1} - 1 \right)^{-1/2}.$$

Using the substitution $z = B_+ - B$ and setting $N(z, w) = (1 - zT(-2w))e^z - 1$, we obtain $\Theta_2 \circ \rho(w) = \int_0^{2w-1} N(z, w)^{-1/2} dz$, hence:

$$(\Theta_2 \circ \rho)'(w) = - \int_0^{2w-1} \frac{ze^z T'(-2w)}{N(z, w)} \frac{dz}{\sqrt{N(z, w)}} + \frac{1}{\sqrt{N(2w-1, w)}}.$$

Inequality (6.10) now follows from the estimates

$$\begin{aligned} \frac{ze^z T'(-2w)}{N(z, w)} &= \frac{T'(-2w)}{T(-z) - T(-2w)} \geq \frac{T'(-2w)}{1 - T(-2w)} \\ &= \frac{1}{4w^2} \frac{1 - (1 + 2w)e^{-2w}}{1 - (1 - e^{-2w})/2w} \geq \frac{1}{4w^2}, \end{aligned}$$

$$N(2w-1, w) = \frac{1}{e} (1 + (e^{2w} - 1)/2w) - 1 \geq \frac{1}{2w} e^{2w-1} - 1,$$

which are due to the monotonicity of T and the hypothesis $2w \geq 3$.

Step 3. $(\Theta_2 \circ \rho)'(w) \leq -(\Theta_2 \circ \rho(w) - \pi + 6 \cdot 10^{-4})/4w^2$ for $w \in [5.22, \infty)$. The term $4w^2(e^{2w-1}/2w - 1)^{-1/2}$ is easily checked to be monotone decreasing on $[2.5, \infty)$ by taking its logarithmic derivative; hence, calculating its numerical value at $w = 5.22$, we obtain the claimed differential inequality directly from (6.10).

Step 4. Notice that monotonicity of $\Theta_2 \circ \rho$ follows from the previous step by the mean value theorem in its integral form, since

$$\lim_{w \rightarrow \infty} \Theta_2 \circ \rho(w) = \lim_{w \rightarrow \infty} \Theta \circ \rho(w) - \lim_{w \rightarrow \infty} \Theta_1 \circ \rho(w) = \pi.$$

In view of Step 1 we have thus proven part (iii) of the proposition.

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