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SURFACES IN 3-SPACE AND THEIR CONTACT WITH CIRCLES

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Two hundred years ago Meusnier established that for any surface in $\mathbf{R}^3$ the set of osculating circles at a point and with a given tangent direction form a sphere [14]. (An osculating circle is one with at least 3-point contact with the surface.)

In this paper we investigate higher-order contact between circles and generic surfaces using a singularity theory approach. This approach, which developed from an observation by Thom, is by now well established. See for example [1], [2], [4], [11] and [12]. The general idea is as follows. Let M be a parametrized family of 'model' submanifolds of $\mathbf{R}^n$ (spheres, circles, lines or whatever), and for each model submanifold m let f_m: $\mathbf{R}^n \to \mathbf{R}^p$ be a map which cuts out m. We require the map F: $\mathbf{R}^n \times M \to \mathbf{R}^p$, $F(y, m) = f_m(y)$, to be smooth. Let g: $X \hookrightarrow \mathbf{R}^n$ be an immersion of the manifold X. Consider the map

$$(1) \qquad \Phi\colon X \times M \to \mathbf{R}^p, \qquad \Phi(x, m) = \varphi_m(x) = f_m \circ g(x).$$

The contact of m and $g(X)$ (which we henceforth refer to as X) at $g(x)$ is then determined by the singularity type (more precisely, the $\mathscr{K}$-class) of the map φ_m at x [9]. The techniques and results of singularity theory can be used to recognize the contact types that occur in a given setting, and for 'generic' immersions of X both to predict which contact types can be expected and to give some global or semi-global information.

The main results of this article are contained in 5 theorems. Theorems 1 and 2 deal with the contact types that arise at each point of the surface. Theorems 3 and 4 deal with the behavior of circles with 6-point contact with the surface near umbilics. Theorem 5 is a generalization of a theorem of Banchoff, Gaffney and McCrory [2]. The genericity theorem needed for the later theorems is stated here, but is proved in greater generality in [10].

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Three articles [9], the present one and a forthcoming one on surfaces in 4-space form the greater part of my Ph.D. thesis [8] written at Liverpool University under the supervision of I. R. Porteous, whom I would like to thank for his interest, stimulation and guidance.

Background

The standard texts which contain the necessary prerequisite material are mentioned here. For the singularity theory see [5], [7] (where $\mathcal{K}$-equivalence is called V-equivalence), and [15]. For the differential geometry see [14], for example.

In [9] we saw that given two submanifolds X and Y of $\mathbf{R}^n$, with g: $X \hookrightarrow \mathbf{R}^n$ an immersion and Y defined by f: $\mathbf{R}^n \to \mathbf{R}^p$ with f nonsingular at each point of $Y = f^{-1}(0)$, then the contact of X and Y (defined up to diffeomorphisms of $\mathbf{R}^n$) at $g(x)$ is determined by the $\mathcal{K}$-class of the *contact map* $f \circ g$ at x. We also saw, from the symmetry lemma, that this $\mathcal{K}$-class was independent of the particular contact map chosen, and of which submanifold is considered as immersed and which as a zero-set. It follows that if one of the manifolds is one-dimensional then the contact maps involved are maps from $\mathbf{R}$ to $\mathbf{R}^p$ (or a suspension of such if the roles are reversed). Any such map (of finite $\mathcal{K}$-codimension) is $\mathcal{K}$-equivalent to the map $t \to (t^k, 0, 0, \cdots, 0)$ for some k. In the standard notation this has an A_{k-1} singularity at $t = 0$. An A_{k-1} thus corresponds to the classical notion of k-point contact. Indeed if one of the submanifolds is one-dimensional then the k in 'k-point contact' is the only contact invariant (contact defined as in [9]).

The distance-squared functions on $\mathbf{R}^3$ can be parametrized by $(c, s) \in S^3$ ($c \in \mathbf{R}^3$, $s \in \mathbf{R}$, $|c|^2 + s^2 = 1$). Given an immersed surface g: $X \hookrightarrow \mathbf{R}^3$, the distance-squared functions on X are defined by

$$V(x, (c, s)) = c \cdot g(x) - \tfrac{1}{2}s|g(x)|^2.$$

The level sets of $V(\cdot, (c, s))$ are spheres, center $s^{-1}c$ if $s \neq 0$, or planes if $s = 0$. From now on these will all be termed *spheres with center* (c, s), whether they are true spheres with center $s^{-1}c$, or, in the case $s = 0$, planes perpendicular to the vector c. It is also to be understood that if we refer to (c, s) as a point in $\mathbf{R}^3$, then we either mean the point $s^{-1}c$ if $s \neq 0$, or the point at infinity in the c direction if $s = 0$. (Note that (c, s) and $(-c, -s)$ give rise to the same level sets.)

In [11] and [12] Porteous established the correspondences between the extrinsic geometry of the immersed surface and the singularities of the functions $V(\cdot, (c, s))$ (alias, contact with spheres).

TABLE 1: Correspondence between the singularity type of $V(\cdot, (c, s))$ at x and the geometry of the surface at x.

Singularity Type of $V(\cdot,(c,s))$ at x	Geometry at $(x,(c,s))$
A_1	(c, s) is on the normal to the surface at $g(x)$
A_2	Focal point
A_3^-	Fertile rib point
A_3^+	Sterile rib point
A_4	Higher order rib point
D_4^-	Elliptic Umbilical center
D_4^+	Hyperbolic Umbilical center

For Table 1, recall that the normal form for an $A_k^\pm$ singularity is $x^2 \pm y^k$, and for $D_k^\pm$ it is $x^2y \pm y^{k-1}$ (in each case k is the codimension of the singularity). The condition for a function f to exhibit an A_k singularity at x is that there be an immersed curve γ through x such that $(df \circ \gamma)$ has zero $(k-1)$-jet. For a D_4 the condition is $df = 0$, $d^2f = 0$ at x [13]. (Strictly speaking, the condition for an A_k singularity should exclude the possibility of an A_{k+1}, though in this article we ignore this.)

A brief word should be said about the concept of a rib. In [11] Porteous introduced this concept in terms of the exponential map of the normal bundle. The singular points of the restriction of this map to the focal set are the rib points. The projection of these rib points on the surface are called ridge points, and for a generic surface immersion the set of ridge points is a union of smooth curves. There is an alternative (equivalent) definition of these ridges as follows. Consider a line of curvature on the surface. At each point of this curve is the associated principal curvature, giving a smooth function along the curve. Ridge points are the critical points of these functions. We can also define an order for a ridge point according to the order of critical point of the associated principal curvature function, which also corresponds to the type of singularity of the exponential map of the normal bundle. For a generic immersion ridge points are of order at most 2. In Table 1 we see that there are two types of first order ridges: fertile and sterile. This distinction arises from considering the singularities of the distance-squared function and can be interpreted geometrically by looking at the intersection of the surface with the appropriate sphere with center (c, s); for a fertile ridge point this intersection consists of two tangential curves while for a sterile ridge it is an isolated point. The adjectives fertile and sterile arose from studying one-parameter families of surfaces, in particular birth and death of pairs of umbilics. A pair of umbilics can only be born on a fertile ridge (this follows from unfolding theory: in any neighborhood of a transversely presented D_4 singularity is an A_3^- singularity, but not an A_3^+). For more details see [12].

A distinction also arises for umbilics according to the type of singularity of the exponential map of the normal bundle at the umbilical center or, equivalently, according to the type of contact with the umbilical sphere. At any umbilic there is an intrinsic cubic form C: if g is the immersion and κ the principal curvature at the umbilic, then

$$C = n \cdot d^3 g - 3\kappa dg \cdot d^2 g, \tag{2}$$

where n is the unit normal to the surface used to define κ. We say an umbilic is elliptic, hyperbolic or parabolic accordingly as C is elliptic, hyperbolic or parabolic. Again, for more details see [12]. It should be pointed out that the third derivative at x of the distance squared function $V(\cdot, (c, s))$, for (c, s) the umbilical center over x, is a real multiple of the intrinsic cubic C.

Contact of surfaces with circles

We are interested in the higher order contact of circles tangent to the immersed surface X. Let $g\colon X \hookrightarrow \mathbf{R}^3$ be the immersion. Abusing notation somewhat, we refer to $g(x)$ as x and for any tangent vector u to X we refer to its image $dg(x)u$ as u. In our notation the first and second fundamental forms are, respectively, $\mathrm{I} = dg \cdot dg$ and $\mathrm{II} = n \cdot d^2 g$, where n is a unit normal to the surface.

For any nonzero tangent vector u at x we define the *Meusnier sphere* M_u to be the sphere tangent to X at x with center (c, s) on the compactified normal satisfying $c \cdot d^2 g u^2 = s dgu \cdot dgu$. Note that if u is an asymptotic direction, so $\mathrm{II}(u, u) = 0$, then $s = 0$ and M_u is the tangent plane to X at x. Clearly, M_u depends only on the direction of u, not on its magnitude.

In the following theorem, a *u-circle* is a circle tangent to the surface at x, whose tangent is in the u-direction. Note that the only genericity assumption in this theorem appears in the second part of (iv), and that generic immersions are defined before the proof of this theorem.

Theorem 1. *For any immersion $g\colon X \hookrightarrow \mathbf{R}^3$ and with M_u defined as above, the following hold for each nonzero tangent vector u at x.*

(i) *A u-circle has 3-point contact with the surface at x if and only if it lies on M_u (this is just Meusnier's theorem extended to include the case where u is asymptotic).*

(ii) *If u is not a principal direction, then there is a unique u-circle on M_u with at least 4-point contact with the surface at x.*

(iii) *If x is not an umbilic and if u is a principal direction, then*

(a) *If u is not associated to a ridge point there is no u-circle with 4-point contact,*

(b) *If u is associated to a ridge point, then every u-circle on M_u has at least 4-point contact with the surface. Moreover, in this case there are 2, 0 or 1 u-circles on M_u with 5-point contact according as the ridge point is fertile, sterile or of higher order.*

(iv) *If x is an umbilic, so all the M_u coincide (call it M), then a circle on M tangent to the surface has at least 4-point contact if and only if its tangent at the surface is a root of the intrinsic cubic C. Moreover, if u is a root of C (defined in (2)) and the umbilic is generic, then exactly one u-circle on M will have (at least) 5-point contact with the surface. Thus at an elliptic umbilic there are 3 circles with 5-point contact while at a hyperbolic umbilic there is just one.*

To prove this theorem, we need to represent each circle in $\mathbf{R}^3$ as the zero-set of a submersion, as described in the introduction. To this end, let

$$M_e = \{((c_1, s_1, \rho_1), (c_2, s_2, \rho_2)) \in (S^3 \times \mathbf{R}) \times (S^3 \times \mathbf{R}): \\ (c_1, s_1) \neq \pm(c_2, s_2)\}.$$

For $m \in M_e$, the map f_m: $\mathbf{R}^3 \to \mathbf{R}^2$ is defined by

$$f_m(y) = \left(c_1 \cdot y - \tfrac{1}{2}s_1|y|^2 - \rho_1, c_2 \cdot y - \tfrac{1}{2}s_2|y|^2 - \rho_2\right).$$

Note that $f_m^{-1}(0)$ is the intersection of two spheres with centers (c_1, s_1) and (c_2, s_2) (if $s_1 = 0$ then the first sphere will be a plane, similarly for $s_2 = 0$). In the case $s_1 = s_2 = 0$ this circle is a straight line. Note also that M_e is not a 'faithful' representation of the family of circles in $\mathbf{R}^3$ as the choice of spheres intersecting in a given circle gives rise to a two-dimensional redundancy. We define M to be the open subset of M_e for which $f_m^{-1}(0)$ is a genuine circle (nonempty and nonsingular). (There should be no confusion caused by M denoting both the set of circles in $\mathbf{R}^3$ and the Meusnier sphere at an umbilic.)

From the family of maps f_m, we define F: $\mathbf{R}^3 \times M_e \to \mathbf{R}^2$ by $F(x, m) = f_m(x)$, and Φ and φ_m as in (1). The main theorem in [10] has the following as a special case:

Genericity Theorem. *Let W be any $\mathcal{K}$-invariant submanifold of $J^r(X, \mathbf{R}^2)$. Denote by R_W the set*

$$\left\{g \in \operatorname{Imm}(X, \mathbf{R}^3) \colon j_1^r\Phi_g \pitchfork W\right\},$$

where Φ_g is the map Φ, with the dependence on g made explicit, and j_1^r means the r-jet with respect to the first variable, so $j_1^r\Phi_g$: $X \times M_e \to J^r(X, \mathbf{R}^2)$. Then R_W is residual in $\operatorname{Imm}(X, \mathbf{R}^3)$, *and moreover if W is closed then R_W is open and dense.*

Remarks. (i) The $\mathscr{K}$-invariance of W is required only because the circles are allowed to degenerate, so the maps f may be singular.

(ii) The result that R_W is open if W is closed depends on the fact that the set of circles through a point can be represented by a compact set.

(iii) There is also exactly the same result for contact with spheres (where each f_m: $\mathbf{R}^3 \to \mathbf{R}$ is a distance-squared function), though of course in this case W must be a $\mathscr{K}$-invariant submanifold of $J^r(X, \mathbf{R})$.

To determine how high we can expect the $\mathscr{K}$-codimension of the singularities to be, consider the following facts: (i) If W is a $\mathscr{K}$-orbit in $J^r(X, \mathbf{R}^2)$, then the codimension of W (for r sufficiently large) is 2 greater than the $\mathscr{K}$-codimension of the singularity it represents (because of the preferred role of the target zero maps—we consider the circles as zero-sets rather than general level-sets); (ii) $\dim(X \times M_e) = 10$; (iii) there is a two-dimensional redundancy in our parametrization of the set of circles in $\mathbf{R}^3$, as already mentioned. From (ii) and (iii) we expect $j_1^r\Phi$ to meet $\mathscr{K}$-invariant submanifolds of codimension up to 8, so from (i) the maps φ_m will exhibit singularities of $\mathscr{K}$-codimension up to 6.

To use the genericity theorem above, we consider the subset of $J^r(X, \mathbf{R}^2)$ of singularities of $\mathscr{K}$-codimension up to 6. This has a regular stratification by a finite number of $\mathscr{K}$-orbits $\{W_1, \cdots, W_s\}$. The complement of this subset is algebraic, so $\{W_1, \cdots, W_s\}$ can be extended to a regular stratification of $J^r(X, \mathbf{R}^2)$, $\{W_1, \cdots, W_t\}$. Let $R = \bigcap_{i=1}^t R_{W_i}$. This set will be open and dense (even though the individual W_i may not be closed, we still get openness essentially because the closure of each W_i contains a W_j of lower dimension). The same argument can be repeated for contact with spheres (singularities of codimension up to 4 can be expected), giving an open dense subset R' of $\mathrm{Imm}(X, \mathbf{R}^3)$. In this article we will say that the immersion g is *generic* if $g \in R \cap R'$. For such an immersion, all singularities of the φ_m (both for contact with circles and with spheres) will be $\mathscr{K}$-versally deformed by Φ. For more details, see [10].

If $m \in M$, then the only singularities which arise are A_k's. However, interesting features occur for $m \in M_e \setminus M$ where corank 2 singularities can arise. We return to this point later, just pointing out for now that it is because of this phenomenon that we choose to immerse X and consider the circles as zero-sets rather than vice-versa.

Suppose the circle m is tangent to the surface at x. To represent this circle, we can choose any two distinct spheres through x with centers on the axis of the circle. It is evident that m is tangent to the surface at x if and only if its axis passes through the normal to the surface at x. We thus have two preferred points on the axis, namely where the axis meets the normal, which we will denote (c_1, s_1), and where it meets the tangent plane—this point will be

(c_2, s_2). We let

$$(3)\quad V(x) = c_1 \cdot g(x) - \tfrac{1}{2}s_1|g(x)|^2, \qquad W(x) = c_2 \cdot g(x) - \tfrac{1}{2}s_2|g(X)|^2$$

be the appropriate distance squared functions. Thus we now have

$$\varphi_m(x) = (V(x) - \rho_1, W(x) - \rho_2).$$

We denote by V_i and W_i the ith derivatives of V and W at x. The conditions for φ_m to have an A_k singularity at x, $k = 1, 2, 3, 4$, are, cumulatively,

$$(4)\quad \begin{aligned} &A_1\colon W_1u = 0 \text{ (for some } u \neq 0) \ (V_1 = 0 \text{ by choice of } (c_1, s_1)),\\ &A_2\colon V_2u^2 = 0;\ W_2u^2 + W_1v = 0 \text{ (for some } v),\\ &A_3\colon V_3u^3 + 3V_2uv = 0;\ W_3u^3 + 3W_2uv + W_1w = 0 \text{ (for some } w),\\ &A_4\colon V_4u^4 + 6V_3u^2v + 3V_2v^2 + V_2uw = 0;\\ &\qquad W_4u^4 + 6W_3u^2v + 3W_2v^2 + W_2uw + W_1z = 0 \text{ (for some } z). \end{aligned}$$

These follow from the condition that a map-germ $\varphi\colon (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ has an A_k singularity if there is an immersed curve $\gamma\colon (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ such that $\varphi \circ \gamma$ has zero k-jet at 0, but $d\varphi(0) \neq 0$.

Proof of Theorem 1. Recall that an A_k singularity of φ_m corresponds to $(k+1)$-point contact, and that a u-circle is tangent to the surface so the A_1 condition (4) is automatically satisfied, and so (c_2, s_2) is such that $W_1u = 0$.

(i) We require $V_2u^2 = 0$, $W_2u^2 + W_1v = 0$ for some v. The first condition is precisely the condition for (c_1, s_1) to be the center of M_u. (The second merely relates v and (c_2, s_2) so imposes no restrictions on u or (c_1, s_1).)

(ii) We require $V_3u^3 + 3V_2uv = 0$, $W_3u^3 + 3W_2uv + W_1w = 0$ for some w, where v satisfies $W_2u^2 + W_1v = 0$. Now, if u is not principal, then $V_2u \neq 0$ ((c_1, s_1) being the center of M_u from (i)) and the first equation can be solved uniquely for v (modulo u). $W_2u^2 + W_1v = 0$ can then be solved uniquely for (c_2, s_2) (despite the ambiguity in v) which determines the circle uniquely (the 2nd A_3 equation being solved for w).

(iii) In this case we have $V_2u = 0$ (u principal) but $V_2 \neq 0$ (x not an umbilic). The condition for u to be associated to a ridge point is that $V_3u^3 = 0$.

(a) Since $V_3u^3 \neq 0$, $V_2u = 0$ there is no v satisfying $V_3u^3 + 3V_2uv = 0$, thus no circle with 4-point contact with the surface.

(b) Here $V_3u^3 = 0$, so every v satisfies $V_3u^3 + 3V_2uv = 0$ so there is no restriction on (c_2, s_2) in the A_3 condition. The A_4 condition reduces to $V_4u^4 + 6V_3u^2v + 3V_2v^2 = 0$ giving a quadratic equation for v which is precisely the one in the definition of the type of ridge point [12], so there are either 2, 1 or 0 solutions for v and so for (c_2, s_2), accordingly as the ridge is fertile, higher order, or sterile. The second A_4 condition can be solved for z.

(iv) Here $V_2 = 0$ for (c_1, s_1) at the center of the umbilical sphere. The intrinsic cubic C is a scalar multiple of V_3. As in (iii)(a), if $V_3u^3 \neq 0$, there are no solutions to the A_3 condition so every u-circle on the umbilical sphere has 3-point contact. However, if u is a root direction, then the A_3 condition leaves v undetermined so every u-circle has at least 4-point contact. The A_4-condition becomes $V_4u^4 + 6V_3u^2v = 0$ which is uniquely solvable for v provided $V_3u^2 \neq 0$. $V_3u^2 = 0$ is the condition for the umbilic to be parabolic which is non-generic. (This is the only genericity assumption made in this theorem.) q.e.d.

There is a useful intuition for the contact of these u-circles on the Meusnier sphere M_u with the surface. The intersection of M_u with the surface is determined up to diffeomorphism, by which type of point (c_1, s_1) is—see Table 1. For example, if (c_1, s_1) is a fertile rib point, then M_u intersects the surface in a pair of tangential curves. Any u-circle on M_u will then be tangent to both of these curves, so giving $2 + 2 = 4$-point contact with the surface. Each of these curves will have an osculating circle, so giving two circles with $2 + 3 = 5$-point contact. This intuition works whenever the intersection determines the contact between M_u and the surface. For the situation in Theorem 1(ii), M_u intersects the surface at x in two transverse curves, one of which has tangent u. The osculating circle to this branch will be the unique u-circle with 4-point contact with the surface at x. From this situation we can approach the situation (iii)(a) of Theorem 1 where u is a principal direction. In the process the curvature of the curve of intersection becomes infinite (as the transverse curves pull through to a cusp) and the osculating circle shrinks to a point. Since this osculating circle is always on the appropriate Meusnier sphere (the radius of which is bounded away from 0), it follows that the plane spanning this circle approaches the tangent plane as we approach the situation where u is principal. We call this phenomenon *curling up and dying* and it is something we return to when considering umbilics.

We now turn to the situation at an arbitrary point on the surface. Looking back at Theorem 1, we see that at a given point x and for most tangent directions u there is a unique circle with 4-point contact with the surface, and for finitely many directions (usually none) there is a one-dimensional family of such circles with 4-point contact, so we expect there to be finitely many circles with 5-point contact with the surface at any point x. *Question*: how many? We saw in Theorem 1(iv) that at an umbilic there are at most three such circles (or one if it is a hyperbolic umbilic). We now put an upper bound on this number for all points.

Theorem 2. *There is an open dense set of immersions $g\colon X \hookrightarrow \mathbf{R}^3$, such that at any point $x \in X$, there will be at most ten circles with at least 5-point contact* *with the surface at x. As we have already seen this reduces to three or one when x is an elliptic or hyperbolic umbilic.*

Proof. For simplicity we express the immersion in Monge form: $g(x, y) = (x, y, h(x, y))$ with

$$h(x, y) = \tfrac{1}{2}B(x, y)^2 + \tfrac{1}{6}C(x, y)^3 + \tfrac{1}{24}D(x, y)^4 + H(x, y),$$

where B is a quadratic form, C cubic, D quartic and H has zero 4-jet at the origin. We can also choose coordinates so that the coordinate axes correspond to the principal directions, thus $B(x, y)^2 = ax^2 + by^2$. As usual we take (c_1, s_1) to be on the normal and (c_2, s_2) to be on the tangent plane, so $(c_1, s_1) = ((0, 0, r), s_1)$ for some r. Since the A_1 condition in (4) requires (c_2, s_2) to be orthogonal to $u = (x, y)$, we get $(c_2, s_2) = (q(-y, x, 0), s_2)$ for some q. Choosing v and w perpendicular to u, so $v = t(-y, x)$ for some t, the remaining equations in (4) become

$$\begin{aligned}
A_2:&\quad r(ax^2 + by^2) - s_1(x^2 + y^2) = 0;\ s_2 = et,\\
A_3:&\quad rC(x, y)^3 + 3rt(b - a)xy = 0;\ w = 0,\\
A_4:&\quad rD(x, y)^4 - 3s_1(ax^2 + by^2)^2 + 6rtC(x, y)^2(-y, x)\\
&\qquad + 3rt^2(bx^2 + ay^2) - 3s_1t^3(x^2 + y^2) = 0;
\end{aligned}$$

the second A_4 equation is solvable for z, so does not contribute. The second A_2 equation can be solved for s_2 so it too can be ignored. We are thus left with three equations: A_2, A_3 and A_4. We can eliminate (r, s_1) between A_2 and A_4 to obtain the following two equations (note that $r = 0$ is not of interest):

$$\begin{aligned}
&C(x, y)^3 + 3(b - a)txy = 0,\\
(5)\qquad &D(x, y)^4(x^2 + y^2) + 6t(x^2 + y^2)C(x, y)^2(-y, x)\\
&\quad - 3(ax^2 + by^2)^3 + 3(b - a)t^2(x^4 - y^4) = 0.
\end{aligned}$$

These equations are homogeneous in x, y, t of degrees 3 and 6 respectively, so representing two algebraic curves in $\mathbf{CP}^2$, of degrees 3 and 6. It follows from Bezout's theorem that they either have a common component or else have 18 points of intersection, counting multiplicity. Lemma B below shows that the possibility of having a common component is avoided for an open dense set of immersions. Thus there are 18 points of intersection of the two curves. However, the intersection at $[x: y: t] = [0: 0: 1]$ does not represent a circle. We therefore need to know the multiplicity of the intersection at $[0: 0: 1]$, and in

Lemma A this is shown to be 8 for $a \neq b$ (it is 15 if $a = b$, the case for an umbilic). Thus we are left with 10 solutions for $u \neq 0$ (or 3 if x is an umbilic). q.e.d.

Lemma A. *The two curves in* $\mathbf{CP}^2$ *defined by* (5) *intersect with multiplicity* 8 *at* $[x\colon y\colon t] = [0\colon 0\colon 1]$, *provided* $a \neq b$.

Proof. Choose local coordinates about $[0\colon 0\colon 1]$ by putting $t = 1$. We obtain

$$C(x, y)^3 + 3(b - a)xy = 0$$

for the cubic, and

$$D(x, y)^4(x^2 + y^2) + 6(x^2 + y^2)C(x, y)^2(-y, x) \\ -3(ax^2 + by^2)^3 + 3(b - a)(x^4 - y^4) = 0$$

for the sextic. The first of these has a node of multiplicity 2 at the origin, while the second has a node of multiplicity 4. The two curves have no common tangent directions at the origin, so the multiplicity of intersection is the product of the multiplicities of each. q.e.d.

Lemma B. *There is an open dense set of immersions for which at no point of the immersed surface X do the* 2 *curves* (5) *have a common component.*

Proof. From the proof of Lemma A we see that there are no common components passing through $[0\colon 0\colon 1]$. Since the cubic curve has multiplicity 2 at that point, any common component can have degree at most 1. Furthermore, in this case the cubic curve will have three linear components. Now, the condition that the cubic in (5) has three linear factors is that xy should divide the cubic form C. In other words, the principal directions are root directions of C. This will occur at isolated points of a generic surface (which are precisely the flyover points in the terminology of [12]). That the linear component of the cubic curve not passing through $[0\colon 0\colon 1]$ be also a component of the sextic is clearly an additional independent condition, and so is avoided for an open dense set of immersions. q.e.d.

It would be worth knowing whether or not it is possible to realize this number (i.e., 10) of circles with 5-point contact with the surface at a point, or whether some of the solutions of (5) are forced to be complex. The answer is not clear. That it is possible to have precisely six such circles follows from work of Blum [3], where he produces a cyclide with the 6-circle property. That is, there are six distinct circles passing through each point of the surface that lie on the surface. Any such circle will *a fortiori* satisfy (5). The six real solutions to (5) in the case of Blum's cyclide are distinct so by a sufficiently small perturbation of the cyclide we can produce a generic surface with, at each point, six circles with 5-point contact with the surface. If indeed equation (5) can be shown to have at most six real solutions, it is probable that this could lead to a proof of Blum's conjecture: "there are no surfaces with the n-circle property for $6 < n < \infty$".

In the vicinity of an umbilic

Many of these circles with 5-point contact will in fact have at least 6-point contact with the surface. It is not easy to say very much about these in general, though we can obtain some information in the vicinity of umbilics using deformation (unfolding) theory. First, we need to see how the Σ^2 singularities we referred to earlier arise. See [6] for details of maps from $\mathbf{R}^2$ to itself and their versal deformations.

It is clear from (3) that $d\varphi_m(x_0) = 0$ if and only if (c_1, s_1) and (c_2, s_2) both lie on the normal to the surface at x_0. Thus φ_m has a Σ^2 singularity and m will be the isolated point $\{x_0\}$, so $m \in M_e \setminus M$. As before, we can choose (c_1, s_1) and (c_2, s_2) to be any two distinct points on the axis of the 'circle' m. In this case the axis is the normal to the surface at x_0 and we let (c_2, s_2) be the point x_0, which for convenience we take to be at the origin in $\mathbf{R}^3$, so $(c_2, s_2) = (0, -1)$ and $\varphi_m(x) = (V(x, (c_1, s_1)) - \rho_1, \frac{1}{2}|g(x)|^2)$. Then

$$d^2\varphi_m = (V_2, dg \cdot dg).$$

If x_0 is not an umbilic of the immersion g then the two quadratic forms that appear in $d^2\varphi_m$ are linearly independent with one being positive definite and it follows that the Σ^2 singularity that occurs is a $\mathrm{I}_{2,2}$ in Mather's notation, which has normal form $(x^2 + y^2, xy)$. If, on the other hand, x_0 is an umbilic, then the two quadratic forms are linearly dependent and the singularity is of a higher type. Since the quadratic form that does occur is positive definite, the singularity is of type IV_k for some $k \geqslant 3$, which has normal form $(x^2 + y^2, x^k)$. The codimension of a IV_k singularity is $2k$, so the only possibility for a generic surface is $k = 3$. We choose (c_1, s_1) to be the umbilical center, so $d^2\varphi_m = (0, \mathrm{I})$ (recall $\mathrm{I} = dg \cdot dg$ is the first fundamental form). We now get

$$d^3\varphi_m = \left(V_3, 3dg \cdot d^2g\right).$$

From this we see that φ_m has a IV_3 singularity at x_0 if and only if I is not a factor of the cubic form V_3 (recall that V_3 is a scalar multiple of the intrinsic cubic C).

Let us pause a moment to describe the notion of the harmonic representative of a cubic form in the presence of a positive definite quadratic form, Q. Given this form Q we can choose a linear change of coordinates h so that $I = h^*Q$, where $I(x, y)^2 = x^2 + y^2$. Associated to Q is the differential operator $\Delta_Q = (h^*)\Delta_I h^*$, where Δ_I is the usual Laplacian. Note that $\Delta_Q(Q(x, y)^2) = 4$, and for any linear form L, $\Delta_Q(QL) = 8L$. Given, in addition to Q, a cubic form C we can form the pencil of cubic forms $\{C + LQ\}$ as L varies through the space of linear forms. There is a unique linear form L for which $\Delta_Q(C + LQ) = 0$ and this cubic form $(C + LQ)$ is called the *harmonic representative of* C *with respect to* Q. Clearly, the harmonic representative of C with respect to Q is zero if and only if $C = LQ$ for some L. It is also the case that $\Delta_Q(C) = 0$ if and only if the hessian of C is a multiple of Q. Consequently, the harmonic representative of any cubic form has 3 real roots which are distributed harmonically with respect to Q.

We can now rephrase the statement above to say that φ_m has a IV_3 singularity at the umbilic if and only if the intrinsic cubic C has a nonzero harmonic representative with respect to the first fundamental form I. We will find that this harmonic representative is of geometrical significance.

Let f: $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be a IV_3 singularity and let F: $\mathbf{R}^2 \times \mathbf{R}^6 \to \mathbf{R}^2$ be any versal 6-parameter deformation of f (the $\mathscr{K}$-codimension of f is 6). In [6], Lander shows that there are three nonsingular curves passing through the origin in $\mathbf{R}^2 \times \mathbf{R}^6$ at each point of which the deformed map has an A_5-singularity. In our application, these A_5 singularities will occur at points in $X \times M_e$ representing genuine circles, i.e., at points in $X \times M$, as for $m \in M_e \setminus M$ the only Σ^1 singularity that can occur is an A_1.

Consequently, we have through any generic umbilic three curves at each point of which there is a circle with 6-point contact with the surface. As we approach the umbilic along any of these curves, the radius of the circle tends to zero, and the plane spanning the circle tends to the tangent plane at the umbilic. We have already mentioned this phenomenon of curling up and dying.

To study this phenomenon in more depth, we need to take a closer look at the IV_3 singularity. Let f: $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be any map-germ with a IV_3 singularity. Then $df_0 = 0$ and $d^2 f_0$ as a quadratic map $\mathbf{R}^2 \to \mathbf{R}^2$ has rank 1. Let $q \in \operatorname{coker}(d^2 f)$ be nonzero, and let $b \notin \operatorname{coker}(d^2 f)$ and $B = b \cdot d^2 f$.

With f, q, b, and B as above, the *harmonic part* of f, denoted H_f, is defined to be the harmonic representative of $q \cdot d^3 f$ with respect to B. Note that this is defined up to a scalar multiple.

Lemma. *Let* f: $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *be a map-germ with a* IV_3 *singularity. The harmonic representative of* f *is intrinsic: it is a* $\mathscr{K}$*-invariant form on the tangent space to* $\mathbf{R}^2$ *at* 0.

Proof. Suppose f, g:$(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ are IV_3 singularities with harmonic parts H_f and H_g. Since f and g are $\mathscr{K}$-equivalent we can write

$$f \circ h(x) = \theta(x, f(x)), \tag{6}$$

where h is a diffeomorphism-germ of $(\mathbf{R}^2, 0)$ and θ: $(\mathbf{R}^2 \times \mathbf{R}^2, (0,0)) \to (\mathbf{R}^2, 0)$ is a germ satisfying (i) $\theta(x, 0) = 0$; (ii) the first derivative with respect to its second argument is invertible—call this linear map A.

To prove this lemma it suffices to show that if u is a root of H_f, then $(h_1 u)$ is a root of H_g (as usual, we use subscripts to denote successive differentiation, thus $h_1 = dh$). Differentiating (6) twice and thrice, we get

$$g_2 \cdot h_1^2 = A \cdot f_2, \tag{7}$$

$$g_3 \cdot h_1^3 + 3g_2(h_1, h_2) = 3\theta_2(1,0)(0, f_2) + A \cdot f_3 \tag{8}$$

(the first term on the right side of (8) is the cubic form $u \mapsto 3d_x d_y \theta(u)(f_2 u^2)$). We let $q' \in \operatorname{coker} g_2$ and $b' \notin \operatorname{coker} g_2$, so from (6), $q = q' \cdot A \in \operatorname{coker} f_2$ and $b = b' \cdot A \notin \operatorname{coker} f_2$, and let $B' = b' \cdot g_2$, $B = b \cdot f_2$, $C' = q' \cdot g_3$ and $C = q \cdot f_3$. Acting on (8) with q' we get

$$C' h_1^3 = L_1 \cdot B + C, \tag{9}$$

where L_1 is a linear form. Now, let L be the unique linear form for which $\Delta_B(C + LB) = 0$, so $H_f = C + LB$ and let $L' = (L - L_1)h_1^{-1}$. Then from the definitions, and (9), we get $\Delta_{B'}(C' + L'B') = 0$, so $H_g = C' + L'B'$.

Finally, suppose u is a root of H_f, so

$$\begin{aligned} 0 = H_f u^3 &= Cu^3 + Lu \cdot Bu^2 = C'(h_1 u)^3 - L_1 u B u^2 + L u B u^2 \\ &= C'(h_1 u)^3 - L'(h_1 u) B'(h_1 u)^2 = H_g (h_1 u)^3, \end{aligned}$$

that is, $h_1 u$ is a root of H_g as required. q.e.d.

We now examine the standard IV_3 singularity and a versal deformation of it:

$$f(x, y) = (x^2 + y^2, x^3),$$
$$F(x, y, u, v, w, s, l, m) = (x^2 + y^2 + l, x^3 + ux^2 + vxy + wx + sy + m).$$

The three A_5 curves referred to above are parametrized by

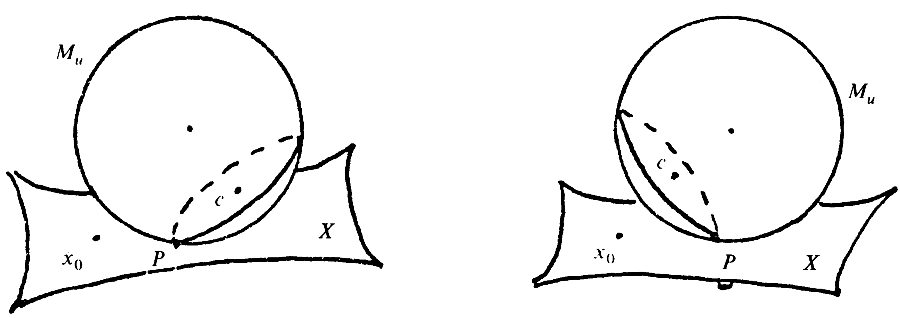
$$\begin{aligned} &t \mapsto (t, 0, -3t, 0, 3t^2, 0, -t^2, -t^3), \\ &t \mapsto (t, \sqrt{3}\,t, -3t, 3\sqrt{3}\,t, -21t^2/2, -15\sqrt{3}\,t^2/2, -4t^2, 26t^3), \\ &t \mapsto (t, -\sqrt{3}\,t, -3t, -3\sqrt{3}\,t, -21t^2/2, 15\sqrt{3}^{\,2}/2, -4t^2, 26t^3). \end{aligned}$$

That is, at each point of each of these curves, the deformed map F_u, with $u = (u, v, w, s, l, m)$ has an A_5-singularity at $(x, y) = (t, 0)$, $(t, \sqrt{3}\,t)$, or $(t, -\sqrt{3}\,t)$ depending on which of the three curves we are considering. The limiting kernel directions of $dF(u)$ as we approach the origin along each of these curves is $(0, 1)$, $(\sqrt{3}, -1)$, and $(\sqrt{3}, 1)$ respectively. Note that the harmonic part of f is $x^3 - 3xy^2$ which has these three directions as root directions. It follows that any versal deformation of any IV_3 singularity has the property that the limiting kernel directions of the A_5 singularities on approaching the IV_3 singularity coincide with the root directions of the harmonic part of the IV_3 singularity.

At an umbilic the root direction of the harmonic part of the intrinsic cubic C (with respect to the first fundamental form) are called the *harmonic directions at the umbilic.* Collating, we have,

Theorem 3. *Through any generic umbilic there are three curves on the surface at each point of which there is a circle with 6-point contact with the surface. As we approach the umbilic along any of these curves, the tangents to the circles approach the harmonic directions at the umbilic. Moreover, the radius of the circles will tend to zero and the plane spanning them will approach the tangent plane to the surface at the umbilic (the circles curl up and die).*

There are two distinct ways in which a circle can curl up and die as it approaches the umbilic x_0. Let P be a point on one of the curves described in Theorem 3—henceforth called A_5-curves, with the appropriate circle called an A_5-circle. Consider the projection of the A_5-circle and its tangent line at P onto the tangent plane at x_0. For P sufficiently close to x_0 these projections will be nonsingular. The distinction between the two types is as follows: for type (i) the projection of the circle and the point x_0 lie on opposite sides of the projection of the tangent line, while for type (ii) they lie on the same side, see Figure 1. Equivalently, if c is the center of the circle, then for types (i) and (ii) $(c - P) \cdot (x_0 - P)$ is respectively negative or positive for P sufficiently close to x_0. More graphically, these two types can be termed 'slipping down; and 'falling over'.



Type (i) Type (ii)

FIGURE 1: The two ways the A_5-circles can curl up and die.

Recall that associated to any umbilic is a half-integer—its index. This number is the index of the singularity at the umbilic of the line field of principal directions, see [12] where the relationship between the intrinsic cubic and the index is elaborated. Briefly, any real cubic form on (x, y) can be written $\alpha z^3 + 3\bar{\beta} z^2 \bar{z} + 3\beta z\bar{z}^2 + \bar{\alpha}\bar{z}^3$, with $z = x + iy$. For a generic umbilic, the intrinsic cubic has nonzero harmonic part, so $\alpha \neq 0$. We can rotate the x-y plane to make $\alpha = 1$, so the cubic forms are parametrized by $\beta \in \mathbf{C}$. The umbilic then has index $1/2$ if $|\beta| > 1$ and index $-1/2$ if $|\beta| < 1$.

Theorem 4. *For a generic immersion, as we approach an umbilic along any of the A_5-curves, the manner in which the A_5-circle curls up and dies corresponds to the index of the umbilic as follows*:

$$\textit{Type (i): index} = 1/2, \qquad \textit{Type (ii): index} = -1/2.$$

Proof. In order to prove this theorem, we reverse the roles of the surface and the circle in expressing the contact: each circle will be immersed while the surface will be the zero-set of a submersion. The composed maps will now be from $\mathbf{R}$ to itself. By the symmetry lemma [9] this will not alter the problem—we are still interested in A_5-singularities.

Let the umbilic be at the origin in $\mathbf{R}^3$, and the surface be given by

$$f(x, y, z) = h(x, y) - z = 0$$

with h as in the proof of Theorem 2. The A_5-circle at P can be immersed by

$$r_P(t) = P + (a, a_3)\sin t + (b, b_3)(1 - \cos t),$$

where $a, b \in \mathbf{R}^2$, $(b, b_3) = c - P$ and (a, a_3) is the vector product of (b, b_3) with the unit normal to the surface at P. This circle has center c and tangent vector (a, a_3) at P. The contact map is then

$$f \circ r_P(t) = h(a\sin t + b(1 - \cos t)) + a_3 \sin t - b_3(1 - \cos t).$$

The A_5-condition is that the first five derivatives vanish at $t = 0$, so

$$\begin{aligned}
A_1&: \quad dha - a_3 = 0,\\
A_2&: \quad d^2ha^2 + dhb - b_3 = 0,\\
A_3&: \quad d^3ha^3 + 3d^2hab = 0,\\
A_4&: \quad d^4ha^4u + 6d^3ha^2b + 3d^2hb^2 - 3d^2ha^2 = 0,\\
A_5&: \quad d^5ha^5 + 10d^4ha^3b + 15d^3ha^3 - 15d^2hab = 0,
\end{aligned}$$

where all derivatives of h are at P.

Now, as the circle curls up and dies, so c, and hence (a, a_3), (b, b_3), tend to 0. Let u, v be vectors in the limiting direction of (a, a_3) and (b, b_3) respectively. Using the A_1 and A_2 conditions we get that u and v are in the tangent plane to the surface at 0, that $u \cdot v = 0$ and that we can take $|u| = |v| = 1$.

Differentiating the A_3, A_4 and A_5 condition three times each and evaluating at 0 we get, with $\hat{p}$ tangent to the A_5-curve,

$$\begin{aligned} &Cu^3 + 3Cuv\hat{p} = 0, \\ &2Cu^2v + Cv^2\hat{p} - Cu^2\hat{p} = 0, \\ &3Cuv^2 - 2Cu^3 - 3Cuv\hat{p} = 0. \end{aligned} \tag{10}$$

Eliminating $Cuv\hat{p}$ between the first and third of these gives $Cu^3 - 3Cuv^2 = 0$ which is precisely the condition that u be a harmonic direction at the umbilic.

For this theorem we are interested in the sign of $\hat{p} \cdot v$ since this determines the limiting sign of $(c - P) \cdot (x_0 - P)$: for type (i) $\hat{p} \cdot v > 0$, for type (ii) $\hat{p} \cdot v < 0$. We express the cubic form as $C(x, y)^3 = z^3 + 3\bar{\beta} z^2 \bar{z} + 3\beta z \bar{z}^Z + \bar{z}^3$ (as described above), and let $\beta = s + it$. Then, for $u = (x, y)$ and $v = (y, -x)$,

$$Cu^3 = (1 + 3s)x^3 + 3tx^2y + (s - 1)xy^2 + 3ty^3,$$

$$Cu^3 - 3Cuv^2 = 4x(x^2 - 3y^2).$$

Thus the harmonic roots are $u = (0,1)$, $\frac{1}{2}(1, \sqrt{3})$ and $\frac{1}{2}(1, -\sqrt{3})$. Without loss of generality, we only look at $u = (0,1)$ (the others follow by similar arguments, or by rotating the x-y plane so that they are each $(0,1)$ in turn). So with $u = (0,1)$, $v = (1,0)$ and $\hat{p} = (x, y)$, (10) become

$$tx + (s - 1)y = t, \qquad (s + 1)x - ty = s - 1.$$

Solving these we get $\hat{p} \cdot v = x = [t^2 + (s - 1)^2]/(s^2 + t^2 - 1)$ and we are done. q.e.d.

A generalization of a theorem of Banchoff, Gaffney and McCrory

This article is concluded by presenting a generalization of (most of) the central theorem in [2] on the cusps of the Gauss map. The singularities of the Gauss map occur at points of zero curvature, this generalization is to points of nonzero curvature.

Let $g\colon X \hookrightarrow \mathbf{R}^3$ be an immersion of a surface X. The Gauss map $G\colon X \to S^2$ (S^2 = the unit sphere in $\mathbf{R}^3$) is singular at points where one of the principal curvatures is zero—the parabolic curve. For $t = [a\colon b] \in \mathbf{RP}^1$, let

$$g_t(X) = [ag(X) + bG(x)\colon a].$$

The map g_t is the parallel map, compactified to include the Gauss map at infinity. Singularities of g_t occur at points x which has a/b as one of its principal curvatures—or, equivalently, when $g_t(x)$ is a point on the focal set of X at x. This suggests that the interesting generalization of the parabolic curve is to curves on the surface along which one of the principal curvature functions is constant—we will call these *curves of constant principal curvature*. Some of the geometry of these curves is discussed in [8].

Theorem 5. *Let g: $X \hookrightarrow \mathbf{R}^3$ be a generic immersion and suppose $x_0 \in X$ is not an umbilic. Let $\kappa(x)$ be one of the principal curvature functions with principal direction u and focal point (c, s) at x_0. Then the following statements are equivalent.*

(i) *x_0 is a ridge point of g, with associated principal curvature $\kappa(x_0)$.*

(ii) *Either $d\kappa(x_0) = 0$ or the line of curvature (associated to κ) is tangent to the curve of constant principal curvature $\kappa(x_0)$ at x_0.*

(iii) *For any neighborhood U of x_0 in X there are three points in U and three concentric spheres each tangent to the surface at one of the three points.*

(iv) *For any neighborhood U of x_0 in X there are two points in U and a sphere which is tangent to the surface at both of these points.*

(v) *Let t be such that $g_t(x_0) = (c, s)$: then x is a swallowtail point of g_t.*

(vi) *There is a circle in $\mathbf{R}^3$ with tangent vector u at x_0 which has at least 4-point contact with the surface (whence it follows that all such circles do).*

(vii) *The osculating sphere at x_0 of any curve through x_0 with tangent u on the surface is tangent to the surface at x_0. If this is true for one such curve, then it is true for all of them.*

Sketch of proof. We show that each statement is equivalent to (i).

(ii): This follows from our definition of ridge point.

(iii) and (iv): The distance-squared function from (c, s) has an A_3 singularity at x if and only if $(x, (c, s))$ is a rib point [12]. Consider the following deformation of the normal form for an $A_3^{\pm}$:

$$f_a(x, y) = x^2 \pm (y^2 - a)^2.$$

f_a is singular at $(0,0)$, $(0, \sqrt{a})$, $(0, -\sqrt{a})$ for $t > 0$, having values a^2 at $(0,0)$ and 0 at both of the other points. Since g is generic, this situation also arises for distance-squared functions from points near (c, s), thus proving that (i) $\Rightarrow$ (iii) and (iv). For the converse it is enough to examine the versal deformation of singularities of lower codimension to see that neither (iii) nor (iv) can arise if $(x, (c, s))$ is not a rib point. Note that (iii) can also be deduced by studying the exponential map of the normal bundle.

(v): The condition for the map g_t: $\mathbf{R}^2 \to \mathbf{R}^3$ to have a swallowtail singularity is

$$dg_t u = 0; \qquad d^2 g_t u^2 + dg_t v = 0$$

for some nonzero vector u, and some vector v. Writing this out in terms of g and G, the first equation is the condition that $(x_0, (c, s))$ is a focal point with associated principal direction u, while the second is precisely the A_3 condition for the distance squared function.

(vi): This follows from Theorem 1(i)–(iv).

(vii): Let us call the curves on the surface through x_0 with tangent u

u-curves. From Meusnier's theorem (see Theorem 1(i)) the osculating circles to u-curves lie on the Meusnier sphere M_u, so their focal lines pass through the center (c, s) of M_u. Now the center of the osculating sphere (also called the center of spherical curvature) of a curve lies on the focal line, so the osculating sphere is tangent to the surface if and only if it coincides with M_u. We show that M_u has 4-point contact with (i.e., is the osculating sphere of) a u-curve if and only if $(x_0, (c, s))$ is a rib point as required. The condition for 4-point contact of a curve with a sphere is

$$V_1 u = V_2 u^2 + V_1 v = V_3 u^3 + 3V_2 uv + V_1 w = 0,$$

where V is the distance squared function from the center of the sphere, and (x_0, u, v, w) is the 3-jet of the curve. For V measured from (c, s), these all reduce to $V_3 u^3 = 0$ which is precisely the condition for $(x_0, (c, s))$ to be a rib point.

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