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NORMAL BUNDLES FOR AN EMBEDDED $\mathbf{R}P^2$ IN A POSITIVE DEFINITE 4-MANIFOLD

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1. Introduction

In this paper we wish to study which normal bundles can occur for differentiable embeddings of the real projective plane $\mathbf{R}P^2$ into a positive definite 4-manifold. The techniques we use were developed by Donaldson [4] to show that only the standard intersection forms can occur for simply connected positive definite 4-manifolds and modified by Fintushel and Stern [5], [6] to reprove part of Donaldson's theorem as well as give results concerning homology 3-spheres and other applications. The starting point for our work is [6]; in particular, the application of their techniques to reprove Kuga's theorem was the main motivation for our approach. Kuga's theorem [13] gives restrictions on which homology classes $S^2 \times S^2$ can be realized by an embedded 2-sphere.

The Fintushel-Stern proof of Kuga's theorem goes roughly as follows. By cutting out a neighborhood of an embedded S^2 realizing a given homology class, one gets a positive definite 4-manifold X with $H^2(X) = \mathbf{Z}$ and boundary a certain lens space. Moreover, there is an $SO(2)$-bundle E over X which restricts to a specified bundle over the lens space so that one can construct a pseudofree orbifold and apply their general theory. However, the crucial point of the construction is not the pseudofree orbifold but rather the bundles involved. Also, their construction utilizes a branched covering which is not essential—it may be replaced by just forming a V-manifold $X \cup c\partial X$, where $c\partial X$ is covered by D^4. The key data needed is the manifold X with ∂X a lens space which is covered by S^3, $H^2(X) \approx \mathbf{Z}$, $H_1(X) = 0$, $X \cup c\partial X$ a positive definite rational homology manifold. Given that data, one can form a V-manifold using X and D^4, where D^4 covers the cone on the lens space, and form an $SO(2)$-bundle E over X corresponding to an Euler class $e \in H^2(X)$.

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Moreover, $SO(2)$-bundles over lens spaces are easily constructed as quotients of the trivial bundle over S^3 via a cyclic action, and this construction may be extended to give a cyclic action on the trivial bundle over D^4 to get a V-bundle over the V-manifold $X \cup c\partial X$ on which we study its moduli space of self-dual V-connections (after stabilization to an $SO(3)$-bundle). This study will lead to a contradiction to the original hypothesis that the given homology class could be realized.

In the situation of an embedded $\mathbf{R}P^2$ in a positive definite 4-manifold M with $H_1M = 0$, one may again excise a neighborhood of the projective plane to get a 4-manifold X and (given a hypothesis on the embedding) $H_1X = 0$, H^2X is free of the same rank as M, and $X \cup c\partial X$ is a positive definite rational homology manifold. Moreover ∂X is (usually) again finitely covered by S^3 and bundles over ∂X can be expressed as quotients of the trivial bundle $S^3 \times \mathbf{R}^2$ by the action of a finite group of quaternions. This allows one to set up a V-bundle over the V-manifold $x \cup c\partial X$ and study the differential geometry of self-dual V-connections on it.

In this paper we will restrict our attention to positive definite 4-manifolds which have the same homology and intersection forms as the connected sum of k copies of $\mathbf{C}P^2$. The assumption on the homology (i.e. $H_1M = 0$) is largely for the convenience of the argument. The free part of H_1M could be surgered away and odd torsion in H_1M could also be dealt with in a manner similar to that used in [5]. As in [5], the modifications for dealing with odd torsion do not work with 2-torsion. As for the assumption that the intersection form is standard, there are at this time no known examples of nonstandard intersection forms. Donaldson's theorem [4] says that the intersection form must be standard if M is simply connected, and [5] rules out many nonstandard intersection forms when H_1M contains no 2-torsion. It is conjectured that in fact only the standard positive definite form can occur. Thus our restriction to this case may in fact be no restriction at all, and it simplifies (and makes much more concrete) the exposition.

Main. Theorem. *Let M be a positive definite 4-manifold with the same homology and intersection form as the connected sum of k copies of $\mathbf{C}P^2$. Write the second homology H_2M as the direct sum of k copies of $\mathbf{Z}$ with generators $y_1,\cdots,y_k$ and Poincaré duals $x_1,\cdots,x_k$ with $x_i^2[M] = 1$, $x_i \cdot x_j = 0$. Let $\bar{y}_1,\cdots,\bar{y}_k$ be the* mod 2 *reductions of these classes. Suppose $\alpha\colon \mathbf{R}P^2 \to M$ is a differentiable embedding so $\alpha\colon H_2(\mathbf{R}P^2;\mathbf{Z}_2) \to H_2(M;\mathbf{Z}_2)$ sends the generator (after possible reordering) to $\bar{y}_1 + \cdots + \bar{y}_l$. Let $e(\nu)$ denote the normal Euler number of the normal bundle of the embedding. Then*

(1) *if $l < 8$, then $e(\nu) \geqslant -2 + l$,*

(2) $e(\nu) \equiv -2 + l \bmod 4$,

(3) *if* $k = l$, $e(\nu) \equiv 2 + l \bmod 16$,

(4) *if* $l = 0$, $-2 \leqslant e(\nu) \leqslant 2 + 4k$,

(5) *for* $M = \#k\mathbf{C}P^2$, *the values between* $-2 + l$ *and* $2 + l + 4(k - l)$ *satisfying* (2) *are all realizable.*

Conditions (3) and (4) follow directly from work of Guillou & Marin [10] and Rochlin [16]. We conjecture that the restriction $l < 8$ can be removed from (1) and the values given in (5) are the most which could be realized for arbitrary M. The removal of $l < 8$ could be achieved if Conjecture 5.7 of [5] holds. To limit the realizable values to ones between $-2 + l$ and $2 + l + 4(k - l)$ requires a limitation on negative indices in our proof which we feel must hold but we have so far been unable to prove. Similar limitations also are plausible and consistent with known examples in [6].

Since the Fintushel-Stern work was not done in the context V-manifolds (although it could have been), we want to describe what adaptations would be necessary to recast it in that framework. In the paper [5] they successfully adapted to $SO(3)$-connections the work of Donaldson, Ulhenbeck, and others that was done for $SU(2)$-connections. Freed & Uhlenbeck [8] also discuss the $SO(3)$ case in their book. Then in [6] they adapt these results to an equivariant setting. The adaptations consist of two types. In some parts (e.g. §8) they take arguments of Donaldson (or, more accurately, Blaine Lawson's excellent exposition [14] of these arguments) and basically just show they work equivariantly when notational changes are made (e.g. replace $\mathcal{O}_{\nabla,\varepsilon}$ by $\mathcal{O}^{\alpha}_{\nabla,\varepsilon}$, etc.). In other parts (e.g. the compactness argument of Theorem 7.5) they recast arguments in an equivariant setting by choosing coverings which respect the group action. Both types of arguments are easily adapted to give analogous results in a V-manifold setting. Thus we will not give proofs for those reformulations but we just refer to [6]. These are some points in the argument, however, where there are significant differences in viewpoint from their presentation (e.g. the compactness argument and the index calculation) and so we will concentrate our efforts on explaining those differences.

The organization of the remainder of the paper is as follows. In §2 we will give an exposition of the adaptations of the Fintushel-Stern work to V-manifolds that we will need. In §3 we will show how the embedding of $\mathbf{R}P^2$ in the positive definite 4-manifold M leads to our V-manifold and then apply the results of §2. In §4 we will then give the index calculation, prove the compactness theorem and then prove our Main Theorem.

I wish to thank Ron Fintushel for patiently explaining his work to me as well as for encouraging me to pursue the adaptations of his work to the problem studied here.

2. Adaptations to V-manifolds

We consider here the adaptatins of the Fintushel-Stern results to the context of V-manifolds. We review briefly the definitions of a V-manifold and V-bundle as given in [9]. (See [9], [12], [11], [2], [17] for more details.) One first defines a local uniformizing system (l.u.s.) for an open set U of a connected metric space B. An l.u.s. for U is a collection $\{\tilde{U}, G, \phi\} \ni$:

(a) $\tilde{U}$ is a connected open set in $\mathbf{R}^n$.

(b) G is a finite group of C^∞ diffeomorphisms of $\tilde{U}$.

(c) ϕ is a continuous map from $\tilde{U}$ onto $U \ni$: $\phi \circ \sigma = \phi$ for any $\sigma \in G$ and ϕ induces a homeomorphism from $\tilde{U}/G$ onto U.

If U, U' are open sets in B and $\{\tilde{U}, G, \phi\}$, $\{\tilde{U}', G', \phi'\}$ are l.u.s., an injection λ: $\{\tilde{U}, G, \phi\} \to \{\tilde{U}', G', \phi'\}$ is a diffeomorphism from $\tilde{U}$ onto an open set in $\tilde{U}'$ such that $\phi = \phi' \circ \lambda$. A C^∞ V-manifold of dimension n is a connected metric space B with a family $\mathscr{A}$ of l.u.s. of dimension n corresponding to an open covering of $B \ni$:

(a) If $\{\tilde{U}, G, \phi\}$ and $\{\tilde{U}', G', \phi'\} \in \mathscr{A}$ and $\phi(\tilde{U}) = \phi'(U')$ there is an injection λ: $\{\tilde{U}, G, \phi\} \to \{\tilde{U}', G', \phi'\}$.

(b) Let $\mathscr{H}$ be a family of open sets U in B for which there exists a l.u.s. $\{\tilde{U}, G, \phi\} \in \mathscr{A}$. $\mathscr{H}$ satisfies the two following conditions:

(i) For any $p \in B$, there exists $U \in \mathscr{H}$ with $p \in U$.

(ii) For $p \in U_1 \cap U_2$, $U_1, U_2 \in \mathscr{H}$, there exists $U_3 \in \mathscr{H}$: $p \in U_3 = U_1 \cap U_2$. Such a family $\mathscr{A}$ is called a defining family.

Example. Although the definition of a V-manifold may seem fairly complicated, the V-manifolds implicit in [6] and used here will be particularly simple. Basically the space B will consist of a 4-manifold with boundary X, with $\partial X = \bigcup \partial X_i$ a disjoint union of a finite number of components. Each component will be a quotient of S^3 by a finite group G. For [6] these quotients are lens spaces and for us they are circle bundles over $\mathbf{R}P^2$. The V-manifold B is just $X \cup (\bigcup c\partial X_i)$. The local uniformizing system consists of a manifold cover for X union an open exterior collar on ∂X together with one copy of int D^4 with appropriate group action extending the action of G on S^3 radially for each boundary component.

We next define a V-vector bundle. Let B and E be two V-manifolds and π: $E \to B$ a C^∞ map. π: $E \to B$ is a C^∞ V-vector bundle with fiber $\mathbf{R}^m$ if there are defining families $\mathscr{A}$ and $\mathscr{A}^*$ of B and $E \ni$:

(i) There is a 1-1 correspondence $\{\tilde{U}, G, \phi\} \Leftrightarrow \{\tilde{U}^*, G^*, \phi^*\}$ between $\mathscr{A}$ and $\mathscr{A}^*$ such that $\tilde{U}^* = \tilde{U} \times R^m$ and $\pi \circ \phi^* = \phi \circ \pi_{\tilde{U}^*}$, where $\pi_{\tilde{U}^*}$ is the canonical projection $U \times \mathbf{R}^m \to \tilde{U}$.

(ii) To each injection λ: $\{\tilde{U}, G, \phi\} \to \{\tilde{U}', G', \phi'\}$ there corresponds an injection λ^*: $\{U^*, G^*, \phi^*\} \to \{\tilde{U}'^*, G'^*, \phi'^*\}$: for $(\tilde{p}, q) \in \tilde{U}^* = \tilde{U} \times \mathbf{R}^m$, $\lambda^*(\tilde{p}, q) = (\lambda(\tilde{p}), g_\lambda(\tilde{p})q)$, where $g_\lambda(\tilde{p}) \in \mathrm{Gl}(m, \mathbf{R})$.

Example. The V-bundle over the V-manifold described above used in [6] and here just consists of an ordinary vector bundle over the manifold with boundary X (union a small collar) together with the trivial bundle $D^4 \times \mathbf{R}^m$ over each D^4 with an extension of the action of G_i on D^4 so that it orthogonally permutes the fibers.

For the work here the following facts about V-manifolds and V-bundles will be used. (See [9], [11], [12], [2] for details.) First one can define rational characteristic classes of the V-bundle using invariant connections on l.u.s. of the bundle. Moreover, there are versions of de Rham's theorem and the Hodge decomposition theorem for V-manifolds. There is also a theory of elliptic (pseudo-) differential operators on V-manifolds and a version of the Atiyah-Singer Index Theorem. An orientable V-manifold Y possesses a fundamental class and is a rational homology manifold.

We now give V-manifold versions of results in [6]—all are proved by straightforward adaptations of the arguments given there. The standing assumption is that we have an orientable V-manifold Y with $SO(2)$ V-bundle E over it. This bundle has an Euler class $e \in H^2(Y; Q)$ which characterizes it—e is the analogue of the pseudofree Euler class in [6]. Y will have the structure of a V-manifold as in Example 1 and E will be a corresponding V-bundle, as in Example 2. E is then stabilized to an $SO(3)$-bundle E as in [6].

We will assume that E' and Y have Riemannian metrics (in the sense of V-manifolds) on them and there is a Riemannian connection defined on the V-bundle E'. We let $\Omega^k_V(F) = \Gamma_V(\Lambda^k T^* \otimes F)$ be the k-forms on B with values in F (as defined for V-manifolds and V-bundles). Then we can view the connection as a linear map $\Omega^0_V(E') \to \Omega^1_V(E')$. We form $\mathscr{C}_V$, the space of all Riemannian connections, and the gauge group $\mathscr{G}_V$—these are the analogues of Ω_α, $\mathscr{C}^\alpha$, $\mathscr{G}^\alpha$ in [6]. One defines self-duality and forms the moduli space $\mathscr{M}_V = \mathscr{A}_V/\mathscr{G}_V$. Again one works with Sobolev spaces—for example, $\mathscr{C}^V_k = \{\nabla_0 + A | A \in L^2_k(\Omega^1_V(\mathfrak{G}_{E'}))\}$, $\nabla_0 \in \mathscr{C}_V$ a fixed base connection, and $\mathscr{A}^V_k = \{\nabla_0 + A | A \in L^2_k(\Omega^1_V(\mathfrak{G}_{E'})),\ d^\nabla_-(A) + [A, A]_- = 0\}$, $\nabla_0 \in \mathscr{A}_V$ a fixed base connection. The gauge group of $\mathscr{C}^V_k$ is

$$\mathscr{G}^V_{k+1} = L^2_{k+1}\Big(\Omega^0_V\big(\mathrm{Aut}_{SO(3)}(E')\big)\Big).$$

One defines reducible and irreducible connections on E' as before and gets that π: $\mathscr{C}^{V*}_3 \to \mathscr{B}^{V*}_3$ is a principal bundle projection and $\mathscr{B}^{V*}_3$ is a smooth

Hausdorff Hilbert manifold with local charts

$$\mathcal{O}^V_{\nabla,\varepsilon} = \left\{\nabla + A | A \in L^2_3\left(\Omega^1(\mathfrak{G}_E)\right),\ \delta^\nabla A = 0,\ \|A\|_{L^2_3} < \varepsilon\right\}$$

for ε sufficiently small. If ∇ is reducible, then $\Gamma_\nabla \simeq SO(2)$ and $\pi'\colon \mathcal{O}^V_{\nabla,\varepsilon}/\Gamma_\nabla \to \mathscr{B}$ is a homeomorphism onto a neighborhood $\pi(\nabla)$.

One applies Hodge theory for V-manifolds as in the proof of 5.3 in [6] to get:

Proposition 1. *Suppose Y has a positive definite intersection form and $H^1(Y,\mathbf{R}) = 0$. Then each $SO(2)$ V-bundle E over Y has a unique gauge equivalence class of self-dual connections.*

Now suppose E is an $SO(2)$ V-bundle with Euler class e which stabilizes to an $SO(3)$ V-bundle E'. Let $\alpha(e)$ denote the number of reductions of E' to $SO(2)$ bundles (up to orientation). For [6], $\alpha(e)$ is expressed in terms of $\mu(e)$ and $|H_1(DX)|$; $\alpha(e)$ will be 1 in our application. The analogue of 5.4 in [6] is

Proposition 2. *Suppose Y has a positive definite intersection form, $H^1(Y,\mathbf{R}) = 0$, and E' is the stabilization of an $SO(2)$ bundle with Euler class e. Then there are exactly $\alpha(e)$ gauge equivalence classes of reducible self-dual connections on E'.*

As in [6], one can define the fundamental elliptic complex

$$0 \to \Omega^0_V(\mathfrak{G}_{E'}) \xrightarrow{\nabla} \Omega^1_V(\mathfrak{G}_{E'}) \xrightarrow{d^\nabla_-} \Omega^2_{V,-}(\mathfrak{G}_{E'}) \to 0.$$

This is the analogue of the $\mathbf{Z}_\alpha$-invariant complex in [6]. This is an elliptic complex with cohomology groups $H^{0,V}_\nabla$, $H^{1,V}_\nabla$, $H^{2,V}_\nabla$ which may be identified with spaces of harmonic forms. In §4 we will compute the index of this complex for our applications. We now denote this index as $I(e)$.

We get analogues of Propositions 6.2 and 6.3 in [6].

Proposition 3. *For a reducible $\nabla \in \mathscr{A}_V$, $\dim H^{0,V} = 1$ and $H^{1,V}_\nabla$ and $H^{2,V}_\nabla$ are even dimensional vector spaces with Γ^V_∇ acting via the standard action of $SO(2)$ (leaving only 0 fixed). Thus $I(e)$ is an odd integer.*

We defer the question of compactness of the moduli space until §4. Let us suppose we are in a situation where the hypotheses of Proposition 1 hold, $I(e) > 0$ and we know the moduli space is compact. Then the argument leading to Theorem 8.3 of [6] may be modified to give:

Proposition 4. *There is a compact perturbation $\psi = R_- + \sigma$ of the self-duality equations on $\mathscr{C}_V$ so the new moduli space $\mathscr{M}'_V = \{\nabla \in \mathscr{C}_V | \psi(\nabla) = 0\}$ is a compact smooth $I(e)$-dimensional manifold with $\alpha(e)$ singular points such that each has a neighborhood which is the cone on the complex projective space $\mathbf{CP}(\frac{1}{2}(I(e) - 1))$.*

3. Embeddings of $\mathbf{R}P^2$

In this section we will show how an embedding of $\mathbf{R}P^2$ into a positive definite 4-manifold may lead to a V-manifold and a V-bundle. For computational reasons we restrict our attention to the case where the 4-manifold M has the same homology and intersection form as the connected sum of k copies of $\mathbf{C}P^2$. However, many of our arguments are valid in the more general case of M being a positive definite 4-manifold.

For $\#k\mathbf{C}P^2$ there are standard generators of H_2 and H^2 which are both Poincaré dual and Hom dual. Since M has the same homology and intersection form as $\#k\mathbf{C}P^2$, we can find similar generators for M. Write the second homology group of M as the direct sum of k copies of $\mathbf{Z}$ with generators $y_1,\cdots,y_k$ with Poincaré duals $x_1,\cdots,x_k$ so $x_i^2[M]=1$, $x_ix_j=0$, $i\neq j$. Let $\bar{y}_1,\cdots,\bar{y}_k$ be the mod 2 reductions of $y_1,\cdots,y_k$.

Suppose α: $\mathbf{R}P^2\to M$ is a differentiable embedding; consider α_*: $H_2(\mathbf{R}P^2;\mathbf{Z}_2)\to H_2(M;\mathbf{Z}_2)$. Either this is the zero map or its image is the sum of l of the generators $\bar{y}_1,\cdots,\bar{y}_k$. By reordering these generators, we can assume it is of the form $\alpha_*[\mathbf{R}P^2]=\bar{y}_1+\cdots+\bar{y}_l$, where $0\leqslant l\leqslant k$ and $l=0$ corresponds to α_* being the zero map. We first treat separately the case $l=0$. Here we may apply the results of Rochlin [16]. His result 7.2 can be restated in our situation as $|e(\nu)/2-k|\leqslant k+1$ and $e(\nu)$ is even. Thus we get the bounds $-2\leqslant e(\nu)\leqslant 2+4k$. From the proof one can deduce that possible values for $e(\nu)$ must differ by multiples of 4. Thus we have part (4) of the Main Theorem.

For the remainder of the paper we will assume $l>0$, i.e., α_* is injective. We will also assume l is odd—we are able to reduce to this case by taking $M\#\,\mathbf{C}P^2$ and modifying α by taking the connected sum of $\mathbf{C}P^1=\mathbf{C}P^2$.

Since α_*: $H_2(\mathbf{R}P^2;\mathbf{Z}_2)\to H_2(M;\mathbf{Z}_2)$ maps the generator to $\bar{y}_1+\cdots+\bar{y}_l$, the diagram

$$\begin{array}{ccc} H^2(M) & \longrightarrow & H^2(\mathbf{R}P^2)\approx\mathbf{Z}_2 \\ \downarrow & & \downarrow\approx \\ H^2(M;\mathbf{Z}_2) & \longrightarrow & H^2(\mathbf{R}P^2;\mathbf{Z}_2)\approx\mathbf{Z}_2 \end{array}$$

shows that the map $H^2(M)\to H^2(\mathbf{R}P^2)$ is surjective, with $x_1,\cdots,x_l$ mapping to the generator and $x_{l+1},\cdots,x_k$ mapping to 0. Since l is odd, $x_1+\cdots+x_l$ will also map to the generator. The long exact sequence of $(M,\alpha(\mathbf{R}P^2))$ implies $H^3(M,\alpha(\mathbf{R}P^2))=0$ and

$$\begin{aligned} H^2(MS,\alpha(\mathbf{R}P^2)) &\approx \ker\left(H^2(M)\overset{\alpha^*}{\to}H^2(\mathbf{R}P^2)\right) \\ &\approx \left\{(a_1,\cdots,a_l)\in l\mathbf{Z}\colon a_1+\cdots+a_l\equiv 0 \bmod 2\right\}\oplus(k-l)\mathbf{Z}\approx k\mathbf{Z}. \end{aligned}$$

Let N be a tubular neighborhood of $\alpha(\mathbf{R}P^2)$, $M = X \cup N$. Then $H^*(X, \partial X) \approx H^*(M, \alpha(\mathbf{R}P^2))$, so $H^3(X, \partial X) \approx 0$, $H^2(X, \partial X) \approx k\mathbf{Z}$.

In homology we have the exact sequence

$$0 = H_2(\mathbf{R}P^2) \to H_2(M) \to H_2\big(M, \alpha(\mathbf{R}P^2)\big) \to H_1(\mathbf{R}P^2) \approx \mathbf{Z}_2 \to 0.$$

This implies $H_2(M, \alpha(\mathbf{R}P^2)) \approx k\mathbf{Z}$ or $k\mathbf{Z} \oplus \mathbf{Z}_2$. With $\mathbf{Z}_2$ coefficients we get

$$0 \to \mathbf{Z}_2 \to k\mathbf{Z}_2 \to H_2\big(M, \alpha(\mathbf{R}P^2); \mathbf{Z}_2\big) \to \mathbf{Z}_2 \to 0$$

so $H_2(M, \alpha(\mathbf{R}P^2); \mathbf{Z}_2) \approx k\mathbf{Z}_2$; thus we must have

$$H^2(X) \approx H_2(X, \partial X) \approx H_2\big(M, \alpha(\mathbf{R}P^2)\big) \approx k\mathbf{Z}.$$

We are concerned with what the normal bundle of our embedded $\mathbf{R}P^2$ could be. That normal bundle is classified by its normal Euler number $e(\nu)$. If the normal Euler number is $p \neq 0$, one can give the following explicit description of ∂N as a quotient of S^3. This arises from the fact that the double cover of the normal disk bundle is a disk bundle over S^2, and nontrivial circle bundles over S^2 are quotients of the Hopf bundle. Thus if $e(\nu) = p$, then ∂N is the quotient of S^3 (viewed as the unit quaternions) by the subgroup Q_{4p} generated by $\omega = \cos(\pi/|p|) + i\sin(\pi|p|)$ and j. Note $\omega^p = j^2 = -1$ and $j\omega j^{-1} = \omega^{-1}$. This is a binary dihedral group. For p odd, its abelianization is $\mathbf{Z}_4$, generated by j with j^2 and ω representing the same element in the abelianization. For $p > 0$, the orientation is induced from $S^4 = \partial D^4$. For $p < 0$, the opposition orientation is used.

We next want to see that the hypothesis that l is odd will imply $e(\nu)$ is odd, so $H^2(\partial N) \approx \mathbf{Z}_4$. We consider the following diagram:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
 & & 0 = H^2(M,X) \arrow[r] \arrow[d] & H^2(M,X;\mathbf{Z}_2) \arrow[r, "\approx"] \arrow[d] & H_2(N;\mathbf{Z}_2) \approx \mathbf{Z}_2 \arrow[d] \\
 & & H^2(M) \arrow[r] \arrow[d, "j^*"] & H^2(M,\mathbf{Z}_2) \arrow[r, "\approx"] \arrow[d] & H_2(M;\mathbf{Z}_2) \approx k\mathbf{Z} \arrow[d] \\
0 \arrow[r] & H^2(X) \arrow[r, "\times 2"] & H^2(X) \arrow[r, "r_2"] \arrow[d, "\delta"] & H^2(X,\mathbf{Z}_2) \arrow[r, "\approx"] \arrow[d] & H_2(M,N;\mathbf{Z}_2) \arrow[d] \\
 & & H^3(M,X) \arrow[r, "\approx"] & H^3(M,X;\mathbf{Z}_2) \arrow[r, "\approx"] & H_1(N,\mathbf{Z}_2) \approx \mathbf{Z}_2 \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

The isomorphisms of the two right columns arises by Lefschetz duality. The first three terms of the third row are part of the coefficient sequence for $\mathbf{Z}_2$ coefficients. Note $H^2(X) \to H^2(X, \mathbf{Z}_2)$ is onto since $H^3(X) \approx H_1(X, \partial X) \approx H_1(M, N) \approx 0$. Similarly $H_1(X) \approx H^3(X, \partial X) \approx H^3(M, N) \approx 0$. Now consider $x_1 + \cdots + x_l \in H^2(M)$. When we map it to $H^2(M; \mathbf{Z}_2)$; and follow by the map to $H^2(X; \mathbf{Z}_2)$, it goes to zero since its Poincaré dual is $\bar{y}_1 + \cdots + \bar{y}_l$ which is in the image of the map $H_2(N; \mathbf{Z}_2) \to H_2(M; \mathbf{Z}_2)$. This means that $r_2 j^*(x_1 + \cdots + x_l) = 0$. Thus $j^*(x_1 + \cdots + x_l) = 2\bar{e}$ for a unique class $\bar{e} \in H^2(X)$. Note $r_2\bar{e} \neq 0$; also $\bar{e}$ is not the image of j^* since that would imply $x_1 + \cdots + x_l = 2x$, $x \in H^2(M)$, a contradiction. Thus $\delta\bar{e}$ is the generator of $H^3(M, X)$ and $r_2\bar{e}$ is a generator of $H^2(X, \mathbf{Z}_2)$.

Next notice that $H^2(N) \to H^2(\partial N)$ is injective since it is dual to $H_2(N, \partial N) \to H_1(\partial N)$ and $H_2(N) = 0$. Now consider the diagram:

$$\begin{array}{ccc} H^2(M) & \longrightarrow & H^2(N) \\ \downarrow & & \downarrow \\ \bar{e} \in H^2(X) & \xrightarrow{r} & H^2(\partial X) \end{array}$$

Since l is odd, $x_1 + \cdots + x_l$ maps to the generator of $H^2(N)$, which is then injected into $H^2(\partial X)$. Thus the element $rj^*(x_1 + \cdots + x_l)$ is nonzero in $H^2(\partial X)$, i.e., $2r(\bar{e}) \neq 0$. But this implies $H^2(\partial X) \approx \mathbf{Z}_4$ and $r(\bar{e})$ is a generator.

Note that ∂X is a rational homology sphere and $X \cup c\partial X$ is a rational homology manifold with $H^2(X \cup c\partial X; Q) \to H^2(X; Q)$ an isomorphism. Now consider the diagram:

$$\begin{array}{ccccc} H^2(X \cup c\partial X; Q) & \times & H^2(X \cup c\partial X; Q) & \longrightarrow & H^4(X \cup c\partial X; Q) \\ \uparrow & & \uparrow & & \uparrow \\ H^2(X \cup c\partial X, c\partial X; Q) & \times & H^2(X \cup c\partial X; Q) & \longrightarrow & H^4(X \cup c\partial X, c\partial X; Q) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(X, \partial X; Q) & \times & H^2(X; Q) & \longrightarrow & H^4(X, \partial X; Q) \\ \uparrow & & \uparrow & & \uparrow \\ H^2(M, N; Q) & \times & H^2(M; Q) & \longrightarrow & H^4(M, N; Q) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(M; Q) & \times & H^2(M; Q) & \longrightarrow & H^4(M; Q) \end{array}$$

All vertical maps are isomorphisms via excisions or long exact sequences. The horizontal maps are cup products. Thus rationally, $X \cup c\partial X$ has the same cup product structure as M. Since $j^*(x_1 + \cdots + x_l) = 2\bar{e}$, this implies that if we let $e \in H^2(X \cup c\partial X; Q)$ denote the element which corresponds to $\bar{e}$ under the maps

$$\begin{array}{ccc} e \in H^2(X \cup c\partial X; Q) & \xrightarrow{\approx} & H^2(X; Q) \\ & & \uparrow \\ & & \bar{e} \in H^2(X; \mathbf{Z}) \end{array}$$

we get $e^2[X \cup c\partial X] = l/4$.

We now specify the V-manifold structure which we wish to put on $X \cup c\partial X$. We can regard this as being formed from X union an open collar on ∂X (which is the complement in M of a disk neighborhood of $\alpha(\mathbf{R}P^2)$ in M), together with the quotient of int D^4 by the extension of the orthogonal action of Q_{4p} on S^3 by right multiplication. Now the V-manifold structure on $X \cup c\partial X$ comes from the manifold structure on M and the structure on $c\partial X$ from $(\operatorname{int} E^4, Q_{4p})$. Note that we have made a fixed choice of a diffeomorphism f: $S^3/Q_{4p} \to \partial N$ or, more precisely, a fixed covering $\tilde{f}$: $S^3 \to \partial N$ which induces f. Here $\tilde{f}$ is chosen to preserve orientation. The two pieces are identified using an extension of f from $S^3 \times (\frac{1}{2}, 1)$ to a collar on ∂N. We denote this V-manifold by Y.

We next wish to construct an $SO(2)$ V-bundle over Y with fiber $\mathbf{R}^2$. This will come from a regular bundle $\bar{E}$ over $X \cup$ collar, a trivial bundle over D^4 on which Q_{4p} acts linearly and a covering map of $SO(2)$-bundles f: $(D^4 \setminus \frac{1}{2}D^4) \times \mathbf{R}^2 \to \bar{E}|$collar extending $\tilde{f}$. The bundle $\bar{E}$ will be induced from the universal bundle using a representative of (the Euler class) $\bar{e} \in H^2(X)$ discussed above. To get F, we use the fact that e maps to a generator of $H^2(\partial X) \approx \mathbf{Z}_4$. Denote by i the abelianization map i: $Q_{4p} \to \mathbf{Z}_4 = \{1, a, a^2, a^3\}$. Then all homomorphisms of Q_{4p} to $\mathbf{Z}_4$ are of the form i^k, $k = 0, \pm 1, 2$. The four isomorphism classes of $SO(2)$ vector bundles over S^3/Q_{4p} are just $S^3 \times D^2/Q_{4p} \to S^3/Q_{4p}$, where Q_{4p} acts on S^3 by right multiplication as before and acts on $\mathbf{R}^2$ via $\mathbf{Z}_4$ acting as rotations, using the homomorphisms i^k: $Q_{4p} \to \mathbf{Z}_4$. Since e maps to a generator $H^2(\partial X)$, $\bar{E}|\partial X$ must be isomorphic to $S^3 \times \mathbf{R}^2/Q_{4p} \to S^3/Q_{4p}$, where $k = \pm 1$. Now our V-bundle E will consist of $\bar{E}|X \cup$ collar together with $(D^4 \times \mathbf{R}^2, Q_{4p})$ as above, where we use the covering F: $(D^4 \setminus \frac{1}{2}D^4) \times \mathbf{R}^2 \to \bar{E}|$collar to piece these together. We next wish to see that the V-isomorphism class of this V-bundle does not depend on our choice of F. We are assuming F extends the fixed $\tilde{f}$, so we suppress $\tilde{f}$ and identify $\bar{E}$ as a bundle over a collar on S^3/Q_{4p}. We restrict also to $F|S^3 \times \mathbf{R}^2$ as the behavior here will determine it on the collar up to equivalence. Suppose G: $S^3 \times \mathbf{R}^2 \to \bar{E}|S^3/Q_{4p}$ is another covering which extends the covering $S^3 \to S^3/Q_{4p}$. Then we have a commutative diagram

$$\begin{tikzcd} S^3 \times \mathbf{R}^2 \arrow[rr, "H"] \arrow[dr, "F"'] & & S^3 \times \mathbf{R}^2 \arrow[dl, "G"] \\ & E|S^3/Q_{4p} & \end{tikzcd}$$

where H can be chosen to be the identity on $S^3 \times 0$ and is an equivariant bundle map. Equivariant bundle maps are classified by $H^1(S^3/Q_{4p}) \approx 0$, so H extends to an equivariant bundle map $D^4 \times \mathbf{R}^2 \to D^4 \times \mathbf{R}^2$. Thus the V-bundle E is independent (up to V-bundle isomorphism) of the choice of F. Since $H^2(Y; Q) \to H^2(X; Q)$ is an isomorphism, the Euler class of E is determined by its restriction to X, so is the class $e \in H^2(Y; Q)$ discussed earlier. Thus $e^2[Y] = l/4$.

We now add on a trivial bundle to form E' from E. Our convention will be that on $D^4 \times \mathbf{R}^3$, Q_{4p} acts trivially on the first coordinate and acts as before on the last two coordinates. Note that $p_1(E')[Y] = l/4$ and $w_2(E'|X)$ is the mod 2 reduction of $\bar{e}$, in particular $w_2(E'|\partial X) \neq 0$. Note also that the actions on $\mathbf{R}^3$ of Q_{4p} corresponding to $k = \pm 1$ are equivalent as $SO(3)$ actions via $(t, z) \to (-t, \bar{z})$, $(t, z) \in \mathbf{R} \times \mathbf{C} = \mathbf{R}^3$. Thus we may assume $k = 1$.

We next wish to see that there is a unique reduction of E' to an $SO(2)$ bundle. For this we use the fact that if we had another reduction with Euler class e' coming from an integral class $\bar{e}' \in H^2(X)$, we would have

$$\begin{aligned} \bar{e}' &= \bar{e} + a_1 j^*(x_1) + \cdots + a_k j^*(x_k) \\ &= (2a_1 + 1)\bar{e} + (a_2 - a_1) j^*(x_2) + \cdots + (a_l - a_1) j^*(x_l) \\ &\quad + a_{l+1} j^*(x_{l+1}) + \cdots + a_k j^*(x_k). \end{aligned}$$

But $\bar{e}$, $j^*(x_2), \cdots, j^*(x_l)$ reduce mod 2 to give generators of $H^2(X; \mathbf{Z}_2)$ and so the condition that $w_2|X$ must be the same implies $a_1 \equiv a_2 \equiv \cdots \equiv a_l$ mod 2 and $a_{l+1} \equiv \cdots \equiv a_k \equiv 0$ mod 2. For e', the corresponding Pontrjagin class is $l/4 + (a_1^2 + a_1) + \cdots + (a_l^2 + a_l) + a_{l+1}^2 + \cdots + a_x^2$. For this to equal $l/4$, we need $a_{l+1} = \cdots = a_k = 0$ and $a_1, \cdots, a_l = 0, -1$. But $a_1 \equiv a_2 \equiv \cdots \equiv a_l$ mod 2 implies all values are 0 or all values are -1. If they are all 0, $e' = e$ and if they are all -1, $e' = -e$. Thus up to orientation, there is a unique reduction.

We now show that the conditions $p_1(E')[Y]$ and $w_2|X$ characterize E' up to V-isomorphism among V-bundles over Y. First consider $E'|X$. Since X has the homotopy type of a 3-complex, $E'|X$ is determined up to isomorphism by $w_2|X$ (cf. [3], [8]). Next consider the equivariant Q_{4p} bundle over D^4. As a vector bundle, ignoring the Q_{4p} action, this bundle is equivalent to $D^4 \times \mathbf{R}^3$.

The equivariant equivalence class is determined by the action on $0 \times \mathbf{R}^3$, so the bundle over D^4 is determined by a representation $Q_{4p} \to SO(3)$. Up to equivalence, this representation is determined by knowing the finite subgroup of $SO(3)$ which is the image. By [21], the only possibilities are a subgroup of $\mathbf{Z}_4$ or a subgroup of the dihedral group D_{2p} of order $2p$. In the latter case, this representation would lift to $S^3 = \mathrm{Spin}(3)$ and so the bundle which would be determined over ∂X would have $w_2 = 0$. Thus $Q_{4p} \to SO(3)$ must factor through $\mathbf{Z}_4$ and $w_2(E'|\partial X) \neq 0$ implies that the equivariant bundle over D^4 is equivalent to $E'|D^4$. Thus if E'' is another $SO(3)$ V-bundle over Y with $w_1(E'|X) = w_1(E''|X)$, then $E'|X \approx E''|X$ and the equivariant bundles over D^4 determined by E' and E'' are equivalent.

We next must worry about how these pieces are glued together:

$$E' = E'|X \cup_F D^4 \times \mathbf{R}^3/Q_{4p}, \qquad E'' = E'|X \cup_G D^4 \times \mathbf{R}^3/Q_{4p},$$

where F, G are covering bundle maps $S^3 \times D^3 \to S^3/Q_{4p}$ which extend the covering $S^3 \to S^3/Q_{4p}$. Here we are identifying ∂X with S^3/Q_{4p} and abusing notation somewhat in glueing along S^3/Q_{4p} rather than on a collar. There is a commutative diagram

$$\begin{tikzcd} S^3 \times \mathbf{R}^3 \arrow[rr, "H"] \arrow[dr, "F"'] & & S^3 \times \mathbf{R}^3 \arrow[dl, "G"] \\ & S^3/Q_{4p} & \end{tikzcd}$$

where H is an equivariant bundle map covering the identity on S^3. Equivariance is measured with respect to the same action on both sides, i.e. using i: $Q_{4p} \to \mathbf{Z}_4$. H is then determined by a map h: $S^3 \to SO(3)$ which satisfies $h(x \cdot q) = I(q)^{-1}h(x)I(q)$, where $x \in S^3$, $q \in Q_{4p}$ and $I(q)$ is determined by I: $Q_{4p} \to \mathbf{Z}_4 = \{\pm 1, \pm i\} = S^1 = \mathbf{C}$ and $i \in \mathbf{Z}_4$ acts via $(t, z)i = (t, zi)$ (i.e. rotation by $\pi/2$ in the last two coordinates). The standard covering $S^3 \to SO(3)$ is given by $x \to f(x)$, where $f(x) \cdot y = xyx^{-1}$, $y \in \mathbf{R}^3 \subset \mathbf{R}^4 = H$, with $\mathbf{R}^3$ the subspace with basis i, j, k. Then h: $S^3 \to SO(3)$ lifts to $\tilde{h}$: $S^3 \to S^3$. Now $4|p|(p_1(E') - p_1(E''))$ is the obstruction to extending H to an equivariant isomorphism $D^4 \times \mathbf{R}^3 \to D^4 \times \mathbf{R}^3$ and this obstruction is also given by -4 times the degree of $\tilde{h}$ (cf. [8]). This is just the Pontrjagin class of the equivariant bundle $(D^4 \times \mathbf{R}^3) \cup_H (D^4 \times \mathbf{R}^3)$ over S^4. The assumption $p_1(E') = p_1(E'')$ shows that there is such an extension so $E' \approx E''$ as V-bundles. Moreover, the argument shows $w_2(E'|X) = w_2(E''|X)$ implies $p_1(E') - p_1(E'') = (-1/|p|)\deg \tilde{h}$.

We next see what degrees can occur for $\deg \tilde{h}$, using the fact that h is equivariant. First note that since $I(\omega^2) = 1$, the Q_{4p}-equivariant map h factors as $S^3 \xrightarrow{p} X = S^3/\langle\omega^2\rangle \xrightarrow{h_1} SO(3)$ and the latter map can be regarded as a $\mathbf{Z}_4$-equivariant map. Thus $\tilde{h} = \tilde{h}_1 p$, where $\tilde{h}_1$ is a lift of h_1. Now $\tilde{h}_1$ satisfies $h_1(x \cdot [j]) = \pm\tilde{I}(j)^{-1}h_1(x)I(j)$, where $I(j) = 1/\sqrt{2} + 1/\sqrt{2i}$. Thus if we look at the action of $[j^2] = [-1]$ we get that h_1 is a $\mathbf{Z}_2$-equivariant map. Any two such equivariant maps have congruent degrees modulo 2 by [7]. Since the constant map sending X to 1 is equivariant, we get that the degree of $\tilde{h}_1$ is a multiple of 2. Thus the degree of $\tilde{h}$ is a multiple of $2p$.

We now summarize the results of this section in Theorem 5.

Theorem 5. *Let M be a positive definite 4-manifold as above. There is a V-manifold structure on $Y = X \cup c(\partial X)$, which is a positive definite rational homology manifold with the same rational cohomology structure as M. There is a class $e \in H^2(Y; Q)$ so that it is the Euler class of an $SO(2)$ V-bundle E over Y, which stabilizes to an $SO(3)$ V-bundle E' over Y. E' admits a unique reduction (up to orientation) to an $SO(2)$ V-bundle. E' is characterized by $w_2|X$ and $p_1(E')[Y] = l/4$. If E' is another $SO(3)$ V-bundle over Y with the same w_2, then $p_1(E'') = l/4 + 2n$, $n \in \mathbf{Z}$.*

4. The index calculation, compactness of the moduli space, and the proof of the Main Theorem

We now apply the discussion of §2 to the V-manifold and V-bundle constructed in §3. We first consider the fundamental elliptic complex corresponding to a reducible V-connection constructed as in Propositions 1 and 2; note that its gauge equivalence class is unique by Proposition 2. Proposition 3 guarantees that this index is an odd integer. We will apply the version of the Atiyah-Singer Index Theorem for V-manifolds given in [12] to calculate this index and see how it depends on the normal Euler class $e(\nu) = p$.

To compute the index, we follow the method given in [1] and [6] and reduce to the equivalent problem of computing the index of the twisted Dirac operator

$$D\colon \Gamma(V_+ \otimes V_- \otimes \mathfrak{G}_{E'}) \to \Gamma(V_- \otimes V_- \otimes \mathfrak{G}_{E'}).$$

This computation is done using the formula

$$\operatorname{ind}_V(u) = (-1)^{\dim Y}\langle \operatorname{ch}(u)\mathscr{I}(Y), [\tau_V Y]\rangle + \sum \frac{(-1)^{\dim \Sigma_i}}{m_i}\langle \operatorname{ch}^{\Sigma}(u)\mathscr{I}^{\Sigma}(Y), [\tau_V \Sigma_i]\rangle$$

given in [12, p. 139] with $u = \sigma(D)$. (See [12] for an explanation of the notation.) In our situation the first term becomes

$$\operatorname{ch}(\mathfrak{G}_{E'} \otimes \mathbf{C})\operatorname{ch}(V_-)\hat{A}(Y)[Y] = 2e^2[Y] + \tfrac{3}{2}(\tfrac{1}{3}p_1(\tau) - e(\tau))[Y],$$

where the evaluation is interpreted in the V-manifold sense. As in [6] this leads to the formula

$$2e^2[Y] + \frac{3}{2}\left[(\sigma(Y) - \chi(Y)) - \frac{1}{4|p|}\left(\sum_{\substack{g\in Q_{4p}\\ g\neq 1}} \sigma_g(Y) - \chi_g(Y)\right)\right],$$

where

$$\sigma_g(Y) = -\cot\frac{r(g)}{2}\cot\frac{s(g)}{2}, \qquad \chi_g(Y) = 1;$$

here $r(g)$ and $s(g)$ are the rotation numbers corresponding to the action of $g \in Q_{4p}$ on D^4 at 0 (cf. [18], [6]). They are determined as follows: $r(\omega) = \pi/p$, $r(j) = \pi/4$; if $p < 0$, $s(g) = r(g)$ and if $p > 0$, $s(g) = -r(g)$. The signature defect and Euler characteristic defect terms occur since we are performing integrations in the cover D^4 over cQ_{4p} when evaluating on $[Y]$. Using $2e^2[Y] = l/2$ and $\sigma(Y) - \chi(Y) = -2$, the first term is

$$\frac{l}{2} - 3 + \frac{3}{8|p|}((4|p| - 1)) + \sum_{g\neq 1} \cot\left(\frac{r(g)}{2}\right)\cot\left(\frac{s(g)}{2}\right).$$

For the second term, one can simplify the formula somewhat since Q_{4p} acts trivially on $0 \in D^4$. It then becomes

$$\frac{1}{4|p|}\,\frac{\Sigma\,\mathrm{ch}_g(V_+ - V_-)\mathrm{ch}_g(V_-)(N^g \otimes \mathbf{C})\mathrm{ch}_g(\mathfrak{G}_E)[\text{point}]}{\mathrm{ch}_g(\Lambda_{-1})}$$

as in [6, p. 24]. Using analogous calculations to those on pp. 24, 25 of [6], this simplifies to

$$\frac{-1}{8|p|}\sum_{g\neq 1}\left(1 + \cot\left(\frac{r(g)}{2}\right)\cot\left(\frac{s(g)}{2}\right)\right)\left(3 - 4\sin^2\left(\frac{t(g)}{2}\right)\right),$$

where $t(g)$ gives the rotation number on the fiber $0 \times \mathbf{R}^3$. $t(g)$ is determined by $t(j) = \pi/2$, $t(\omega) = \pi$. When we add these two contributions, there is cancellation and the index becomes

$$\frac{l}{2} - 3 + \frac{1}{2|p|}\left(\sum_{g\neq 1}\cot\left(\frac{r(g)}{2}\right)\cot\left(\frac{s(g)}{2}\right)\sin^2\left(\frac{t(g)}{2}\right) + \sum_{g\neq 1}\sin^2\left(\frac{t(g)}{2}\right)\right).$$

The elements of Q_{4p} can be divided into the cyclic group of order $2p$ generated by ω and the product of those elements with j. For the first elements, the terms in $\Sigma_{g\neq 1}\sin^2(t(g)/2)$ add up to $|p|$. For the second ones, these terms add to $2|p|(\frac{1}{2}) = |p|$. Thus $\Sigma_{g\neq 1}\sin^2(t(g)/2) = 2|p|$, and so the index is

$$\frac{l}{2} - 2 + \frac{1}{2|p|}\left(\sum_{g\neq 1} \cot\left(\frac{r(g)}{2}\right)\cot\left(\frac{s(g)}{2}\right)\sin^2\left(\frac{t(g)}{2}\right)\right).$$

We again look at the two types of terms in the last sum. First consider the case where $p < 0$. Then $r(g) = s(g)$. For group elements in the subgroup generated by ω,

$$\sin^2\left(\frac{t(g)}{2}\right) = \begin{cases} 0 & \text{if } g = \omega^r,\ r \text{ even},\\ 1 & g = \omega^r,\ r \text{ odd}.\end{cases}$$

Thus

$$\sum_{\substack{g\in\langle\omega\rangle\\ g\neq 1}} \cot\left(\frac{r(g)}{2}\right)\cot\left(\frac{s(g)}{2}\right)\sin^2\left(\frac{t(g)}{2}\right) = \sum_{k=1}^{|p|} \cot\left(\frac{\pi(2k-1)}{2|p|}\right)$$
$$= |p|(|p|-1).$$

For $q \notin \langle\omega^2\rangle$, each term gives $\frac{1}{2}$, so these terms sum to $|p|$. Thus the final sum is

$$\frac{l}{2} - 2 + \frac{1}{2|p|}(|p|(|p|-1) + |p|) = \frac{l}{2} - 2 + \frac{|p|}{2}.$$

For $p > 0$, each sign is changed in the sum since $s(g) = -r(g)$, giving $l/2 - 2 - |p|/2$. Thus the formula for all p is $(l-p)/2 - 2$, i.e., $I = (l-p)/2 - 2$. Note that since I is an odd integer, we must have $e(\nu) = p \equiv -2 + l \bmod 4$, proving (2) of the Main Theorem.

For $p < -2 + l$, the index is positive. We wish to study the moduli space of all self-dual connections on E' in this case. If we knew that it was compact, then Proposition 4 would show that it could be perturbed into a compact smooth manifold of dimension I with one singularity which is a cone on a complex projective space of dimension $(I-1)/2$. This then leads to a contradiction as in [6].

To see when the moduli space is compact we apply the Bubble Theorem and Removability of Singularities Theorem of Uhlenbeck [19], [20] as in [6, pp. 27–30]. The adaptations for V-manifolds are done by the same equivariant covering arguments as in [6, p. 30]. The main point of the argument is that a sequence $\{\nabla_i\}$ of self-dual connections on E' has a subsequence $\{\nabla_i'\}$ and gauge equivalent connections $\{\tilde{\nabla}_i\}$ which converge to a self-dual connection ∇_∞ on $E'|Y_0$, $Y_0 = Y\setminus\{x_1,\cdots,x_k\}$. The Removability of Singularities Theorem then allows $\{\nabla_\infty\}$ to be extended to a self-dual connection on a possibly different bundle E''. Since $E'|Y_0 \approx E''|Y_0$, we must have the same w_2. Moreover, self-duality and integral estimates imply $0 \leqslant p_1(E'') \leqslant p_1(E') = l/4$. But Theorem 5 implies that $p_1(E'') = l/4 + 2n$, $n \in \mathbf{Z}$. Thus if $l < 8$, we must have $E' \approx E''$, giving compactness of the moduli space. Thus for $l < 8$ the moduli space will be compact, yielding our contradiction when $p < -2 + l$. We have thus proved parts (1), (2), and (4) of the Main Theorem. For part (3), the imbedded $\mathbf{R}P^2$ will be characteristic. One then can apply the nonorientable Rochlin theorem of Guillou & Marin [10] to get (3). When M is $\#k\mathbf{C}P^2$, we need to give embeddings of $\mathbf{R}P^2$ into $\#k\mathbf{C}P^2$ with normal Euler classes realizing all values between $-2 + l$ and $-2 + l + 4(k - l)$. We start with the two standard embeddings of $\mathbf{R}P^2$ in S^4 with normal Euler numbers -2 and 2. For the first l copies of $\mathbf{C}P^2$, we take $S^2 = \mathbf{C}P^1 = \mathbf{C}P^2$. For the last $k - l$ copies, we use the decomposition of $\mathbf{C}P^2$ as $N_{-1}(\mathbf{R}P^2) \cup T_4(S^2)$ described in [15]. The S^2 is embedded with normal Euler number 4. Taking the connected sum of the $\mathbf{R}P^2$ and the $\mathbf{C}P^1$'s in the connected sum of S^4 with the l copies of $\mathbf{C}P^2$ yields the base embedding. To get the various normal Euler numbers claimed in (5), we just take the connected sum of m copies of the S^2's with normal Euler number 4 in the last $(k - l)$ copies of $\mathbf{C}P^2$, where $0 \leqslant m \leqslant (k - l)$.

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