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PLANES WITHOUT CONJUGATE POINTS

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1. Problems involving perturbations of canonical Riemannian metrics arise frequently. If one requires the resulting geometry to satisfy various other conditions, the possible perturbations often become quite limited. An interesting case of this is the following

Theorem. *Let g be a smooth Riemannian metric on $\mathbf{R}^2$ which differs from the canonical flat metric g_0 at most on a compact set. If $(\mathbf{R}^2, g)$ has no conjugate points, then it is isometric to $(\mathbf{R}^2, g_0)$.*

In physical terms, one may think of the Riemannian metric g as a very general type of lens, made of optically anisotropic material. A pair of conjugate points occurs when an appropriately positioned point source of light emits rays which converge at the second point. The theorem states that this refocusing of light must occur for any nontrivial lens.

R. Michel has indicated a proof of a similar result with the additional hypothesis that every geodesic of $(\mathbf{R}^2, g)$ coincides, outside a compact set, with a straight line [4]. In our case, although it is clear that positive and negative subrays of a geodesic are parts of straight lines, we do not assume that these lines are the same, or even that they are parallel. Nonetheless, our proof resembles Michel's in the strategy of showing that E. Hopf's flat torus theorem can be applied (compare Michel [5, §3.3]). Hopf showed that any Riemannian metric without conjugate points on the two-dimensional torus has vanishing Gauss curvature [3]. In a private communication to one of the authors, Michel has suggested an approach along the lines of §6 of [4] which uses Hopf's calculations but avoids appealing to the full torus theorem.

The hypotheses of the theorem are to be interpreted in the following way: If g_0 is the canonical Euclidean metric on $\mathbf{R}^2$, then we assume that $g - g_0$ has compact support. We do not assert, and it is not true, that a Riemannian metric without conjugate points on $\mathbf{R}^2$ which is *locally* Euclidean outside a compact set must be flat. For an example, one may smooth out the vertex of a

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"négacône" (see J.-P. Petit [6, p. 10]). An explicit C^∞ example is $g = dr^2 + h(r)^2\,d\theta^2$ in polar coordinates, with $h(r) = 2r - \frac{1}{2} + \frac{1}{2}\exp(2r/(r-1))$ for $0 \leqslant r \leqslant 1$ and $h(r) = 2r - \frac{1}{2}$ for $r \geqslant 1$. Such examples in two dimensions are extraordinarily general: given an arbitrary nonpositive smooth even function $K(r)$, there is a radially symmetric Riemannian metric in the plane with Gauss curvature $K(r)$ at all points on the circle of radius r (see [2, Proposition 4.2A]).

It should be remarked that we have made no hypotheses on sign or magnitude of the Gaussian curvature of g; if $K \leqslant 0$, for example, then the Theorem follows immediately from the Gauss-Bonnet formula applied to a triangle containing the support of $g - g_0$ in its interior.

2. Let K be the Gaussian curvature of the metric g and Ω the compact support of $g - g_0$. $(\mathbf{R}^2, g)$ is clearly complete, and for any geodesic γ, each component of the intersection of γ with $\mathbf{R}^2 - \Omega$ is a piece of a Euclidean straight line. Call each such straight piece of γ a straight component; γ is either a complete Euclidean straight line or among its straight components precisely two are unbounded.

Lemma 1. *Any unbounded straight components of a geodesic in* $(\mathbf{R}^2, g)$ *are parallel.*

Proof. The simple connectivity of $\mathbf{R}^2$ and our hypothesis of no conjugate points imply that $(\mathbf{R}^2, g)$ is a "straight space", in the sense of Busemann [1]; namely, that any two points can be joined by a unique geodesic. Let r_1 and r_2 be the unbounded straight components of the geodesic γ in question. (We need ony examine the case when $r_1 \neq r_2$.) If r_1 and r_2 are not parallel, then they contain points p_i on r_i, $i = 1, 2$, which can be joined by an ordinary Euclidean segment lying entirely in $\mathbf{R}^2 - \Omega$, contradicting the uniqueness of the geodesic connecting them. q.e.d.

Now let γ be a geodesic which intersects Ω, with $\gamma((-\infty, s_1])$, $\gamma(\,s_2\,, \infty))$ its unbounded straight components. Any perpendicular Jacobi field along γ is characterized by a single solution of the differential equation

$$y''(s) + K(s)y(s) = 0, \tag{J}$$

where $K(s)$ stands for the Gauss curvature of g at $\gamma(s)$. Since $K(s) = 0$ for $s \leqslant s_1$ or $s \geqslant s_2$, $y(s)$ is linear in these intervals.

Lemma 2. *Let* $y(s) = as + b$ *for* $s \leqslant s_1$ *and* $cs + d$ *for* $s \geqslant s_2$. *Then* $a = c$.

Proof. This is an immediate corollary of Lemma 1, using the fact that any Jacobi field can be produced by a one-parameter variation of γ through geodesics.

We may now turn to the proof of the Theorem. Choose a large rectangular region D which contains Ω in its interior. Then D is geodesically convex in the $(\mathbf{R}^2, g)$ geometry. Define a metric $\bar{g}$ on $\mathbf{R}^2$ by tiling the plane with copies of D; $(\mathbf{R}^2, \bar{g})$ is then the Riemannian covering surface of the torus T obtained by identifying opposite edges of D. If we prove that $(\mathbf{R}^2, \bar{g})$ has no conjugate points, Hopf's theorem [3] will then apply to T, so that $\bar{g}$, and therefore g, have Gauss curvature zero.

Let $\bar{\gamma}$ be a geodesic of $(\mathbf{R}^2, \bar{g})$. We may assume that $\bar{\gamma}$ does not coincide with any of the straight lines in the grid created by translates of the boundary of D. Hence there exists an increasing sequence $\{s_i\}$, with $\lim_{i \to \pm\infty} s_i = \pm\infty$, such that $\bar{\gamma}([s_i, s_{i+1}])$ lies entirely in the closure of a single translate of D, say D_i. Note that $\bar{\gamma}|_{[s_i, s_{i+1}]}$ is part of a geodesic for the translated metric $(\mathbf{R}^2, g_i)$, defined in the obvious way to be an image of $(\mathbf{R}^2, g)$. The Jacobi equation along $\bar{\gamma}$ is

$$(\bar{\mathrm{J}}) \qquad y''(s) + \bar{K}(s)y(s) = 0,$$

and it has the following properties:

(i) $\bar{K}(s)$ is zero in a neighborhood of each s_i.

(ii) Set $K_i(s) = \bar{K}(s)$ for $s \in [s_i, s_{i+1}]$, zero elsewhere. No solution of

$$(\mathrm{J}_i) \qquad w''(s) + K_i(s)w(s) = 0$$

has more than one zero (for $(\mathbf{R}^2, g_i)$ is isometric to $(\mathbf{R}^2, g)$ and therefore has no conjugate points).

(iii) If y is a solution of $(\bar{\mathrm{J}})$, then $y'(s_i)$ is constant as i ranges over the integers (for Lemma 2 may be applied to each of the equations (J_i) in turn).

We can now prove that $\bar{\gamma}$ has no conjugate points. Otherwise there is a nontrivial solution y of $(\bar{\mathrm{J}})$ with two successive zeros, say $y(t) = y(t') = 0$. Observe that t and t' cannot both lie in one interval $[s_i, s_{i+1}]$, by property (ii). Let $t \in [s_i, s_{i+1})$, $t' \in (s_j, s_{j+1}]$, $i < j$, and set $a = y'(s_i)$. Without loss of generality, $a \geqslant 0$, and by (iii), $a = y'(s_{i+1}) = y'(s_j) = y'(s_{j+1})$.

Suppose $a > 0$. Then $y(s_{i+1}) > 0$. In fact, the restriction of y to $[s_i, s_{i+1}]$ may be extended to a solution w of (J_i) by defining

$$w(s) = y(s_i) + a(s - s_i), \qquad s \leqslant s_i,$$
$$w(s) = y(s_{i+1}) + a(s - s_{i+1}), \qquad s_{i+1} \leqslant s.$$

If $y(s_{i+1}) < 0$, then w has two zeros, at t and at $s = s_{i+1} - y(s_{i+1})/a$, contradicting (ii). If $y(s_{i+1}) = 0$, then $w(t) = w(s_{i+1}) = 0$ again contradicts (ii). This shows that $y(s_{i+1}) > 0$. By the analogous argument, we see that $y(s_j) < 0$. But y has no zeros between t and t', and in particular none in $[s_{i+1}, s_j]$, a contradiction.

Hence we need only eliminate the possibility that $a = 0$. In this case, the uniqueness of solutions of ($\bar{\mathrm{J}}$) forces $s_i < t < s_{i+1}$, and $\pm y|_{[s_i, s_{i+1}]}$ may be extended as before to a solution w of (J_i) with the properties

$$w(s) = \pm y(s_i) < 0 \quad \text{for } s \leqslant s_i,$$
$$w(s) = \pm y(s_{i+1}) > 0 \quad \text{for } s_{i+1} \leqslant s,$$

and $w(t) = 0$. By the continuous dependence of solutions on initial conditons, we may perturb w to a solution v of (J_i) such that

$$v(s_i) = w(s_i), \qquad v'(s_i) < 0$$

and $v(\tau) = 0$ for some τ in (t, s_{i+1}). In fact, $v'(s) = v'(s_i)$ for all $s \leqslant s_i$ and for all $s > s_{i+1}$. But then v has a zero to the left of s_i, contradicting the disconjugacy property (ii).

It now follows from Hopf's theorem that $(\mathbf{R}^2, g)$ has Gauss curvature identically zero. The conclusion of our theorem is a consequence of the well-known theorem of Cartan-Ambrose-Hicks, or of the elementary arguments in the Corollary below.

Remark. The reader can supply a proof in terms of intersecting geodesics, of which the above argument is an infinitesimal version.

Corollary. *Let g be a Riemannian metric on $\mathbf{R}^2$, without conjugate points, so that $g = g_0$ outside a compact set Ω. Given any connected component U of $\mathbf{R}^2 - \Omega$, there is a diffeomorphism φ of $\mathbf{R}^2$, equal to the identity on U, such that $\varphi^* g_0 = g$.*

Proof. Consider Riemannian normal coordinates (x, y) from a point $p \in U$. Since the Gauss curvature of g vanishes identically, the coordinates (x, y) are defined on all of $\mathbf{R}^2$, and $g = dx^2 + dy^2$ everywhere. For any point q in $\mathbf{R}^2$ with coordinates $(x(q), y(q))$, we may define $\varphi(q)$ to be the point which has coordinates $X = x(q)$, $Y = y(q)$ in terms of the corresponding Euclidean coordinate system (X, Y) centered at p. Then φ agrees with the identity mapping on a neighborhood of p. Now any other point $q \in U$ may be reached by a polygonal path Γ, starting at p and lying entirely in U. But φ preserves the angles and sidelengths of Γ, which implies that the Euclidean coordinates (X, Y) and the Riemannian normal coordinates (x, y) remain equal along Γ, and in particular, that $\varphi(q) = q$. This shows that φ equals the identity mapping on the connected component U.

3. The last part of the proof (the case $a = 0$) may be reformulated in terms of focal points. Its extension to dimension n is also true, namely, that for a metric on $\mathbf{R}^n$, whose geodesics have no conjugate points and which agrees with the Euclidean metric outside a compact connected set Ω, no geodesic orthogonal to a hyperplane disjoint from Ω has a point focal to that hyperplane. Lemma 1 and a version of Lemma 2 are also true in higher dimensions, but other arguments with Jacobi fields do not have obvious generalizations. And of course, E. Hopf's theorem is still open for a torus of dimension greater than two. Nonetheless, the generalization of the Theorem to higher dimensions remains plausible and forms an interesting conjecture.

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