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INVARIANT POLYNOMIALS OF THE AUTOMORPHISM GROUP OF A COMPACT COMPLEX MANIFOLD

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1. Introduction

Let M be a compact complex manifold of dimension n, $H(M)$ the complex Lie group of all automorphisms of M, and $h(M)$ the complex Lie algebra of all holomorphic vector fields of M. When $c_1(M)$ is positive, the first author defined in [13] a character f: $h(M) \to \mathbf{C}$ which is intrinsically defined, vanishes if M admits a Kähler-Einstein metric, and has its origin in Kazdan-Warner's integrability condition for Nirenberg's problem [16].

In this note we give a better understanding of f along the lines of the classical works by Bott and the recent works in symplectic geometry by Duistermaat-Heckman [12], Berline-Vergne [3, 4], and Atiyah-Bott [1]. We begin by rephrasing Theorem 2.18 of Berline-Vergne [3] in the following way; there exists a linear map F: $I^{n+k}(\mathrm{GL}(n,\mathbf{C})) \to I^k(H(M))$ where, for a complex Lie group G, $I^p(G)$ denotes the set of all holomorphic G-invariant symmetric polynomials of degree p. The character f coincides with $F(c_1^{n+1})$ up to a constant. By a proof identical to Bott [5, 6] we have a localization formula of the elements of the image of F. The main result of this note is to show explicitly that the linear map F corresponds to the Gysin map in the context of equivariant cohomology (Theorem 4.1 and Corollary 4.2).

We also give another interpretation of f in terms of secondary characteristic classes of Chern-Simons [11] and Cheeger-Simons [10]. More precisely we find that f appears as the so-called Godbillon-Vey invariant of certain complex foliations which are defined naturally.

The linear map F, which depends only on the complex structure of M, may be regarded as a generalization of f. There is another type of generalization of f ([14], [9], [2]) which depends on a fixed Kähler class. We think that this latter one also deserves further study.

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2. Definition of F

Let P be a complex analytic fiber bundle over M with the right action of a complex Lie group G. Suppose $H(M)$ acts on P from the left complex analytically and commuting with the action of G. Let θ be any type $(1,0)$ connection and Θ the curvature form of θ. Since $H(M)$ acts on P, $X \in h(M)$ defines a vector field on P, which we shall denote by the same letter X. Then since $\phi(-\theta(X) + \frac{i}{2\pi}\Theta)$, $\phi \in I^{n+k}(G)$, is horizontal and G-invariant it projects to a form on M. We define

$$f_\phi(X) = \int_M \phi\left(-\theta(X) + \frac{i}{2\pi}\Theta\right).$$

The following is a complex version of Theorem 2.18 of [3] and is proved similarly.

Proposition 2.1. *The definition of f_ϕ does not depend on the choice of the type $(1,0)$ connection θ. Furthermore f_ϕ is invariant under the coadjoint action of $H(M)$. So we obtain a linear map F: $I^{n+k}(G) \to I^k(H(M))$.*

Let θ be a type $(1,0)$ connection of the holomorphic tangent bundle of M which is associated by the frame bundle of M. Let D be the covariant differentiation and put $L(X) = L_X - D_X$ for $X \in h(M)$ where L_X is the Lie differentiation by X. For $\phi \in I^{n+k}(\mathrm{GL}(n, \mathbf{C}))$ we define

$$f_\phi(X) = \int_M \phi\left(-L(X) + \frac{i}{2\pi}\Theta\right).$$

From Proposition 2.1 together with Lemma 1.10 in [4] we obtain the same conclusion for the new f_ϕ with $G = \mathrm{GL}(n, \mathbf{C})$. This conclusion also follows from Bott's localization theorem. We say that X is nondegenerate if zeros of X are isolated and if at each zero p the linear map $L(X)_p$: $T_pM \to T_pM$ is nondegenerate.

Proposition 2.2 (*Bott* [5]). *If X is nondegenerate then*

$$f_\phi(X) = \sum_p \phi(L(X)_p)/\det L(X)_p.$$

Now we assume that $c_1(M)$ is positive. We put $c_1^+(M)$ to be the set of all positive $(1,1)$ forms representing $c_1(M)$. Choose any $\omega \in c_1^+(M)$ which is regarded as a Kähler form. Denote by γ_ω the Ricci form which also represents $c_1(M)$. Since $\gamma_\omega - \omega$ is a real exact $(1,1)$ form there exists a real smooth function F_ω, uniquely determined up to an additive constant, such that $\gamma_\omega - \omega = \frac{1}{2}(i/\pi)\partial\bar{\partial}F_\omega$. By definition ω is Kähler-Einstein iff F_ω is constant.

We define a linear function $f\colon h(M) \to \mathbf{C}$ by

$$f(X) = \int_M XF_\omega \omega^n.$$

In [13] we proved that the definition of f does not depend on the choice of $\omega \in c_1^+(M)$.

Proposition 2.3. $f_{c_1^{n+1}} = (n+1)f$.

Proof. By the Calabi-Yau theorem [18] there exists a unique Kähler form $\eta \in c_1^+(M)$ such that $\gamma_\eta = \omega$. Therefore we may assume $F_\omega = -\log(\omega^n/\eta^n)$. It then follows from the divergence theorem with respect to the Kähler form η that

$$\begin{aligned} f(X) &= -\int_M X\log(\omega^n/\eta^n)\,\omega^n = -\int_M X(\gamma_\eta^n/\eta^n)\,\eta^n \\ &= -\int_M \delta'' X \gamma_\eta^n \\ &= \int_M \operatorname{trace}(DX)\left(\operatorname{trace}\left(\frac{i}{2\pi}\Theta\right)\right)^n \end{aligned}$$

where D and Θ is the covariant differentiation and the curvature form with respect to η and trace(DX) makes sense because DX is a section of $T^*M \otimes TM \simeq \operatorname{End}(TM)$. Since η is Kähler we have $L(X) = -DX$. This proves the proposition.

3. Secondary characteristic classes of complex foliations

First we review some known facts about complex foliations. Let W be a smooth manifold. A complex foliation $\mathscr{F}$ of codimension q is an open covering $\{U_\alpha\}$ of W such that

(1) there exist submersions $\gamma_\alpha\colon U_\alpha \to V_\alpha \subset \mathbf{C}^q$,

(2) on $U_\alpha \cap U_\beta \neq \varnothing$, $\gamma_{\alpha\beta} = \gamma_\alpha \circ \gamma_\beta^{-1}\colon \gamma_\beta(U_\alpha \cap U_\beta) \to \gamma_\alpha(U_\alpha \cap U_\beta)$ is a complex analytic diffeomorphism.

We may choose local coordinates $(t_\alpha^1,\cdots,t_\alpha^p, x_\alpha^1, y_\alpha^1,\cdots,x_\alpha^q, y_\alpha^q)$ on U_α so that $\gamma_\alpha(t_\alpha, x_\alpha, y_\alpha) = (z_\alpha^1,\cdots,z_\alpha^q)$ where $z_\alpha^i = x_\alpha^i + iy_\alpha^i$. It is easy to observe that the covectors dz_α^i, $i = 1,\cdots,q$, span a well defined subbundle of $T^*W \otimes \mathbf{C}$ which we will denote by $T^*W^{1,0}$. We denote by $TW^{0,1}$ the subbundle of $TW \otimes \mathbf{C}$ spanned by vectors annihilated by the covectors in $T^*W^{1,0}$. Clearly $TW^{0,1}$ is spanned by vectors $\partial/\partial t_\alpha^i$ and $\partial/\partial \bar{z}_\alpha^i$. The quotient bundle $\nu(\mathscr{F}) = TW \otimes \mathbf{C}/TW^{0,1}$ is called the normal bundle of $\mathscr{F}$. A connection ∇ of $\nu(\mathscr{F})$ is called a Bott connection if for $X \in TW^{0,1}$ and $Y \in C^\infty(\nu(\mathscr{F}))$ we have

$$\nabla_X Y = \pi[\tilde{X}, \tilde{Y}] \tag{3.1}$$

where $\tilde{X} \in C^\infty(TW^{0,1})$ is an arbitrary extension of X, $\tilde{Y} \in C^\infty(TW \otimes \mathbf{C})$ is an arbitrary lift of Y, and π: $TW \otimes \mathbf{C} \to \nu(\mathscr{F})$ is the projection. It is easy to check that this definition is well defined. Roughly speaking a Bott connection is a type $(1,0)$ connection. So by the type reasons:

Theorem (*Bott* [7]). *Let ∇ be a Bott connection and Θ the curvature form of ∇. Then $\phi(\Theta) = 0$ for $\phi \in I^j(\mathrm{GL}(n, \mathbf{C})), j > q$.*

Now we put $I_0^j(\mathrm{GL}(n,\mathbf{C})) = I^j(\mathrm{GL}(n,\mathbf{C})) \cap \mathbf{Z}[c_1, \cdots, c_n]$. By the argument of Cheeger-Simons [10] we can define a class $S_\phi(\mathscr{F}, \nabla) \in H^{2j-1}(W; \mathbf{C}/\mathbf{Z})$ for $\phi \in I_0^j(\mathrm{GL}(n,\mathbf{C}))$, $j > q$. When $j = q+1$, it is known that $S_\phi(\mathscr{F}, \nabla)$ is independent of the choice of the Bott connection; so we shall write it $S_\phi(\mathscr{F})$. And $S_{c_1^{n+1}}(\mathscr{F})$ is known as the Godbillon-Vey class.

Let M be a compact complex manifold and $W = M \times S^1$ where $S^1 = \mathbf{R}/\mathbf{Z}$. Consider a vector field $Y = \partial/\partial t + 2\,\mathrm{Re}(X)$ on W where $\mathrm{Re}(X)$ is the real part of $X \in h(M)$ and t is the coordinate of S^1. Then the flow generated by Y defines a complex foliation $\mathscr{F}$ of codimension n.

Theorem 3.1. *For any $\phi \in I_0^{n+1}(\mathrm{GL}(n,\mathbf{C}))$ we have*

$$S_\phi(\mathscr{F})[W] = \frac{i}{2\pi} f_\phi(X) \bmod \mathbf{Z}.$$

Proof. We denote by $\mathscr{F}_\lambda$ the foliation obtained by replacing Y by $Y_\lambda = \partial/\partial t + 2\,\mathrm{Re}(\lambda X)$ for any $\lambda \in \mathbf{R}$. Then we have $\nu(\mathscr{F}_\lambda) \simeq \pi^* TM$ where π: $M \times S^1 \to M$ and TM is the holomorphic tangent bundle of M. Let h be any Hermitian metric of TM and D its connection. We define a Bott connection ∇^λ by

$$\nabla^\lambda_{\partial/\partial \bar{z}^i} \frac{\partial}{\partial z^j} = (\pi^* D)_{\partial/\partial z^i} \frac{\partial}{\partial z^j}, \qquad \nabla^\lambda_{\partial/\partial \bar{z}^i} \frac{\partial}{\partial z^j} = 0, \tag{3.2}$$

$$\nabla^\lambda_{Y_\lambda} \frac{\partial}{\partial z^j} = \pi_*\left[Y_\lambda, \frac{\partial}{\partial z^j}\right] = \lambda\left[X, \frac{\partial}{\partial z^j}\right].$$

Then from (3.2) we have

$$\nabla^\lambda_{\partial/\partial t} \frac{\partial}{\partial z^j} = \nabla^\lambda_{Y_\lambda - \lambda X} \frac{\partial}{\partial z^j} = \lambda L(X) \frac{\partial}{\partial z^j}. \tag{3.3}$$

Denoting by θ^λ and θ the connection forms of ∇^λ and D

$$\frac{d}{d\lambda}\theta^\lambda = \frac{d}{d\lambda}\left(\pi^*\theta + \lambda L(X)\, dt\right) = L(X)\, dt. \tag{3.4}$$

Moreover the curvature form Θ^λ of θ^λ is computed as

$$\Theta^\lambda = d\theta^\lambda + \theta^\lambda \wedge \theta^\lambda = \pi^*\Theta^\lambda \bmod dt. \tag{3.5}$$

It follows from (3.4), (3.5), and Proposition 2.9 in [10] that

$$\begin{aligned}\frac{d}{d\lambda}S_\phi(\mathscr{F}_\lambda)[W] &= (n+1)\int_W \phi\left(\frac{i}{2\pi}\frac{d}{d\lambda}\theta^\lambda, \frac{i}{2\pi}\Theta^\lambda, \cdots, \frac{i}{2\pi}\Theta^\lambda\right) \bmod \mathbf{Z}\\ &= \frac{(n+1)i}{2\pi}\int_M \phi\left(L(X), \frac{i}{2\pi}\Theta, \cdots, \frac{i}{2\pi}\Theta\right) \bmod \mathbf{Z} \qquad (3.6)\\ &= \frac{1}{2\pi i}f_\phi(X) \bmod \mathbf{Z}.\end{aligned}$$

Since the right-hand side does not depend on λ we obtain Theorem 3.2 by integrating the both sides of (3.6) over $[0,1]$ with respect to λ.

4. Relation to equivariant cohomology

For brevity we shall write H for $H(M)$. Let $EH \to BH$ be the universal H-bundle. We put $MH = EH \times_H M$. Let P be as in §2. Then $PH = EH \times_H P$ is a principal G-bundle over MH.

Theorem 4.1. *The following diagram commutes*:

$$\begin{array}{ccc} I^{n+k}(G) & \xrightarrow{\Phi = (i/2\pi)^k F} & I^k(H) \\ \big\downarrow W & & \big\downarrow W \\ H^{2n+2k}(MH) & \xrightarrow{\pi_*} & H^{2k}(BH) \end{array}$$

where two W's are Weil maps corresponding to $PH \to MH$ and $EH \to BH$, and π_* is the Gysin map of π: $MH \to BH$.

Proof. We may prove it for a principal H-bundle E over a finite-dimensional base space B instead of $EH \to BH$. Let κ be a connection form of $E \to B$ and V the horizontal distribution. Let $X_\#$ be a right invariant horizontal (local) vector field of E.

Lemma 4.2. *$X_\#$ defines a well-defined vector field X on PH. In particular V defines a distribution V' in $T(PH)$ whose dimension is equal to* dim B.

Proof. Let ξ_t be the flow generated by $X_\#$. Then by the right invariance of $X_\#$ we have $\xi_t(eh) = \xi_t(e)h$ for any $e \in P$ and $h \in H$. We put $X = (d/dt)(\xi_t(e), p) \in T(PH), p \in P$. This is well defined because

$$\left(\xi_t(eh), h^{-1}p\right) = \left(\xi_t(e)h, h^{-1}p\right) = \left(\xi_t(e), p\right). \qquad \text{q.e.d.}$$

Let π_2: $PH \to B$ be the projection and $T(\pi_2)$ the vector bundle consisting of all vectors tangent to the fibers of π_2. Then clearly $T(PH) = T(\pi_2) \oplus V'$. Let κ': $T(PH) \to T(\pi_2)$ be the projection defined by this splitting. On the other hand PH is considered as a differentiable family of complex analytic principal bundle over B. We may choose a differentiable family $\tilde{\theta}$ of type $(1,0)$ connections on PH. So $\tilde{\theta}$ is just defined on each fibers and depends smoothly on the base space B. We define a connection ψ of the G-bundle $PH \to MH$ by $\psi = \tilde{\theta} \circ \kappa'$. Let K, $\tilde{\Theta}$, and Ψ be the curvature forms of κ, $\tilde{\theta}$, and ψ respectively. Let $\lambda\colon E \times P \to PH$ be the projection and $\lambda(e, p) = q$. Clearly $d\lambda(V \oplus 0) = V'$.

Lemma 4.3. *For* $X, Y \in T_q(\pi_2)$, $\Psi(X, Y) = \tilde{\Theta}(X, Y)$.

Lemma 4.4. *For* $X, Y \in V_q'$, $\Psi(X, Y) = \tilde{\theta}(d\lambda_{(e,p)}K(X_\sharp, Y_\sharp)_*)$ *where, for* $X \in h(M)$, X_* *denotes the basic vector field of* E.

Lemma 4.5. *For* $X \in T_q(\pi_2)$ *of type* $(0,1)$ *and* $Y \in V'$, $\Psi(X, Y) = 0$.

Lemma 4.3 follows immediately from $\Psi = d\psi + \frac{1}{2}[\psi, \psi]$. For $X = d\lambda(X_\sharp)$ and $Y = d\lambda(Y_\sharp)$, we also have $\Psi(X, Y) = -\tilde{\theta}(\kappa'[X, Y])$. On the other hand since

$$\kappa\big([X_\sharp, Y_\sharp] - (\kappa[X_\sharp, Y_\sharp])_*\big) = 0 \quad \text{and} \quad K(X_\sharp, Y_\sharp) = -\kappa[X_\sharp, Y_\sharp]$$

we have $\kappa'(d\lambda([X_\sharp, Y_\sharp] + K(X_\sharp, Y_\sharp)_*)) = 0$. Hence

$$\begin{aligned}\Psi(X, Y) &= -\tilde{\theta}(\kappa'[X, Y]) = -\tilde{\theta}\big(\kappa'\big(d\lambda\big[X_\sharp, Y_\sharp\big]\big)\big) \\ &= \tilde{\theta}\big(\kappa' d\lambda\big(K(X_\sharp, Y_\sharp)_*\big)\big) = \tilde{\theta}\big(d\lambda\big(K(X_\sharp, Y_\sharp)_*\big)\big).\end{aligned}$$

This proves Lemma 4.4.

We now assume that X is a section of $T(\pi_2)$ of type $(0,1)$ and that $Y = d\lambda(Y_\sharp)$. Let ξ_t be the flow generated by $Y_\sharp$. We consider a trivialization $U \times H$ of E on an open set $U \subset B$. We may write $\xi_t(b, h) = (\xi_t^1(b), \xi_t^2(h))$ for $b \in U$ and $h \in H$. Putting $\xi_t^2(1) = \rho_t$, by the right invariance of ξ_t we have $\xi_t^2(h) = \rho_t h$. Let $U \times P$ be a trivialization of $PH \times B$. Then $\lambda\colon E \times P \to PH$ is given over U by $\lambda\colon U \times H \times P \to U \times P$, $\lambda(b, h, p) = (b, hp)$. Therefore the flow η_t generated by $Y = d\lambda(Y_\sharp)$ is expressed by $\eta_t(b, p) = (\xi_t^1(b), \rho_t p)$. Since ρ_t is an automorphism of P, $\eta_{t*}(X)$ is also a section of $T(\pi_2)$ of type $(0,1)$. Therefore

$$[X, Y]_q = \lim_{t \to 0} t^{-1}\big((\eta_{t*}Y)_q - Y_q\big)$$

is also type $(0,1)$. Then we obtain Lemma 4.5 from

$$\Psi(X, Y) = -\tilde{\theta}\big(\kappa'[X, Y]\big) = -\tilde{\theta}([X, Y]) = 0.$$

Returning to the proof of Theorem 4.1, the curvature form Ψ restricted to a fiber does not have type $(2,0)$ part. This fact together with Lemma 4.5 shows that only the $(1,1)$ part of $\tilde{\Theta}$ contributes to the integration over the fiber of $\phi(\Psi)$, $\phi \in I^{n+k}(G)$. Thus we obtain from Lemma 4.3, Lemma 4.4 and

Proposition 2.1 that

$$\pi_*\phi(\Psi) = \binom{n+k}{k}\int_M \phi\left(\frac{i}{2\pi}\theta(K),\cdots,\frac{i}{2\pi}\theta(K),\frac{i}{2\pi}\Theta,\cdots,\frac{i}{2\pi}\Theta\right)$$

$$= \left(\frac{i}{2\pi}\right)^k f_\phi(K).$$

This proves Theorem 4.1.

Now let H^δ be the same group as H but equipped with the discrete topology. As before let $EH^\delta \to BH^\delta$ be the universal H^δ-bundle and put $MH^\delta = EH^\delta \times_{H^\delta} M$. The structure group of the bundle $M \to MH^\delta \to BH^\delta$ is the discrete group H^δ which acts on M holomorphically. Hence MH^δ admits a complex foliation $\mathscr{F}_M$ of codimension n whose leaves are transverse to the fibers. The normal bundle $\nu(\mathscr{F}_M)$ of $\mathscr{F}_M$ is naturally isomorphic to the subbundle of $T(MH^\delta)$ consisting of vectors which are tangent to the fibers. We can define a homomorphism S: $I_0^{n+k}(\mathrm{GL}(n,\mathbf{C})) \to H^{2n+2k-1}(MH^\delta; \mathbf{C}/\mathbf{Z})$ as follows. For an element $\phi \in I_0^{n+k}(\mathrm{GL}(n,\mathbf{C}))$, $S(\phi) \in H^{2n+2k-1}(MH^\delta$: $\mathbf{C}/\mathbf{Z})$ is the Simons class [10] defined by applying the Bott vanishing theorem to $\nu(\mathscr{F}_M)$. On the other hand consider the element $\Phi_0(\phi) \in I^k(H(M))$, where Φ_0: $I_0^{n+k}(\mathrm{GL}(n,\mathbf{C})) \to I^k(H(M))$ is the restriction of Φ. By Theorem 4.1 the cohomology class $W\Phi_0(\phi)$ is equal to $\pi_*W(\phi)$. Hence it is the reduction of the integral cohomology class $\pi'_*W(\phi) \in H^{2k}(BH;\ \mathbf{Z})$, where π'_*: $H^{2n+2k}(MH;\ \mathbf{Z}) \to H^{2k}(BH;\ \mathbf{Z})$ is the Gysin map. Now $EH^\delta \to BH^\delta$ is a flat H-bundle so that $W\Phi_0(\phi) = 0$ in $H^{2k}(BH^\delta;\ \mathbf{C})$. Hence we can define the Simons class $S_{\Phi_0(\phi),\,\pi'_*W(\phi)} \in H^{2k-1}(BH^\delta;\ \mathbf{C}/\mathbf{Z})$. The above procedure defines a homomorphism μ: Image $\phi_0 \to H^{2k-1}(BH^\delta;\ \mathbf{C}/\mathbf{Z})$ and we have

Corollary 4.6. *The following diagram commutes*:

$$\begin{array}{ccc} I_0^{n+k}(\mathrm{GL}(n,\mathbf{C})) & \xrightarrow{\Phi_0} & \text{Image}\,\Phi_0 \\ \downarrow S & & \downarrow \mu \\ H^{2n+2k-1}(MH^\delta;\ \mathbf{C}/\mathbf{Z}) & \xrightarrow{\pi_*} & H^{2k-1}(BH^\delta;\ \mathbf{C}/\mathbf{Z}). \end{array}$$

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